GENERALIZED HAMMING WEIGHTS OF PROJECTIVE REED–MULLER-TYPE CODES OVER GRAPHS

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Abstract. Let G be a connected graph and let $\mathbb{X}$ be the set of projective points defined by the column vectors of the incidence matrix of G over a field K of any characteristic. We determine the generalized Hamming weights of the Reed–Muller-type code over the set $\mathbb{X}$ in terms of graph theoretic invariants. As an application to coding theory we show that if G is non-bipartite and K is a finite field of $\mathrm{char}(K)\neq 2$, then the r-th generalized Hamming weight of the linear code generated by the rows of the incidence matrix of G is the r-th weak edge biparticity of G. If $\mathrm{char}(K)=2$ or G is bipartite, we prove that the r-th generalized Hamming weight of that code is the r-th edge connectivity of G.

1. Introduction

In this work we study basic parameters of projective Reed–Muller-type codes over graphs using an algebraic geometric approach via graph theory and commutative algebra, and give some applications to linear codes whose generator matrices are incidence matrices of graphs.

Let K be a field of characteristic $p\geq 0$, let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $t_1,\ldots,t_s$ and $f_1,\ldots,f_m$ be the vertices and edges of G, respectively. The *incidence matrix* of G, over the field K, is the $s\times m$ matrix $A=(a_{ij})$ given by $a_{ij}=1$ if $t_i\in f_j$ and $a_{ij}=0$ otherwise. The *edge biparticity* of G, denoted $\varphi(G)$, is the minimum number of edges whose removal makes the graph bipartite, and maybe not connected. The r-th *weak edge biparticity* of G, denoted $\upsilon_r(G)$, is the minimum number of edges whose removal results in a graph with r bipartite components, and maybe some non-bipartite components. If $r=1$, $\upsilon_1(G)$ is the *weak edge biparticity* of G and is denoted by $\upsilon(G)$. The r-th *edge connectivity* of G, denoted $\lambda_r(G)$, is the minimum number of edges whose removal results in a graph with $r+1$ connected components. If $r=1$, $\lambda_1(G)$ is the edge connectivity of G and is denoted by $\lambda(G)$. We will use these invariants to study the minimum distance and the Hamming weights of Reed-Muller-type codes over graphs.

The edge biparticity and the edge connectivity are well studied invariants of a graph [16, 34]. In Section 2 we give an algebraic method for computing the edge biparticity (Proposition 2.3). For a discussion of computational and algorithmic aspects of edge bipartization problems we refer to [26].

The set of columns $\{P_1,\ldots,P_m\}$ of A can be regarded as a set of points $\mathbb{X}=\{[P_1],\ldots,[P_m]\}$ in a projective space $\mathbb{P}^{s-1}$ over the field K. Consider a polynomial ring $S=K[t_1,\ldots,t_s]=\bigoplus_{d=0}^{\infty}S_d$ over the field K with the standard grading. The *vanishing ideal* $I(\mathbb{X})$ of $\mathbb{X}$ is the graded ideal of S generated by the homogeneous polynomials of S that vanish at all points of $\mathbb{X}$. Fix integers

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$d \geq 1$ and $r \geq 1$. The aim of this work is to determine the following number in terms of the combinatorics of the graph G:

$$\delta_{\mathbb{X}}(d,r) := \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(F)| : F = \{f_i\}_{i=1}^r \subset S_d,\, \dim_K(\{\overline{f}_i\}_{i=1}^r) = r\},$$

where $V_{\mathbb{X}}(F)$ is the set of zeros or projective variety of F in $\mathbb{X}$, and $\overline{f}_i = f_i + I(\mathbb{X})$ is the class of f_i modulo $I(\mathbb{X})$. This is equivalent to determine

$$\text{hyp}_{\mathbb{X}}(d,r) := \max\{|V_{\mathbb{X}}(F)| : F = \{f_i\}_{i=1}^r \subset S_d,\, \dim_K(\{\overline{f}_i\}_{i=1}^r) = r\}$$

because $\delta_{\mathbb{X}}(d,r) = |\mathbb{X}| - \text{hyp}_{\mathbb{X}}(d,r)$.

A *projective Reed–Muller-type code* of degree d on $\mathbb{X}$ [8, 13], denoted $C_{\mathbb{X}}(d)$, is the image of the following evaluation linear map

$$\text{ev}_d\colon S_d \to K^m, \quad f \mapsto (f(P_1), \ldots, f(P_m)).$$

The motivation to study $\delta_{\mathbb{X}}(d,r)$ comes from algebraic coding theory because—over a finite field—the r-th generalized Hamming weight of the Reed–Muller-type code $C_{\mathbb{X}}(d)$ of degree d is equal to $\delta_{\mathbb{X}}(d,r)$ [11, Lemma 4.3(iii)].

Generalized Hamming weights were introduced by Helleseth, Kløve and Mykkeltveit [18, 22] and were first used systematically by Wei [31]. For convenience we recall this notion. Let $K = \mathbb{F}_q$ be a finite field and let C be a $[m, k]$ *linear code* of *length* m and *dimension* k, that is, C is a linear subspace of K^m with $k = \dim_K(C)$. Let $1 \leq r \leq k$ be an integer. Given a linear subspace D of C, the *support* of D is the set

$$\chi(D) := \{i \,|\, \exists\, (a_1, \ldots, a_m) \in D,\, a_i \neq 0\}.$$

The r-th *generalized Hamming weight* of C, denoted $\delta_r(C)$, is given by

$$\delta_r(C) := \min\{|\chi(D)|\colon D \text{ is a subspace of } C,\, \dim_K(D) = r\}.$$

The set $\{\delta_1(C), \ldots, \delta_k(C)\}$ is called the *weight hierarchy* of the code C. The following *duality* of Wei [31, Theorem 3] is a classical result in this area that shows a strong relationship between the weight hierarchies of C and its dual $C^\perp$:

$$\{\delta_i(C) \,|\, i = 1, \ldots, k\} = \{1, \ldots, m\} \setminus \{m + 1 - \delta_i(C^\perp) \,|\, i = 1, \ldots, m-k\}.$$

These numbers are a natural generalization of the notion of minimum distance and they have several applications from cryptography (codes for wire–tap channels of type II), t–resilient functions, trellis or branch complexity of linear codes, and shortening or puncturing structure of codes; see [1, 4, 6, 10, 11, 12, 17, 20, 25, 28, 29, 31, 32, 33] and the references therein. If $r = 1$, we obtain the *minimum distance* $\delta(C)$ of C which is the most important parameter of a linear code. In this paper we give combinatorial formulas for the weight hierarchy of $C_{\mathbb{X}}(d)$ for $d \geq 1$.

Our main results are:

Theorems 2.10, 2.11, 2.12 *Let G be a connected graph with s vertices, m edges, r-th weak edge biparticity $\upsilon_r(G)$, r-th edge connectivity $\lambda_r(G)$, and let A be the incidence matrix of G over a field K of characteristic p. If $\mathbb{X}$ is the set of column vectors of A, then*

$$\delta_{\mathbb{X}}(d,r) = \delta_r(C_{\mathbb{X}}(d)) = \begin{cases} \upsilon_r(G) & \text{if } d = 1,\ p \neq 2,\ G \text{ is non-bipartite}, 1 \leq r \leq s, \\ \lambda_r(G) & \text{if } d = 1,\ p = 2,\ 1 \leq r \leq s-1, \\ \lambda_r(G) & \text{if } d = 1,\ G \text{ is bipartite}, 1 \leq r \leq s-1, \\ r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m. \end{cases}$$

Thus computing $\upsilon_r(G)$ and $\lambda_r(G)$ is equivalent to computing the r-th generalized Hamming weight of $C_{\mathbb{X}}(1)$ for $K = \mathbb{F}_2$ or $K = \mathbb{F}_3$. These are the only cases that matter.

The *incidence matrix code* of a graph G over a finite field K of characteristic p, denoted $C_p(G)$, is the linear code generated by the rows of the incidence matrix of G. As an application to coding theory we obtain the following combinatorial formulas for the generalized Hamming weights of $C_p(G)$ when G is connected (Corollary 2.13).

$$\delta_r(C_p(G)) = \begin{cases} \upsilon_r(G) & \text{if } p \neq 2,\ G \textit{ is non-bipartite}, 1 \leq r \leq s, \\ \lambda_r(G) & \text{if } p = 2,\ 1 \leq r \leq s-1, \\ \lambda_r(G) & \text{if } G \textit{ is bipartite}, 1 \leq r \leq s-1. \end{cases}$$

The *minimum distance* of the incidence matrix code of the graph G is defined as

$$\delta(C_p(G)) := \min\{\omega(a)\colon a \in C_p(G) \setminus \{0\}\},$$

where $\omega(a)$ is the Hamming weight of the vector a, that is, the number of non-zero entries of a. The minimum distance of $C_p(G)$ is $\delta_1(C_p(G))$, the 1st Hamming weight of this code. Then we can recover the combinatorial formulas of Dankelmann, Key and Rodrigues [5, Theorems 1–3] for the minimum distance of $C_p(G)$ in terms of the weak edge biparticity $\upsilon(G)$ and the edge connectivity $\lambda(G)$ of G (Corollary 2.14).

Using Macaulay 2 [14], SageMath [27], and Wei's duality [31, Theorem 3], we can compute the weight hierarchy of $C_p(G)$. In Sections 3 and 4, we illustrate this with some examples and procedures. There are algebraic methods that can be used to obtain lower bounds for $\delta_r(C_p(G))$ or equivalently for $\lambda_r(G)$ and $\upsilon_r(G)$ [11, Theorem 4.9].

For all unexplained terminology and additional information we refer to [3, 7, 16] (for graph theory), [24, 29] (for the theory of error-correcting codes and linear codes), and [9, 23, 30] (for commutative algebra and Hilbert functions).

2. Reed–Muller-type codes over connected graphs

In this section we present our main results. To avoid repetitions, we continue to employ the notations and definitions used in Section 1.

Lemma 2.1. *Let G be a connected graph and let $e_1, \ldots, e_r$ be a minimum set of edges whose removal makes the graph bipartite. Then there is $\omega\colon V(G) \to \{+,-\}$ such that the edges of G whose vertices have the same sign are precisely $e_1, \ldots, e_r$.*

Proof. If G is bipartite, there is nothing to prove. If G is non-bipartite, pick a bipartition V_1, V_2 of the graph $G \setminus \{e_1, \ldots, e_r\}$. Setting $\omega(v) = +$ if $v \in V_1$ and $\omega(v) = -$ if $v \in V_2$, note that the vertices of each e_i have the same sign. Indeed if the vertices of e_i have different sign, then $G \setminus \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\}$ is bipartite, a contradiction. □

The edge biparticity of a graph G can be easily expressed by considering all possible ways of making G a vertex-signed graph.

Lemma 2.2. *Let G be a connected graph, let $\mathcal{F}$ be the set of surjective maps $\omega\colon V(G) \to \{+,-\}$, and let E_ω be the set of edges of G whose vertices have the same sign. Then*

$$\varphi(G) = \min\{|E_\omega| : \omega \in \mathcal{F}\}.$$

Proof. If G is bipartite, $\varphi(G) = 0$ and there is nothing to prove. Assume that G is non-bipartite. Then $E_\omega \neq \emptyset$ for $\omega \in \mathcal{F}$. By Lemma 2.1, there is $\omega \in \mathcal{F}$ such that $\varphi(G) = |E_\omega|$. Thus, one has the inequality "$\geq$". To show the reverse inequality take ω in $\mathcal{F}$. It suffices to show that $\varphi(G) \leq |E_\omega|$. The vertex set of G can be partitioned as $V(G) = V^+ \cup V^-$, where V^+ (resp.

V^-) is the set of vertices of G with positive (resp. negative) sign. Then $G \setminus E_\omega$ is bipartite with bipartition V^+, V^-. Thus $\varphi(G) \leq |E_\omega|$. □

This lemma can be used to compute $\varphi(G)$. Let K be a field of $\mathrm{char}(K) \neq 2$. Each ω in $\mathcal{F}$ defines a linear polynomial

$$h_\omega = \sum_{\omega(t_i)=+} t_i - \sum_{\omega(t_i)=-} t_i.$$

The number of points of $\mathbb{X}$ where h_ω does not vanish is equal to $|E_\omega|$. As a consequence one obtains the following algebraic formula for the edge biparticity.

Proposition 2.3. *Let G be a connected non-bipartite graph over a field of $\mathrm{char}(K) \neq 2$. Then*

$$\varphi(G) = \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(h)| \colon\ h = a_1t_1 + \cdots + a_st_s,\ a_i \in \{1, -1\},\ \forall\, i\}.$$

Proof. Any $h = a_1t_1 + \cdots + a_st_s$, $a_i \in \{1, -1\}$ for all i, $h \neq \pm(t_1 + \cdots + t_s)$, can be written as $h = h_\omega$ for some $\omega \in \mathcal{F}$. As $|E_\omega| = |\mathbb{X}\setminus V_{\mathbb{X}}(h_\omega)|$ for $\omega \in \mathcal{F}$, the result follows from Lemma 2.2. □

This result can be used in practice to compute $\varphi(G)$ using *Macaulay*2 [14] (see the examples and procedures of Sections 3 and 4).

Remark 2.4. If we allow $a_1, \ldots, a_s$ to be in $\{0, 1, -1\}$ such that not all of them are zero, we obtain the minimum distance of $C_p(G)$. This follows from [11, Lemma 4.3(iii)].

The following result is well known.

Proposition 2.5. [2, 15, 21] *Let G be a connected graph with s vertices and let A be its incidence matrix over a field K. Then*

$$\mathrm{rank}(A) = \begin{cases} s & \text{if } \mathrm{char}(K) \neq 2 \text{ and } G \text{ is non-bipartite,} \\ s-1 & \text{if } \mathrm{char}(K) = 2 \text{ or } G \text{ is bipartite.} \end{cases}$$

Corollary 2.6. *Let G be a connected graph with s vertices and m edges and let $C = C_p(G)$ (resp. $C^\perp$) be the code (resp. dual code) of G. Then*

(a) *C (resp. $C^\perp$) is an $[m, s]$ (resp. $[m, m-s]$) code if $p \neq 2$ and G is non-bipartite.*
(b) *C (resp. $C^\perp$) is an $[m, s-1]$ (resp. $[m, m-s+1]$) code if $p = 2$ or G is bipartite.*

Proof. This follow from Proposition 2.5 noticing that $\dim(C) + \dim(C^\perp) = m$. □

Lemma 2.7. *Let G be a connected graph and let K be a field. The following hold.*

(a) *If $\mathrm{char}(K) \neq 2$, G is non-bipartite and h is a linear form in $I(\mathbb{X})$, then $h = 0$.*
(b) *If $\mathrm{char}(K) = 2$ and $h \neq 0$ is a linear form in $I(\mathbb{X})$, then $h = c\sum_{i=1}^{s} t_i$, for some $c \in K$.*
(c) *If $\mathrm{char}(K) = 2$ and h is a linear form in $I(\mathbb{X})$ in $s-1$ variables, then $h = 0$.*

Proof. Let ψ be the linear map $\psi\colon K^s \to K^m$, $x \mapsto xA$. Fix a linear from $h = \sum_{i=1}^{s} a_it_i$ of S_1 and set $v_h = (a_1, \ldots, a_s)$. Then v_h is in $\ker(\psi)$ if and only if $h \in I(\mathbb{X})$. For use below notice that $s = \dim(\ker(\psi)) + \mathrm{rank}(A)$.

(a): By Proposition 2.5, $\ker(\psi) = (0)$. Thus $v_h = 0$, that is, $h = 0$.

(b): From Proposition 2.5 we get that $\ker(\psi)$ has dimension 1, and since $\mathbf{1} = (1, \ldots, 1) \in \ker(\psi)$ the result follows.

(c): It is a consequence of (b). □

Lemma 2.8. *Let G be a connected bipartite graph with bipartition V_1, V_2. The following hold.*

(a) *If K is a field and $h \neq 0$ is a linear form of S that vanishes at all points of $\mathbb{X}$, then $h = c(\sum_{t_i \in V_1} t_i - \sum_{t_i \in V_2} t_i)$ for some $c \in K$.*

(b) *If t_i and t_j are in V_1, then $G \cup \{t_i, t_j\}$ contains an odd cycle.*

Proof. (a): It follows adapting the proof of Lemma 2.7.

(b): As G is connected and bipartite, there is a path $\mathcal{P}$ in G of even length joining t_i and t_j. Then, adding the new edge $\{t_i, t_i\}$ to the path $\mathcal{P}$, gives an odd cycle of $G \cup \{t_i, t_j\}$. □

Lemma 2.9. *Let G be a connected non-bipartite graph. If $\ell = \upsilon_r(G)$ and $f_1, \ldots, f_\ell$ are edges of G, then the graph $H = G \setminus \{f_1, \ldots, f_\ell\}$ has at most r bipartite connected components.*

Proof. Let $H_1, \ldots, H_n$ be the connected components of H. We proceed by contradiction assuming that $H_1, \ldots, H_{r+1}$ are bipartite. Consider the graph $G' = G \setminus \{f_2, \ldots, f_\ell\}$. If $f_1 \subset V(H_i)$ for some i, then G' has r bipartite components, a contradiction. Thus $f_1 \not\subset V(H_i)$ for $i = 1, \ldots, n$. Hence, f_1 joins H_i and H_j for some i, j with $i < j$. If $j \leq r + 1$, the graph $H_i \cup H_j \cup \{f_1\}$ is bipartite and connected, and G' has r bipartite components, a contradiction. Thus $j > r + 1$ and in this case G' has r bipartite components, a contradiction. □

We come to one of our main results.

Theorem 2.10. *Let G be a connected non-bipartite graph with s vertices and m edges, let K be a field of $\operatorname{char}(K) \neq 2$, and let A be the incidence matrix of G. If $\mathbb{X}$ is the set of column vectors of A and $\upsilon_r(G)$ is the r-th weak edge biparticity of G, then*

$$\delta_{\mathbb{X}}(d, r) = \begin{cases} \upsilon_r(G) & \text{if } d = 1 \text{ and } 1 \leq r \leq s = \dim_K(C_{\mathbb{X}}(d)), \\ r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_{\mathbb{X}}(d)). \end{cases}$$

Proof. Assume $d = 1$. First we show the inequality $\delta_{\mathbb{X}}(1, r) \geq \upsilon_r(G)$. We proceed by contradiction assuming that $\upsilon_r(G) > \delta_{\mathbb{X}}(1, r)$. Then $\upsilon_r(G) > |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$ for some set F consisting of r linear forms $h_1, \ldots, h_r$ which are linearly independent, over K, modulo $I(\mathbb{X})$. Let $[P_1], \ldots, [P_\ell]$ be the points in $\mathbb{X} \setminus V_{\mathbb{X}}(F)$ and let $f_1, \ldots, f_\ell$ be the edges of G corresponding to these points. Consider the graph $H = G \setminus \{f_1, \ldots, f_\ell\}$. Let $H_1, \ldots, H_n$ be the bipartite connected components of H. Since $\upsilon_r(G) > \ell$, n is at most $r - 1$. Let $\mathbb{X}_H$ be the set of points corresponding to the columns of the incidence matrix of H. Note that h_i vanishes at all points of $\mathbb{X}_H$ for $i = 1, \ldots, r$. Then, by Lemma 2.7, $h_1, \ldots, h_r$ are linear forms in the variables $V(H_1) \cup \cdots \cup V(H_n)$. For each $1 \leq j \leq n$, let A_1^j, A_2^j be the bipartition of H_j and set $g_j = \sum_{t_i \in A_1^j} t_i - \sum_{t_i \in A_2^j} t_i$. Then, by Lemma 2.8, $F = \{h_1, \ldots, h_r\}$ is in the K-linear space generated by $g_1, \ldots, g_n$, a contradiction because F is linearly independent over K and $n < r$.

Now we show the inequality $\delta_{\mathbb{X}}(1, r) \leq \upsilon_r(G)$. Note that, by Lemma 2.7, any minimal generator of $I(\mathbb{X})$ has degree at least 2. Hence, it suffices to find a set $F = \{h_1, \ldots, h_r\}$ of linearly independent forms of degree 1 such that $\upsilon_r(G) = |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. We set $\ell = \upsilon_r(G)$. There are edges $f_1, \ldots, f_\ell$ of G such that the graph

$$H = G \setminus \{f_1, \ldots, f_\ell\}$$

has exactly r connected bipartite components (see Lemma 2.9). We denote the connected components of H by $H_1, \ldots, H_n$, where $H_1, \ldots, H_r$ are bipartite. Consider a bipartition A_1^j, A_2^j of H_j for $j = 1, \ldots, r$ and set

$$h_j = \sum_{t_i \in A_1^j} t_i - \sum_{t_i \in A_2^j} t_i.$$

Let P_i be the point in $\mathbb{P}^{s-1}$ that corresponds to f_i for $i = 1, \ldots, \ell$. To complete the proof of the case $d = 1$ we need only show the equality $\{[P_1], \ldots, [P_\ell]\} = |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. To show the inclusion "$\subset$" fix an edge f_k with $1 \leq k \leq \ell$ and set

$$H' = \bigcup_{i=1}^{r} H_i, \quad H'' = \bigcup_{i=r+1}^{n} H_i \ \text{ and } \ G' = G \setminus \{f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_\ell\}.$$

Note that $f_k \not\subset V(H_j)$ for $r < j$, otherwise G' has r bipartite components. As a consequence f_k intersects $V(H')$, otherwise $f_k \subset V(H'')$, f_k joins H_i and H_j for some $r < i < j$, and the graph G' has r bipartite components, a contradiction.

Case (1): $f_k \subset V(H_j)$ for some $1 \leq j \leq r$. As $V(H_j) = A_1^j \cup A_2^j$, either $f_k \subset A_1^j$ or $f_k \subset A_2^j$, otherwise the graph G' has r bipartite components, a contradiction. Hence, as $\operatorname{char}(K) \neq 2$, we get that $h_j(P_k) \neq 0$. Thus $[P_k] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

Case (2): $f_k \cap V(H_i) \neq \emptyset$ and $f_k \cap V(H_j) \neq \emptyset$ for some $i < j \leq r$. Then using the bipartitions of H_i and H_j we get $h_i(P_k) \neq 0$ and $h_j(P_k) \neq 0$. Thus $[P_k] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

Case (3): $f_k \cap V(H_i) \neq \emptyset$ for some $1 \leq i \leq r$ and $f_k \cap V(H'') \neq \emptyset$. Then using the bipartition of H_i we get $h_i(P_k) \neq 0$. Thus $[P_k] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

To show the inclusion "$\supset$" take $[P] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$ and denote by f its corresponding edge in G. Then there is $1 \leq j \leq n$ such that $h_j(P) \neq 0$. We proceed by contradiction assuming $[P] \notin \{[P_1], \ldots, [P_\ell]\}$, that is, $f \neq f_i$ for $i = 1, \ldots \ell$. Then f is an edge of H. Thus f is an edge of H_k for some $1 \leq k \leq n$. If $r < k$, then $h_i(P) = 0$ for $i = 1, \ldots, r$ by construction of the h_i's, a contradiction. Thus $1 \leq k \leq r$. If $f \subset A_1^k$ or $f \subset A_2^k$, then f would not be an edge of H_k, a contradiction. Hence f joins A_1^k with A_2^k, and consequently $h_i(P) = 0$ for $i = 1, \ldots, r$ by construction of the h_i's, a contradiction. Thus $P = P_i$ for some $1 \leq i \leq \ell$, as required.

Assume $d \geq 2$. We claim that in this case the evaluation function ev_d is surjective. Indeed, taking all m-tuples of the form $\operatorname{ev}_d(t_i t_j^{d-1})$, where $\{t_i, t_j\}$ is an edge of G, one gets the canonical basis of K^m. Therefore $C_{\mathbb{X}}(d) = K^m$ and $\delta_{\mathbb{X}}(d, r) = r$ for $1 \leq r \leq m$. □

We come to another of our main results.

Theorem 2.11. *Let G be a connected graph with s vertices and m edges, let K be a field of $\operatorname{char}(K) = 2$, and let A be the incidence matrix of G. If $\mathbb{X}$ is the set of column vectors of A and $\lambda_r(G)$ is the r-th edge connectivity of G, then*

$$\delta_{\mathbb{X}}(d, r) = \begin{cases} \lambda_r(G) & \text{if } d = 1 \text{ and } 1 \leq r \leq s - 1 = \dim_K(C_{\mathbb{X}}(d)), \\ r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_{\mathbb{X}}(d)). \end{cases}$$

Proof. Assume $d = 1$. First we show the inequality $\delta_{\mathbb{X}}(1, r) \geq \lambda_r(G)$. We proceed by contradiction assuming that $\lambda_r(G) > \delta_{\mathbb{X}}(1, r)$. Then $\lambda_r(G) > |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$ for some set F consisting of r linear forms $h_1, \ldots, h_r$ which are linearly independent modulo $I(\mathbb{X})$. We set $\ell = |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. Let $[P_1], \ldots, [P_\ell]$ be the points in $\mathbb{X} \setminus V_{\mathbb{X}}(F)$ and let $f_1, \ldots, f_\ell$ be the edges of G corresponding to these points. Consider the graph $H = G \setminus \{f_1, \ldots, f_\ell\}$ and denote by $H_1, \ldots, H_n$ its connected components. Since $\lambda_r(G) > \ell$, H cannot have $r + 1$ components, that is, $n \leq r$. Let $\mathbb{X}_H$ be the set of points corresponding to the columns of the incidence matrix of H. Note that h_i vanishes at all points of $\mathbb{X}_H$ for $i = 1, \ldots, r$. Indeed, take a point $[P]$ in $\mathbb{X}_H$, then its corresponding edge f is in H_k for some k, then $f \neq f_j$ for $j = 1, \ldots, \ell$, that is, $[P] \notin \mathbb{X} \setminus V_{\mathbb{X}}(F)$. Thus $h_i(P) = 0$. We set $g_j = \sum_{t_i \in V(H_j)} t_i$ for $j = 1, \ldots, n$. As $h_i \in I(\mathbb{X}_H)$, by Lemma 2.7, h_i is a linear combination of $g_1,\ldots,g_n$ for $i=1,\ldots,r$. Therefore

$$Kh_1\oplus\cdots\oplus Kh_r\subset Kg_1\oplus\cdots\oplus Kg_n,$$

and consequently $r\leq n$. Thus $r=n$ and the inclusion above is an equality. Therefore taking classes modulo $I(\mathbb{X})$, we get

$$K\overline{h}_1\oplus\cdots\oplus K\overline{h}_r=K\overline{g}_1\oplus\cdots\oplus K\overline{g}_n.$$

As $\overline{h}_1,\ldots,\overline{h}_r$ are linearly independent, so are $\overline{g}_1,\ldots,\overline{g}_n$ because $r=n$, a contradiction because by construction of the g_i's and since $\mathrm{char}(K)=2$, one has $\sum_{i=1}^n\overline{g}_i=\sum_{i=1}^s\overline{t}_i=\overline{0}$.

Next we show the inequality $\delta_{\mathbb{X}}(1,r)\leq\lambda_r(G)$. It suffices to find a set $F=\{h_1,\ldots,h_r\}$ of forms of degree 1 whose image $\overline{F}=\{\overline{h}_1,\ldots,\overline{h}_r\}$ in $S/I(\mathbb{X})$ is linearly independent over K and $\lambda_r(G)=|\mathbb{X}\setminus V_{\mathbb{X}}(F)|$. We set $\ell=\lambda_r(G)$. There are edges $f_1,\ldots,f_\ell$ of G such that the graph

$$H=G\setminus\{f_1,\ldots,f_\ell\}$$

has exactly $r+1$ connected components $H_1,\ldots,H_{r+1}$. For $j=1,\ldots,r$, we set

$$h_j=\sum_{t_i\in V(H_j)}t_i.$$

Note that h_i and h_j have no common variables for $i\neq j$ and any sum of the polynomials $h_1,\ldots,h_r$ is a linear form in $s-1$ variables. Hence, by Lemma 2.7(b), $\overline{F}$ is linearly independent.

Let $[P_i]$ be the point in $\mathbb{P}^{s-1}$ that corresponds to f_i for $i=1,\ldots,\ell$. To complete the proof of the case $d=1$ we need only show the equality $\{[P_1],\ldots,[P_\ell]\}=|\mathbb{X}\setminus V_{\mathbb{X}}(F)|$. To show the inclusion "$\subset$" fix an edge f_k with $1\leq k\leq\ell$ and set

$$G'=G\setminus\{f_1,\ldots,f_{k-1},f_{k+1},\ldots,f_\ell\}.$$

Note that $f_k\not\subset V(H_j)$ for $j=1,\ldots,r+1$, otherwise G' has $r+1$ components, a contradiction. As a consequence f_k joins H_i and H_j for some $i<j$. Thus $h_i(P_k)\neq 0$ and $[P_k]\in\mathbb{X}\setminus V_{\mathbb{X}}(F)$.

To show the inclusion "$\supset$" take $[P]\in\mathbb{X}\setminus V_{\mathbb{X}}(F)$ and denote by f its corresponding edge in G. Then there is $1\leq j\leq r$ such that $h_j(P)\neq 0$. We proceed by contradiction assuming $[P]\notin\{[P_1],\ldots,[P_\ell]\}$, that is, $f\neq f_i$ for $i=1,\ldots\ell$. Then f is an edge of H. As $\mathrm{char}(K)=2$, we get $h_i(P)=0$ for $i=1,\ldots,r$ by construction of h_i, a contradiction.

If $d\geq 2$, the equality $\delta_{\mathbb{X}}(d,r)=r$ for $1\leq r\leq m=\dim_K(C_{\mathbb{X}}(d))$ follows from the proof of Theorem 2.10. □

The next result is a hybrid of Theorems 2.10 and 2.11 and is characteristic free.

Theorem 2.12. *Let G be a connected bipartite graph with s vertices and m edges, let K be a field of any characteristic, and let A be the incidence matrix of G. If $\mathbb{X}$ is the set of column vectors of A and $\lambda_r(G)$ is the r-th edge connectivity of G, then*

$$\delta_{\mathbb{X}}(d,r)=\begin{cases}\lambda_r(G) & \text{if } d=1 \text{ and } 1\leq r\leq s-1=\dim_K(C_{\mathbb{X}}(d)),\\ r & \text{if } d\geq 2 \text{ and } 1\leq r\leq m=\dim_K(C_{\mathbb{X}}(d)).\end{cases}$$

Proof. Let V_1, V_2 be the bipartition of G. Consider the set $\mathbb{Y}$ of all points $[\mathbf{e}_i-\mathbf{e}_j]$ in $\mathbb{P}^{s-1}$ such that $\{t_i,t_j\}$ is an edge of G with $t_i\in V_1$ and $t_j\in V_2$, where $\mathbf{e}_i$ is the i-th unit vector in K^s. Noticing that the polynomial $h=t_1+\cdots+t_s$ vanishes at all points of $\mathbb{Y}$ and the equality $C_{\mathbb{X}}(1)=C_{\mathbb{Y}}(1)$, the result follows adapting Lemma 2.7 and the proof of Theorem 2.11 with $\mathbb{Y}$ playing the role of $\mathbb{X}$. □

The main application to coding theory is the following.

Corollary 2.13. *Let $C_p(G)$ be the incidence matrix code of a connected graph G with s vertices, m edges, r-th weak edge biparticity $\upsilon_r(G)$, r-th edge connectivity $\lambda_r(G)$, over a finite field K of* $\mathrm{char}(K)=p$. *Then the r-th generalized Hamming weight of $C_p(G)$ is given by*

$$\delta_r(C_p(G))=\begin{cases}\upsilon_r(G) & \text{if } p\neq 2,\ G \text{ is non-bipartite}, 1\leq r\leq s,\\ \lambda_r(G) & \text{if } p=2,\ 1\leq r\leq s-1,\\ \lambda_r(G) & \text{if } G \text{ is bipartite}, 1\leq r\leq s-1.\end{cases}$$

Proof. Note that the linear code $C_p(G)$ is the image of S_1—the vector space of linear forms of S—under the evaluation map $\mathrm{ev}_1\colon S_1\rightarrow K^m$, $f\mapsto (f(P_1),\ldots,f(P_m))$. The image of the linear function t_i, under the map ev_1, gives the i-th row of the incidence matrix of G. This means that $C_p(G)$ is the Reed–Muller-type code $C_{\mathbb{X}}(1)$. Hence, the result follows using the equality $\delta_{\mathbb{X}}(1,r)=\delta_r(C_p(G))$ and Theorems 2.10, 2.11, and 2.12. $\square$

As a consequence we recover the following result.

Corollary 2.14. [5, Theorems 1–3] *Let $C_p(G)$ be the incidence matrix code of a connected graph G with s vertices, m edges, weak edge biparticity $\upsilon(G)$, edge connectivity $\lambda(G)$, over a finite field K of* $\mathrm{char}(K)=p$. *Then the minimum distance of $C_p(G)$ is given by*

$$\delta(C_p(G))=\begin{cases}\upsilon(G) & \text{if } p\neq 2,\ G \text{ is non-bipartite}, 1\leq r\leq s,\\ \lambda(G) & \text{if } p=2,\ 1\leq r\leq s-1,\\ \lambda(G) & \text{if } G \text{ is bipartite}, 1\leq r\leq s-1.\end{cases}$$

Proof. It follows from Corollary 2.13 making $r=1$. $\square$

3. Examples

Let G be a connected graph and let $C_p(G)$ be the incidence matrix code of G over a finite field $K=\mathbb{F}_q$ of characteristic p. As an application of our main results we can use *Macaulay*2 [14], SageMath [27], and Wei's duality [31, Theorem 3], to compute the weight hierarchy of $C_p(G)$. Hence, by Corollary 2.13, we can compute the corresponding higher weak biparticity and edge connectivity numbers of the graph. We do not claim, however, to have found an efficient way to compute the weight hierarchy of an incidence matrix code.

Conversely any algorithm that computes these graph invariants can be used to compute the weight hierarchy of $C_p(G)$. Using Proposition 2.3, we can also compute the edge biparticity of G using the field of rational numbers.

The weight hierarchy of $C_p(G)$ can also be computed using a formula of Johnsen and Verdure [19] for the Hamming weights in terms of the Betti numbers of the Stanley–Reisner ring whose faces are the independent sets of the vector matroid of a parity check matrix of $C_p(G)$.

We illustrate how to use our results in practice with some examples.

Example 3.1. Let G be the graph of Figure 1. Recall that the dimension of $C_p(G)$ is 6 if $p=3$ and is 5 if $p=2$ (Corollary 2.6). For use below we denote the dual code by $C_p(G)^{\perp}$.

Using Procedure 4.1, together with, Wei's duality [31, Theorem 3] we obtain Table 1 with the weight hierarchy of $C_p(G)$. The edge biparticity of this graph is 2, the weak edge biparticity is 2, and the edge connectivity is 3.

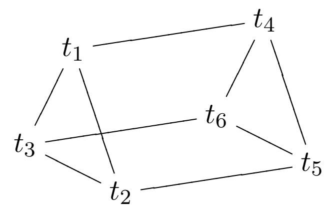

FIGURE 1. Non-bipartite graph G.

r	1	2	3	4	5	6
$\delta_r(C_2(G))$	3	5	6	8	9	
$\delta_r(C_2(G)^\perp)$	3	6	8	9		
$\delta_r(C_3(G))$	2	4	5	7	8	9
$\delta_r(C_3(G)^\perp)$	4	7	9			

TABLE 1. Weight hierarchy of $C_p(G)$ for the graph of Figure 1.

Example 3.2. Let G be the Petersen graph of Figure 2. Recall that the dimension of $C_p(G)$ (resp. $C_p(G)^\perp$) is 9 (resp. 6) if $p = 2$, and the dimension of $C_p(G)$ (resp. $C_p(G)^\perp$) is 10 (resp. 5) if $p \neq 2$ (Corollary 2.6).

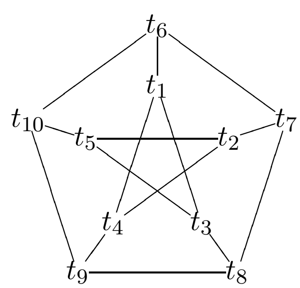

FIGURE 2. Petersen graph G.

Using Procedure 4.2, together with, Wei's duality [31, Theorem 3] we obtain Table 2 describing the weight hierarchy of $C_p(G)$. The edge biparticity, the weak edge biparticity, and the edge connectivity of the Petersen graph are equal to 3.

r	1	2	3	4	5	6	7	8	9	10
$\delta_r(C_2(G))$	3	5	7	9	10	12	13	14	15	
$\delta_r(C_2(G)^\perp)$	5	8	10	12	14	15				
$\delta_r(C_3(G))$	3	5	7	8	9	11	12	13	14	15
$\delta_r(C_3(G)^\perp)$	6	10	12	14	15					

TABLE 2. Weight hierarchy of $C_p(G)$ for the graph of Figure 2.

4. Procedures for Macaulay2 and SageMath

Procedure 4.1. Computing the weight hierarchies using *Macaulay*2 [14], SageMath [27], and Wei's duality [31]. This procedure corresponds to Example 3.1. It could be applied to any connected graph G to obtain the generalized Hamming weights of $C_p(G)$. The next procedure for *Macaulay*2 uses the algorithms of [11] to compute generalized minimum distance functions.

```
--Procedure for Macaulay2
input "points.m2"
q=3, R = ZZ/q[t1,t2,t3,t4,t5,t6]--p=char(K)=3
A = transpose(matrix{{1,1,0,0,0,0},{0,1,1,0,0,0},{1,0,1,0,0,0},
{0,0,0,1,1,0},{0,0,0,0,1,1},{0,0,0,1,0,1},{1,0,0,1,0,0},
{0,1,0,0,0,1},{0,0,1,0,1,0}})
I=ideal(projectivePointsByIntersection(A,R)), M=coker gens gb I
genmd=(d,r)->degree M-max apply(apply(subsets(apply(apply(apply
(toList (set(0..q-1))^**(hilbertFunction(d,M))
-(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),r),ideal),x-> if #set flatten entries
mingens ideal(leadTerm gens x)==r and not quotient(I,x)==I
then degree(I+x) else 0)
--The following are the first two generalized Hamming weights
genmd(1,1), genmd(1,2)
```

```
#Procedure for SageMath
A = transpose(matrix(GF(3),[[1,1,0,0,0,0],[0,1,1,0,0,0],[1,0,1,0,0,0],
[0,0,0,1,1,0],[0,0,0,0,1,1],[0,0,0,1,0,1],[1,0,0,1,0,0],[0,1,0,0,0,1],
[0,0,1,0,1,0]]))
C = codes.LinearCode(A)
C.parity_check_matrix()
C.generator_matrix()
#the next line Gives the minimum distance of the dual code
C.dual_code().minimum_distance()
```

Procedure 4.2. Computing the weight hierarchies and the edge biparticity using *Macaulay*2 [14], SageMath [27], and Wei's duality [31]. This procedure corresponds to Example 3.2. The next procedure for *Macaulay*2 uses the algorithms of [11] to compute generalized footprint functions. The footprint gives easy to compute lower bounds for the generalized Hamming weights of projective Reed–Muller-type codes.

```
--Procedure for Macaulay2 for Petersen graph
input "points.m2"
R = QQ[t1,t2,t3,t4,t5,t6,t7,t8,t9,t10]
--Incidence matrix to compute the edge biparticity
A = transpose matrix{{1,1,0,0,0,0,0,0,0,0},{0,1,1,0,0,0,0,0,0,0},
{0,0,1,1,0,0,0,0,0,0},{0,0,0,1,1,0,0,0,0,0},{1,0,0,0,1,0,0,0,0,0},
{1,0,0,0,0,1,0,0,0,0},{0,1,0,0,0,0,1,0,0,0},{0,0,1,0,0,0,0,1,0,0},
{0,0,0,1,0,0,0,0,1,0},{0,0,0,0,1,0,0,0,0,1},{0,0,0,0,0,1,0,1,0,0},
{0,0,0,0,0,0,0,1,0,1},{0,0,0,0,0,0,1,0,0,1},{0,0,0,0,0,0,1,0,1,0},
{0,0,0,0,0,1,0,0,1,0}}
q=2, R = ZZ/q[t1,t2,t3,t4,t5,t6,t7,t8,t9]
--Generator matrix computed with Sage to find Hamming weights.
```

```
A1=matrix({{1,0,0,0,1,0,0,0,0,0,1,0,0,0,1},
{0,1,0,0,1,0,0,0,0,0,1,0,1,1,1},{0,0,1,0,1,0,0,0,0,0,0,1,1,1,1},
{0,0,0,1,1,0,0,0,0,0,0,1,1,0,0},{0,0,0,0,0,1,0,0,0,0,1,0,0,0,1},
{0,0,0,0,0,0,1,0,0,0,0,0,1,1,0},{0,0,0,0,0,0,0,1,0,0,1,1,0,0,0},
{0,0,0,0,0,0,0,0,1,0,0,0,0,1,1},{0,0,0,0,0,0,0,0,0,1,0,1,1,0,0}})
q=2, R = ZZ/q[t1,t2,t3,t4,t5,t6]
--Parity check matrix computed with Sage to find
--the Hamming weights of dual code
A2=matrix({{1,0,0,0,0,1,1,0,0,0,0,0,0,1,1},
{0,1,0,0,0,0,1,1,0,0,0,1,1,0,0},{0,0,1,0,0,0,0,1,1,0,0,1,1,1,0},
{0,0,0,1,0,0,0,0,1,1,0,0,1,1,0},{0,0,0,0,1,1,0,0,0,1,0,0,1,1,1},
{0,0,0,0,0,0,0,0,0,0,1,1,1,1,1}})
--The following functions can be applied to A, A1, A2
I=ideal(projectivePointsByIntersection(A,R)), M=coker gens gb I
--This function computes the edge biparticity of Petersen graph
--using the incidence matrix over the rational numbers
genmd1=(d,r)->degree M-max apply(apply(subsets(apply(apply(apply
(toList (set(1,-1))^**(hilbertFunction(d,M))
-(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),r),ideal),x-> if #set flatten entries
mingens ideal(leadTerm gens x)==r and not quotient(I,x)==I
then degree(I+x) else 0)
--To compute the r-th Hamming weight of the dual code
--use genmd(1,r) of the previous procedure:
genmd(1,1),genmd(1,2),genmd(1,3),genmd(1,4),genmd(1,5)
--To compute the edge biparticity use genmd1(1,1)
init=ideal(leadTerm gens gb I), degree M
er=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
--This is the footprint function
fpr=(d,r)->degree M - max apply(apply(apply(subsets(flatten
entries basis(d,M),r),toSequence),ideal),er)
--To find lower bounds for Hamming weights use the footprint:
fpr(1,1),fpr(1,2),fpr(1,3),fpr(1,4),fpr(1,5),fpr(1,6),fpr(1,7),fpr(1,8)
```

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