FANO DEFORMATION RIGIDITY OF RATIONAL HOMOGENEOUS SPACES OF SUBMAXIMAL PICARD NUMBERS

QIFENG LI

Abstract. We study the question whether rational homogeneous spaces are rigid under Fano deformation. In other words, given any smooth connected family $\pi : \mathcal{X} \to \mathcal{Z}$ of Fano manifolds, if one fiber is biholomorphic to a rational homogeneous space S, whether is π an S-fibration? The cases of Picard number one were studied in a series of papers by J.-M. Hwang and N. Mok. For higher Picard number cases, we notice that the Picard number of a rational homogeneous space G/P satisfies $\rho(G/P) \leq \text{rank}(G)$. Recently A. Weber and J. A. Wiśniewski proved that rational homogeneous spaces G/P with Picard numbers $\rho(G/P) = \text{rank}(G)$ (i.e. complete flag manifolds) are rigid under Fano deformation. In this paper we show that the rational homogeneous space G/P is rigid under Fano deformation, providing that G is a simple algebraic group of type ADE, the Picard number $\rho(G/P) = \text{rank}(G) - 1$ and G/P is not biholomorphic to $\mathbb{F}(1, 2, \mathbb{P}^3)$ or $\mathbb{F}(1, 2, Q^6)$. The variety $\mathbb{F}(1, 2, \mathbb{P}^3)$ is the set of flags of projective lines and planes in $\mathbb{P}^3$, and $\mathbb{F}(1, 2, Q^6)$ is the set of flags of projective lines and planes in 6-dimensional smooth quadric hypersurface. We show that $\mathbb{F}(1, 2, \mathbb{P}^3)$ have a unique Fano degeneration, which is explicitly constructed. The structure of possible Fano degeneration of $\mathbb{F}(1, 2, Q^6)$ is also described explicitly. To prove our rigidity result, we firstly show that the Fano deformation rigidity of a homogeneous space of type ADE can be implied by that property of suitable homogeneous submanifolds. Then we complete the proof via the study of Fano deformation rigidity of rational homogeneous spaces of small Picard numbers. As a byproduct, we also show the Fano deformation rigidity of other manifolds such as $\mathbb{F}(0, 1, \ldots, k_1, k_2, k_2 + 1, \ldots, n - 1, \mathbb{P}^n)$ and $\mathbb{F}(0, 1, \ldots, k_1, k_2, k_2 + 1, \ldots, n, Q^{2n+2})$ with $0 \leq k_1 < k_2 \leq n - 1$.

Keywords: Fano deformation rigidity, Symbol algebras, Minimal rational curves.

MSC2010: 14M15, 14D06, 53B15.

Contents

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. A Fano manifold M is said to be rigid under Fano deformation if any smooth connected family $\pi : \mathcal{X} \to \mathcal{Z}$ of Fano manifolds with M being a fiber must be an M-fibration. If the fiber $\mathcal{X}_z := \pi^{-1}(z)$ at some point $z \in \mathcal{Z}$ is not biholomorphic to M, we say $\mathcal{X}_t$ is a Fano degeneration of M.

Our interest in this paper is the Fano deformation rigidity of rational homogeneous spaces. The Fano deformation rigidity of rational homogeneous spaces of Picard number one is studied by J.-M. Hwang and N. Mok in [5][8][9][10]. Among the rational homogeneous spaces of Picard number one, $\mathbb{F}(1, Q^5)$ is the only variety that is not rigid under Fano deformation, where $\mathbb{F}(1, Q^5)$ is the family of projective lines on a 5-dimensional smooth quadric hypersurface. Moreover, the variety $\mathbb{F}(1, Q^5)$ has a unique Fano degeneration, see [14] and [7]. In particular, we have

Theorem 1.1. [5][8][9][10] *Let $\boldsymbol{S}$ be a rational homogeneous space of Picard number one. If $\boldsymbol{S} \ncong \mathbb{F}(1, Q^5)$, then $\boldsymbol{S}$ is rigid under Fano deformation.*

To our knowledge the first result on higher Picard number cases is due to J. A. Wiśniewski [17].

Theorem 1.2. [17] *The variety $F(1, n, \mathbb{C}^{n+1})$ is rigid under Fano deformation, where $F(1, n, \mathbb{C}^{n+1})$ is the set of flags of 1-dimensional and n-dimensional vector subspaces in $\mathbb{C}^{n+1}$.*

A rational homogeneous space is denoted by G/P, where G is a semisimple algebraic group and P is a parabolic subgroup of G. The Picard number of G/P satisfies that $\rho(G/P) \leq \mathrm{rank}(G)$, where $\mathrm{rank}(G)$ is the dimensional of any maximal torus of G. Recently A. Weber and J. A. Wiśniewski [16] verified Fano deformation rigidity of the cases with $\rho(G/P) = \mathrm{rank}(G)$. More precisely,

Theorem 1.3. [16] *The rational homogenous space G/B is rigid under Fano deformation, where G is a semisimple algebraic group and B is a Borel subgroup.*

Motivated by Theorem 1.1 and Theorem 1.3, one naturally ask what about the intermediate cases? A previous result of the author [13, Theorem 1] shows that product structure is preserved under Fano deformation. In particular, a rational homogeneous space $\mathbf{S}$, satisfying $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2$, is rigid under Fano deformation if and only if so are $\mathbf{S}_1$ and $\mathbf{S}_2$. It reduces the problem to the case when G is simple.

Our main result is on the cases with submaximal Picard number, i.e. $\rho(G/P) = \mathrm{rank}(G) - 1$. More precisely, we have the following

Theorem 1.4. *Let G be a simple algebraic group of type ADE and P be a parabolic subgroup of G such that the Picard number $\rho(G/P) = rank(G) - 1$. If G/P is not biholomorphic to $\mathbb{F}(1, 2, \mathbb{P}^3)$ or $\mathbb{F}(1, 2, Q^6)$, then it is rigid under Fano deformation, where $\mathbb{F}(1, 2, \mathbb{P}^3)$ (resp. $\mathbb{F}(1, 2, Q^6)$) is the set of flags of projective lines and planes on $\mathbb{P}^3$ (resp. a 6-dimensional smooth quadric hypersurface).*

It was observed by A. Weber and J. A. Wiśniewski [16] that $F^d(1, 2; \mathbb{C}^4)$ is a Fano deformation of $\mathbb{F}(1, 2, \mathbb{P}^3)$, where $F^d(1, 2; \mathbb{C}^4)$ is defined as follows.

Construction 1.5. Let ω be a symplectic form on $\mathbb{C}^4$, i.e. ω is a nondegenerate antisymmetric form on $\mathbb{C}^4$. Denote by $\mathcal{L}^\omega \subset T\mathbb{P}^3$ the associated contact distribution on $\mathbb{P}^3 := \mathbb{P}(\mathbb{C}^4)$, and write $\mathcal{L}_\sigma := T\mathbb{P}^3/\mathcal{L}^\omega$. We define $F^d(1, 2; \mathbb{C}^4) := \mathbb{P}(\mathcal{L}_\sigma \oplus \mathcal{L}^\omega)$.

Indeed we can show moreover the following

Theorem 1.6. *The variety $F^d(1, 2; \mathbb{C}^4)$ is the unique Fano degeneration of $\mathbb{F}(1, 2, \mathbb{P}^3)$.*

We also describe the structure possible Fano degeneration of $\mathbb{F}(1, 2, Q^6)$, see Proposition 4.65.

The strategy to prove Theorem 1.4 is as follows. Firstly, we show that the Fano deformation rigidity of a rational homogenous space is implied the that property of a suitable class of its homogeneous submanifolds. Then we show the Fano deformation rigidity of these homogeneous submanifolds.

To explain the sketch, we need some convention on notations. Given a simple algebraic group G and a Borel subgroup B. Denote by R the set of simple roots and Γ the Dynkin diagram. There is a one to one correspondence between subsets I of R and parabolic subgroups P_I containing B such that $P_R = B$, $P_\emptyset = G$ and $P_I \subset P_{I'}$ if and only if $I' \subset I$. There is a one to one correspondence between rational homogeneous spaces G/P_I and marked Dynkin diagrams defined by marking nodes I in the Dynkin diagram of G. One can see the diagrams on page 5 intuitively. The following proposition reduces the Fano deformation rigidity of G/P_I to that property of its homogeneous submanifolds.

Proposition 1.7. *Let G be a simple algebraic group of type ADE, and I be a subset of R with cardinality $|I| \geq 3$. Suppose that for any $\alpha \neq \beta \in I$, there exists a subset $A \subset I$ such that $\alpha, \beta \in A$ and the rational homogeneous space $P_{I\setminus A}/P_I$ is rigid under Fano deformation. Then G/P_I is rigid under Fano deformation.*

Note that in the proposition above the variety $P_{I\setminus A}/P_I$ is a rational homogeneous space whose Picard number is $|A| \leq |I| = \rho(G/P_I)$. By Proposition 1.7, an easy analysis of marked Dynkin diagrams shows that Theorem 1.4 is a direct consequence of Theorems 1.1, 1.2, 1.3 and the following

Proposition 1.8. *The rational homogeneous spaces $A_4/P_{I'}$ and $D_5/P_{I''}$ are rigid under Fano deformation, where $|I'| = 3$ and $|I''| = 4$ respectively.*

As an example we analysis the Fano deformation rigidity of D_4/P_I with $I = \{\alpha_1, \alpha_3, \alpha_4\}$. Given any two different roots $\alpha, \beta \in I$, the rational homogeneous space $P_{I\setminus\{\alpha,\beta\}}/P_I$ is biholomorphic to $A_3/P_{\{\alpha_1,\alpha_3\}}$, which is rigid under Fano deformation. Hence, D_4/P_I is rigid under Fano deformation.

Our argument to Fano deformation rigidity of $A_4/P_{\{\alpha_1,\alpha_2,\alpha_4\}}$, which is a special case of Proposition 1.8, works equally well for $A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ with $m \geq 3$. In other words, we have

Proposition 1.9. *The rational homogeneous spaces $A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ with $m \geq 3$ are rigid under Fano deformation.*

Applying Proposition 1.7, we have the following consequence.

Theorem 1.10. *Let G be a simple algebraic group of type ADE, Γ be the Dynkin diagram of G, and I be a subset of the set of simple roots R. Denote by $J := R \setminus I$ and $\bar{\alpha}$ the node with three branches in Γ (of type DE). Suppose J contains no end nodes of Γ, the subdiagram with nodes J are connected, and there is at most one $\beta \in J$ with Cartan pairing $\langle \beta, \bar{\alpha} \rangle \neq 0$. Then the rational homogeneous space G/P_I is rigid under Fano deformation.*

If G is of type AD in Theorem 1.10, the manifolds G/P_I are exact $\mathbb{F}(0, 1, \ldots, k_1, k_2, k_2 + 1, \ldots, n-1, \mathbb{P}^n)$ and $\mathbb{F}(0, 1, \ldots, k_1, k_2, k_2 + 1, \ldots, n, Q^{2n+2})$ with $0 \leq k_1 < k_2 \leq n-1$.

Now let us explain the proof of Propositions 1.7, 1.8 and 1.9. It is well-known that the local deformation rigidity of rational homogeneous spaces follows from the vanishing $H^1(G/P_I, T_{G/P_I}) = 0$, which is a consequence of Borel-Weil-Bott theorem. So we only need to discuss in the following Setting 1.11 and show $\mathcal{X}_0 \cong \mathbf{S}$ in each corresponding case.

Setting 1.11. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a holomorphic map such that $\mathcal{X}_t \cong \mathbf{S} := G/P_I$ for $t \neq 0$ and $\mathcal{X}_0$ is a connected Fano manifold, where G is a connected simple algebraic group of ADE type and $I \subset R$. Here R is the set of simple roots and we define $J := R \setminus I$.

The key point to prove Propositions 1.7, 1.8 and 1.9 is the study of symbol algebras. Given a distribution $\mathcal{V}$ on a complex manifold Y, the weak derived system $\mathcal{V}^{-k}$ gives rise to a filtration $\mathcal{V}^0 \subset \mathcal{V}^{-1} \subset \mathcal{V}^{-2} \subset \cdots$, where $\mathcal{V}^0 := 0$, $\mathcal{V}^{-1} := \mathcal{V}$, and $\mathcal{V}^{-k-1} := \mathcal{V}^{-k} + [\mathcal{V}^{-1}, \mathcal{V}^{-k}]$ for $k \geq 1$. In an open neighborhood of a general point $y \in Y$ these $\mathcal{V}^{-k}$'s are subbundles of TY. The graded vector space $\mathrm{Symb}_y(\mathcal{V}) := \oplus_{k\geq 1}\mathcal{V}_y^{-k}/\mathcal{V}_y^{-k+1}$ is a graded nilpotent Lie algebra, and called the symbol algebra of $\mathcal{V}$ at y.

Let $\mathfrak{g}_{-1}(\mathbf{S})$ be the sum of all G-invariant minimal distributions on $\mathbf{S}$. The subscript -1 in the notation $\mathfrak{g}_{-1}(\mathbf{S})$ comes from the grading induced by I, see Subsection 2.1. There is a meromorphic distribution $\mathfrak{g}_{-1}(\mathcal{X}) \subset T^\pi$ such that its singular locus is a (possibly reducible) proper closed subvariety of $\mathcal{X}_0$ and its restriction on $\mathcal{X}_t$ with $t \neq 0$ coincides with the distribution $\mathfrak{g}_{-1}(\mathbf{S})$.

It is known that $\mathrm{Symb}_s(\mathfrak{g}_{-1}(\mathbf{S})) \cong \mathfrak{g}_-(I)$, where s is any point of $\mathbf{S}$, $\mathfrak{g}_-(I)$ is the nilradical of the Lie algebra of P_I^-, and P_I^- the opposite parabolic group of P_I. By the works of A. Čap and H. Schichl [2] and K. Yamaguchi [19], we can conclude the following

Proposition 1.12. *Suppose in Setting 1.11 that* $|I| \geq 3$ *and* $Symb_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) \cong \mathfrak{g}_-(I)$ *at general points* $x \in \mathcal{X}_0$. *Then* $\mathcal{X}_0 \cong \boldsymbol{S}$.

We can complete the proof of Propositions 1.7 and 1.8 by applying Proposition 1.12 and the following

Proposition 1.13. *If the manifold* $\boldsymbol{S}$ *in Setting 1.11 is the variety* G/P_I *in Propositions 1.7, 1.8 or 1.9, then* $Symb_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) \cong \mathfrak{g}_-(I)$ *at general points* $x \in \mathcal{X}_0$.

To prove Proposition 1.13, we need the algebraic and geometric feature of each situation. As an example, we suppose $\mathbf{S} = A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ in Setting 1.11. It can be shown that any two points in $\mathcal{X}_0$ can be jointed by chains of rational curves tangent to $\mathfrak{g}_{-1}(\mathcal{X}_0)$. Hence the tangent bundle $T\mathcal{X}_0$ is k-th weak derivative of $\mathfrak{g}_{-1}(\mathcal{X}_0)$ for some k. In particular, $\dim \mathrm{Symb}_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) = \dim \mathcal{X}_0 = \dim \mathfrak{g}_-(I)$ at a general point $x \in \mathcal{X}_0$. One the other hand, if the symbol algebra $\mathrm{Symb}_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) \ncong \mathfrak{g}_-(I)$, then an easy calculation of Lie algebras shows that $\dim \mathrm{Symb}_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) < \dim \mathfrak{g}_-(I)$. The contradiction implies that $\mathrm{Symb}_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) \cong \mathfrak{g}_-(I)$.

The organization of this paper is as follows. In Section 2 by studying the G-action on family of rational curves and the G-invariant minimal distributions on $\mathbf{S}$ we give a characterization of $\mathfrak{g}_-(I)$, which is a variation of Serre's theorem on simple Lie algebras. In Section 3 we firstly study the basic properties of Fano deformations and symbol algebras in Setting 1.11, and then prove Proposition 1.12. With the help of Proposition 1.12 and the characterization of $\mathfrak{g}_-(I)$, we give the proof of Proposition 1.7 in Section 3. In Section 4 we prove the rigidity results by applying Proposition 1.7. In Subsection 4.1 we prove Theorems 1.4 and 1.10 by assuming Propositions 1.8 and 1.9. The proof of Proposition 1.9 is given in Subsection 4.4. Theorem 1.6 is proved in Section 4.3, and with the help of this theorem we prove Proposition 1.8 in Subsection 4.4. Finally we analysis the possible Fano degeneration of $\mathbb{F}(1, 2, Q^6)$.

2. Geometry on rational homogeneous spaces

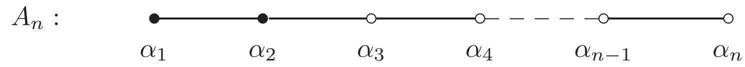
2.1. **Distributions and families of lines.** In this subsection, we collect the geometric properties on rational homogeneous spaces, which are useful in this paper. These results are classical, and most of them are stated without proof.

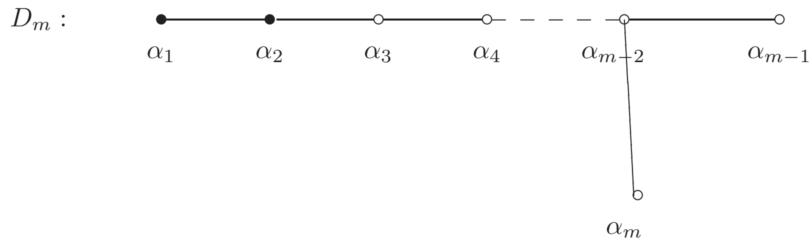
Setting 2.1. Let G be a connected semisimple algebraic group of adjoint type such that each simple factor is of type ADE, B be a Borel subgroup, and R be the set of simple roots. Fix a subset I of R and denote by $J := R \setminus I$.

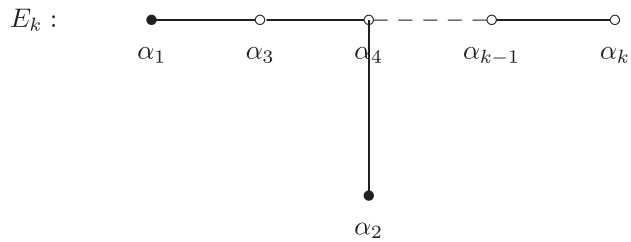
Denote by $P_I := \bigcap_{\alpha \in I} P_\alpha$, where P_α is the associated maximal parabolic subgroup of G which contains B. Denote by P_I^- the opposite parabolic subgroup of P_I, and $G_0 := P_I \cap P_I^-$.

Definition 2.2. Denote by $\mathfrak{g}$ the Lie algebra of G. Let Λ be the set of all roots of G and $\mathfrak{h}$ the fixed Cartan subalgebra of $\mathfrak{g}$. For each $\eta \in \Lambda$, denote by $\mathfrak{g}_\eta$ the 1-dimensional linear subspace of $\mathfrak{g}$ with weight η. We can write $\eta = \sum_{\alpha \in R} n_\alpha \alpha$, where either all n_α are nonnegative integers or all n_α are nonpositive integers. Define $\deg_I \eta = \sum_{\alpha \in I} n_\alpha$. For each $k \in \mathbb{Z}$ denote by $\Lambda_k(I)$ the set of elements $\eta \in \Lambda$ with $\deg_I(\eta) = k$. Equip a grading on $\mathfrak{g}$ such that $\mathfrak{g}_k(I) := \bigoplus_{\eta \in \Lambda_k(I)} \mathfrak{g}_\eta$ for $k \neq 0$ and $\mathfrak{g}_0(I) := \mathfrak{h} \oplus (\bigoplus_{\eta \in \Lambda_0(I)} \mathfrak{g}_\eta)$. Then $\mathfrak{g}$ becomes a graded Lie algebra. Moreover $\mathfrak{g}_0$, $\mathfrak{p}_I := \bigoplus_{k \geq 0} \mathfrak{g}_k$ and $\mathfrak{p}_I^- := \bigoplus_{k \leq 0} \mathfrak{g}_k$ are Lie algebras of G_0, P_I and P_I^- respectively. When there is no confusion, we omit I in the expressions, for example $\mathfrak{g}_k := \mathfrak{g}_k(I)$. In case $I = R$, we may also write $\mathfrak{g}_-(G) := \mathfrak{g}_-(R)$ in order to emphasize on the group G.

A rational homogeneous space can be expressed by a marked Dynkin diagram. To explain the order of simple roots and the way to express a rational homogeneous space, we draw the marked Dynkin diagram corresponding to $G/P_{\{\alpha_1,\alpha_2\}}$ as follows, where $G = A_n, D_m$ or E_k with $n \geq 2$, $m \geq 4$ and $k = 6, 7, 8$ respectively.

(2.1) 

(2.2) 

(2.3) 

Since we assume G to be of adjoint type, the restriction of Adjoint representation induces an injective homomorphism $G_0 \subset GL(\mathfrak{g}_-(I))$, where $\mathfrak{g}_-(I) := \bigoplus_{k \geq 1} \mathfrak{g}_{-k}(I)$.

Notation 2.3. Given any $\alpha \in R$, denote by $N(\alpha)$ the set of simple roots that are next to α in the Dynkin diagram Γ_R of G, and set $N_J(\alpha) := N(\alpha) \cap J$. For each subset A of R, denote by a semisimple subgroup G_A of G associated to the Dynkin subdiagram Γ_A of Γ_R.

Definition 2.4. Set $\mathbf{S} := G/P_I$. Given a subset $A \subset I$, denote by $\mathbf{S}^A$ the central fiber of $\Phi^A : \mathbf{S} \to G/P_{I \setminus A}$, which is a rational homogeneous space of Picard number $|A|$. Given any $\alpha \in I$, each fiber of Φ^α is biholomorphic to $\mathbf{S}^\alpha$, which is covered by lines under its minimal embedding. Denote by $\mathcal{K}^\alpha(\mathbf{S})$ the family of these lines (associated with α) on $\mathbf{S}$. Indeed $\mathcal{K}^\alpha(\mathbf{S}) = G/P_{(I \cup N(\alpha)) \setminus \{\alpha\}}$, which can be concluded from the following commutative diagram of Tits fibrations

$$\begin{array}{ccc} G/P_{I \cup N(\alpha)} & \xrightarrow{\text{ev}} & G/P_I \\ \downarrow{\scriptstyle \mu} & & \downarrow \\ G/P_{(I \cup N(\alpha)) \setminus \{\alpha\}} & \longrightarrow & G/P_{I \setminus \{\alpha\}}. \end{array} \tag{2.4}$$

Denote by $\mathcal{C}^\alpha(\mathbf{S}) \subset \mathbb{P}(T\mathbf{S})$ the variety of tangent directions of $\mathcal{K}^\alpha(\mathbf{S})$, i.e. at each point $x \in \mathbf{S}$,

$$\mathcal{C}^\alpha_x(\mathbf{S}) = \bigcup_{C \in \mathcal{K}^\alpha_x(\mathbf{S})} \mathbb{P}(T_x C) \subset \mathbb{P}(T_x(\mathbf{S})),$$

Denote by $\mathbf{Z}^\alpha := \mathcal{C}^\alpha_p(\mathbf{S}) \subset \mathbb{P}(T_p\mathbf{S})$, where p is the base point of $\mathbf{S}$.

Remark 2.5. Tits [15] studied diagrams in the style of (2.4) and he called $\mu(ev^{-1}(z))$ the shadow of $z \in G/P_I$. This variety is biholomorphic to $\mathcal{C}^\alpha_x(\mathbf{S})$ for $x \in ev^{-1}(z) \subset G/P_I$. The notation $\mathcal{C}^\alpha_x(\mathbf{S})$ called the variety of minimal rational tangents (VMRT for short) at x of the minimal rational component $\mathcal{K}^\alpha(\mathbf{S})$. One could find more details about minimal rational components and VMRT in [6]. For more details about the properties of those $\mathcal{C}^\alpha$, one can consult [12].

The following results are straight-forward.

Lemma 2.6. *(i) The group G_J is the semisimple part of the reductive group G_0.*
(ii) The simple factor of G_J is of type ADE;
(iii) There is a natural G_0-action on $\boldsymbol{C}^\alpha$, and the action is transitive.

Lemma 2.7. *The tangent bundle of $\boldsymbol{S}$ is identified with $G \times^{P_I} (\mathfrak{g}/\mathfrak{p}_I)$. For each $\alpha \in I$ there exists a unique G-invariant holomorphic distribution*

$$\mathfrak{g}^\alpha(\boldsymbol{S}) := G \times^{P_I} ((\mathfrak{g}_{-1}(\alpha) + \mathfrak{p}_I)/\mathfrak{p}_I).$$

The G-invariant holomorphic distribution

$$\mathfrak{g}_{-1}(\boldsymbol{S}) := G \times^{P_I} ((\mathfrak{g}_{-1}(I) + \mathfrak{p}_I)/\mathfrak{p}_I)$$

satisfies that

$$\mathfrak{g}_{-1}(\boldsymbol{S}) = \bigoplus_{\alpha \in I} \mathfrak{g}^\alpha(\boldsymbol{S}) = \sum_{\alpha \in I} \mathfrak{g}^\alpha(\boldsymbol{S}) \subset T\boldsymbol{S}.$$

Lemma 2.8. *Take any $\alpha \in I$. Then*
(1) $\mathcal{C}^\alpha(\boldsymbol{S}) \subset \mathbb{P}(\mathfrak{g}^\alpha(\boldsymbol{S}))$;
(2) The inclusion $\boldsymbol{Z}^\alpha \subset \mathbb{P}(\mathfrak{g}_{-1}(\alpha))$ is G_0-equivariant;
(3) $\boldsymbol{Z}^\alpha$ is the unique closed G_0-orbit in $\mathbb{P}(\mathfrak{g}_{-1}(\alpha))$;
(4) $\boldsymbol{Z}^\alpha$ is nondegenerate in $\mathbb{P}(\mathfrak{g}_{-1}(\alpha))$.
(5) The G_J-action on $\boldsymbol{Z}^\alpha$ induces the isomorphism

$$\boldsymbol{Z}^\alpha \cong G_J/P_{N_J(\alpha)} \cong \prod_{\beta \in N_J(\alpha)} G_J/P_\beta.$$

Particularly each $\boldsymbol{Z}^\alpha_\beta := G_J/P_\beta \cong G_0/P_\beta$ is a rational homogeneous space of Picard number one, where $\alpha \in I$ and $\beta \in N_J(\alpha)$.

Remark 2.9. Given $\alpha \in I$ and $\beta \in N_J(\alpha)$, as in Definition 2.4 we have a family of rational curves $\mathcal{K}^\beta(\mathbf{Z}^\alpha)$ and its associated variety of tangent directions $\mathcal{C}^\beta(\mathbf{Z}^\alpha)$ on the rational homogeneous space $\mathbf{Z}^\alpha$. As in Lemma 2.7 we can construct the distribution $\mathfrak{g}^\beta(\mathbf{Z}^\alpha)$ on $\mathbf{Z}^\alpha$, which is the minimal G_0-invariant (hence G_J-invariant) distribution associated with the root $\beta \in N_J(\alpha) \subset J$. Denote by $\widehat{\mathbf{Z}}^\alpha \subset \mathfrak{g}_{-1}(\alpha)$ the affine cone of $\mathbf{Z}^\alpha \subset \mathbb{P}(\mathfrak{g}_{-1}(\alpha))$. Then can define $\widehat{\mathcal{C}}^\alpha(\mathbf{S})$, $\mathcal{C}^\beta(\widehat{\mathbf{Z}}^\alpha)$, $\widehat{\mathcal{C}}^\beta(\widehat{\mathbf{Z}}^\alpha)$ and $\mathfrak{g}^\beta(\widehat{\mathbf{Z}}^\alpha)$ in an obvious way.

Notation 2.10. Write $J = \bigcup_{1 \le i \le \tau} J_i$, which is a disjoint union such that $\Gamma_{J_1}, \ldots, \Gamma_{J_\tau}$ are the connected components of the Dynkin diagram of Γ_J.

The following is straight-forward.

Lemma 2.11. *Take any $\alpha \in I$, and any $\beta \in N_J(\alpha)$. Then there exists a unique J_i containing β. Moreover, β is an end vertex of the Dynkin diagram Γ_{J_i}, and $\boldsymbol{Z}^\alpha_\beta \cong G_{J_i}/P_\beta$.*

The following result on automorphism groups of rational homogeneous spaces is straight-forward.

Lemma 2.12. *The natural homomorphism $G \to Aut^o(\boldsymbol{S})$ is bijective. Take $\alpha \in I$ and let $Aut^o(\widehat{\boldsymbol{Z}}^\alpha, \mathfrak{g}_{-1}(\alpha))$ be the identity component of*

$$Aut(\widehat{\boldsymbol{Z}}^\alpha, \mathfrak{g}_{-1}(\alpha)) := \{\varphi \in GL(\mathfrak{g}_{-1}(\alpha)) \mid \varphi \cdot \widehat{\boldsymbol{Z}}^\alpha = \widehat{\boldsymbol{Z}}^\alpha\}.$$

Then the natural homomorphism $G_0 \to Aut^o(\widehat{\boldsymbol{Z}}^\alpha)$ is surjective and $Aut^o(\widehat{\boldsymbol{Z}}^\alpha) = Aut^o(\widehat{\boldsymbol{Z}}^\alpha, \mathfrak{g}_{-1}(\alpha))$. Take any $\beta \in N_J(\alpha)$. Then the distribution $\mathfrak{g}^\beta(\boldsymbol{Z}^\alpha)$ on $\boldsymbol{Z}^\alpha$ is $Aut^o(\boldsymbol{Z}^\alpha)$-invariant.

Given $\alpha, \beta \in I$, $k \geq 1$ and $[C^\alpha] \in \mathcal{K}^\alpha(\mathbf{S})$, we can describe the splitting type along $C^\alpha \cong \mathbb{P}^1$ of distributions $\mathfrak{g}^\beta_{-k}(\mathbf{S})$ on $\mathbf{S}$. To obtain it, we need to apply Grothendiecks splitting theorem for principal bundles on $\mathbb{P}^1$ with reductive structure groups and associated vector bundles [3].

Proposition 2.13. *(Grothendieck). Let $\mathcal{O}(1)^*$ be the $\mathbb{C}^*$-principal bundle on $\mathbb{P}^1$ corresponding to the line bundle $\mathcal{O}(1)$. Let L be a reductive complex Lie group. Up to conjugation, any L-principal bundle on $\mathbb{P}^1$ is associated to $\mathcal{O}(1)^*$ by a group homomorphism from $\mathbb{C}^*$ to a maximal torus of L. If E is the coroot of $sl(2)$, such a group homomorphism is determined by the image of E in $\mathfrak{h}$, a fixed Cartan subalgebra of L. Given a representation of L with weights $\mu_1, \dots, \mu_\ell \in \mathfrak{h}^*$, the associated vector bundle on $\mathbb{P}^1$ splits as $\mathcal{O}(\mu_1(E)) \oplus \cdots \oplus \mathcal{O}(\mu_\ell(E))$, where $\mu_j(E)$ denotes the value of μ_j on the image of E in $\mathfrak{h}$.*

Given $\beta \in I$ and $[C^\alpha] \in \mathcal{K}^\alpha(\mathbf{S})$, we can identify C^α with $\exp(sl_\beta(2))/\exp(\mathfrak{g}_\beta \oplus [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}])$, where $sl_\beta(2) := \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta} \oplus [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \subset \mathfrak{g}$ is a subalgebra isomorphic to $sl(2)$, and $\mathfrak{g}_\beta \oplus [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] = \mathfrak{p}_I \cap sl_\beta(2)$ is a Borel subalgebra. Then as a direct consequence of Proposition 2.13, we have the following result.

Proposition 2.14. *Given $\alpha, \beta \in I$, $k \geq 1$ and $[C^\alpha] \in \mathcal{K}^\alpha(\boldsymbol{S})$, we have*

$$\mathfrak{g}^\beta_{-k}(\boldsymbol{S})|_{C^\alpha} = \bigoplus_{\gamma \in \Lambda_k(\beta)} \mathcal{O}_{\mathbb{P}^1}(\langle \gamma, \alpha \rangle).$$

2.2. **Characterization of the nilradical of a parabolic subalgebra.** We want to give a description of the algebra $\mathfrak{g}_-(I) := \bigoplus_{k \geq 1} \mathfrak{g}^I_{-k}$. When $I = R$, it is described by Serre's theorem on semisimple Lie algebra in the following way.

Proposition 2.15. [4, Section 18] *Let R be a set of simple roots for $\mathfrak{g}$ and choose a nonzero element $x_\alpha \in \mathfrak{g}_{-\alpha}$ for each $\alpha \in R$. Then the subalgebra $\mathfrak{g}_-(R)$ of $\mathfrak{g}$ is the quotient of the free Lie algebra generated by $\{x_\alpha \mid \alpha \in R\}$ by the relations*

$$ad(x_\alpha)^{-\langle \beta, \alpha \rangle + 1}(x_\beta) = 0 \text{ for all } \alpha \neq \beta \in R.$$

Proposition 2.16. *Denote by $\mathbb{F}(\mathfrak{g}_{-1}(I))$ the free graded Lie algebra generated by $\mathfrak{g}_{-1}(I)$. Fix an arbitrary $z_\alpha \in \widehat{\boldsymbol{Z}}^\alpha \setminus \{0\}$ for each $\alpha \in I$. Let $\mathcal{I} := \mathcal{I}(z_\alpha, \alpha \in I)$ be the ideal of $\mathbb{F}(\mathfrak{g}_{-1}(I))$ generated by the following relations:*

(i) for all $\alpha' \neq \alpha'' \in I$ and all $(v', v'') \in G_0 \cdot (z_{\alpha'}, z_{\alpha''}) \in \widehat{\boldsymbol{Z}}^{\alpha'} \times \widehat{\boldsymbol{Z}}^{\alpha''}$,

$$(adv')^{-\langle \alpha'', \alpha' \rangle + 1}(v'') = 0;$$

(ii) for all $\alpha \in I$, $\beta \in N_J(\alpha)$, $v \in \widehat{\boldsymbol{Z}}^\alpha \setminus \{0\}$, and $u \in \mathfrak{g}^\beta_v(\widehat{\boldsymbol{Z}}^\alpha)$,

$$(adv)^{-\langle \beta, \alpha \rangle}(u) = 0.$$

Then $\mathfrak{g}_-(I) := \bigoplus_{i \geq 1} \mathfrak{g}_{-i}$ is isomorphic to $\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}$ as graded nilpotent Lie algebra. In particular, up to isomorphism $\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}(z_\alpha, \alpha \in I)$ is independent of the choice of those $z_\alpha \in \widehat{\boldsymbol{Z}}^\alpha \setminus \{0\}$.

Proof. Step 1. We will show that $\mathfrak{g}_-(I)$ satisfies conditions (i) and (ii).

The inclusion $\mathfrak{g}_-(R) \subset \mathfrak{p}_I^-$ induces a semidirect product decomposition of Lie algebra structure $\mathfrak{g}_-(R) = \mathfrak{n}_0 \rtimes \mathfrak{g}_-(I)$, where $\mathfrak{n}_0 := \mathfrak{g}_-(R) \cap \mathfrak{g}_0(I)$. For each $\alpha \in R$ we choose a nonzero element $x_\alpha \in \mathfrak{g}_{-\alpha}$.

For those $\alpha \in I$, we write $z_\alpha := x_\alpha$. Since the point $\mathbb{P}(\mathfrak{g}_{-\alpha}) \in \mathbf{Z}^\alpha \subset \mathbb{P}(\mathfrak{g}^\alpha_{-1})$, we have $z_\alpha \in \widehat{\mathbf{Z}}^\alpha \setminus \{0\}$. For those $\beta \in J := R \setminus I$, we have $x_\beta \in \mathfrak{n}_0$. Denote by $\pi : \mathfrak{n}_0 \subset \mathfrak{g}_0(I) \to \mathrm{aut}(\mathfrak{g}_-(I))$ the homomorphism induced by the adjoint representation, and write $\eta_\beta := \pi(x_\beta) \in \mathrm{aut}(\mathfrak{g}_-(I))$. Then by Proposition 2.15, we have

$$(\mathrm{ad} z_{\alpha'})^{-\langle \alpha'', \alpha' \rangle + 1}(z_{\alpha''}) = 0, \qquad \text{for all } \alpha' \neq \alpha'' \in I, \tag{2.5}$$

$$(\mathrm{ad} z_\alpha)^{-\langle \beta, \alpha \rangle}(\eta_\beta(z_\alpha)) = 0, \quad \text{for all } \alpha \in I \text{ and } \beta \notin I. \tag{2.6}$$

Since the Lie algebra $\mathfrak{g}_-(I)$ is a G_0-module, the conclusion (2.5) implies the condition (i) in the statement of Proposition 2.16.

Now let us check the condition (ii). By (2.6), $\eta_\beta(z_\alpha)=0$ for $\beta\in J\setminus N_J(\alpha)$. Now suppose that $\beta\in N_J(\alpha)$. Then $\eta_\beta(v_\alpha)=[x_\beta,x_\alpha]$ is a nonzero vector in $\mathfrak{g}_{-1}(\alpha)=\sum\limits_{\gamma\in\Lambda_{-1}(\alpha)}\mathfrak{g}_\gamma$. Moreover, $\mathbb{P}(\eta_\beta(z_\alpha))$ is a point in $H^\beta_{[z_\alpha]}(\mathbf{Z}^\alpha)\subset\mathbb{P}(\mathfrak{g}_{-1}(\alpha))$. Since $\mathfrak{g}_-(I)$ is a G_0-module, we have

$$(\mathrm{ad}(\varphi\cdot z_\alpha))^{-\langle\beta,\alpha\rangle}(\varphi\cdot\eta_\beta(z_\alpha))=0\quad\text{ for all }\varphi\in G_0.$$

Denote by $P'_\beta:=P_\beta\cap G_0$. Then $P'_\beta\cdot z_\alpha\subset\mathbb{C}z_\alpha$, and $P'_\beta\cdot\eta_\beta(z_\alpha)=\widehat{\mathbf{Z}}^\beta_{z_\alpha}(\widehat{\mathbf{Z}}^\alpha)$. Since $\widehat{\mathbf{Z}}^\beta_{z_\alpha}(\widehat{\mathbf{Z}}^\alpha)$ is nondegenerate in the subspace $\mathfrak{g}^\beta_{z_\alpha}(\widehat{\mathbf{Z}}^\alpha)$ of $\mathfrak{g}_{-1}(\alpha)$, we have

$$\mathrm{ad}(z_\alpha)^{\langle\beta,\alpha\rangle}(u)=0\text{ for all }u\in\mathfrak{g}^\beta_{z_\alpha}(\widehat{\mathbf{Z}}^\alpha).$$

It follows that

$$(\mathrm{ad}(\varphi\cdot z_\alpha))^{-\langle\beta,\alpha\rangle}(\varphi\cdot u)=0\text{ for all }\varphi\in G_0\text{ and all }u\in\mathfrak{g}^\beta_{z_\alpha}(\widehat{\mathbf{Z}}^\alpha).$$

Since $\mathbf{Z}^\alpha$ is G_0-transitive and $\mathfrak{g}^\beta(\widehat{\mathbf{Z}}^\alpha)$ is G_0-equivariant, the condition (ii) holds.

Step 2. Show that the isomorphism $\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}(z_\alpha,\alpha\in I)$ is independent of the choice of those $z_\alpha\in\widehat{\mathbf{Z}}^\alpha\setminus\{0\}$.

Now take any $z'_\alpha\in\widehat{\mathbf{Z}}^\alpha\setminus\{0\}$ for each $\alpha\in I$. Since the inclusion $\widehat{\mathbf{Z}}^\alpha\subset\mathfrak{g}_{-1}(\alpha)$ is G_0-equivariant and $\widehat{\mathbf{Z}}^\alpha\setminus\{0\}$ is a single G_0-orbit, there exists an isomorphism $\varphi^\alpha:\mathfrak{g}_{-1}(\alpha)\to\mathfrak{g}_{-1}(\alpha)$ of G_0-modules sending $\widehat{\mathbf{Z}}^\alpha$ onto itself and $\varphi^\alpha(z_\alpha)=z'_\alpha$. These φ^α induce an isomorphism $\mathbb{F}(\mathfrak{g}_{-1}(I))\to\mathbb{F}(\mathfrak{g}_{-1}(I))$ whose restriction sending $\mathcal{I}(z_\alpha,\alpha\in I)$ onto $\mathcal{I}(z'_\alpha,\alpha\in I)$. Hence we have an isomorphism

$$\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}(z_\alpha,\alpha\in I)\cong\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}(z'_\alpha,\alpha\in I).$$

Step 3. Show the isomorphism $\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}\cong\mathfrak{g}_-(I)$.

By Step 1 and Step 2 we can set $z_\alpha:=x_\alpha\in\widehat{\mathbf{Z}}^\alpha\setminus\{0\}$ for each $\alpha\in I$ and get a surjective homomorphism of G_0-modules $\psi:\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I}\to\mathfrak{g}_-(I)$. It should be noticed that

$$\begin{aligned}&\mathcal{F}:=\mathfrak{g}_0(I)\oplus(\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I})\cong(\mathfrak{g}_0(I)\oplus\mathbb{F}(\mathfrak{g}_{-1}(I)))/\mathcal{I},\text{ and}\\&\mathfrak{p}^-_I:=\mathfrak{g}_0(I)\oplus\mathfrak{g}_-(I)=\bigoplus_{i\leq0}\mathfrak{g}_i(I)\end{aligned}$$

are both graded Lie algebras as well as G_0-modules. Moreover, ψ induces a surjective homomorphism between Lie algebras as well as between G_0-modules: $\psi':\mathcal{F}\to\mathfrak{p}^-_I$.

Similarly as in Step 1 let $\mathfrak{n}_0$ be the Lie subalgebra of $\mathfrak{g}_0(I)$ generated by those x_β with $\beta\in J$. We have $\mathfrak{g}_-(R)=\mathfrak{n}_0\oplus\mathfrak{g}_-(I)\subset\mathfrak{p}^-_I$, and set $\widetilde{\mathcal{F}}:=\mathfrak{n}_0\oplus(\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I})\subset\mathcal{F}$. Then the restriction of ψ' induces a surjective homomorphism of Lie algebras

$$\widetilde{\psi}:\widetilde{\mathcal{F}}\to\mathfrak{g}_-(R).$$

Denote by $\widetilde{\mathbb{F}}$ the free graded Lie algebra generated by those x_γ with $\gamma\in R$. Let $\widetilde{\mathcal{I}}$ be the ideal of $\widetilde{\mathbb{F}}$ generated by the set

$$\{(\mathrm{ad}x_{\gamma'})^{-\langle\gamma'',\gamma'\rangle+1}(x_{\gamma''})\mid\gamma'\neq\gamma''\in R\}.$$

There is a commutative diagram of Lie algebras as follows:

$$\begin{array}{ccc}\widetilde{\mathbb{F}} & & \\ {\scriptstyle\theta_1}\downarrow & \searrow^{\theta_2} & \\ \widetilde{\mathcal{F}} & \xrightarrow[\widetilde{\psi}]{} & \mathfrak{g}_-(R).\end{array}$$

We claim that $\theta_1(\widetilde{\mathcal{I}})=0$. Equivalently we claim that for all $\gamma'\neq\gamma''\in R$,

$$\theta_1((\mathrm{ad}x_{\gamma'})^{-\langle\gamma'',\gamma'\rangle+1}(x_{\gamma''}))=0.\tag{2.7}$$

Case 1. Suppose $\gamma', \gamma'' \in I$. Recall our definition of $\mathcal{I}$ for $z_\alpha := x_\alpha \in \widehat{\mathbf{Z}}^\alpha \setminus \{0\}$. Then in this case (2.7) follows from the condition (i) of Proposition 2.16.

Case 2. Suppose $\gamma' \in I$ and $\gamma'' \in J := R \setminus I$. The condition (ii) of Proposition 2.16 implies (2.7) under the additional assumption that $\gamma'' \in N_J(\gamma')$ by noting that $\theta_1(x_{\gamma'}) \in \mathfrak{g}_{-1}(I)$ and $\theta_1(x_{\gamma''}) \in \mathfrak{n}_0$.

Now for $\gamma' \in I$ and $\gamma'' \in J \setminus N_J(\gamma')$, we have $\langle \gamma'', \gamma' \rangle = 0$. The (2.7) becomes that $[\theta_1(x_{\gamma''}), \theta_1(x_{\gamma'})]_{\widetilde{\mathcal{F}}} = 0$. The latter can be deduced from the $\mathfrak{g}_0$-action (hence the $\mathfrak{n}_0$-action) on $\mathfrak{g}_{-1}(I)$.

Case 3. Suppose $\gamma' \in J := R \setminus I$ and $\gamma'' \in I$. Similarly in this case (2.7) can also be deduced from the $\mathfrak{g}_0$-action (hence the $\mathfrak{n}_0$-action) on $\mathfrak{g}_{-1}(I)$.

Case 4. Suppose $\gamma', \gamma'' \in J := R \setminus I$. In this case (2.7) can also be deduced from the Lie algebra structure of $\mathfrak{n}_0$ (coming from that of $\mathfrak{g}_0$).

In summary, the claim $\theta_1(\widetilde{\mathcal{I}}) = 0$ holds. Then it induces a homomorphism

$$\widetilde{\theta}_1 : \widetilde{\mathbb{F}}/\widetilde{\mathcal{I}} \to \widetilde{\mathcal{F}}.$$

By the construction of $\widetilde{\mathcal{F}}$, the morphism θ_1 is surjective. Hence $\widetilde{\theta}_1$ is surjective. By Proposition 2.15, θ_2 induces an isomorphism $\widetilde{\theta}_2 : \widetilde{\mathbb{F}}/\widetilde{\mathcal{I}} \cong \mathfrak{g}_-(R)$. Hence $\widetilde{\psi}$ is an isomorphism preserving gradings, and its restriction gives an isomorphism of graded nilpotent Lie algebras $\mathbb{F}(\mathfrak{g}_{-1}(I))/\mathcal{I} \cong \mathfrak{g}_-(I)$. □

3. Fano deformation of rational homogeneous spaces

From now on, we study the family $\mathcal{X}$ over Δ in Setting 1.11. The organization of this section is as follows. In subsection 3.1, we study the basic property of minimal rational curves and Cartier divisors on the family $\mathcal{X}/\Delta$. In subsection 3.2, we study the property of symbol algebras and prove Proposition 1.12, which is reformulated in Proposition 3.19. In subsection 3.3, we prove Theorem 3.22, which implies Proposition 1.7 as a corollary.

3.1. Minimal rational curves on the family.

The following result is due to Wiśniewski [18].

Proposition 3.1. [18, Theorem 1] *We can identify the Mori cones $\overline{NE}(\mathcal{X}/\Delta) = \overline{\mathcal{X}_t}$ for all $t \in \Delta$.*

The following is a classical result on the rational homogeneous space $\mathbf{S} := G/P_I$.

Lemma 3.2. *The Mori cone $\overline{NE}(\boldsymbol{S})$ is a simplicial cone generated by those $R_\alpha := \mathbb{R}^+[\mathcal{K}^\alpha(\boldsymbol{S})]$ with $\alpha \in I$ i.e. $\dim \overline{NE}(\boldsymbol{S})$ equals to the cardinality of I, and $\overline{NE}(\boldsymbol{S}) = \sum_{\alpha \in I} R_\alpha$, where $\mathcal{K}^\alpha(\boldsymbol{S})$ is as in Definition 2.4. The set of extremal faces of $\overline{NE}(\boldsymbol{S})$ can be identified with the set of subsets of I.*

As a direct consequence of Proposition 3.1 and Lemma 3.2, we have the following result.

Proposition-Definition 3.3. *For each $A \subset I$, denote by $\Phi^A : \boldsymbol{S} \to G/P_{I\setminus A}$ the Mori contraction associated with the extremal face $\sum_{\alpha \in A} R_\alpha$ of $\overline{NE}(\boldsymbol{S})$. We can extend it to be a relative Mori contraction $\pi^A : \mathcal{X} \to \mathcal{X}^A$. We denote by $\pi^A_t := \pi^A|_{\mathcal{X}_t}$ for each $t \in \Delta$.*

Notation 3.4. Given a subset $A \subset I$ and a point $x \in \mathcal{X}$, denote by F^A_x the fiber of $\pi^A : \mathcal{X} \to \mathcal{X}^A$ passing through x. In particular, if $x \notin \mathcal{X}_0$ then $F^A_x \cong \mathbf{S}^A$, where $\mathbf{S}^A$ is defined in Definition 2.4.

Proposition 3.5. *Take any $\alpha \in I$. Then $F^\alpha_x \cong \boldsymbol{S}^\alpha$ for $x \in \mathcal{X}_0$ general.*

Proof. The fiber F^α_x is a smooth Fano deformation of $\mathbf{S}^\alpha$. Then the conclusion follows from the Fano deformation rigidity of $\mathbf{S}^\alpha$, which is obtained by J.-M. Hwang and N. Mok [9, Main Theorem]. □

By Proposition 3.1, Proposition 3.5 and intersection theory on rational homogeneous spaces, we have the following result.

Proposition-Definition 3.6. *Take any $\alpha \in I$. Denote by $\mathcal{K}^\alpha(\mathcal{X})$ the irreducible component of Chow$(\mathcal{X})$ extending $\mathcal{K}^\alpha(\boldsymbol{S})$. Take any $[C] \in \mathcal{K}^\alpha(\mathcal{X})$. Then C is an irreducible and reduced rational curve on $\mathcal{X}_t$ for a unique $t \in \Delta$. If either $t \neq 0$ or $[C]$ is general in $\mathcal{K}^\alpha(\mathcal{X}_0)$, then $C \cong \mathbb{P}^1$. Moreover, there exists a unique $\mathcal{L}^\alpha \in Pic(\mathcal{X}/\Delta)$ such that $(\mathcal{L}^\alpha \cdot \mathcal{K}^\beta(\mathcal{X})) = \delta_{\alpha\beta}$ for all $\beta \in I$.*

Proposition 3.7. *Any two points $x, y \in \mathcal{X}_0$ can be connected by chains of elements in $\bigcup_{\alpha\in I} \mathcal{K}^\alpha(\mathcal{X}_0)$.*

Proof. Consider the rational map $\psi : \mathcal{X}_0 \dashrightarrow Z$ defined by equivalence relation induced by $\bigcup_{\alpha\in I} \mathcal{K}^\alpha(\mathcal{X}_0)$. For the existence and the property of such a rational map, see [11, Theorem IV.4.16]. Suppose $\dim Z \geq 1$. Take a general divisor $D \subset Z$ and a general point $x \in Z \setminus D$. Then $\psi^{-1}(z)$ is a closed subvariety of $\mathcal{X}_0$ which has empty intersection with the indeterminant locus of ψ. Thus $\psi^{-1}(z) \cap E = \emptyset$, where $E := \overline{\psi^{-1}(D)} \subset \mathcal{X}_0$ is an effective divisor. For each $x \in \psi^{-1}(z)$ and each $\alpha \in I$, we have $\mathcal{K}^\alpha_x(\mathcal{X}_0) \neq \emptyset$. By definition $C^\alpha_x \subset \psi^{-1}(z)$ (hence $C^\alpha_x \cap E = \emptyset$) for all $[C^\alpha_x] \in \mathcal{K}^\alpha_x(\mathcal{X}_0)$. It follows that $(E \cdot \mathcal{K}^\alpha(\mathcal{X}_0)) = 0$ for all $\alpha \in I$, which implies that $E = 0$. It contradicts the choice of E. Then the conclusion follows. □

3.2. **Properties of symbol algebras.**

Definition 3.8. Given a distribution $\mathcal{V}$ on a complex manifold Y, define the weak derived system $\mathcal{V}^{-k}$ inductively by

$$\begin{aligned}
&\mathcal{V}^0 := 0,\\
&\mathcal{V}^{-1} := \mathcal{V},\\
&\mathcal{V}^{-k-1} := \mathcal{V}^{-k} + [\mathcal{V}^{-1}, \mathcal{V}^{-k}], \quad k \geq 1.
\end{aligned}$$

Denote by $\mathcal{V}^{-\infty} := \lim_{k\to\infty} \mathcal{V}^{-k}$. There exists a positive integer d such that $\mathcal{V}^{-d+i} = \mathcal{V}^{-d}$ for all $i \geq 0$. In particular, $\mathcal{V}^{-\infty} = \mathcal{V}^{-d}$ and it is integrable on Y. In an open neighborhood of a general point $y \in Y$ these $\mathcal{V}^{-k}$'s are subbundles of TY. We define the symbol algebra of $\mathcal{V}$ at y as the graded nilpotent Lie algebra $\mathrm{Symb}_y(\mathcal{V}) := \bigoplus_{1\leq k\leq d} \mathcal{V}^{-k}_y/\mathcal{V}^{-k+1}_y$. We say $\mathcal{V}$ is bracket-generating if $\mathcal{V}^{-\infty} = TY$. When $\mathcal{V}$ is bracket-generating, $\dim \mathrm{Symb}_y(\mathcal{V}) = \dim T_yY = \dim Y$.

Notation 3.9. Take a subset $A \subset I$. The distribution

$$\mathfrak{g}^A_{-1}(\mathbf{S}) := G \times^{P_I} \Big(\sum_{\alpha\in A} (\mathfrak{g}_{-1}(\alpha) + \mathfrak{p}_I)/\mathfrak{p}_I\Big)$$

on $\mathbf{S}$ can be extended to be a meromorphic distribution $\mathcal{D}^A$ on $\mathcal{X}$, which is well-defined on general points of $\mathcal{X}_0$ and all points of $\bigcup_{t\neq 0} \mathcal{X}_t$. Take a general point $x \in \mathcal{X}_0$. Denote by $\mathfrak{m}_x(A)$ the symbol algebra of $\mathcal{D}^A_x$, i.e. $\mathfrak{m}_x(A) := \mathrm{Symb}_x(\mathcal{D}^A)$. We say $\mathfrak{m}_x(A)$ is standard if it is isomorphic to the symbol algebra of the distribution $\mathfrak{g}^A_{-1}(\mathbf{S})$ on $\mathbf{S}$. Otherwise, we say $\mathfrak{m}_x(A)$ is degenerate. When $A = I$, we omit the superscript I and write $\mathcal{D} := \mathcal{D}^I$ briefly.

Proposition 3.10. *The unique integrable meromorphic distribution on $\mathcal{X}_0$ containing $\mathcal{D}$ is the tangent bundle. Consequently, Then the distribution $\mathcal{D}$ is bracket-generating on $\mathcal{X}_0$ and $\dim \mathfrak{m}_x(I) = \dim \mathcal{X}_0$ for $x \in \mathcal{X}_0$ general.*

Proof. Let $\mathcal{V}$ be an integrable meromorphic distribution on $\mathcal{X}_0$ containing $\mathcal{D}$, and M be a general leaf. Take $\alpha \in I$ and $[C] \in \mathcal{K}^\alpha(\mathcal{X}_0)$ with $C \cap M \neq \emptyset$. Then at a point $x \in C \cap M$ we have $T_xC \subset \mathcal{D}_x \subset \mathcal{V}_x$. Thus C is contained in the analytic closure of M. By Proposition 3.7, the leaf closure of M is $\mathcal{X}_0$, completing the proof. □

To continue, we need to recall some concepts and results related with Cartan connections.

Definition 3.11. Fix a positive integer ν. Let $\mathfrak{l}_- = \mathfrak{l}_{-1} \oplus \cdots \oplus \mathfrak{l}_{-\nu}$ be a graded nilpotent Lie algebra. Denote by $\mathrm{grAut}(\mathfrak{l}_-)$ the group of Lie algebra automorphisms of $\mathfrak{l}_-$ preserving the gradation and by $\mathrm{graut}(\mathfrak{l}_-)$ its Lie algebra. Fix a connected algebraic subgroup $L_0 \subset \mathrm{grAut}(\mathfrak{l}_-)$ and its Lie algebra $\mathfrak{l}_0 \subset \mathrm{graut}(\mathfrak{l}_-)$. For each positive integer i, the i-th prolongation of $\mathfrak{l}_0$ is inductively defined as

$$\mathfrak{l}_i := \Big\{ \phi \in \mathrm{Hom}(\mathfrak{l}_-, \bigoplus_{-\nu \le j < i} \mathfrak{l}_j)_i := \bigoplus_{k=1}^{\nu} \mathrm{Hom}(\mathfrak{l}_{-k}, \mathfrak{l}_{-k+i}),$$
$$\phi([v_1, v_2]_{\mathfrak{l}_-}) = \phi(v_1)(v_2) - \phi(v_2)(v_1), \quad \text{for any} \quad v_1, v_2 \in \mathfrak{l}_- \Big\}.$$

Here $[\quad,\quad]_{\mathfrak{l}_-}$ denotes the Lie bracket on $\mathfrak{l}_-$ and, if $\phi(v_1) \in \mathfrak{l}_-$, then

$$\phi(v_1)(v_2) := [\phi(v_1), v_2]_{\mathfrak{l}_-}.$$

For convenience, we put $\mathfrak{l}_{-\nu-j} = 0$ for every positive integer j and write

$$\mathfrak{l}_- = \bigoplus_{k \in \mathbb{N}} \mathfrak{l}_{-k}.$$

The graded vector space

$$\mathfrak{l} := \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}_k$$

is a graded Lie algebra and called the universal prolongation of $(\mathfrak{l}_0, \mathfrak{l}_-)$.

The following result on prolongations is due to K. Yamaguchi [19].

Proposition 3.12. [19, Theorem 5.2] *Suppose in Setting 2.1 that G is simple.*

(*i*) *Suppose G/P_I is not biholomorphic to a projective space. Then $\mathfrak{g}$ is the universal prolongation of $(\mathfrak{g}_-(I), \mathfrak{g}_0(I))$.*

(*ii*) *Suppose $|I| \ge 2$ and (G, I) is not one of the following:*

$$(A_m, \{\alpha_1, \alpha_i\}), \quad 2 \le i \le m; \tag{3.1}$$
$$(A_m, \{\alpha_i, \alpha_m\}), \quad 1 \le i \le m-1; \tag{3.2}$$

Then $\mathfrak{g}_0(I)$ is isomorphic to $\mathrm{aut}(\mathfrak{g}_-(I))$*, the Lie algebra of* $\mathrm{gr}Aut(\mathfrak{g}_-)$.

Definition 3.13. Let L be a connected algebraic group and $L^0 \subseteq L$ be a connected algebraic subgroup. Let $\mathfrak{l}^0 \subset \mathfrak{l}$ be their Lie algebras. A Cartan connection of type (L, L^0) on a complex manifold M with $\dim M = \dim L/L^0$ is a principal L^0-bundle $E \to M$ with a $\mathfrak{l}$-valued 1-form Υ on E with the following properties.

(i) For $A \in \mathfrak{l}^0$, denote by ζ_A the fundamental vector field on E induced by the right L^0-action on E. Then $\Upsilon(\zeta_A) = A$ for each $A \in \mathfrak{l}^0$.
(ii) For $a \in L^0$, denote by $R_a : E \to E$ the right action of a. Then $R_a^* \Upsilon = \mathrm{Ad}(a^{-1}) \circ \Upsilon$ for each $a \in L^0$.
(iii) The linear map $\Upsilon_y : T_y E \to \mathfrak{l}$ is an isomorphism for each $y \in E$.

The Cartan connection $(E \to M, \Upsilon)$ is flat if the curvature $\kappa := d\Upsilon + \frac{1}{2}[\Upsilon, \Upsilon]$ vanishes.

Example 3.14. Let L and L^0 be as in Definition 3.13, and denote by ω^{MC} the Maurer-Cartan form on L. Then $(L \to L/L^0, \omega^{MC})$ is a flat Cartan connection of type (L, L^0).

Definition 3.15. Let $\mathfrak{l}_- = \oplus_{k \in \mathbb{N}} \mathfrak{l}_{-k}$ be a graded nilpotent Lie algebra with $\mathfrak{l}_{-j} = 0$ for all j larger than for a fixed positive integer ν. A filtration of type $\mathfrak{l}_-$ on a complex manifold M is a filtration $(F^j M, j \in \mathbb{Z})$ on M such that

(i) $F^k M = 0$ for all $k \ge 0$;
(ii) $F^{-k} M = TM$ for all $k \ge \nu$; and
(iii) for any $x \in M$, the symbol algebra

$$\mathrm{gr}_x(M) := \bigoplus_{i \in \mathbb{N}} F_x^{-i} M / F_x^{-i+1} M$$

is isomorphic to $\mathfrak{l}_-$ as graded Lie algebras.

The graded frame bundle of the manifold M with a filtration of type $\mathfrak{l}_-$ is the $\mathrm{grAut}(\mathfrak{l}_-)$-principal bundle $\mathrm{grFr}(M)$ on M whose fiber at x is the set of graded Lie algebra isomorphisms from $\mathfrak{l}_-$ to $\mathrm{gr}_x(M)$. Let $L_0 \subset \mathrm{grAut}(\mathfrak{l}_-)$ be a connected algebraic subgroup. An L_0-structure (subordinate to the filtration) on M means an L_0-principal subbundle $E \subset \mathrm{grFr}(M)$.

Remark 3.16. Now let us summarize the work of A. Čap and H. Schichl [2] on the construction of Cartan connections of type (G, P_I). For more detail of our summarization, see Sections 3.20-3.23 in [2]. Let G/P_I be as in Setting 2.1 and suppose that $\mathfrak{g}$ is the universal prolongation of $(\mathfrak{g}_-(I), \mathfrak{g}_0(I))$. Suppose there is a differential system $\mathcal{V}$ and a principal bundle E on a complex manifold M such that the weak derivatives of $\mathcal{V}$ induces a filtration of type $\mathfrak{g}_-(I)$ and $E \subset \mathrm{grFr}(M)$ is an G_0-structure on M. Then we can construct a Cartan connection of type of (G, P_I) on M. The construction is canonical in the sense that it works well for a family, which will be explained in the proof of Proposition 3.18, and that the Cartan connection we construct on G/P_I itself is $(G \to G/P_I, \omega^{MC})$.

Now we state a setting that is slightly more general than Setting 1.11.

Setting 3.17. Suppose in Setting 2.1 that G is simple and G/P_I is not biholomorphic to a projective space. Let $\psi : \mathcal{Y} \to \Delta \ni 0$ be a holomorphic map from an irreducible analytic variety $\mathcal{Y}$ to Δ such that $\mathcal{Y}_t \cong G/P_I$ for $t \neq 0$ and $\mathcal{Y}_0$ is an irreducible reduced projective variety.

Proposition 3.18. *Suppose in Setting 3.17 that there exists a proper closed algebraic subset $Z \subset \mathcal{Y}_0$ and a holomorphic fiber bundle $\mathcal{E} \to \mathcal{Y} \setminus Z$ such that $\mathfrak{m}_x(I) \cong \mathfrak{g}_-(I)$ for all $x \in \mathcal{Y}_0 \setminus Z$ and $\mathcal{E}_t \to \mathcal{Y}_t \setminus Z$ is an G_0-structure for all $t \in \Delta$. Then $\mathcal{Y}_0 \cong G/P_I$.*

Proof. By Proposition 3.12 the Lie algebra $\mathfrak{g}$ is the universal prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$. By Sections 3.20 – 3.23 in [2] we can construct a Cartan connection of type (G, P_I) in the neighborhood of a general point $x \in \mathcal{Y}_0$. Furthermore, the construction works well for the family $\mathcal{Y}$ over Δ. In other words, there exists an analytic open subset $\mathcal{Y}^o$ of $\mathcal{Y}$, a principal P_I-bundle $\Psi : \mathcal{P} \to \mathcal{Y}^o$, and a holomorphic 1-form $\omega : T\mathcal{P} \to \mathfrak{g}$ such that

(1) $\mathcal{Y}^o \supset \mathcal{Y}_t$ for all $t \neq 0$;
(2) $\mathcal{Y}^o_0 := \mathcal{Y}^o \cap \mathcal{Y}_0$ is an analytic open neighborhood of the general point $x \in \mathcal{Y}_0$;
(3) for each $t \in \Delta$ (including $t = 0$), (Ψ_t, ω_t) is a Cartan connection of type (G, P_I);
(4) for each $t \neq 0$, the Cartan connection (Ψ_t, ω_t) is flat.

By the continuity on $t \in \Delta$ of the curvature $\kappa_t := d\omega_t + \frac{1}{2}[\omega_t, \omega_t]$, the Cartan connection (Ψ_0, ω_0) is also flat. By [19, Corollary 5.4] the Lie algebra of infinitesimal automorphisms of $\mathcal{Y}_0$, which preserves the symbol algebras on $\mathcal{Y}^o_0$ and the G_0-structure, is isomorphic to $\mathfrak{g}$.

By upper semi-continuity of $\dim H^0(\mathcal{Y}_t, T\mathcal{Y}_t)$, $\dim \mathrm{aut}(\mathcal{Y}_0) \geq \dim \mathfrak{g}$, where $\mathrm{aut}(\mathcal{Y}_0)$ is the Lie algebra of automorphism group of $\mathcal{Y}_0$. Hence $\mathrm{aut}(\mathcal{Y}_0) \cong \mathfrak{g}$ and G acts on $\mathcal{Y}_0$ with isotropy subgroup at a general point $x \in \mathcal{Y}_0$ being conjugate to P_I. It follows that $\mathcal{Y}_0 \cong G/P_I$. □

Proposition 3.19. *In Setting 3.17 suppose $|I| \geq 2$ and (G, I) is neither* (3.1) *nor* (3.2) *listed in Proposition 3.12. Then the followings are equivalent:*
(i) $\mathcal{Y}_0 \cong G/P_I$;
(ii) $\mathfrak{m}_x(I)$ is standard at general points $x \in \mathcal{Y}_0$.

Proof. It is straight-forward to see $(i) \Rightarrow (ii)$. Now let us prove $(ii) \Rightarrow (i)$. Let $\mathcal{Y}^o$ be the open subset of $\mathcal{Y}$ where the symbol algebras of $\mathcal{D}$ are isomorphic to $\mathfrak{g}_-(I)$. In particular, $\mathcal{Y}_t \subset \mathcal{Y}^o$ for all $t \neq 0$ and $\mathcal{Y}^o_0$ is a dense open subset of $\mathcal{Y}_0$. Denote by $\mathcal{F}$ a connected component of the graded frame bundle of the family $\mathcal{Y}^o$ over Δ.

By Proposition 3.12 the group $G_0 \cong \mathrm{grAut}^o(\mathfrak{g}_-(I))$. Thus the G_0-structure $\mathcal{F}_t$ on $\mathcal{Y}_t$ with $t \neq 0$ is holomorphically extended to be the G_0-structure $\mathcal{F}_0$ on $\mathcal{Y}^0_0$. The conclusion follows from Proposition 3.18. □

The key point to obtain $\mathcal{Y}_0 \cong G/P_I$ in Setting 3.17 is invariance of symbol algebras. Once this is done, it is not hard to extend the G_0-structure $E \subset \mathrm{grFr}(G/P_I)$ holomorphically to general points on $\mathcal{Y}_0$, even in case (3.1) or (3.2) listed in Proposition 3.12. For instance we have the following result.

Proposition 3.20. *Suppose in Setting 3.17 that* $\boldsymbol{S} \cong A_m/P_I$ *and* $\mathfrak{m}_x(\alpha_1, \alpha_2) \cong \mathfrak{g}_-(I)$*, where* $m \geq 2$*,* $I = \{\alpha_1, \alpha_2\}$*, and* $x \in \mathcal{Y}_0$ *is general. Then* $\mathcal{Y}_0 \cong A_m/P_I$*.*

Proof. The distributions $\mathcal{D}^{\alpha_1}$ and $\mathcal{D}^{\alpha_2}$ are integrable on $\mathcal{Y}_0$. Thus the isomorphism $\mathfrak{m}_x(\alpha_1, \alpha_2) \cong \mathfrak{g}_-(I)$ implies that $F : \mathcal{D}^{\alpha_1} \otimes \mathcal{D}^{\alpha_2} \to T^\pi/\mathcal{D}$ is surjective on the general point $x \in \mathcal{Y}_0$, where F is the restriction of the Frobenius bracket of $\mathcal{D} = \mathcal{D}^{\alpha_1} + \mathcal{D}^{\alpha_2} \subset T^\pi$.

Denote by $Z \subset \mathcal{Y}$ the set of points z such that $\mathfrak{m}_x(I) \ncong \mathfrak{g}_-(I)$. Then Z is a proper closed algebraic subset of $\mathcal{Y}_0$. Take any $y \in \mathcal{Y} \setminus Z$ and define $\mathcal{E}_y$ to be the set of grading preserving isomorphisms $\varphi : \mathfrak{m}_x(I) \to \mathfrak{g}_-(I)$ such that $\varphi(\mathcal{D}^{\alpha_i}_y) = \mathfrak{g}_{-1}(\alpha_i)$ for $i = 1, 2$. Then $\mathcal{E}$ is an G_0-structure on the family $\mathcal{Y} \setminus Z$ over Δ, and the conclusion follows from Proposition 3.18. □

3.3. **Reduction to homogeneous submanifolds.** The following is straight-forward.

Lemma-Definition 3.21. *Take* $\alpha \neq \beta \in I$ *in Setting 2.1. Then the followings are equivalent:*
(i) *the manifold* $\boldsymbol{S}^{\alpha,\beta} \cong \boldsymbol{S}^\alpha \times \boldsymbol{S}^\beta$*;*
(ii) *the roots* α *and* β *lie in different connected components of the Dynkin diagram of* $G_{J\cup\{\alpha,\beta\}}$*.*
If (i) *and* (ii) *do not hold, we say* (α, β) *is a* J*-connected pair.*

Our main aim in this subsection is to show that

Theorem 3.22. *In Setting 1.11 suppose* $|I| \geq 3$ *and* $F^{\alpha,\beta}_x \cong \boldsymbol{S}^{\alpha,\beta}$ *for any* J*-connected pair* $\alpha \neq \beta \in I$ *and general* $x \in \mathcal{X}_0$*. Then the manifold* $\mathcal{X}_0 \cong \boldsymbol{S}$*.*

As a direct consequence of Theorem 3.22, we have the following result.

Corollary 3.23. *In Setting 2.1 suppose* $|I| \geq 2$ *and that for any* $\alpha \neq \beta \in I$*, there exists a subset* $A \subset I$ *such that* $\alpha, \beta \in A$ *and the rational homogeneous space* $\boldsymbol{S}^A$ *is rigid under Fano deformation. Then* G/P_I *is rigid under Fano deformation.*

Proof. By Proposition 3.24 in the following, we can assume the group G is simple. Then we can discuss in Setting 1.11. Given any subset $A \subset I$, a general fiber of $\pi^A_0 : \mathcal{X}_0 \to \mathcal{X}^A_0$ is a Fano deformation of $\mathbf{S}^A$. Then the conclusion follows from Theorem 3.22. □

Proposition 3.24. [13, Theorem 1] *Let* $\phi : \mathcal{Z} \to \Delta \ni 0$ *be a holomorphic map with all fiber being connected Fano manifolds. Suppose that* $\mathcal{Z}_0 \cong \mathcal{Z}'_0 \times \mathcal{Z}''_0$*. Then there are holomorphic maps* $\phi' : \mathcal{W}' \to \Delta$ *and* $\phi'' : \mathcal{W}'' \to \Delta$ *such that all fiber of* ϕ' *and* ϕ'' *are connected Fano manifolds,* $\mathcal{W}'_0 \cong \mathcal{Z}'_0$*,* $\mathcal{W}''_0 \cong \mathcal{Z}''_0$*, and* $\mathcal{Z} = \mathcal{W}' \times_\Delta \mathcal{W}''$*.*

Now we turn to the proof of Theorem 3.22. By Proposition 3.19, it suffices to show that the symbol algebra $\mathfrak{m}_x(I)$ is standard for $x \in \mathcal{X}_0$ general. To verify it, we will apply Proposition 2.16.

Lemma 3.25. *In Setting 1.11 the followings hold at general points* $x \in \mathcal{X}_0$*:*

$$\sum_{\alpha\in I} T_x F^\alpha_x = \bigoplus_{\alpha\in I} T_x F^\alpha_x \subset T_x\mathcal{X}_0, \tag{3.3}$$

$$\mathcal{D}_x = \sum_{\alpha\in I} \mathcal{D}^\alpha_x = \bigoplus_{\alpha\in I} \mathcal{D}^\alpha_x \subset T_x\mathcal{X}_0, \tag{3.4}$$

where the distributions $\mathcal{D}^\alpha$ *and* $\mathcal{D}$ *are as in Notation 3.9.*

Proof. The relative Mori contractions $\pi^\alpha : \mathcal{X} \to \mathcal{X}^\alpha$ and $\pi^{I\setminus\{\alpha\}} : \mathcal{X} \to \mathcal{X}^{I\setminus\{\alpha\}}$ induce a morphism

$$\begin{aligned} \pi' : \quad & \mathcal{X}_0 \to \mathcal{X}^\alpha_0 \times \mathcal{X}^{I\setminus\{\alpha\}}_0 \\ & x \mapsto (\pi^\alpha(x), \pi^{I\setminus\{\alpha\}}(x)), \end{aligned}$$

which contracts no curves. Then $T_x F^\alpha_x \cap T_x F^{I\setminus\{\alpha\}}_x = \{0\}$ for $\alpha \in I$ and $x \in \mathcal{X}_0$, which implies (3.3). Now (3.4) follows from the inclusion $\mathcal{D}^\alpha_x \subset T_x F^\alpha_x$ and (3.3). □

Lemma 3.26. *Take* $\alpha \in I$ *and* $x \in \mathcal{X}_0$ *general in setting of Theorem 3.22. Then* $\mathcal{C}^\alpha_x \subset \mathbb{P}(\mathcal{D}^\alpha_x)$ *is projectively equivalent to* $\boldsymbol{Z}^\alpha \subset \mathbb{P}(\mathfrak{g}_{-1}(\alpha))$*.*

Proof. By Proposition 3.5, the fiber F^α_x at a general point $x \in \mathcal{X}_0$ is biholomorphic to $\mathbf{S}^\alpha$. Thus $\mathcal{C}^\alpha_x \cong \mathbf{Z}^\alpha$. □

Lemma 3.27. *In setting of Theorem 3.22, take $\alpha \in I$ and $\beta \in N_J(\alpha)$ and a general point $x \in \mathcal{X}_0$. Then the distribution $\mathfrak{g}^\beta(\widehat{\boldsymbol{Z}}^\alpha)$ is extended a holomorphic distribution $\mathcal{D}^\beta(\widehat{\mathcal{C}}^\alpha_x)$ on $\mathcal{C}^\alpha_x$, and under the identification $\widehat{\boldsymbol{Z}}^\alpha = \widehat{\mathcal{C}}^\alpha_x$ we have $\mathfrak{g}^\beta(\widehat{\boldsymbol{Z}}^\alpha) = \mathcal{D}^\beta(\widehat{\mathcal{C}}^\alpha_x)$.*

Proof. It is a direct consequence of Lemma 3.26 and Lemma 2.12. □

Now we are ready to check condition (ii) of Proposition 2.16, while condition (i) is to be checked later.

Lemma 3.28. *In setting of Theorem 3.22, take $x \in \mathcal{X}_0$ general, and any $\alpha \in I$, $\beta \in N_J(\alpha)$, $v \in \widehat{\boldsymbol{Z}}^\alpha_x \setminus \{0\}$ and $u \in \mathfrak{g}^\beta_v(\widehat{\boldsymbol{Z}}^\alpha_x)$, we have*

$$(adv)^{-\langle\beta,\alpha\rangle}(u) = 0 \ \text{in} \ \mathfrak{m}_x(I). \tag{3.5}$$

Proof. Let $\gamma \in I \setminus \{\alpha\}$ be any root that is J-connected with α. By assumption of Theorem 3.22,

$$(\mathrm{ad}v)^{-\langle\beta,\alpha\rangle}(u) = 0 \text{ in } \mathfrak{m}_x(\alpha, \gamma)$$

Then the inclusion $\mathcal{D}^{\alpha,\gamma} \subset \mathcal{D}^I$ implies that (3.5) holds. □

To check the condition (i) of Proposition 2.16, we need to write I as a disjoint union $I(j)$ in a special way.

Construction 3.29. Fix any element $\bar{\alpha} \in I$ and define $I(1) := \{\bar{\alpha}\}$. Now for each $j \geq 1$ define by induction that

$$I(j+1) := \{\alpha \in I \setminus \bigcup_{s \leq j} I(s) \mid (\alpha, \beta) \text{ is } J\text{-connected for some } \beta \in I(j)\}.$$

Lemma 3.30. *In setting of Construction 3.29, the followings hold.*

(1) The set I is the disjoint union of $I(j), j \geq 1$.

(2) Given any $j \geq 2$ with $I(j) \neq \emptyset$ and any $\alpha \in I(j)$, there exists a unique $\beta \in (\bigcup_{s \leq j} I(s)) \setminus \{\alpha\}$ such that (α, β) is J-connected. Moreover, this unique β belongs to $I(j-1)$.

(3) Given any J-connected pair (α, β), there exists a unique $j \geq 1$ such that $\{\alpha, \beta\} \subset I(j) \cup I(j+1)$. Moreover, either $\alpha \in I(j), \beta \in I(j+1)$ or $\beta \in I(j), \alpha \in I(j+1)$.

Proof. The assertion (1) holds because the Dynkin diagram $\Gamma_{I \cup J}$ is connected. To prove (2), it suffices to notice that $\Gamma_{I \cup J}$ contains no loop and each element in $\bigcup_{2 \leq s \leq j} I(s)$ is connected with the unique element $\bar{\alpha} \in I(1)$ by the elements in $J \cup (\bigcup_{s \leq j} I(s))$. The assertion (3) is a direct consequence of (1) and (2). □

Now we are ready to check the condition (i) of Proposition 2.16 in our situation.

Lemma 3.31. *In setting of Theorem 3.22, take a general point $x \in \mathcal{X}_0$. We can define a G_0-representation on $\mathcal{D}^\alpha_x$ and fix some $0 \neq v_\alpha \in \widehat{\mathcal{C}}^\alpha_x$ for each $\alpha \in I$ such that for all $\alpha' \neq \alpha'' \in I$ and all $(v', v'') \in G_0 \cdot (v_{\alpha'}, v_{\alpha''}) \in \widehat{\boldsymbol{Z}}^{\alpha'} \times \widehat{\boldsymbol{Z}}^{\alpha''}$,*

$$(adv')^{-\langle\alpha'',\alpha'\rangle+1}(v'') = 0 \ \text{in} \ \mathfrak{m}_x(I). \tag{3.6}$$

Proof. Now we will define a G_0-representation on $\mathcal{D}^\alpha_x$ and fix some $0 \neq v_\alpha \in \widehat{\mathcal{C}}^\alpha_x$ for each $\alpha \in I = \bigcup_{j \geq 1} I(j)$ and show they satisfy (3.6) by induction on $j \geq 1$.

By our construction, $I(1) = \{\bar{\alpha}\}$ consists of a unique element. Since $|I| > 1$, the set $I(2) \neq \emptyset$. Fix any $\bar{\beta} \in I(2)$. By definition $(\bar{\alpha}, \bar{\beta})$ is J-connected. By assumption of Theorem 3.22, $F^{\bar{\alpha},\bar{\beta}}_x$ with $x \in \mathcal{X}_0$ general is biholomorphic to $P_{I \setminus \{\bar{\alpha},\bar{\beta}\}}/P_I$. This is also biholomorphic to $G_{J \cup \{\bar{\alpha},\bar{\beta}\}}/P_{\{\bar{\alpha},\bar{\beta}\}}$, see Notation 2.3. By Lemma 2.12, $G_{J \cup \{\bar{\alpha},\bar{\beta}\}} \to \mathrm{Aut}^o(F^{\bar{\alpha},\bar{\beta}}_x)$ is a surjective homomorphism with a finite kernel. Then we obtain the $G_0(J \cup \{\bar{\alpha}, \bar{\beta}\})$ representations on $\mathcal{D}^{\bar{\alpha}}_y$ and $\mathcal{D}^{\bar{\beta}}_y$ on any point $y \in F^{\bar{\alpha},\bar{\beta}}_x$, which preserves $\widehat{\mathcal{C}}^{\bar{\alpha}}_y$ and $\widehat{\mathcal{C}}^{\bar{\beta}}_y$. Here $G_0(J \cup \{\bar{\alpha}, \bar{\beta}\})$ is the Lie subgroup of $G_{J \cup \{\bar{\alpha},\bar{\beta}\}}$ associated with Lie subalgebra $\mathfrak{g}_0 \subset \mathrm{Lie}(G_{J \cup \{\bar{\alpha},\bar{\beta}\}})$. The

$G_0(J \cup \{\bar\alpha, \bar\beta\})$ representations induce the required $G_0(= G_0(R))$ representations on $\mathcal{D}_y^{\bar\alpha}$ and $\mathcal{D}_y^{\bar\beta}$ respectively. Note that $G_0(J \cup \{\bar\alpha, \bar\beta\})$ is the quotient group of $G_0 := G_0(R)$ by some torus in the center. This torus acts trivially on $\mathcal{D}_y^{\bar\alpha}$ and $\mathcal{D}_y^{\bar\beta}$. Denote by

$$\varphi^{\bar\alpha}_{\{\bar\alpha,\bar\beta\}} : G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha}) \subset GL(\mathcal{D}_x^{\bar\alpha}),$$
$$\varphi^{\bar\beta}_{\{\bar\alpha,\bar\beta\}} : G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\beta}, \mathcal{D}_x^{\bar\beta}) \subset GL(\mathcal{D}_x^{\bar\beta}).$$

Applying Proposition 2.16 to $F_x^{\bar\alpha,\bar\beta} \cong P_{I\setminus\{\bar\alpha,\bar\beta\}}/P_I$, we can conclude that there exists $0 \neq v_{\bar\alpha} \in \widehat{\mathcal{C}}_x^{\bar\alpha}$ and $0 \neq v_{\bar\beta} \in \widehat{\mathcal{C}}_x^{\bar\beta}$ such that for any $(w_{\bar\alpha}, w_{\bar\beta}) \in G_0 \cdot (v_{\bar\alpha}, v_{\bar\beta}) \in \widehat{\mathcal{C}}_x^{\bar\alpha} \times \widehat{\mathcal{C}}_x^{\bar\beta}$,

$$(\mathrm{ad} w_{\bar\alpha})^{-\langle\bar\beta,\bar\alpha\rangle+1}(w_{\bar\beta}) = 0 \text{ in } \mathfrak{m}_x(\bar\alpha,\bar\beta),$$
$$(\mathrm{ad} w_{\bar\beta})^{-\langle\bar\alpha,\bar\beta\rangle+1}(w_{\bar\alpha}) = 0 \text{ in } \mathfrak{m}_x(\bar\alpha,\bar\beta).$$

Then the inclusion $\mathcal{D}^{\bar\alpha,\bar\beta} \subset \mathcal{D} := \mathcal{D}^I$ implies that

$$(\mathrm{ad} w_{\bar\alpha})^{-\langle\bar\beta,\bar\alpha\rangle+1}(w_{\bar\beta}) = 0 \text{ in } \mathfrak{m}_x(I),$$
$$(\mathrm{ad} w_{\bar\beta})^{-\langle\bar\alpha,\bar\beta\rangle+1}(w_{\bar\alpha}) = 0 \text{ in } \mathfrak{m}_x(I).$$

In case $I(2)$ consists of the unique element $\bar\beta$, we have constructed the G_0-representation for both $I(1)$ and $I(2)$.

Now suppose (for the moment) that $|I(2)| \geq 2$. By Lemma 2.12, $G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}^{\bar\alpha}) = \mathrm{Aut}^o(\widehat{\mathcal{C}}^{\bar\alpha}, \mathfrak{g}^{\bar\alpha}_{-1})$ is surjective. Take any $\gamma \in I(2) \setminus \{\bar\beta\}$. Then as previous argument for $(\bar\alpha, \beta)$ we get G_0-representations

$$\varphi^{\bar\alpha}_{\{\bar\alpha,\gamma\}} : G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha}) \subset GL(\mathcal{D}_x^{\bar\alpha}),$$
$$\varphi^{\gamma}_{\{\bar\alpha,\gamma\}} : G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\gamma}, \mathcal{D}_x^{\gamma}) \subset GL(\mathcal{D}_x^{\gamma}).$$

There is an automorphism

$$\psi(\bar\alpha; \bar\beta, \gamma) : \quad \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha}) \to \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha})$$

such that the following diagram commutes:

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi^{\bar\alpha}_{\{\bar\alpha,\bar\beta\}}} & \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha}) \\ & \searrow^{\varphi^{\bar\alpha}_{\{\bar\alpha,\gamma\}}} & \downarrow{\scriptstyle \psi(\bar\alpha;\bar\beta,\gamma)} \\ & & \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha}). \end{array}$$

Since G_0 is reductive, there is an automorphism $\theta(\bar\alpha; \bar\beta, \gamma) : G_0 \to G_0$ such that the following diagram commutes

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi^{\bar\alpha}_{\{\bar\alpha,\gamma\}}} & \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha}) \\ {\scriptstyle \theta(\bar\alpha;\bar\beta,\gamma)}\downarrow & \nearrow_{\varphi^{\bar\alpha}_{\{\bar\alpha,\bar\beta\}}} & \\ G_0. & & \end{array}$$

In other words, we lift the automorphism $\psi(\bar\alpha; \bar\beta, \gamma)$ of $\mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\bar\alpha}, \mathcal{D}_x^{\bar\alpha})$ to an automorphism $\theta(\bar\alpha; \bar\beta, \gamma)$ of G_0.

Define $\tau(\gamma) := \varphi^{\gamma}_{\{\bar\alpha,\gamma\}} \circ \theta(\bar\alpha; \bar\beta, \gamma)^{-1} : G_0 \to G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}_x^{\gamma}, \mathcal{D}_x^{\gamma}) \subset GL(\mathcal{D}_x^{\gamma})$ and $\tau(\bar\alpha) := \varphi^{\bar\alpha}_{\{\bar\alpha,\gamma\}} \circ \theta(\bar\alpha; \bar\beta, \gamma)^{-1}$. In particular, we have $\tau(\bar\alpha) = \varphi^{\bar\alpha}_{\{\bar\alpha,\bar\beta\}}$. Applying Proposition 2.16 to $F_x^{\bar\alpha,\gamma} \cong P_{I\setminus\{\bar\alpha,\gamma\}}/P_I$, we can conclude that there exists $0 \neq v'_{\bar\alpha} \in \widehat{\mathcal{C}}_x^{\bar\alpha}$ and $0 \neq v'_{\gamma} \in \widehat{\mathcal{C}}_x^{\gamma}$ such that for any $(w_{\bar\alpha}, w_\gamma) \in G_0 \cdot (v'_{\bar\alpha}, v'_\gamma) \in \widehat{\mathcal{C}}_x^{\bar\alpha} \times \widehat{\mathcal{C}}_x^{\gamma}$

(under the representation $\varphi^{\bar{\alpha}}_{\{\bar{\alpha},\gamma\}}$ and $\varphi^{\gamma}_{\{\bar{\alpha},\gamma\}}$),

$$(\mathrm{ad}w_{\bar{\alpha}})^{-\langle\gamma,\bar{\alpha}\rangle+1}(w_\gamma) = 0 \text{ in } \mathfrak{m}_x(\bar{\alpha},\gamma), \tag{3.7}$$

$$(\mathrm{ad}w_{\gamma})^{-\langle\bar{\alpha},\gamma\rangle+1}(w_{\bar{\alpha}}) = 0 \text{ in } \mathfrak{m}_x(\bar{\alpha},\gamma). \tag{3.8}$$

Denote by $R(\bar{\alpha},\gamma) := \varphi_{\{\bar{\alpha},\gamma\}}(G_0)\cdot([v'_{\bar{\alpha}}],[v'_\gamma]) \subset \mathcal{H}^{\bar{\alpha}}_x \times \mathcal{H}^{\gamma}_x$. Then $R(\bar{\alpha},\gamma)$ is a closed G_0-orbit, and the two projections $R(\bar{\alpha},\gamma) \to \mathcal{H}^{\bar{\alpha}}_x$ and $R(\bar{\alpha},\gamma) \to \mathcal{H}^{\gamma}_x$ are surjective. In particular, for the previously chosen element $0 \neq v_{\bar{\alpha}} \in \widehat{\mathcal{C}}^{\bar{\alpha}}_x$ there exists $0 \neq v_\gamma \in \widehat{\mathcal{C}}^{\gamma}_x$ such that $([v_{\bar{\alpha}}],[v_\gamma]) \in R(\bar{\alpha},\gamma)$. Furthermore,

$$R(\bar{\alpha},\gamma) = \varphi_{\{\bar{\alpha},\gamma\}}(G_0)\cdot([v'_{\bar{\alpha}}],[v'_\gamma]) = \varphi_{\{\bar{\alpha},\gamma\}}(G_0)\cdot([v_{\bar{\alpha}}],[v_\gamma]) \subset \mathcal{H}^{\bar{\alpha}}_x \times \mathcal{H}^{\gamma}_x.$$

Since $\theta := \theta(\bar{\alpha};\bar{\beta},\gamma)$ is an automorphism of G_0, we know that

$$\tau_{\{\bar{\alpha},\gamma\}}(G_0) = \varphi_{\{\bar{\alpha},\gamma\}}(\theta^{-1}(G_0)) = \varphi_{\{\bar{\alpha},\gamma\}}(G_0),$$

where $\tau_{\{\bar{\alpha},\gamma\}}(G_0) := (\tau(\bar{\alpha}),\tau(\gamma))$. It follows that

$$\tau_{\{\bar{\alpha},\gamma\}}(G_0)\cdot([v_{\bar{\alpha}}],[v_\gamma]) = \varphi_{\{\bar{\alpha},\gamma\}}(G_0)\cdot([v_{\bar{\alpha}}],[v_\gamma]) = R(\bar{\alpha},\gamma).$$

Hence for all $(w_{\bar{\alpha}},w_\gamma) \in \tau_{\{\bar{\alpha},\gamma\}}(G_0)\cdot(v_{\bar{\alpha}},v_\gamma) \subset \widehat{\mathcal{C}}^{\bar{\alpha}}_x \times \widehat{\mathcal{C}}^{\gamma}_x$ the formulae (3.7) and (3.8) hold. Then the inclusion $\mathcal{D}^{\bar{\alpha},\gamma} \subset \mathcal{D} := \mathcal{D}^I$ implies that for all $(w_{\bar{\alpha}},w_\gamma) \in \tau_{\{\bar{\alpha},\gamma\}}(G_0)\cdot(v_{\bar{\alpha}},v_\gamma) \subset \widehat{\mathcal{C}}^{\bar{\alpha}}_x \times \widehat{\mathcal{C}}^{\gamma}_x$,

$$(\mathrm{ad}w_{\bar{\alpha}})^{-\langle\gamma,\bar{\alpha}\rangle+1}(w_\gamma) = 0 \text{ in } \mathfrak{m}_x(I),$$
$$(\mathrm{ad}w_{\gamma})^{-\langle\bar{\alpha},\gamma\rangle+1}(w_{\bar{\alpha}}) = 0 \text{ in } \mathfrak{m}_x(I).$$

Now we have obtained G_0-representations on $\mathcal{D}^\alpha_x$ and chosen $0 \neq v_\alpha \in \widehat{\mathcal{C}}^\alpha$ for all $\alpha \in I(1) \cup I(2)$ such that (3.6) holds for J-connected pair $\alpha',\alpha'' \in I(1) \cup I(2)$. Repeat the argument above, we can obtain $\tau_\alpha : G_0 \to \mathrm{Aut}^o(\widehat{\mathcal{C}}^\alpha_x, \mathcal{D}^\alpha_x) \subset GL(\mathcal{D}^\alpha_x)$ and choose $0 \neq v_\alpha \in \widehat{\mathcal{C}}^\alpha_x$ for all $\alpha \in I = \bigcup_{j\geq 1} I(j)$ such that (3.6) holds for all J-connected pair $(\alpha',\alpha'') \in I \times I$.

Now take any pair $\alpha \neq \beta \in I \times I$ which is not J-connected. By Lemma-Definition 3.21, $F^{\alpha,\beta}_y = F^\alpha_y \times F^\beta_y$ at any $y \in \bigcup_{t\neq 0} \mathcal{X}_t$. By Proposition 3.24,

$$F^{\alpha,\beta}_x = F^\alpha_x \times F^\beta_x \text{ at any } x \in \mathcal{X}_0. \tag{3.9}$$

Now for $x \in \mathcal{X}_0$ general, $\mathcal{D}^\alpha_x$, $\mathcal{D}^\beta_x$ and $\mathcal{D}_x$ are well-extended. By (3.9) the Levi bracket of vector fields satisfies

$$[\mathcal{D}^\alpha_x, \mathcal{D}^\beta_x] \subset \mathcal{D}^\alpha_x + \mathcal{D}^\beta_x \subset \mathcal{D}_x,$$

which implies that for any $(w_\alpha, w_\beta) \in \widehat{\mathcal{C}}^\alpha_x \times \widehat{\mathcal{C}}^\beta_x \subset \mathcal{D}^\alpha_x \times \mathcal{D}^\beta_x$

$$[w_\alpha, w_\beta] = 0 \text{ in } \mathfrak{m}_x(I).$$

In summary, (3.6) holds for all pairs $(\alpha',\alpha'') \in I \times I$ with $\alpha' \neq \alpha''$. □

Now we are ready to complete the proof of Theorem 3.22

Proof of Theorem 3.22. Take a general point $x \in \mathcal{X}_0$. By Lemma 3.28 and Lemma 3.31, the symbol algebra $\mathfrak{m}_x(I)$ satisfies conditions (i) and (ii) in Proposition 2.16. Then by Proposition 2.16 the symbol algebra $\mathfrak{m}_x(I)$ is a quotient algebra of $\mathfrak{g}_-(I)$. By Proposition 3.10, $\dim \mathfrak{m}_x(I) = \dim \mathfrak{g}_-(I)$, which implies $\mathfrak{m}_x(I) \cong \mathfrak{g}_-(I)$. Then the conclusion follows from Proposition 3.19. □

4. Rigidity and degeneration under Fano deformation

4.1. **Proof of Main results.** Now we will prove Theorems 1.4 and 1.10 by assuming Propositions 1.8 and 1.9. It is devoted to the proof of Proposition 1.8 from next subsection until the end of the paper.

Proof of Theorem 1.4. By assumption we can write the rational homogeneous space to be $\mathbf{S} := G/P_{R\setminus\{\beta_0\}}$, where β_0 is a root in R. When $\rho(\mathbf{S}) \leq 3$, $\mathbf{S}$ is biholomorphic to $\mathbb{P}^2$, $\mathbb{F}(1,2,\mathbb{P}^3)$, $\mathbb{F}(1,2,Q^6)$ or $\mathbb{F}(0,2,Q^6)$. It remains to check the Fano deformation rigidity of $\mathbb{F}(0,2,Q^6) = D_4/P_I$ with $I = \{\alpha_1,\alpha_3,\alpha_4\}$. Take any two different roots $\beta_1,\beta_2 \in I$. The manifold $\mathbf{S}^{\beta_1,\beta_2}$ is biholomorphic to $\mathbb{P}(T_{\mathbb{P}^3})$, which is rigid under Fano deformation by Theorem 1.2. By Corollary 3.23 $\mathbb{F}(0,2,Q^6)$ is rigid under Fano deformation.

Now we will apply Corollary 3.23 to $\mathbf{S}$ with $\rho(G/P_{R\setminus\{\beta_0\}}) \geq 4$. Take any J-connected pair $(\beta_1,\beta_2) \in I\times I$. By our assumption, one of the followings hold:

(i) the Dynkin diagram $\Gamma_{\beta_0,\beta_1,\beta_2} = \Gamma_{\beta_0} \cup \Gamma_{\beta_1,\beta_2}$ is of type $A_1 \times A_2$;

(ii) the Dynkin diagram $\Gamma_{\beta_0,\dots,\beta_3}$ is of type A_4 for some $\beta_3 \in I\setminus\{\beta_1,\beta_2\}$;

(iii) the Dynkin diagram $\Gamma_{\beta_0,\dots,\beta_4}$ is of type D_5 for some $\beta_3,\beta_4 \in I\setminus\{\beta_1,\beta_2\}$.

By Theorem 1.3 and Proposition 1.8, the manifolds $\mathbf{S}^{\beta_1,\beta_2}$, $\mathbf{S}^{\beta_1,\beta_2,\beta_3}$ and $\mathbf{S}^{\beta_1,\dots,\beta_4}$ corresponding to (i), (ii) and (iii) respectively are rigid under Fano deformation. Then so is $G/P_{R\setminus\{\beta_0\}}$ by Corollary 3.23. □

Proof of Theorem 1.10. In this situation for any J-connected pair $(\alpha,\beta) \in I\times I$, the unique connected component of the Dynkin diagram $\Gamma_{J\cup\{\alpha,\beta\}}$ containing both α and β is one of the following types:

(i) $(A_m, \{\alpha_1,\alpha_m\})$ with $m \geq 2$;

(ii) $(A_m, \{\alpha_1,\alpha_2\})$ with $m \geq 3$.

By our assumption, in case (ii) there exists $\gamma \in I\setminus\{\alpha,\beta\}$ such that the unique connected component of the Dynkin diagram $\Gamma_{J\cup\{\alpha,\beta,\gamma\}}$ containing all of α, β and γ is of type $(A_{m+1}, \{\alpha_1,\alpha_2,\alpha_{m+1}\})$ up to symmetry. Then the conclusion follows from Corollary 3.23. □

Indeed by a careful analysis of Dynkin diagrams we can apply the same proof to deduce the following rigidity result.

Theorem 4.1. *Let G be a simple algebraic group of type ADE, $I \subset R$ be a subset and $J := R\setminus I$. Write I as the disjoint union $\cup I_i$, where each Γ_{I_i} is a connected component of Γ_I. Suppose that*

(1) *the end nodes of Dynkin diagram of G is contained in I,*

(2) *each I_i satisfies that either $I_i \cap \partial R \neq \emptyset$ or its cardinality $|I_i| \geq 3$,*

(3) *in case G is of type D or E, there exists at most one $\beta \in J$ such that $\langle \beta, \bar{\alpha}\rangle \neq 0$, where $\bar{\alpha}$ is the node in Dynkin diagram of G with three branches.*

Then the rational homogeneous space G/P_I is rigid under Fano deformation.

Remark 4.2. As a direct consequence of Proposition 3.24, we can know that $\mathbf{S}$ is rigid under Fano deformation if $\mathbf{S} = \mathbf{S}_1 \times \cdots \times \mathbf{S}_k$ and each $\mathbf{S}_i$ is as in the statement of one of Theorems 1.3, 1.4, 1.10 or 4.1.

4.2. **Rigidity of $A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$.** The aim of this subsection is to show the following rigidity property.

Theorem 4.3. *The flag manifold $A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ is rigid under Fano deformation.*

In other words, we want to prove $\mathcal{X}_0 \cong A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ in Setting 1.11 under additional assumption that $\mathbf{S} = A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$. Firstly, we have the following rigidity result on fibers.

Proposition 4.4. *Suppose $\boldsymbol{S} = A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ in Setting 1.11. Then the followings hold for $x \in \mathcal{X}_0$ general:*

$$F_x^{\alpha_1} \cong \mathbb{P}^1, \quad F_x^{\alpha_2} \cong \mathbb{P}^{m-2}, \quad F_x^{\alpha_m} \cong \mathbb{P}^{m-2}; \tag{4.1}$$

$$F_x^{\alpha_1,\alpha_m} \cong F_x^{\alpha_1} \times F_x^{\alpha_m} \cong \mathbb{P}^1 \times \mathbb{P}^{m-2}; \tag{4.2}$$

$$F_x^{\alpha_2,\alpha_m} \cong P_{\alpha_1}/P_{\{\alpha_1,\alpha_2,\alpha_m\}}. \tag{4.3}$$

Proof. The conclusions (4.1) and (4.3) follows from Fano deformation rigidity of projective spaces and $A_k/P_{\{\alpha_1,\alpha_k\}}$ respectively, see Theorem 1.2. The conclusion (4.2) follows from (4.1) and Proposition 3.24. □

As a direct consequence of Proposition 4.4, we have the following result.

Corollary 4.5. *Suppose* $\boldsymbol{S} = A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ *in Setting 1.11. Then the followings hold for* $x \in \mathcal{X}_0$ *general.*

(1) The symbol algebras $\mathfrak{m}_x(\alpha_1), \mathfrak{m}_x(\alpha_2)$ *and* $\mathfrak{m}_x(\alpha_m)$ *are standard. More precisely, they are abelian algebras of dimension* $1, m-2$ *and* $m-2$ *respectively.*

(2) The symbol algebras $\mathfrak{m}_x(\alpha_1,\alpha_m)$ *and* $\mathfrak{m}_x(\alpha_2,\alpha_m)$ *are standard. More precisely,*

(i) there is a decomposition of abelian algebra $\mathfrak{m}_x(\alpha_1,\alpha_m) = \mathfrak{m}_x(\alpha_1)\oplus\mathfrak{m}_x(\alpha_m)$*;*

(ii) $\dim \mathfrak{m}_{-2}(\alpha_2,\alpha_m) = 1$ *and the bilinear map*

$$\mathfrak{m}_x(\alpha_2)\times\mathfrak{m}_x(\alpha_m) \to (\mathfrak{m}_x(\alpha_2,\alpha_m))_{-2}$$
$$(x,y)\mapsto [x,y]$$

induces an isomorphism of vector spaces $\mathfrak{m}_x(\alpha_2) \cong Hom(\mathfrak{m}_x(\alpha_m), (\mathfrak{m}_x(\alpha_2,\alpha_m))_{-2})$*.*

Proposition 4.6. *Suppose* $\boldsymbol{S} \cong A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ *in Setting 1.11. Then* $F^{\alpha_1,\alpha_2} \cong P_{I\setminus\{\alpha_1,\alpha_2\}}/P_I$ *for* $x \in \mathcal{X}_0$ *general.*

Proof. Take $x \in \mathcal{X}_0$ general. We claim that

$$\text{the symbol algebra } \mathfrak{m}_x(\alpha_1,\alpha_2) \text{ is standard.} \tag{4.4}$$

For the simplicity of discussion, we omit the subscript x in the notations of symbol algebras such as $\mathfrak{m}_x(\alpha_1,\alpha_2)$ and $\mathfrak{m}_x(\alpha_m)$.

Now suppose that $\mathfrak{m}(\alpha_1,\alpha_2)$ is not standard. Then there exists $0 \neq v_2 \in \mathfrak{m}(\alpha_2)$ such that $[\mathfrak{m}(\alpha_1), v_2] = 0$. Since $\mathfrak{m}(\alpha_2,\alpha_m)$ is standard, there exists $0\neq v_3 \in \mathfrak{m}(\alpha_m)$ such that $v_4 := [v_2,v_3] \neq 0$ and $\mathfrak{m}_{-2}(\alpha_2,\alpha_m) = \mathbb{C}v_4$. In particular, there is a decomposition of vector spaces

$$\mathfrak{m}(\alpha_2,\alpha_m) = \mathfrak{m}(\alpha_2)\oplus\mathfrak{m}(\alpha_m)\oplus\mathbb{C}v_4.$$

Take $0 \neq v_1 \in \mathfrak{m}(\alpha_1)$. Then we have

$$[v_1,v_4] = [v_1,[v_2,v_3]] = [[v_1,v_2],v_3] + [v_2,[v_1,v_3]] = 0. \tag{4.5}$$

In other words, $[\mathfrak{m}(\alpha_1), \mathbb{C}v_4] = 0$. Let $\mathcal{A}(\alpha_1,\alpha_2,\alpha_m)$ be the vector subspace of $\mathfrak{m}(\alpha_1,\alpha_2,\alpha_m)$ generated by $\mathfrak{m}(\alpha_1,\alpha_2)$, $\mathfrak{m}(\alpha_m)$ and $\mathbb{C}v_4$. Denote by

$$\mathfrak{m}(1;\alpha_1,\alpha_2) := \mathfrak{m}(\alpha_1)\oplus\mathfrak{m}(\alpha_2),$$
$$\mathfrak{m}(k;\alpha_1,\alpha_2) := [\mathfrak{m}(1;\alpha_1,\alpha_2), \mathfrak{m}(k-1;\alpha_1,\alpha_2)] \text{ for each } k\geq 2.$$

Thus $\mathfrak{m}(\alpha_1,\alpha_2) = \sum_{k=1}^{\infty}\mathfrak{m}(k;\alpha_1,\alpha_2)$. We claim that (when (4.4) fails),

$$\mathcal{A}(\alpha_1,\alpha_2,\alpha_m) \text{ is a Lie subalgebra of } \mathfrak{m}(\alpha_1,\alpha_2,\alpha_m). \tag{4.6}$$

Indeed by Corollary 4.5 we already know that

$$\mathcal{A}(\alpha_1,\alpha_2,\alpha_m) = \mathfrak{m}(\alpha_1,\alpha_2) + \mathfrak{m}(\alpha_1,\alpha_m) + \mathfrak{m}(\alpha_2,\alpha_m).$$

It follows that

$$[\mathfrak{m}(\alpha_m)+\mathbb{C}v_4, \mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \subset \mathfrak{m}(\alpha_2,\alpha_m) \subset \mathcal{A}(\alpha_1,\alpha_2,\alpha_m).$$

Hence to prove the claim (4.6) it remains to show that

$$[\mathfrak{m}(k;\alpha_1,\alpha_2), \mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \subset \mathcal{A}(\alpha_1,\alpha_2,\alpha_m) \text{ for all } k \geq 1. \tag{4.7}$$

Now let us prove (4.7) by induction on k. The case $k=1$ of (4.7) follows from

$$[\mathfrak{m}(\alpha_1), \mathfrak{m}(\alpha_m)+\mathbb{C}v_4] = 0, \tag{4.8}$$
$$[\mathfrak{m}(\alpha_2), \mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \subset \mathfrak{m}(\alpha_2,\alpha_m) \subset \mathcal{A}(\alpha_1,\alpha_2,\alpha_m), \tag{4.9}$$

where in the first equality we apply Corollary 4.5 and (4.5).

Now we assume that $k \geq 2$ and

$$[\mathfrak{m}(i;\alpha_1,\alpha_2), \mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \subset \mathcal{A}(\alpha_1,\alpha_2,\alpha_m) \text{ for all } 1\leq i\leq k-1.$$

Then by the definition of $\mathfrak{m}(k;\alpha_1,\alpha_2)$ we have

$$
\begin{aligned}
(4.10)\qquad & [\mathfrak{m}(k;\alpha_1,\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \\
\subset & \sum_{j=1,2}[[\mathfrak{m}(\alpha_j),\mathfrak{m}(k-1;\alpha_1,\alpha_2)],\mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \\
\subset & \sum_{j=1,2}\big([[\mathfrak{m}(\alpha_j),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4],\mathfrak{m}(k-1;\alpha_1,\alpha_2)] \\
& +[\mathfrak{m}(\alpha_j),[\mathfrak{m}(k-1;\alpha_1,\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4]]\big).
\end{aligned}
$$

We analyse term by term. By (4.8) we have

$$
(4.11)\qquad [[\mathfrak{m}(\alpha_1),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4],\mathfrak{m}(k-1;\alpha_1,\alpha_2)]=0.
$$

On one hand, we have

$$
\begin{aligned}
(4.12)\qquad & [\mathfrak{m}(\alpha_1),[\mathfrak{m}(k-1;\alpha_1,\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4]] \\
\subset\ & [\mathfrak{m}(\alpha_1),\mathcal{A}(\alpha_1,\alpha_2,\alpha_m)] \\
=\ & [\mathfrak{m}(\alpha_1),\mathfrak{m}(\alpha_1,\alpha_2)]+[\mathfrak{m}(\alpha_1),\mathfrak{m}(\alpha_m)]+[\mathfrak{m}(\alpha_1),\mathbb{C}v_4] \\
\subset\ & \mathfrak{m}(\alpha_1,\alpha_2) \\
\subset\ & \mathcal{A}(\alpha_1,\alpha_2,\alpha_m).
\end{aligned}
$$

By Corollary 4.5 we have

$$
[\mathfrak{m}(\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4]\subset\mathfrak{m}(\alpha_2)+\mathfrak{m}(\alpha_m)+\mathbb{C}v_4,
$$

which implies that

$$
\begin{aligned}
(4.13)\qquad & [[\mathfrak{m}(\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4],\mathfrak{m}(k-1;\alpha_1,\alpha_2)] \\
\subset\ & [\mathfrak{m}(\alpha_2),\mathfrak{m}(k-1;\alpha_1,\alpha_2)]+[\mathfrak{m}(\alpha_m)+\mathbb{C}v_4,\mathfrak{m}(k-1;\alpha_1,\alpha_2)] \\
\subset\ & \mathfrak{m}(k;\alpha_1,\alpha_2)+\mathcal{A}(\alpha_1,\alpha_2,\alpha_m) \\
=\ & \mathcal{A}(\alpha_1,\alpha_2,\alpha_m).
\end{aligned}
$$

Meanwhile by induction we have

$$
\begin{aligned}
& [\mathfrak{m}(k-1;\alpha_1,\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4] \\
\subset\ & \mathcal{A}(\alpha_1,\alpha_2,\alpha_m) \\
=\ & \mathfrak{m}(\alpha_1,\alpha_2)+\mathfrak{m}(\alpha_m)+\mathbb{C}v_4,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(4.14)\qquad & [\mathfrak{m}(\alpha_2),[\mathfrak{m}(k-1;\alpha_1,\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4]] \\
\subset\ & [\mathfrak{m}(\alpha_2),\mathfrak{m}(\alpha_1,\alpha_2)]+[\mathfrak{m}(\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4]+[\mathfrak{m}(\alpha_2),\mathbb{C}v_4] \\
\subset\ & \mathfrak{m}(\alpha_1,\alpha_2)+\mathfrak{m}(\alpha_2,\alpha_m) \\
\subset\ & \mathcal{A}(\alpha_1,\alpha_2,\alpha_m).
\end{aligned}
$$

By (4.10)–(4.14) we have $[\mathfrak{m}(k;\alpha_1,\alpha_2),\mathfrak{m}(\alpha_m)+\mathbb{C}v_4]\subset\mathcal{A}(\alpha_1,\alpha_2,\alpha_m)$. In other words (4.7) holds. Then the claim (4.6) holds.

Now $\mathcal{A}(\alpha_1,\alpha_2,\alpha_m)$ is a Lie subalgebra of $\mathfrak{m}(\alpha_1,\alpha_2,\alpha_m)$ that contains $\mathfrak{m}(\alpha_1)+\mathfrak{m}(\alpha_2)+\mathfrak{m}(\alpha_m)$. Recall that $\mathfrak{m}(\alpha_1,\alpha_2,\alpha_m)$ is a Lie algebra generated by $\mathfrak{m}(\alpha_1)+\mathfrak{m}(\alpha_2)+\mathfrak{m}(\alpha_m)$. Then we have $\mathcal{A}(\alpha_1,\alpha_2,\alpha_m)=\mathfrak{m}(\alpha_1,\alpha_2,\alpha_m)$. This contradicts the fact that

$$
\dim\mathcal{A}(\alpha_1,\alpha_2,\alpha_m)=3m-4=\dim\mathfrak{m}(\alpha_1,\alpha_2,\alpha_m)-1,
$$

where the dimension of $\mathfrak{m}(\alpha_1,\alpha_2,\alpha_m)$ is obtained by Proposition 3.10. Hence we conclude that $\mathfrak{m}_x(\alpha_1,\alpha_2)$ is standard for $x\in\mathcal{X}_0$ general, verifying the claim 4.4. Then the conclusion follows from Proposition 3.20. □

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Suppose $\mathbf{S} \cong A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ in Setting 1.11. By Proposition 4.4 and Proposition 4.6, $F_x^{\alpha,\beta} \cong P_{I\setminus\{\alpha,\beta\}}/P_I$ for all $\alpha \neq \beta \in I$ and general points $x \in \mathcal{X}_0$. Then by Theorem 3.22 $\mathcal{X}_0 \cong A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$. In other words, the manifold $A_m/P_{\{\alpha_1,\alpha_2,\alpha_m\}}$ is rigid under Fano deformation. □

4.3. **Fano degeneration of** $A_3/P_{\{\alpha_1,\alpha_2\}}$**.** The aim of this section is to prove Theorem 1.6, namely the manifold $F^d(1,2;\mathbb{C}^4)$ in Construction 1.5 is the unique Fano degeneration of $A_3/P_{\{\alpha_1,\alpha_2\}}$. Throughout Section 4.3, we always discuss under the following assumption.

Assumption 4.7. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a holomorphic map such that $\mathcal{X}_t \cong A_3/P_{\{\alpha_1,\alpha_2\}}$ for $t \neq 0$, $\mathcal{X}_0$ is a connected Fano manifold, and $\mathcal{X}_0 \ncong A_3/P_{\{\alpha_1,\alpha_2\}}$.

By definition $F^d(1,2;\mathbb{C}^4) := \mathbb{P}(\mathcal{L}_\sigma \oplus \mathcal{L}^\omega)$. Then the restriction of the $\mathbb{P}^2$-bundle $F^d(1,2;\mathbb{C}^4) \to \mathbb{P}^3$ gives a biholomorphic map $\mathbb{P}(\mathcal{L}_\sigma) \cong \mathbb{P}^3$. Moreover the hyperplane bundle $\mathbb{P}(\mathcal{L}^\omega)$ is biholomorphic to the complete flag manifold C_2/B.

The outline to show $\mathcal{X}_0 \cong F^d(1,2;\mathbb{C}^4)$ is as follows. Firstly, the Mori contraction $\pi_0^{\alpha_2} : \mathcal{X}_0 \to \mathcal{X}_0^{\alpha_2}$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. We know that at a general point $x \in \mathcal{X}_0$, the family $\mathcal{K}_x^{\alpha_1}(\mathcal{X}_0)$ consists a single element, denoted by $[C_x]$. An irreducible component of the locus $\{x \in \mathcal{X}_0 \mid \dim \mathcal{K}_x^{\alpha_1}(\mathcal{X}_0) \geq 1\}$ gives a meromorphic section $\sigma : \mathbb{P}^3 \dashrightarrow \mathcal{X}_0$. Let H be an effective divisor on $\mathcal{X}_0$ which is a general element in a linear system satisfying $(H \cdot \mathcal{K}^{\alpha_1}) = 0$ and $(H \cdot \mathcal{K}^{\alpha_2}) = 1$. The restriction of $\pi_0^{\alpha_2}$ on H is a fibration over $\mathbb{P}^3$, whose general fiber is a line in $\mathbb{P}^2$. Then we show that σ is a holomorphic section, $H \to \mathbb{P}^3$ is a $\mathbb{P}^1$-bundle and $H \cap \sigma(\mathbb{P}^3) = \emptyset$. Finally we show $H \cong C_2/B$ and $\mathcal{X}_0 \cong F^d(1,2;\mathbb{C}^4)$.

Now we sketch how to show $H \cong C_2/B$, which is the key point of the argument in this section. Denote by $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ the closure in the Chow scheme of $\mathbb{P}^3$ of the set of those $\pi_0^{\alpha_2}(C_x)$, where $x \in \mathcal{X}_0$ general and $\mathcal{K}_x^{\alpha_1}(\mathcal{X}_0) = \{[C_x]\}$. By considering the symbol algebra of $\mathcal{D} = \mathcal{D}^{\alpha_1} + \mathcal{D}^{\alpha_2}$ on $\mathcal{X}_0$, we obtain that a meromorphic distribution $\mathcal{E}$ of rank two on $\mathbb{P}^3$ satisfying that $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is the family of lines on $\mathbb{P}^3$ that are tangent to $\mathcal{E}$. This gives an antisymmetric form ω on $\mathbb{C}^4$ – which is shown to be a symplectic form later – such that $\mathcal{E}$ coincides with the induced contact form $\mathcal{L}^\omega$ on $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$.

This section is organized as follows. In the part 4.3.1, by studying splitting types of various meromorphic vector bundles along a general element in $\mathcal{K}^{\alpha_2}(\mathcal{X}_0)$, we obtain the symbol algebra of $\mathcal{D} = \mathcal{D}^{\alpha_1} + \mathcal{D}^{\alpha_2}$ on $\mathcal{X}_0$. In the part 4.3.2, we obtain the meromorphic section σ by studying splitting types of various meromorphic vector bundles along a general element in $\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$. In the part 4.3.3, we study the property of the family $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$. In the part 4.3.4, we complete the proof of Theorem 1.6 by studying the property of divisor H explained above. In the part 4.3.5, we summarize some properties of the manifold $F^d(1,2;\mathbb{C}^4)$, which will be useful in Subsections 4.4 and 4.5.

4.3.1. *Type of symbol algebra.*

Convention 4.8. In Section 4.3, we denote by $\mathscr{D}^{\alpha_i}$, $\mathscr{D}$ and $\mathscr{D}^{-i}$ the restriction of $\mathcal{D}^{\alpha_i}$, $\mathcal{D}$ and $\mathcal{D}^{-i}$ on $\mathcal{X}_0$ respectively, where the latter is defined in Notation 3.9.

Lemma 4.9. *Under Assumption 4.7, there exists a unique meromorphic line bundle $\mathcal{N} \subset T^{\pi^{\alpha_2}}\mathcal{X}_0$ such that $[\mathcal{N}, \mathscr{D}] \subset \mathscr{D}$, where $\mathscr{D} := T^{\pi_0^{\alpha_1}} + T^{\pi_0^{\alpha_2}} = \mathcal{D}|_{\mathcal{X}_0}$. Moreover rank$\mathscr{D}^{-2} = 4$ and $\mathscr{D}^{-3} = T\mathcal{X}_0$.*

Proof. The restriction of the Frobenius bracket of $\mathscr{D}$ induces a homomorphism $F : \mathscr{D}^{\alpha_1} \otimes \mathscr{D}^{\alpha_2} \to T\mathcal{X}_0/\mathscr{D}$. The image of F is $\mathscr{D}^{-2}/\mathscr{D}$ on $\mathcal{X}_0$, whose rank is at most two. By Proposition 3.10, rank$(\mathscr{D}^{-2}/\mathscr{D}) \geq 1$. If rank$(\mathscr{D}^{-2}/\mathscr{D}) = 2$, then $\mathfrak{m}_x(\alpha_1,\alpha_2) \cong \mathfrak{g}_-(\alpha_1,\alpha_2)$ for $x \in \mathcal{X}_0$. Then by Proposition 3.20 $\mathcal{X}_0 \cong A_3/P_{\{\alpha_1,\alpha_2\}}$, contradicting Assumption 4.7. Hence rank$(\mathscr{D}^{-2}/\mathscr{D}) = 1$. By Proposition 3.10 rank$(\mathscr{D}^{-3}/\mathscr{D}^{-2}) \geq 1$, implying that $\mathscr{D}^{-3} = T\mathcal{X}_0$. □

Lemma 4.10. *Under Assumption 4.7, there exists a unique meromorphic vector subbundle $\mathcal{W} \subset \mathscr{D}^{-1}$ of rank two such that $[\mathcal{W}, \mathscr{D}^{-2}] \subset \mathscr{D}^{-2}$. Furthermore, $\mathcal{N} \subset \mathcal{W}$.*

Proof. The conclusion follows from the two facts that rank$\mathscr{D}^{-3} =$ rank$\mathscr{D}^{-2}+1$ and that $[\mathcal{N}, \mathscr{D}^{-1}] \subset \mathscr{D}^{-1}$. □

The following result is important to the proof of Theorem 1.6.

Proposition 4.11. *We have $\mathcal{W} = \mathscr{D}^{\alpha_2}$.*

In summary of the description of symbol algebras $\mathrm{Symb}(\mathscr{D})_x$ studied in Lemma 4.9, Lemma 4.10 and Proposition 4.11, we have the following result.

Corollary 4.12. *The symbol algebra $\mathfrak{m}_x(\alpha_1, \alpha_2) := Symb(\mathscr{D})_x$ at general point $x \in \mathcal{X}_0$ is isomorphic to $\mathfrak{g}_-(C_2 \times A_1)$, where $\mathfrak{g}_-(C_2 \times A_1)$ is defined in Definition 2.2. More precisely, there exists a nonempty Zariski open subset Ω of $\mathcal{X}_0$ such that*

(i) there is an isomorphism on Ω:

$$\mathscr{D} \cong \mathscr{D}^{\alpha_1} \oplus \mathscr{D}^{\alpha_2}; \tag{4.15}$$

(ii) the Frobenius bracket of $\mathscr{D}$ induces a surjective homomorphism on Ω:

$$\wedge^2 \mathscr{D}^{-1} \to \mathscr{D}^{\alpha_1} \otimes (\mathscr{D}^{\alpha_2}/\mathcal{N}) \cong (\mathscr{D}^{-2}/\mathscr{D}^{-1}), \tag{4.16}$$

where $\mathscr{D}^{-1} := \mathscr{D}$ by definition;

(iii) the restriction of the Frobenius bracket of $\mathscr{D}^{-2}$ induces a surjective homomorphism on Ω:

$$\mathscr{D}^{-1} \otimes (\mathscr{D}^{-2}/\mathscr{D}^{-1}) \to \mathscr{D}^{\alpha_1} \otimes (\mathscr{D}^{-2}/\mathscr{D}^{-1}) \cong (\mathscr{D}^{-3}/\mathscr{D}^{-2}), \tag{4.17}$$

(iv) the derivative $\mathscr{D}^{-3}$ of is the whole tangent bundle of $\mathcal{X}_0$, i.e. $\mathscr{D}^{-3} = T\mathcal{X}_0$.

Remark 4.13. (i) The isomorphisms in (4.15) (4.16) and (4.17) hold on Ω instead of on the whole holomorphic loci of corresponding meromorphic vector bundles. Meanwhile as meromorphic vector bundles over $\mathcal{X}_0$, we have injective homomorphisms

$$\begin{aligned}
&\mathscr{D}^{\alpha_1} \oplus \mathscr{D}^{\alpha_2} \hookrightarrow \mathscr{D},\\
&\mathscr{D}^{\alpha_1} \otimes (\mathscr{D}^{\alpha_2}/\mathcal{N}) \hookrightarrow \mathscr{D}^{-2}/\mathscr{D}^{-1},\\
&\mathscr{D}^{\alpha_1} \otimes (\mathscr{D}^{-2}/\mathscr{D}^{-1}) \hookrightarrow \mathscr{D}^{-3}/\mathscr{D}^{-2}.
\end{aligned}$$

(ii) The Lie algebra $\mathrm{Symb}(\mathscr{D})_x \cong \mathfrak{g}_-(C_2 \times A_1)$ can be descried explicitly as the following graded Lie algebra $\mathfrak{m}_- := \bigoplus_{k \geq 1} \mathfrak{m}_{-k}$:

$$\begin{aligned}
&\mathfrak{m}_{-1} := \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3,\\
&\mathfrak{m}_{-2} := \mathbb{C}v_{12},\\
&\mathfrak{m}_{-3} := \mathbb{C}v_{121},\\
&\mathfrak{m}_{-k} := 0, \quad \text{for all } k \geq 4.
\end{aligned}$$

where $v_{12} := [v_1, v_2]$ and $v_{121} := [v_{12}, v_1]$. In the identification $\mathrm{Symb}(\mathscr{D})_x = \mathfrak{m}_-$, we have

$$\begin{aligned}
&\mathfrak{m}_x(\alpha_1) = \mathscr{D}_x^{\alpha_1} = \mathbb{C}v_1,\\
&\mathcal{N}_x = \mathbb{C}v_3 \subset \mathscr{D}_x^{\alpha_2},\\
&\mathfrak{m}_x(\alpha_2) = \mathscr{D}_x^{\alpha_2} = \mathbb{C}v_2 \oplus \mathbb{C}v_3.
\end{aligned}$$

The rest of the part 4.3.1 is devoted to the proof of Proposition 4.11. Firstly, the following conclusion is straight-forward.

Lemma 4.14. *There exists a closed variety $Y_1 \subset \mathcal{X}_0$ such that $codim_{\mathcal{X}_0}(Y_1) \geq 2$, $\mathscr{D}^{\alpha_1}$, $\mathscr{D}^{\alpha_2}$, $\mathcal{N}$, $\mathcal{W}$ and $\mathscr{D}$ are holomorphic vector bundles over $\mathcal{X}_0 \backslash Y_1$. Moreover, for $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ general and $[C_2] \in \mathcal{K}^{\alpha_2}(\mathcal{X}_0)$ general, $C_1 \cap Y_1 = \emptyset$ and $C_2 \cap Y_1 = \emptyset$.*

To continue, we need a useful result in [1] due to L. Bonavero, C. Casagrande and S. Druel.

Proposition 4.15. [1, Proposition 1] *Let Y be a normal $\mathbb{Q}$-factorial projective variety, and $\mathcal{F}$ be a quasi-unsplit covering family of 1-cycles on Y. Denote by $E_{\mathcal{F}} \subset Y$ the union of all $\mathcal{F}$-equivalence classes of dimension larger than m, where m is the dimension of a general $\mathcal{F}$-equivalence class. Then*

(i) $E_{\mathcal{F}}$ is a Zariski closed subset of Y, and $\dim E_{\mathcal{F}} \leq \dim Y - 2$;

(ii) there exists a normal variety Z and a surjective morphism $\varphi : Y \backslash E_{\mathcal{F}} \to Z$ such that fibers $\varphi^{-1}(z)$, $z \in Z$ are $\mathcal{F}$-equivalence classes on Y.

Remark 4.16. (i) In the setting of Proposition 4.15, the meaning of $\mathcal{F}$ being a quasi-unsplit family is that all irreducible components of the cycles parameterized by $\mathcal{F}$ are numerically proportional.

(ii) In the setting of Proposition 4.15, two points in Y are defined to be $\mathcal{F}$-equivalent if they are connected by a chain of elements in $\mathcal{F}$.

(iii) In our situation of Assumption 4.7, both $\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ and $\mathcal{K}^{\alpha_2}(\mathcal{X}_0)$ are unsplit (hence quasi-unsplit) covering family of rational curves on the complex projective manifold $\mathcal{X}_0$. In particular, the conditions in Proposition 4.15 is satisfied by both families $\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ and $\mathcal{K}^{\alpha_2}(\mathcal{X}_0)$.

Applying Proposition 4.15 to $\mathcal{X}_0$, we obtain the following result immediately.

Corollary 4.17. *Denote by*

$$\Psi_0^{\alpha_1} : \mathcal{X}_0\backslash\mathcal{E}_0^{\alpha_1} \to \mathcal{Z}_0^{\alpha_1}, \quad \Psi_0^{\alpha_2} : \mathcal{X}_0\backslash\mathcal{E}_0^{\alpha_2} \to \mathcal{Z}_0^{\alpha_2}.$$

morphisms in Proposition 4.15 corresponding to $\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ *and* $\mathcal{K}^{\alpha_2}(\mathcal{X}_0)$ *respectively. We can take* $Y_2 \subset \mathcal{X}_0$ *to be* $Y_1 \cup sing(\Psi_0^{\alpha_1}) \cup sing(\Psi_0^{\alpha_2}) \cup \mathcal{E}_0^{\alpha_1} \cup \mathcal{E}_0^{\alpha_2}$ *where* Y_1 *is as in Lemma 4.14, and* $sing(\Psi_0^{\alpha_i}) \subset \mathcal{X}_0\backslash\mathcal{E}_0^{\alpha_i}$ *is the singular locus of the morphism* $\Psi_0^{\alpha_i}$*. Then* $\dim Y_2 \leq \dim \mathcal{X}_0 - 2 = 3$*.*

Proof. The existence of $\Psi_0^{\alpha_1}$ and $\Psi_0^{\alpha_2}$ follows from Proposition 4.15. The rest follows from the generic smoothness and the equi-dimensionality of $\Psi_0^{\alpha_i}$, $i = 1, 2$. □

Proposition 4.18. *Take* $[C_2] \in \mathcal{K}^{\alpha_2}(\mathcal{X}_0)$ *general. Then* $\mathscr{D}^{\alpha_1}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathscr{D}^{\alpha_2}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$*.*

Proof. The curve C_2 is a line in a general fiber $F_x^{\alpha_2} \cong \mathbb{P}^2$ of the elementary Mori contraction $\pi_0^{\alpha_2}$, where x is a general point in C_2. Thus,

$$\mathscr{D}^{\alpha_2}|_{C_2} = TF_{C_2}^{\alpha_2}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Now take a general local section of $\mathcal{K}^{\alpha_2}(\mathcal{X}) \to \Delta$ passing through $[C_2] \in \mathcal{K}^{\alpha_2}(\mathcal{X}_0) \subset \mathcal{K}^{\alpha_2}(\mathcal{X})$. We obtain a holomorphic family $\{A^t\}_{t\in\Delta}$ (by shrinking Δ if necessary) such that $\mathcal{S} := \bigcup_{t\in\Delta} A^t \subset \mathcal{X}$ is a complex manifold of dimension two, and $A^0 = C_2 \subset \mathcal{X}_0$, $A^t \subset \mathcal{X}_t$. Moreover, $\mathcal{S} \cap Y_2 = C_2 \cap Y_2 = \emptyset$ by Corollary 4.17. Thus for any $x \in \mathcal{S}$, there exists a unique $[l_x] \in \mathcal{K}^{\alpha_1}(\mathcal{X})$ such that $x \in l_x$. Furthermore, x is a smooth point of l_x. Denote by $\mathcal{L} := \bigcup_{x\in\mathcal{S}} T_x l_x$ which is a holomorphic line bundle over $\mathcal{S}$. By Proposition 2.14 we know that for any $t \neq 0$,

$$\mathcal{L}|_{A^t} = T^{\pi_t^{\alpha_1}}|_{A^t} \cong \mathcal{O}_{\mathbb{P}^1}(\langle\alpha_1, \alpha_2\rangle) = \mathcal{O}_{\mathbb{P}^1}(-1).$$

It follows that $\mathcal{L}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Thus $\mathscr{D}^{\alpha_1}|_{C_2} \cong \mathcal{L}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. □

Proposition 4.19. *Take* $[C_2] \in \mathcal{K}^{\alpha_2}(\mathcal{X}_0)$ *general. Then* $\mathscr{D}^{\alpha_1}, \mathscr{D}^{\alpha_2}, \mathscr{D}, \mathscr{D}^{-2}, \mathscr{D}^{-3}, \mathcal{N}, \mathcal{W}$ *are holomorphic in an open neighborhood of* $C_2 \subset \mathcal{X}_0$*, and*

$$\begin{aligned}
&\mathcal{N}|_{C_2} = \mathcal{O}(1),\\
&\mathscr{D}^{\alpha_2}/\mathcal{N}|_{C_2} = \mathcal{O}(2),\\
&\mathscr{D}^{-2}/\mathscr{D}|_{C_2} = \mathcal{O}(1),\\
&\mathscr{D}^{-3}/\mathscr{D}^{-2}|_{C_2} = \mathcal{O},\\
&\mathscr{D}|_{C_2} = \mathscr{D}^{\alpha_1}|_{C_1} \oplus \mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1),\\
&\mathscr{D}^{-2}|_{C_2} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}^2,\\
&\mathscr{D}^{-3}|_{C_2} = T\mathcal{X}_0|_{C_2} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}^3.
\end{aligned}$$

Proof. By the generality of $[C_2] \in \mathcal{K}^{\alpha_2}(\mathcal{X}_0)$, $T\mathcal{X}_0|_{C_2} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}^3$. Then by Proposition 4.18 and the injectivity of $\mathscr{D}^{\alpha_1} \oplus \mathscr{D}^{\alpha_2} \to \mathscr{D} \subset T\mathcal{X}_0$ in an open neighborhood of $C_2 \subset \mathcal{X}_0$, either

$$\begin{aligned}
&\mathscr{D}|_{C_2} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}, \quad \mathscr{D}/\mathscr{D}^{\alpha_2}|_{C_2} = \mathcal{O}, \text{ or}\\
&\mathscr{D}|_{C_2} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1), \quad \mathscr{D}/\mathscr{D}^{\alpha_2}|_{C_2} = \mathcal{O}(-1),
\end{aligned}$$

By Lemma 4.9, $[\mathcal{N},\mathscr{D}]\subset\mathscr{D}$, $[\mathscr{D}^{\alpha_1},\mathscr{D}^{\alpha_2}]\subset\mathscr{D}^{\alpha_2}\subset\mathscr{D}$ and $[\mathscr{D},\mathscr{D}]\nsubseteq\mathscr{D}$. Then the Frobenius bracket $\Lambda^2\mathscr{D}\to T\mathcal{X}_0/\mathscr{D}$ induces an nonzero homomorphism:

$$f:\quad \left(\mathscr{D}/\mathscr{D}^{\alpha_2}\right)\otimes\left(\mathscr{D}^{\alpha_2}/\mathcal{N}\right)\to T\mathcal{X}_0/\mathscr{D}. \tag{4.18}$$

Note that $\deg(\mathscr{D}/\mathscr{D}^{\alpha_2})|_{C_2}\geq -1$, $\deg(\mathscr{D}^{\alpha_2}/\mathcal{N})|_{C_2}\geq 2$ and that the degree of each factor of $T\mathcal{X}_0/\mathscr{D}|_{C_2}$ is at most one. Since f in (4.18) is a nonzero morphism, $\mathscr{D}/\mathscr{D}^{\alpha_2}|_{C_2}=\mathcal{O}(-1)$, $\mathscr{D}^{\alpha_2}/\mathcal{N}|_{C_2}=\mathcal{O}(2)$ and $\mathscr{D}^{-2}/\mathscr{D}|_{C_2}=\mathcal{O}(1)$. It follows that $\mathcal{N}|_{C_2}=\mathcal{O}(1)$, and $\mathscr{D}|_{C_2}=\mathcal{O}(2)\oplus\mathcal{O}(1)\oplus\mathcal{O}(-1)$. Since $\mathscr{D}^{-2}/\mathscr{D}^{\alpha_2}|_{C_2}\subset T\mathcal{X}_0/\mathscr{D}^{\alpha_2}=\mathcal{O}^3$, and $\deg(\mathscr{D}^{-2}/\mathscr{D}^{\alpha_2}|_{C_2})=\deg(\mathscr{D}^{-2}/\mathscr{D}|_{C_2})+\deg(\mathscr{D}/\mathscr{D}^{\alpha_2}|_{C_2})=0$, we have $\mathscr{D}^{-2}|_{C_2}=\mathcal{O}(2)\oplus\mathcal{O}(1)\oplus\mathcal{O}^2$, and $\mathscr{D}^{-3}/\mathscr{D}^{-2}|_{C_2}=\mathcal{O}$. □

Now we can complete the proof of Proposition 4.11.

Proof of Proposition 4.11. By definition of $\mathscr{D}^{-2}$, we have $[\mathscr{D}^{\alpha_2},\mathscr{D}^{-1}]\subset\mathscr{D}^{-2}$. Then the Frobenius bracket $\Lambda^2\mathscr{D}^{-2}\to T\mathcal{X}_0/\mathscr{D}^{-2}$ induces a homomorphism of meromorphic vector bundles over $\mathcal{X}_0$ as follows:

$$\psi:\mathscr{D}^{\alpha_2}\otimes(\mathscr{D}^{-2}/\mathscr{D})\to T\mathcal{X}_0/\mathscr{D}^{-2}.$$

Recall that $\mathscr{D}^{\alpha_2},\mathscr{D}^{-2}/\mathscr{D},T\mathcal{X}_0/\mathscr{D}^{-2}$ are holomorphic in an open neighborhood of $C_2\subset\mathcal{X}_0$, where $[C_2]\in\mathcal{K}^{\alpha_2}(\mathcal{X}_0)$ is a general element. By Proposition 4.18 and Proposition 4.19, $\mathscr{D}^{\alpha_2}|_{C_2}=\mathcal{O}(2)\oplus\mathcal{O}(1)$, $(\mathscr{D}^{-2}/\mathscr{D}^{-1})|_{C_2}=\mathcal{O}(1)$ and $(T\mathcal{X}_0/\mathscr{D}^{-2})|_{C_2}=\mathcal{O}$. Thus, $\psi|_{C_2}=0$. By the general choice of C_2, $\psi=0$. In other words, $[\mathscr{D}^{\alpha_2},\mathscr{D}^{-1}]\subset\mathscr{D}^{-2}$. By the uniqueness of $\mathcal{W}$ in Lemma 4.10, we have $\mathcal{W}=\mathscr{D}^{\alpha_2}$. □

4.3.2. *The meromorphic section σ.* Let us firstly recall a result of A. Weber and J. A. Wiśniewski in [16], in which paper they studied Fano deformation rigidity of complete flag manifolds.

Proposition 4.20. [16, Corollary 1.4, Corollary 3.3] *In the setting 1.11 let α be an element of I such that $\Phi^\alpha: G/P_I\to G/P_{I\setminus\{\alpha\}}$ is a $\mathbb{P}^k$-bundle for some $k\geq 1$. Suppose either*
(i) $H^*(G/P_{I\setminus\{\alpha\}},\mathbb{Q})$ is generated by $H^2(G/P_{I\setminus\{\alpha\}},\mathbb{Q})$; or
(ii) $\mathcal{X}_0^\alpha$ is smooth.
Then $\pi_0^\alpha:\mathcal{X}_0\to\mathcal{X}_0^\alpha$ is also a $\mathbb{P}^k$-bundle.

As a consequence of Proposition 4.20, we have the following result.

Proposition 4.21. *There exists a unique vector bundle of rank 3 over $\mathbb{P}^3$, denoted by $\mathcal{V}$, such that*
(i) $\mathcal{X}_0$ is biholomorphic to $\mathbb{P}(\mathcal{V})$ and $\mathcal{X}_0^{\alpha_2}$ is biholomorphic to $\mathbb{P}^3$;
(ii) $\pi_0^{\alpha_2}:\mathcal{X}_0\to\mathcal{X}_0^{\alpha_2}$ coincides with the projective bundle $\phi:\mathbb{P}(\mathcal{V})\to\mathbb{P}^3$;
(iii) the distribution $\mathscr{D}^{\alpha_2}=T^\phi$, which is holomorphic on $\mathcal{X}_0$;
(iv) $\phi(C_1)$ is a line in $\mathbb{P}^3$ for each $[C_1]\in\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$.
(v) along any line l in $\mathbb{P}^3$, $4\leq\deg(\mathcal{V}|_l)\leq 6$.

Proof. By Proposition 4.20, there exists a vector bundle $\mathcal{V}$ on $\mathbb{P}^3$ satisfying the properties (i) and (ii). Hence $\mathscr{D}^{\alpha_2}=T^\phi$, verifying ($iii$). By Proposition 3.6, $\phi(C_1)$ is a line in $\mathbb{P}^3$, verifying (iv). Since $\deg(\mathcal{V}\otimes\mathcal{O}(k))|_l=\deg(\mathcal{V}|_l)+3k$, we obtain the uniqueness of $\mathcal{V}$ with property (v). □

Notation 4.22. In the rest of Section 4.3, we fix the vector bundle $\mathcal{V}$ as in Proposition 4.21. We use $\phi:\mathbb{P}(\mathcal{V})\to\mathbb{P}^3$ to represent $\pi_0^{\alpha_2}:\mathcal{X}_0\to\mathcal{X}_0^{\alpha_2}$. For $t\in\mathbb{P}^3$ general, we denote by $\mathbb{P}^2_t:=\phi^{-1}(t)$.

Now let us check the splitting types of various meromorphic vector bundles along general elements in $\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$.

Proposition 4.23. *Take $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ general. Then $\mathscr{D}^{\alpha_1}, \mathscr{D}^{\alpha_2}, \mathscr{D}, \mathscr{D}^{-2}, \mathscr{D}^{-3}, \mathcal{N}$ are holomorphic in an open neighborhood of $C_1 \subset \mathcal{X}_0$, and*

$$\begin{aligned}
&\mathcal{N}|_{C_1} = \mathcal{O},\\
&\mathscr{D}^{\alpha_1}|_{C_1} = \mathcal{O}(2),\\
&\mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(-2) \oplus \mathcal{O},\\
&\mathscr{D}^{\alpha_2}/\mathcal{N}|_{C_1} = \mathcal{O}(-2),\\
&\mathscr{D}^{-2}/\mathscr{D}|_{C_1} = \mathcal{O},\\
&\mathscr{D}^{-3}/\mathscr{D}^{-2}|_{C_1} = \mathcal{O}(2),\\
&\mathscr{D}|_{C_1} = \mathscr{D}^{\alpha_1}|_{C_1} \oplus \mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(2) \oplus \mathcal{O}(-2) \oplus \mathcal{O},\\
&\mathscr{D}^{-3}|_{C_1} = T\mathcal{X}_0|_{C_1} = \mathcal{O}(2) \oplus \mathcal{O}^4.
\end{aligned}$$

Proof. The restriction $\mathscr{D}^{\alpha_2}|_{C_1} = TC_1 = \mathcal{O}(2)$. Choose a holomorphic family $[l_t] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_t)$, $t \in \Delta$ satisfying $[l_0] = [C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$. By Proposition 4.21$(ii)$,

$$\deg(\mathscr{D}^{\alpha_2}|_{C_1}) = \deg(T^{\pi_0^{\alpha_2}}|_{l_0}) = \deg(T^{\pi_t^{\alpha_2}}|_{l_t}) \text{ for all } t \in \Delta.$$

By Proposition 2.14, we have

$$\deg(T^{\pi_t^{\alpha_2}}|_{l_t}) = \langle \alpha_2, \alpha_1 \rangle + \langle \alpha_2 + \alpha_3, \alpha_1 \rangle = -2 \text{ for } t \neq 0.$$

It follows that $\deg(\mathscr{D}^{\alpha_2}|_{C_1}) = -2$. Then can write $\mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$, where $a_1 + a_2 = -2$. Since $\mathscr{D}^{-3}|_{C_1} = T\mathcal{X}_0|_{C_1} = \mathcal{O}(2) \oplus \mathcal{O}^4$ and $\mathscr{D}^{\alpha_1}|_{C_1} = \mathcal{O}(2)$, we know that $a_1 \leq 0, a_2 \leq 0$. Hence

$$\text{either} \quad \mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(-1)^2, \text{ or } \mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(-2) \oplus \mathcal{O}. \tag{4.19}$$

It follows that

$$\big(\mathscr{D}^{\alpha_2}/\mathcal{N}\big)|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(a), \text{ where } a \geq -2. \tag{4.20}$$

The injectivity of the homomorphism $\mathscr{D}^{\alpha_1} \otimes \big(\mathscr{D}^{\alpha_2}/\mathcal{N}\big) \to \mathscr{D}^{-2}/\mathscr{D} \subset T\mathcal{X}_0/\mathscr{D}$ in an open neighborhood of $C_1 \subset \mathcal{X}_0$ implies that

$$\mathscr{D}^{-2}/\mathscr{D}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(b), \text{ where } b \geq a + 2. \tag{4.21}$$

The injectivity of $\mathscr{D}^{\alpha_1} \otimes \big(\mathscr{D}^{-2}/\mathscr{D}\big) \to T\mathcal{X}_0/\mathscr{D}^{-2}$ in an open neighborhood of $C_1 \subset \mathcal{X}_0$ implies that

$$\mathscr{D}^{-3}/\mathscr{D}^{-2}|_{C_1} = T\mathcal{X}_0/\mathscr{D}^{-2}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(c), \text{ where } c \geq b + 2. \tag{4.22}$$

On the other hand, the injectivity of $\mathscr{D}^{\alpha_2} \to \mathscr{D}/\mathscr{D}^{\alpha_1} \subset T\mathcal{X}_0/\mathscr{D}^{\alpha_1}$ in an open neighborhood of $C_1 \subset \mathcal{X}_0$ implies that

$$\deg(\mathscr{D}/\mathscr{D}^{\alpha_1})|_{C_1} \geq \deg(\mathscr{D}^{\alpha_2}|_{C_1}) = -2. \tag{4.23}$$

We also have

$$\begin{aligned}
&\deg(T\mathcal{X}_0|_{C_1}) - \deg(\mathscr{D}^{\alpha_1})|_{C_1} - \deg(\mathscr{D}^{-1}/\mathscr{D}^{\alpha_1})|_{C_1}\\
= \;&\deg(\mathscr{D}^{-2}/\mathscr{D})|_{C_1} + \deg(\mathscr{D}^{-3}/\mathscr{D}^{-2})|_{C_1}.
\end{aligned} \tag{4.24}$$

By (4.20)–(4.24), we have

$$\begin{aligned}
2 \;&\geq\; -\deg(\mathscr{D}/\mathscr{D}^{\alpha_1})|_{C_1} = \deg(\mathscr{D}^{-2}/\mathscr{D})|_{C_1} + deg(\mathscr{D}^{-3}/\mathscr{D}^{-2})|_{C_1}\\
&=\; b + c \geq 2b + 2 \geq 2a + 6 \geq 2.
\end{aligned}$$

Hence $\deg(\mathscr{D}/\mathscr{D}^{\alpha_1})|_{C_1} = -2$, $a = -2$, $b = 0$ and $c = 2$. By (4.19) and the fact

$$\deg(\mathscr{D}^{\alpha_2}/\mathcal{N})|_{C_1} = a = -2 = \deg(\mathscr{D}/\mathscr{D}^{\alpha_1})|_{C_1},$$

we know that $\mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(-2) \oplus \mathcal{O} \cong (\mathscr{D}/\mathscr{D}^{\alpha_1})|_{C_1}$. The rest of the conclusion follows immediately. ∎

Proposition 4.24. *In setting of Proposition 4.21, $\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)} = \mathcal{O}(2)^2 \oplus \mathcal{O}$ for $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ general.*

Proof. Take $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ general. Denote by $\mathcal{L}(C_1)$ the line subbundle of $\mathcal{V}$ over the line $\pi_0^{\alpha_2}(C_1) \subset \mathbb{P}^3$ such that $C_1 = \mathbb{P}(\mathcal{L}(C_1)) \subset \mathbb{P}(\mathcal{V}) = \mathcal{X}_0$. Then the relative tangent bundle $T^{\pi_0^{\alpha_2}}|_{C_1} = \mathcal{L}(C_1)^* \otimes \big(\mathcal{V}|_{C_1}/\mathcal{L}(C_1)\big)$. By Proposition 4.21$(iii)$ and Proposition 4.23, $T^{\pi_0^{\alpha_2}}|_{C_1} = \mathscr{D}^{\alpha_2}|_{C_1} = \mathcal{O}(-2) \oplus \mathcal{O}$. Then $\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)} = \mathcal{O}(k)^2 \oplus \mathcal{O}(k-2)$, where $k := \deg \mathcal{L}(C_1)$. By Proposition 4.21$(v)$, $k = 2$ and the conclusion follows. □

Proposition 4.25. *Let $\mathcal{V}$ be as in Proposition 4.21. Then the following holds.*

(i) *Over any line $l \subset \mathbb{P}^3$, either $\mathcal{V}|_l = \mathcal{O}(2) \oplus \mathcal{O}(1)^2$ or $\mathcal{V}|_l = \mathcal{O}(2)^2 \oplus \mathcal{O}$.*

(ii) *Take any $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$. Then $\mathcal{L}(C_1) = \mathcal{O}_{\mathbb{P}^1}(2)$, where $\mathcal{L}(C_1)$ is the unique line subbundle of $\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)}$ such that $C_1 = \mathbb{P}(\mathcal{L}(C_1)) \subset \mathbb{P}(\mathcal{V}) = \mathcal{X}_0$.*

Proof. Since $\mathcal{X}_0 \cong \mathbb{P}(\mathcal{V})$ by Proposition 4.21, any $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ must be a section over the line $\pi_0^{\alpha_2}(C_1) \subset \mathbb{P}^3$ with largest degree. The degree of this section over the line $\pi_0^{\alpha_2}(C_1)$, is independent of the choice of $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$. Then the assertion (ii) follows from proposition 4.24.

Take any line $l \subset \mathbb{P}^3$. Then by Proposition 4.21, $\mathcal{V}|_l$ is a deformation of $\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)} = \mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1}$. Thus we can write $\mathcal{V}|_l = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3)$, where

$$a_1 + a_2 + a_3 = 4, \quad \text{and} \quad a_1 \geq a_2 \geq a_3. \tag{4.25}$$

By the maximality of $\deg \mathcal{L}(C_1)$ among sections of $\mathcal{V}$ over lines in $\mathbb{P}^3$, we have

$$a_1 \leq \deg \mathcal{L}(C_1) = 2. \tag{4.26}$$

The assertion (i) follows from (4.25) and (4.26). □

Corollary 4.26. *Let $\mathcal{V}$ be as in Proposition 4.21. Then there exists a nonempty Zariski open subset $\mathcal{U} \subset \mathbb{P}^3$ and a section of $\pi_0^{\alpha_2} : \mathcal{X}_0 = \mathbb{P}(\mathcal{V}) \to \mathbb{P}^3$ over $\mathcal{U}$, denoted by $\sigma : \mathcal{U} \to \mathcal{X}_0$, such that for any $x \in (\pi_0^{\alpha_2})^{-1}(U) \setminus \sigma(U)$,*

(i) *$\mathcal{N}$ is holomorphic at x;*

(ii)*$\mathcal{N}_x = T_x l_x$, where $l_x := \langle x, \sigma(\pi_0^{\alpha_2}(x))\rangle$ is the line in $(\pi_0^{\alpha_2})^{-1}(\pi_0^{\alpha_2}(x)) \cong \mathbb{P}^2$ joining x and $\sigma(\pi_0^{\alpha_2}(x))$;*

(iii) *the leaf of $\mathcal{N}$ at x is the affine line $l_x \setminus \{\sigma(\pi_0^{\alpha_2}(x))\}$.*

Proof. Take $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ general. Then $\pi_0^{\alpha_2}(C_1)$ is a line in $\mathbb{P}^3$ and $\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)} = \mathcal{O}(2)^2 \oplus \mathcal{O}$. The curve C_1 is identified with $\mathbb{P}(\mathcal{L}(C_1)) \subset \mathbb{P}(\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)}) = (\pi_0^{\alpha_2})^{-1}(\pi_0^{\alpha_2}(C_1))$, where $\mathcal{L}(C_1) \cong \mathcal{O}_{\mathbb{P}^1}(2) \subset \mathcal{V}|_{\pi_0^{\alpha_2}(C_1)}$ is as in Proposition 4.25(ii). We know $\mathcal{N} \subset \mathscr{D}^{\alpha_2}$, $\mathcal{N}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}(-2) \oplus \mathcal{O} = \mathscr{D}^{\alpha_2}|_{C_1} \cong \mathcal{L}(C_1)^* \otimes (\mathcal{V}|_{\pi_0^{\alpha_2}(C_1)}/\mathcal{L}(C_1))$. It follows that $\mathcal{N}|_{C_1} = \bigcup_{x \in C_1} T_x \mathbb{P}(\mathcal{V}|^+_{\pi_0^{\alpha_2}(x)})$, where $\mathcal{V}|^+_{\pi_0^{\alpha_2}(C_1)} = \mathcal{O}(2)^2 \subset \mathcal{V}|_{\pi_0^{\alpha_2}(C_1)} = \mathcal{O}(2)^2 \oplus \mathcal{O}$ and $T_{C_1}\mathbb{P}(\mathcal{V}|^+_{\pi_0^{\alpha_2}(C_1)})$ is the relative tangent bundle of $\mathbb{P}(\mathcal{V}|^+_{\pi_0^{\alpha_2}(C_1)}) \to \pi_0^{\alpha_2}(C_1)$ along $C_1 \subset \mathbb{P}(\mathcal{V}|^+_{\pi_0^{\alpha_2}(C_1)})$. In other words, at any point $x \in C_1$, $\mathcal{N}_x = T_x \mathbb{P}(\mathcal{O}_{\pi_0^{\alpha_2}(C_1)}(2)^2|_{\pi_0^{\alpha_2}(x)})$, where $\mathcal{O}_{\pi_0^{\alpha_2}(C_1)}(2)^2|_{\pi_0^{\alpha_2}(x)} \subset \mathcal{V}_{\pi_0^{\alpha_2}(x)}$ is the fiber of $\mathcal{O}(2)^2 \subset \mathcal{V}|_{\pi_0^{\alpha_2}(C_1)}$ at the point $\pi_0^{\alpha_2}(x) \in \pi_0^{\alpha_2}(C_1)$.

Note that $\mathbb{P}_{\pi_0^{\alpha_2}(C_1)}(\mathcal{O}(2)^2) \cong \mathbb{P}^1 \times \pi_0^{\alpha_2}(C_1) \cong \mathbb{P}^1$. It follows that given any $x \in C_1$ and any $y \in \mathbb{P}(\mathcal{O}_{\pi_0^{\alpha_2}(C_1)}(2)^2|_{\pi_0^{\alpha_2}(x)})$ lying in the regular locus of $\mathcal{N}$, there exists $[C_y] \in \mathcal{K}^{\alpha_1}_y(\mathcal{X}_0)$ such that $\pi_0^{\alpha_2}(C_y) = \pi_0^{\alpha_2}(C_1)$ and $\mathcal{N}_y = T_y \mathbb{P}(\mathcal{O}_{\pi_0^{\alpha_2}(C_1)}(2)^2|_{\pi_0^{\alpha_2}(x)})$. Hence, the closure of the leaf at $x \in C_1 \subset \mathcal{X}_0$ is the line $l_x = \mathbb{P}(\mathcal{O}_{\pi_0^{\alpha_2}(C_1)}(2)^2_{\pi_0^{\alpha_2}(x)})$.

Take $t \in \mathbb{P}^3$ general and denote by $\mathbb{P}^2_t := (\pi_0^{\alpha_2})^{-1}(t) \cong \mathbb{P}^2 \subset \mathcal{X}_0$. Let $A \subset (\mathbb{P}^2_t)^*$ be the closure of the family of lines $l_x := \mathbb{P}(\mathcal{O}_{\pi_0^{\alpha_2}(C_x)}(2)^2|_{\pi_0^{\alpha_2}(x)})$, where x runs over the set of general points on $\mathbb{P}^2_t$ such that $\mathcal{K}^{\alpha_1}_x(\mathcal{X}_0)$ consists of a unique element $[C_x]$ and $\mathcal{N}$ is holomorphic at x. For a general point $x \in \mathbb{P}^2_t$, $E_x := \{[l] \in (\mathbb{P}^2_t)^*\}$ is a line in $(\mathbb{P}^2_t)^*$ and $E_x \cap A$ consist of a single point, namely $[l_x]$, in $(\mathbb{P}^2_t)^*$. Since E_x could be a general line in $(\mathbb{P}^2_t)^*$, the intersection number $(E_x \cdot A) = 1$. It follows that A is a line in $(\mathbb{P}^2_t)^*$ and there exists a unique point $\sigma(t) \in \mathbb{P}^2_t$ such that $A = \{[l] \in (\mathbb{P}^2_t)^* \mid \sigma(t) \in l\}$.

It turns out that $\mathcal{N}$ is well-defined on $\mathbb{P}^2_t \setminus \{\sigma(t)\}$, and at any $x \in \mathbb{P}^2_t \setminus \{\sigma(t)\}$, the line $\langle x, \sigma(t)\rangle$ is the leaf closure of $\mathcal{N}$ at x. The conclusion follows. □

4.3.3. *A subset of family of lines on* $\mathbb{P}^3$.

Notation 4.27. For $t \in \mathbb{P}^3$ general, denote by $\mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)$ the Zariski closure of

$$\{[\pi^{\alpha_2}_0(C_x)] \in \mathbb{G}(1,\mathbb{P}^3) \mid x \in (\pi^{\alpha_2}_0)^{-1}(t) \text{ general }, \mathcal{K}^{\alpha_1}_x(\mathcal{X}_0) = \{[C_x]\}\}$$

in $\mathbb{G}(1,\mathbb{P}^3)$, and set

$$\mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) := \bigcup_{[l]\in\mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)} \mathbb{P}(T_t l) \subset \mathbb{P}(T_t\mathbb{P}^3).$$

Here $\mathbb{G}(1,\mathbb{P}^3)$ is the family of lines in $\mathbb{P}^3$. Denote by

$$\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) := \text{ Zariski closure of } \bigcup_{t\in\mathbb{P}^3 \text{ general}} \mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) \text{ in } \mathbb{G}(1,\mathbb{P}^3),$$

$$\mathcal{C}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) := \text{ Zariski closure of } \bigcup_{t\in\mathbb{P}^3 \text{ general}} \mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) \text{ in } \mathbb{P}(T\mathbb{P}^3).$$

Let $U^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ be the inverse image of $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ under the natural morphism $F(1,2;\mathbb{C}^4) \to \mathbb{G}(1,\mathbb{P}^3) \supset \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$.

Lemma 4.28. *Take* $t \in \mathbb{P}^3$ *general. Then* $\mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)$ *is an irreducible rational curve. Take any* $[l] \in \mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)$. *There exists* $[C] \in \mathcal{K}^{\alpha_1}_{\sigma(t)}(\mathcal{X}_0)$ *such that* $\pi^{\alpha_2}_0(C) = l$.

Proof. Take $t \in \mathbb{P}^3$ general and $x \in \mathbb{P}^2_t := (\pi^{\alpha_2}_0)^{-1}(t)$ general. Then $\mathcal{K}^{\alpha_1}_x$ consists of a single element, written as $[C_x]$. Furthermore, $C_x \cong \mathbb{P}^1$ and $\pi^{\alpha_2}_0$ sends C_x biholomorphically onto a line in $\mathbb{P}^3$. Since $\mathcal{V}|_{\pi^{\alpha_2}_0(C_x)} = \mathcal{O}(2)^2 \oplus \mathcal{O}$ and the line $\langle x, \sigma(t)\rangle$ in $\mathbb{P}^2_t$ coincides with the fiber $\mathbb{P}(\mathcal{O}_{\pi^{\alpha_2}_0(C_x)}(2)^2|_t)$, there exists a unique $[C_{t,x}] \in \mathcal{K}^{\alpha_1}_{\sigma(t)}(\mathcal{X}_0)$ such that $\pi^{\alpha_2}_0(C_{t,x}) = \pi^{\alpha_2}_0(C_x)$. Take $y \in \mathbb{P}^2_t \setminus \langle x, \sigma(x)\rangle$ general. Then the fact

$$\mathbb{P}(\mathcal{O}_{\pi^{\alpha_2}_0(C_x)}(2)^2|_t) \cap \mathbb{P}(\mathcal{O}_{\pi^{\alpha_2}_0(C_y)}(2)^2|_t) = \langle x, \sigma(t)\rangle \cap \langle y, \sigma(t)\rangle = \{\sigma(t)\}$$

implies that $\pi^{\alpha_2}_0(C_x) \neq \pi^{\alpha_2}_0(C_y)$ (and hence $C_{t,x} \neq C_{t,y}$). This induces injective rational maps (hence injective morphisms)

$$\begin{aligned} \xi: \quad & \mathbb{P}^1 \cong \{[l] \in (\mathbb{P}^2_t)^* \mid \sigma(t) \in l\} \dashrightarrow \mathcal{K}^{\alpha_1}_{\sigma(t)}(\mathcal{X}_0) \\ & \langle x, \sigma(t)\rangle \mapsto [C_{t,x}], \\ \eta: \quad & \mathbb{P}^1 \cong \{[l] \in (\mathbb{P}^2_t)^* \mid \sigma(t) \in l\} \dashrightarrow \mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) \\ & \langle x, \sigma(t)\rangle \mapsto [\pi^{\alpha_2}_0(C_x)]. \end{aligned}$$

By definition $\mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)$ is the closure of the image of η. Then the conclusion follows immediately from these morphisms ξ and η. □

The following can also be deduced from the proof of Lemma 4.28.

Lemma 4.29. *Take* $t \in \mathbb{P}^3$ *general. Denote by* $\mathbb{P}^2_t := (\pi^{\alpha_2}_0)^{-1}(t) \subset \mathcal{X}_0$. *Define*

$$\begin{aligned} \psi: \quad & \mathbb{P}^2_t \dashrightarrow \mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) \subset \mathbb{G}(1,\mathbb{P}^3) \\ & x \longmapsto [\pi^{\alpha_2}_0(C_x)], \end{aligned}$$

where $x \in \mathbb{P}^2_t$ *general and* $[C_x]$ *is the unique element of* $\mathcal{K}^{\alpha_1}_x(\mathcal{X}_0)$. *Then* ψ *coincides with the linear projection of* $\mathbb{P}^2_t$ *with center* $\sigma(t)$. *In other words, for* $x, y \in Dom(\psi)$, $\psi(x) = \psi(y)$ *if and only if* $\langle x, \sigma(t)\rangle = \langle y, \sigma(t)\rangle$.

Construction 4.30. Take $x \in \mathcal{X}_0$ general. Recall two elementary Mori contractions:

$$\begin{aligned} \pi^{\alpha_1}_0: \quad & \mathcal{X}_0 \to \mathcal{X}^{\alpha_1}_0, \text{ and} \\ \pi^{\alpha_2}_0: \quad & \mathcal{X}_0 = \mathbb{P}(\mathcal{V}) \to \mathcal{X}^{\alpha_2}_0 = \mathbb{P}^3 \end{aligned}$$

Set $\Sigma_0(x) := \{x\}$. For each $k \geq 0$ let $\Sigma_{2k+1}(x)$ be the unique irreducible component of $(\pi^{\alpha_2}_0)^{-1}(\pi^{\alpha_2}_0(\Sigma_{2k}(x)))$ dominating $\pi^{\alpha_2}_0(\Sigma_{2k}(x))$, and $\Sigma_{2k+2}(x)$ be the unique irreducible component of $(\pi^{\alpha_1}_0)^{-1}(\pi^{\alpha_1}_0(\Sigma_{2k+1}(x)))$ dominating $\pi^{\alpha_1}_0(\Sigma_{2k+1}(x))$.

Lemma 4.31. *In setting of Construction 4.30, we have*

$$\dim \Sigma_k(x) = k+1, \text{ where } 1 \leq k \leq 4.$$

In particular, $\Sigma_4(x) = \mathcal{X}_0$.

Proof. By construction, $\Sigma_1(x) = (\pi_0^{\alpha_2})^{-1}(\pi_0^{\alpha_2}(x)) \cong \mathbb{P}^2$, which has dimension 2. Now we claim that for each $k \geq 1$, either $\Sigma_k(x) = \mathcal{X}_0$ or $\dim \Sigma_{k+1}(x) \geq \dim \Sigma_k(x) + 1$.

Suppose $\dim \Sigma_{k+1}(x) = \dim \Sigma_k(x)$ for some $k \geq 1$. Then $\Sigma_{k+1}(x) = \Sigma_k(x)$. By construction of $\Sigma_k(x)$ and $\Sigma_{k+1}(x)$, $C_y^1 \subset \Sigma_k(x)$ and $C_y^2 \subset \Sigma_k(x)$ for $y \in \Sigma_k(x)$ general, $[C_y^1] \in \mathcal{K}_y^{\alpha_1}(\mathcal{X}_0)$ and $[C_y^2] \in \mathcal{K}_y^{\alpha_2}(\mathcal{X}_0)$ general. By Proposition 3.7, we have $\Sigma_k(x) = \mathcal{X}_0$, and the claim holds.

By general choice of $x \in \mathcal{X}_0$ and the construction of $\Sigma_k(x)$, for each $i \geq 1$ we have

$$\dim \Sigma_{2i+1}(x) \leq \dim \pi_0^{\alpha_2}(\Sigma_{2i}(x)) + 2 \leq \dim \Sigma_{2i}(x) + 2,$$
$$\dim \Sigma_{2i}(x) \leq \dim \pi_0^{\alpha_1}(\Sigma_{2i-1}(x)) + 1 \leq \dim \Sigma_{2i-1}(x) + 1.$$

Note that $\pi_0^{\alpha_2}(\Sigma_2(x)) = \bigcup_{[l] \in \mathcal{K}^{\alpha_1}_{\pi_0^{\alpha_2}(x)}(\mathcal{X}_0/\mathbb{P}^3)} l$, which has dimension 2 by Lemma 4.28. Then the conclusion follows from the inequalities above. □

Lemma 4.32. *Take $t \in \mathbb{P}^3$ general, and set*

$$\Lambda_1(t) := \bigcup_{[l] \in \mathcal{K}_t^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)} l \subset \mathbb{P}^3,$$
$$\Lambda_2(t) := \text{ Zariski closure of } \bigcup_{[l] \in \mathcal{K}^g_{\Lambda_1(t)}} l \text{ in } \mathbb{P}^3,$$

where we define

$$\mathcal{K}^g_{\Lambda_1(t)} := \bigcup_{z \in \Lambda_1(t) \text{ general}} \mathcal{K}_z^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3).$$

Then $\Lambda_2(t) = \mathbb{P}^3$.

Proof. Take $x \in \mathbb{P}_t^2 := (\pi_0^{\alpha_2})^{-1}(t)$ general, then by construction we have

$$\pi^{\alpha_2}(\Sigma_{2k}(x)) = \Lambda_k(t), \quad k = 1, 2,$$

where $\Sigma_{2k}(x)$ is as in Construction 4.30. By Lemma 4.31, $\Sigma_4(x) = \mathcal{X}_0$, which implies the conclusion. □

Lemma 4.33. *Let $\mathcal{L}_\sigma \subset \mathcal{V}$ be the meromorphic line subbundle of $\mathcal{V}$ over $\mathbb{P}^3$ defining the meromorphic section σ of $\pi_0^{\alpha_2} : \mathcal{X}_0 = \mathbb{P}(\mathcal{V}) \to \mathbb{P}^3$, and S_σ be the singular locus of σ. Then $\dim S_\sigma \leq 1$ and there exist nonempty Zariski open subsets $U'' \subset U' \subset \mathbb{P}^3 \setminus S_\sigma$ such that*

(i) $C_1 \subset \mathbb{P}(\mathcal{L}_\sigma)$ for any $t \in U'$ and any $[C_1] \in \mathcal{K}^{\alpha_1}_{\sigma(t)}(\mathcal{X}_0)$;

(ii) given any $t \in U''$ we have $M_2(t) = \mathbb{P}(\mathcal{L}_\sigma)$, where

$$M_1(t) := \bigcup_{[C] \in \mathcal{K}^{\alpha_1}_{\sigma(t)}(\mathcal{X}_0)} C \subset \mathbb{P}(\mathcal{L}_\sigma),$$
$$M_2(t) := \text{ Zariski closure of } \bigcup_{[C] \in \mathcal{K}^{\alpha_1}_{M_1(t) \cap \sigma(U')}} C \subset \mathbb{P}(\mathcal{L}_\sigma),$$

where we define

$$\mathcal{K}^{\alpha_1}_{M_1(t) \cap \sigma(U')} := \bigcup_{x \in M_1(t) \cap \sigma(U')} \mathcal{K}_x^{\alpha_1}(\mathcal{X}_0).$$

Proof. Being the singular locus of a meromorphic section, the dimension of S_σ is less or equal to $\dim \mathbb{P}^3 - 2 = 1$. By Lemma 4.28, $\dim \mathcal{K}^{\alpha_1}_{\sigma(t)}(\mathcal{X}_0) \geq 1$ for $t \in \mathbb{P}^3$ general. By semicontinuity of the dimension function, $\dim \mathcal{K}^{\alpha_1}_x(\mathcal{X}_0) \geq 1$ for all $x \in \mathbb{P}(\mathcal{L}_\sigma)$. Hence $\mathbb{P}(\mathcal{L}_\sigma) \subset E(\mathcal{K}^{\alpha_1})$, where $E(\mathcal{K}^{\alpha_1}) \subset \mathcal{X}_0$ is the union of $\mathcal{K}^{\alpha_1}(\mathcal{X}_0)$-equivalence classes that are of dimension at least two. By Proposition 4.15, $E(\mathcal{K}^{\alpha_1})$ is a Zariski closed subset of $\mathcal{X}_0$ and $\dim E(\mathcal{K}^{\alpha_1}) \leq \dim \mathcal{X}_0 - 2 = 3$. By dimension reason the variety $\mathbb{P}(\mathcal{L}_\sigma)$ is an irreducible component of $E(\mathcal{K}^{\alpha_1})$.

Denote by U the nonempty Zariski open subset of $\mathbb{P}(\mathcal{L}_\sigma)$ such that at any $x \in U$, $\mathbb{P}(\mathcal{L}_\sigma)$ is the unique irreducible component of $E(\mathcal{K}^{\alpha_1})$ containing x. Set $U' := \pi_0^{\alpha_2}(U) \setminus S_\sigma$, then the assertion (i) of Lemma 4.33 holds.

By Lemma 4.28, $\pi_0^{\alpha_2}(M_k(t)) = \Lambda_k(t)$ for $k = 1, 2$. Then by Lemma 4.32 $\phi(M_2(t)) = \mathbb{P}^3$, implying that $\dim M_2(t) \geq 3$. Since $M_2(t) \subset \mathbb{P}(\mathcal{L}_\sigma)$ by the assertion (i), we have $M_2(t) = \mathbb{P}(\mathcal{L}_\sigma)$, verifying the assertion (ii). □

Lemma 4.34. *For $t \in \mathbb{P}^3$ general, $\mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)$ is a line in $\mathbb{P}(T_t\mathbb{P}^3)$. Furthermore, $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is a hyperplane section of $\mathbb{G}(1, \mathbb{P}^3) \subset \mathbb{P}^5$.*

Proof. By Proposition 4.11, $[\mathscr{D}^{\alpha_2}, \mathscr{D}^{-2}] \subset \mathscr{D}^{-2}$, where $\mathscr{D}^{-2}$ is the weak derivative of $\mathscr{D} = \mathscr{D}^{\alpha_1} + \mathscr{D}^{\alpha_2}$. It follows that $\mathcal{E} := d\pi_0^{\alpha_2}(\mathscr{D}^{-2})$ is a meromorphic distribution $\mathcal{E}$ on $\mathbb{P}^3$ of rank 2, where $d\pi_0^{\alpha_2} : T(\mathcal{X}_0) \to T(\mathbb{P}^3)$ is the tangent map of $\pi_0^{\alpha_2}$. Take a general element $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$. Then $T(C_1) = \mathscr{D}^{\alpha_1}|_{C_1} \subset \mathscr{D}^{-2}$, which implies that $T(\pi_0^{\alpha_2}(C_1)) \subset \mathcal{E}|_{\pi_0^{\alpha_2}(C_1)}$. Hence at a general point $t \in \mathbb{P}^3$, we have $\mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) \subset \mathbb{P}(\mathcal{E}_t)$. Since $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is a set of lines on $\mathbb{P}^3$, we have $\mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) \cong \mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3)$, which is an irreducible rational curve by Lemma 4.28. Hence $\mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) = \mathbb{P}(\mathcal{E}_t)$ is a line in $\mathbb{P}(T_t\mathbb{P}^3)$. Moreover,

$$\dim \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) = \dim \mathbb{P}^3 + \dim \mathcal{K}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) - 1 = 3.$$

Thus the variety $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is an effective divisor on $\mathbb{G}(1, \mathbb{P}^3)$. Consider

$$\begin{array}{ccc} U^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) & \longrightarrow & \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \\ \downarrow & & \downarrow \\ \mathbb{P}(T\mathbb{P}^3) & \longrightarrow & \mathbb{G}(1, \mathbb{P}^3) \\ \downarrow & & \\ \mathbb{P}^3. & & \end{array} \tag{4.27}$$

Since for $t \in \mathbb{P}^3$ general,

$$\mathcal{C}^{\alpha_1}_t(\mathcal{X}_0/\mathbb{P}^3) = U^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \cap \mathbb{P}(T_t\mathbb{P}^3)$$

is a line in $\mathbb{P}(T_t\mathbb{P}^3)$, we can conclude that $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \in |\iota^*\mathcal{O}_{\mathbb{P}^5}(1)|$, where $\iota : \mathbb{G}(1, \mathbb{P}^3) \to \mathbb{P}^5$ is the Plüker embedding. Since $\mathbb{G}(1, \mathbb{P}^3) \subset \mathbb{P}^5$ is linearly normal, $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is a hyperplane section of $(1, \mathbb{P}^3) \subset \mathbb{P}^5$. □

4.3.4. *Hyperplane bundles of $\mathbb{P}(\mathcal{V})$ over $\mathbb{P}^3$.*

Notation 4.35. Let $\mathcal{L}_0^{\alpha_i}$ be the Cartier divisor on $\mathcal{X}_0$ such that the intersection number $(\mathcal{L}_0^{\alpha_i} \cdot C_j) = \delta_{ij}$, where $[C_j] \in \mathcal{K}^{\alpha_j}(\mathcal{X}_0)$ and $1 \leq i, j \leq 2$. In other words, $\mathcal{L}_0^{\alpha_i} := \mathcal{L}^{\alpha_i}$, where $\mathcal{L}^{\alpha_i}$ is as in Proposition-Definition 3.6. Denote by $|\mathcal{L}_0^{\alpha_i}|$ the corresponding linear system of effective Weil divisors on $\mathcal{X}_0$.

Lemma 4.36. *We have* $\dim |\mathcal{L}_0^{\alpha_1}| = 3$ *and* $\dim |\mathcal{L}_0^{\alpha_2}| \geq 5$.

Proof. Since $\mathcal{X}_0^{\alpha_2} = \mathbb{P}^3$, we have $\mathcal{L}_0^{\alpha_1} = (\pi_0^{\alpha_2})^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $\dim |\mathcal{L}_0^{\alpha_1}| = \dim \mathbb{P}^3 = 3$. There exists a holomorphic line bundle $\mathcal{L}^{\alpha_2}$ on $\mathcal{X}$ such that $\mathcal{L}_0^{\alpha_2} \cong \mathcal{L}^{\alpha_2}|_{\mathcal{X}_0}$ and for $0 \neq t \in \Delta$, the linear system $|\mathcal{L}_t^{\alpha_2}|$ induces the morphism

$$\mathcal{X}_t \cong F(1, 2; \mathbb{C}^4) \to Gr(2, \mathbb{C}^4) \subset \mathbb{P}^5,$$

where $\mathcal{L}_t^{\alpha_2} := \mathcal{L}^{\alpha_2}|_{\mathcal{X}_t}$ By semicontinuity, we have $\dim |\mathcal{L}_0^{\alpha_2}| \geq 5$. □

Notation 4.37. Take any $W \in |\mathcal{L}_0^{\alpha_2}|$. Then for $t \in \mathbb{P}^3$ general, we denote by

$$\mathcal{K}_t^{\alpha_1}(W/\mathbb{P}^3) := \psi(W_t) \subset \mathcal{K}_t^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3),$$

where ψ is as in Lemma 4.29. Set

$$\mathcal{K}^{\alpha_1}(W/\mathbb{P}^3) := \text{ Zariski closure of } \bigcup_{t \in \mathbb{P}^3 \text{ general}} \mathcal{K}_t^{\alpha_1}(W/\mathbb{P}^3) \text{ in } \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3).$$

Lemma 4.38. *In setting of Notation 4.37, there is an injective map*

$$\theta : \quad \{W \in |\mathcal{L}_0^{\alpha_2}| \mid \mathbb{P}(\mathcal{L}_\sigma) \subset W\} \to \{\textit{hyperplane sections of } \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)\}$$
$$W \longmapsto \mathcal{K}^{\alpha_1}(W/\mathbb{P}^3).$$

Proof. Take $t \in \mathbb{P}^3$ general. Then the fact $\sigma(t) \in W$ implies that W_t is a line in $\mathbb{P}_t^2 := (\pi_0^{\alpha_2})^{-1}(t)$ passing through $\sigma(t)$. By Lemma 4.29, $\mathcal{K}_t^{\alpha_1}(W/\mathbb{P}^3)$ consists of a single element. Then $\mathcal{K}^{\alpha_1}(W/\mathbb{P}^3)$ is an effective divisor on $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$. Similarly with the analysis for diagram (4.27), we know that $\mathcal{K}^{\alpha_1}(W/\mathbb{P}^3)$ is a hyperplane section of $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$. □

Lemma 4.39. *Take $W \in |\mathcal{L}_0^{\alpha_2}|$ general. Then $\sigma(t) \notin W$ for $t \in \mathbb{P}^3$ general.*

Proof. By Lemma 4.34 and Lemma 4.38, the space $\{W \in |\mathcal{L}_0^{\alpha_2}| \mid \mathbb{P}(\mathcal{L}_\sigma) \subset W\}$ has dimension at most 4. On the other hand, $\dim |\mathcal{L}_0^{\alpha_2}| \geq 5$ by Lemma 4.36. Then the conclusion follows. □

Lemma 4.40. *Take $W \in |\mathcal{L}_0^{\alpha_2}|$ general, and denote by*

$$S(W) := \{t \in \mathbb{P}^3 \mid (\pi_0^{\alpha_2})^{-1}(t) \subset W\}. \tag{4.28}$$

Then $\dim S(W) \leq 1$ and $W|_{\mathbb{P}^3 \setminus S(W)} \to \mathbb{P}^3 \setminus S(W)$ is a $\mathbb{P}^1$-bundle.

Proof. As a Cartier divisor we have $\mathcal{O}_{\mathcal{X}_0}(W)|_{\mathbb{P}_t^2} \cong \mathcal{O}_{\mathbb{P}_t^2}(1)$ for any $t \in \mathbb{P}^3$, where $\mathbb{P}_t^2 := (\pi_0^{\alpha_2})^{-1}(t)$. Thus for any $t \in \mathbb{P}^3 \setminus S(W)$, the scheme-theoretic intersection of W with $\mathbb{P}_t^2$ is a line. By dimension counting $\dim S(W) \leq \dim W - 2 = 2$. If $\dim S(W) = 2$, then the intersection number $(W \cdot C_1) > 0$ for $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$, contradicting our definition of $\mathcal{L}_0^{\alpha_2}$ in Notation 4.35. □

Lemma 4.41. *Take $W \in |\mathcal{L}_0^{\alpha_2}|$ general, and denote by $S_W := \pi_0^{\alpha_2}(\mathbb{P}(\mathcal{L}_\sigma) \cap W) \subset \mathbb{P}^3$. Then $\dim S_W \leq 1$.*

Proof. Now suppose $\dim S_W \geq 2$. By Lemma 4.39, $S_W \neq \mathbb{P}^3$. Choose any irreducible component $\widetilde{S}_W$ of S_W such that $\dim \widetilde{S}_W = 2$.

We claim that for $\tilde{t} \in \widetilde{S}_W$ general, there exists $t \in U''$ and $[l] \in \mathcal{K}_t^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ such that $\tilde{t} \in l$, where U'' is as in Lemma 4.33 (*ii*).

Suppose the claim holds. By Lemma 4.28 there exists $[C] \in K_{\sigma(t)}^{\alpha_1}(\mathcal{X}_0)$ such that $\pi_0^{\alpha_2}(C) = l$. By Lemma 4.33, $\dim S_\sigma \leq 1$, where $S_\sigma \subset \mathbb{P}^3$ is the singular locus of the section σ. Then the general choice of $\tilde{t}$ in the divisor $\widetilde{S}_W \subset \mathbb{P}^3$ implies that $\tilde{t} \notin S_\sigma$. In particular, $\mathbb{P}(\mathcal{L}_\sigma)_{\tilde{t}} = \sigma(\tilde{t}) \in C \cap W$. Since the intersection number $(W \cdot C) = 0$, we have $C \subset W$, implying that $\sigma(t) \in W$. By Lemma 4.33(*ii*) and the fact $(W \cdot C_1) = 0$ for any $[C_1] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$, we have $\mathbb{P}(\mathcal{L}_\sigma) \subset W$. This contradicts Lemma 4.39. Hence we obtain the conclusion of Lemma 4.41.

Now we turn to prove the claim. Suppose it fails. Let A be the Zariski closure of the union $\bigcup(l \cap \widetilde{S}_W)$ in $\widetilde{S}_W$, where $[l]$ runs over the set $\bigcup_{t \in \mathbb{P}^3 \text{ general}} \mathcal{K}_t^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$. By assumption, $\dim A \leq \dim \widetilde{S}_W - 1 = 1$.

Since every element in $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ has a nonempty intersection with $\widetilde{S}_W$, there is an irreducible component $\widetilde{A}$ of A such that

$$\dim \mathcal{K}_s^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \geq \dim \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) - \dim \widetilde{A} \geq 2 \text{ for each } s \in \widetilde{A}.$$

Since $\mathcal{K}_s^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \cong \mathcal{C}_s^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \subset \mathbb{P}(T_s\mathbb{P}^3)$, we know that

$$\dim \widetilde{A} = 1, \quad \mathcal{K}_s^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \cong \mathcal{C}_s^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) = \mathbb{P}(T_s\mathbb{P}^3)$$

and $[\langle t, s \rangle] \in \mathcal{K}_t^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ for all $s \in \widetilde{A}$ and all $t \in \mathbb{P}^3 \setminus \{s\}$.

Take $t \in \mathbb{P}^3$ general. By Lemma 4.34 and the conclusions above, $\mathcal{K}_t^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) = \{[\langle t, s\rangle] \mid s \in \widetilde{A}\}$, and the join variety $J(t, \widetilde{A}) := \bigcup_{s\in\widetilde{A}} \langle t, s\rangle$ is a plane in $\mathbb{P}^3$. Thus in the notations of Lemma 4.32, we have $\Lambda_1(t) = J(t, \widetilde{A})$. For $t' \in \Lambda_1(t)$ general, the same reason implies that $\Lambda_1(t') = J(t', \widetilde{A}) = J(t, \widetilde{A}) = \Lambda_1(t)$. It follows that $\Lambda_2(t) = \Lambda_1(t) \subsetneq \mathbb{P}^3$, contradicting Lemma 4.32. Hence, the claim holds. □

Lemma 4.42. *There exists a meromorphic vector subbundle $\mathcal{L}_W \subset \mathcal{V}$ of rank two over $\mathbb{P}^3$ and a closed subvariety $S_W \subset \mathbb{P}^3$ such that*

(i) $\dim S_W \le 1$*;*

(ii) *both $\mathcal{L}_\sigma$ and $\mathcal{L}_W$ are holomorphic vector bundles on $\mathbb{P}^3 \setminus S_W$, where $\mathcal{L}_\sigma$ is as in Lemma 4.33;*

(iii) *there is a direct sum decomposition $\mathcal{V}|_{\mathbb{P}^3\setminus S_W} = \mathcal{L}_\sigma|_{\mathbb{P}^3\setminus S_W} \oplus \mathcal{L}_W|_{\mathbb{P}^3\setminus S_W}$;*

(iv) $\mathbb{P}(\mathcal{L}_W) \in |\mathcal{L}_0^{\alpha_2}|$ *is a chosen general divisor.*

Proof. It is a direct consequence of Lemma 4.40, Lemma 4.41 and the fact $\dim S_\sigma \le 1$, where S_σ is the singular locus of the section $\mathbb{P}(\mathcal{L}_\sigma)$. □

To continue, we need to collect a result of decomposition of vector bundles, which can be found on page 409 in [8]. See also [13, Proposition 5] for an explicit statement with a brief proof.

Proposition 4.43. [8, page 409] *Let $\mathcal{E}$ be a vector bundle over a connected complex manifold Y. Suppose there is a complex subvariety $A \subset Y$ and vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over $Y \setminus A$ such that* $\dim A \le \dim Y - 2$ *and $\mathcal{E}|_{Y\setminus A} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Then $\mathcal{E}_1$ and $\mathcal{E}_2$ can be extended uniquely as vector bundles $\mathcal{E}_1'$ and $\mathcal{E}_2'$ over Y such that $\mathcal{E} = \mathcal{E}_1' \oplus \mathcal{E}_2'$.*

As a direct consequence of Lemma 4.42 and Proposition 4.43, we have the following result.

Proposition 4.44. *In setting of Lemma 4.42, both $\mathcal{L}_\sigma$ and $\mathcal{L}_W$ are holomorphic vector bundles on $\mathbb{P}^3$, and $\mathcal{V} = \mathcal{L}_\sigma \oplus \mathcal{L}_W$.*

Lemma 4.45. *In setting of Proposition 4.44, the followings hold.*

(i) *For any $[l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$, $\mathcal{L}_\sigma|_l = \mathcal{O}(2)$ and $\mathcal{L}_W|_l = \mathcal{O}(2) \oplus \mathcal{O}$.*

(ii) *For any $[l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$, there exists a unique $[C_l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ such that $C_l \subset W$ and $\pi_0^{\alpha_2}(C_l) = l$. Moreover, this curve $C_l \cong \mathbb{P}^1$.*

(iii) *For any $x \in W$, $\mathcal{K}_x^{\alpha_1}(\mathcal{X}_0)$ consists of a single element, denoted by $[C_x]$. Moreover, this curve $C_x \subset W$ and $C_x \cong \mathbb{P}^1$.*

Proof. By Proposition 4.24,

$$\mathcal{V}|_l = \mathcal{O}(2)^2 \oplus \mathcal{O}, \quad \text{for } [l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \text{ general.} \tag{4.29}$$

By Proposition 4.25(i), the restriction of $\mathcal{V}$ on any line of $\mathbb{P}^3$ is either $\mathcal{O}(2)^2 \oplus \mathcal{O}$ or $\mathcal{O}(2) \oplus \mathcal{O}(1)^2$. Then by (4.29), we conclude that

$$\mathcal{V}|_l = \mathcal{O}(2)^2 \oplus \mathcal{O}, \quad \text{for any } [l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) = \mathcal{K}^{\alpha_1}(W/\mathbb{P}^3). \tag{4.30}$$

This is because a positive dimensional family of vector bundles over $\mathbb{P}^1$ of type $\mathcal{O}(2)^2 \oplus \mathcal{O}$ can not have a limit of type $\mathcal{O}(2) \oplus \mathcal{O}(1)^2$.

Now take any $[l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$, we have $\mathcal{L}_\sigma|_l = \mathcal{O}(2)$ by Proposition 4.25(ii). Thus by (4.30) and Proposition 4.44, $\mathcal{L}_W|_l = \mathcal{O}(2) \oplus \mathcal{O}$, verifying the assertion (i). It follows that there exists a unique $[C_l] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ such that $C_l \subset W = \mathbb{P}(\mathcal{L}_W)$, and $\pi_0^{\alpha_2}(C_l) = l$. In fact $C_l = \mathbb{P}(\mathcal{O}(2)|_l) \subset \mathbb{P}((\mathcal{O}(2) \oplus \mathcal{O})|_l) = \mathbb{P}(\mathcal{L}_W|_l)$. Moreover $C_l \cong \mathbb{P}^1$, verifying the assertion (ii).

Take any $[C] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$. Since $(W \cdot C) = 0$, either $C \subset W$ or $C \cap W = \emptyset$. Then the assertion (iii) follows from (i) and (ii). □

Lemma 4.46. *In setting of Proposition 4.44, the variety $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is a smooth hyperplane section of $\mathbb{G}(1, \mathbb{P}^3) \subset \mathbb{P}^5$, and $W \cong C_2/(P_{\beta_1} \cap P_{\beta_2})$, where β_1 and β_2 are the short and long simple root of C_2 respectively.*

Proof. By Lemma 4.34, $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is a hyperplane section of $\mathbb{G}(1,\mathbb{P}^3) \subset \mathbb{P}^5$. By Proposition 4.15 and Lemma 4.45, there is a $\mathbb{P}^1$-fibration $\varphi : W \to \mathcal{K}^{\alpha_1}(W) = \mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$, where $\mathcal{K}^{\alpha_1}(W)$ is the set of $[C] \in \mathcal{K}^{\alpha_1}(\mathcal{X}_0)$ such that $C \subset W$. The variety $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$ is smooth because so is W. Then there exists a nondegenerate form $\omega \in \wedge^2(\mathbb{C}^4)^*$ such that $\mathcal{X}_0^{\alpha_2} = \mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$,

$$\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) = \{[A] \in Gr(2,\mathbb{C}^4) \mid \omega(A,A) = 0\}, \tag{4.31}$$

and $\pi_0^{\alpha_2}|_W : W \to \mathbb{P}^3$ is the evaluation morphism of the family $\mathcal{K}^{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3)$. Then the conclusion follows. □

Now we can complete the proof of Theorem 1.6.

Proof of Theorem 1.6. By Proposition 4.21 and Proposition 4.44, $\mathcal{X}_0 \cong \mathbb{P}(\mathcal{V})$, and $\mathcal{V} \cong \mathcal{L}^\sigma \oplus \mathcal{L}_W$. By Proposition 4.25$(ii)$, $\mathcal{L}_\sigma \cong \mathcal{O}(2)$. By Lemma 4.46, $\mathcal{L}_W \cong \mathcal{L}^\omega \otimes \mathcal{O}(k)$ for some $k \in \mathbb{Z}$, where ω is the symplectic form on $\mathbb{C}^4$ satisfying (4.31). Take any line $l \subset \mathbb{P}^3$. We have

$$\deg(\mathcal{L}_W|_l) = \deg(\mathcal{V}|_l) - \deg(\mathcal{L}_\sigma|_l) = 2,$$
$$\deg(\mathcal{L}^\omega|_l) = \deg(T\mathbb{P}^3|_l) - \deg(\mathcal{O}(2)|_l) = 2.$$

Then $k = 0$ and $\mathcal{L}_W \cong \mathcal{L}^\omega$. Hence $\mathcal{V} \cong \mathcal{O}(2) \oplus \mathcal{L}^\omega$ and $\mathcal{X}_0 \cong F^d(1,2;\mathbb{C}^4)$. □

4.3.5. *Properties of $F^d(1,2;\mathbb{C}^4)$.* For the convenience of discussion later, we give several basic properties of the manifold $F^d(1,2;\mathbb{C}^4)$ in Construction 1.5. All these properties are straight-forward from the construction. They have also been proved in a more involved way in the previous arguments in subsection 4.3 by realizing $F^d(1,2;\mathbb{C}^4)$ as the a priori unclear Fano degeneration of $A_3/P_{\{\alpha_1,\alpha_2\}}$, see Lemma 4.10, Corollary 4.12, Corollary 4.26, Lemma 4.29 for the corresponding statements of them.

Notation 4.47. In setting of Construction 1.5, denote by $\phi : F^d(1,2;\mathbb{C}^4) \to \mathbb{P}^3$ the $\mathbb{P}^2$-bundle, and let $\sigma : \mathbb{P}^3 \to \mathbb{P}(\mathcal{L}_\sigma) \subset F^d(1,2;\mathbb{C}^4)$ be the holomorphic section. Given a point $x \in F^d(1,2;\mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$, denote by l_x the line $\langle x, \sigma(\phi(x))\rangle$ in the projective plane $\phi^{-1}(\phi(x)) \cong \mathbb{P}^2$. By abuse of notations (to be compatible with those in Section 4.3), we denote by $\mathscr{D}^{\alpha_1}$ the meromorphic distribution of rank one on $F^d(1,2;\mathbb{C}^4)$ whose general leaves are minimal rational curves biholomorphically sent to isotropic lines in $\mathbb{P}^3$, and by $\mathcal{K}^{\alpha_1}(F^d(1,2;\mathbb{C}^4))$ the closure this family of minimal rational curves. Set $\mathscr{D}^{\alpha_2} := T^\phi$ and $\mathscr{D} := \mathscr{D}^{\alpha_1} + \mathscr{D}^{\alpha_2}$. Denote by $\mathcal{K}^{\alpha_2}(F^d(1,2;\mathbb{C}^4))$ the family of minimal rational curves which are lines in the fibers of ϕ.

The following two propositions are immediate from the constructions.

Proposition 4.48. *At any point $x \in F^d(1,2;\mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$, $\mathcal{K}_x^{\alpha_1}(\mathcal{X}_0)$ consists of a unique element, denoted by $[C_x]$. Two points $y, z \in \mathbb{P}_t^2 \setminus \{\sigma(t)\}$ satisfy $\phi(C_y) = \phi(C_z)$ if and only if the two lines $\langle y, \sigma(t)\rangle$ and $\langle z, \sigma(t)\rangle$ in $\mathbb{P}_t^2$ coincide, where $t \in \mathcal{X}_0^{\alpha_2}$ is an arbitrary point and $\mathbb{P}_t^2 := \phi^{-1}(t)$.*

Proposition 4.49. *In setting of Construction 1.5 the surjective homomorphism $\mathcal{L}_\sigma \oplus \mathcal{L}^\omega \to \mathcal{L}_\sigma$ induces a rational map $F^d(1,2;\mathbb{C}^4) \dashrightarrow \mathbb{P}(\mathcal{L}^\omega) \cong C_2/B$ over $\mathbb{P}^3$. It is a linear projection from $\mathbb{P}_t^2 := \phi^{-1}(t)$ with center $\sigma(t)$ over each $t \in \mathbb{P}^3$.*

Proposition 4.50. *In setting of Notation 4.47, define a meromorphic distribution $\mathcal{N}$ on $F^d(1,2;\mathbb{C}^4)$ such that $\mathcal{N}_x = T_x(l_x)$ at any point $x \in F^d(1,2;\mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$. Then $\mathcal{N}$ is the unique meromorphic line subbundle of $\mathscr{D}$ on $F^d(1,2;\mathbb{C}^4)$ such that $[\mathcal{N},\mathscr{D}] \subset \mathscr{D}$. Moreover, $[\mathcal{N},\mathscr{D}^{\alpha_1}] \subset \mathcal{N} + \mathscr{D}^{\alpha_1}$.*

Proof. The leaf of $\mathcal{N}$ passing through a point $x \in F^d(1,2;\mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$ is $l_x^o := l_x \setminus \{\sigma(t)\}$, where $t := \phi(x)$ and $l_x := \langle x, \sigma(t)\rangle$. The the leaf of $\mathcal{D}^{\alpha_1}$ passing through a point $y \in l_x^o$ is C_y, where $[C_y]$ is the unique element of $\mathcal{K}_y^{\alpha_1}(\mathcal{X}_0)$. Since $\overline{\bigcup_{y \in l_x^o} C_y} \cong \phi(C_x) \times l_x$, we have $[\mathcal{N},\mathscr{D}^{\alpha_1}] \subset \mathcal{N} + \mathscr{D}^{\alpha_1}$. Since $\mathscr{D}^{\alpha_2}$ is integrable and $\mathcal{N} \subset \mathscr{D}^{\alpha_2}$, we have $[\mathcal{N},\mathscr{D}^{\alpha_2}] \subset \mathscr{D}^{\alpha_2}$. It follows that $[\mathcal{N},\mathscr{D}] \subset \mathscr{D}$.

If the uniqueness of $\mathcal{N}$ fails, then the rank three distribution $\mathcal{D}$ has to be integrable. However one can easily check the $F^d(1,2;\mathbb{C}^4)$ is chained-connected by the family $\bigcup_{i=1,2} \mathcal{K}^{\alpha_i}(F^d(1,2;\ C^4))$. It is a contradiction. Hence $\mathcal{N}$ is unique. □

Proposition 4.51. *At each point $x \in F^d(1,2;\mathbb{C}^4)\backslash\mathbb{P}(\mathcal{L}_\sigma)$, the symbol algebra $Symb_x(\mathscr{D}) \cong \mathfrak{g}_-(C_2)\oplus\mathfrak{g}_-(A_1)$. Moreover, this isomorphism is induced by the identification $\mathfrak{g}_-(A_1) = \mathcal{N}_x$, $\mathscr{D}^{\alpha_2}_x = \mathfrak{g}_-(\alpha_2) + \mathfrak{g}_-(A_1)$ and $\mathscr{D}^{\alpha_1}_x = \mathfrak{g}_-(\alpha_1)$, where α_1 and α_2 is the long and short simple root of C_2 respectively.*

Proof. It follows from Proposition 4.50 and Construction 1.5 directly. □

4.4. **Proof of Proposition 1.8.** The main aim of this subsection is to show the following proposition, from which we can complete the proof of Proposition 1.8.

Proposition 4.52. *The manifold $A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$ is rigid under Fano deformation.*

Proof of Proposition 1.8. (i) Consider the Fano deformation rigidity of A_4/P_I with $|I| = 3$. The set of simple roots is $R = \{\alpha_1, \dots, \alpha_4\}$. The manifolds $A_4/P_{R\backslash\{\alpha_1\}}$ and $A_4/P_{R\backslash\{\alpha_4\}}$ are biholomorphic to each other, which are rigid under Fano deformation by Proposition 4.52. The manifolds $A_4/P_{R\backslash\{\alpha_2\}}$ and $A_4/P_{R\backslash\{\alpha_3\}}$ are biholomorphic to each other, which are rigid under Fano deformation by Proposition 1.9.

(ii) Consider the Fano deformation rigidity of $\mathbf{S} := D_5/P_I$ with $|I| = 4$. Set $J := R \setminus I = \{\alpha_i\}$ for some i, where R is the set of simple roots. Take any J-connected pair $\beta_1 \neq \beta_2 \in I$. There exists $\beta_3 \in I \setminus \{\beta_1, \beta_2\}$ such that the manifold $\mathbf{S}^{\beta_1,\beta_2,\beta_3}$ is biholomorphic to $A_4/P_{I'}$ with $|I'| = 3$ or 4. The latter is rigid under Fano deformation by (i) as well as Theorem 1.3. By Corollary 3.23, D_5/P_I is rigid under Fano deformation . □

To prove Proposition 4.52, it suffices to deduce a contradiction in the following setting.

Setting 4.53. Let $\pi : \mathcal{X} \to \Delta$ be a holomorphic map such that $\mathcal{X}_t \cong \mathbf{S}$ for all $t \neq 0$, $\mathcal{X}_0$ is a connected Fano manifold and $\mathcal{X}_0 \ncong \mathbf{S}$, where $\mathbf{S} := A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$.

Remark 4.54. Let us firstly explain the idea to prove Proposition 4.52 in the following, while the rigorous proof is not no so immediate from this idea. In Setting 4.53, $\mathcal{X}_0$ has to be a compactification of the total space of the normal bundle $N_{U/\mathbf{S}}$, where U is the inverse image of some hyperplane section of $A_4/P_{\alpha_2} = Gr(2,\mathbb{C}^5) \subset \mathbb{P}^9$ under the natural morphism $\mathbf{S} \to A_4/P_{\alpha_2}$. On the other hand, we can show that any Fano deformation of $\mathbf{S}$ must be a $\mathbb{P}^2$-bundle over $A_4/P_{\{\alpha_3,\alpha_4\}} = F(3,4;\mathbb{C}^5)$, while the compactification $\mathcal{X}_0$ of $N_{U/\mathbf{S}}$ does not have such a projective bundle structure.

Proposition 4.55. *In Setting 4.53, take a general point $x \in \mathcal{X}_0$. Then $F^{\alpha_2}_x \cong \mathbb{P}^2$, $F^{\alpha_3}_x \cong \mathbb{P}^1$, $F^{\alpha_4}_x \cong \mathbb{P}^1$, $F^{\alpha_2,\alpha_4}_x \cong \mathbb{P}^2 \times \mathbb{P}^1$ and $F^{\alpha_3,\alpha_4}_x \cong \mathbb{P}(T\mathbb{P}^2) = F(1,2;\mathbb{C}^3)$ respectively.*

Proof. The assertions for $F^{\alpha_i}_x$, F^{α_2,α_4}_x and F^{α_3,α_4}_x follow from the rigidity of projective spaces, Proposition 3.24 and Theorem 1.3 respectively. □

Proposition 4.56. *In Setting 4.53, take a general point $x \in \mathcal{X}_0$. Then $F^{\alpha_2,\alpha_3} \cong F^d(1,2;\mathbb{C}^4)$, where $F^d(1,2;\mathbb{C}^4)$ is as in Construction 1.5.*

Proof. By Theorem 1.6, either $F^{\alpha_2,\alpha_3} \cong F(1,2;\mathbb{C}^4)$ or $F^{\alpha_2,\alpha_3} \cong F^d(1,2;\mathbb{C}^4)$. In the former case, $\mathcal{X}_0 \cong A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$ by Theorem 3.22 and Proposition 4.55. This contradicts our assumption in Setting 4.53. □

Proposition 4.57. *In Setting 4.53, the morphism $\pi^{\alpha_2}_0 : \mathcal{X}_0 \to \mathcal{X}^{\alpha_2}_0$ is a $\mathbb{P}^2$-bundle. In particular, the variety $\mathcal{X}^{\alpha_2}_0$ is smooth.*

Proof. By formula (3.4) in [16], the cohomology ring $H^*(A_n/P_{\{\alpha_1,\alpha_2\}},\mathbb{Q})$ is generated by $H^2(A_n/P_{\{\alpha_1,\alpha_2\}},\mathbb{Q})$. Then the conclusion of Proposition 4.57 follows from Proposition 4.20 immediately. □

Convention 4.58. In Subsection 4.4, we denote by $\mathscr{D}^{\alpha_i}$, $\mathscr{D}^{\alpha_i,\alpha_j}$ $\mathscr{D}$ and $\mathscr{D}^{-k}$ the restriction of $\mathcal{D}^{\alpha_i}$, $\mathcal{D}^{\alpha_i,\alpha_j}$, $\mathcal{D}$ and $\mathcal{D}^{-k}$ on $\mathcal{X}_0$ respectively, where the latter is defined in Notation 3.9.

Now let us turn to analysis the symbol algebra Symb$(\mathscr{D})$ on $\mathcal{X}_0$.

Lemma 4.59. *In Setting 4.53, there exists a unique meromorphic distribution $\mathcal{N} \subset \mathscr{D}^{\alpha_2}$ of rank one over $\mathcal{X}_0$ such that the Levi bracket of vector fields $[\mathcal{N}, \mathscr{D}] \subset \mathscr{D}$.*

Proof. By Proposition 4.56, $F_x^{\alpha_2,\alpha_3} \cong F^d(1,2;\mathbb{C}^4)$ for $x \in \mathcal{X}_0$ general. Then by Proposition 4.50 there exists a unique meromorphic line subbundle $\mathcal{N} \subset \mathscr{D}^{\alpha_2}$ over $\mathcal{X}_0$ such that $[\mathcal{N}, \mathscr{D}^{\alpha_3}] \subset \mathcal{N} + \mathscr{D}^{\alpha_3}$. By Proposition 4.55, $F_x^{\alpha_2,\alpha_4} \cong F_x^{\alpha_2} \times F_x^{\alpha_4}$ for $x \in \mathcal{X}_0$ general. Then $[\mathscr{D}^{\alpha_2}, \mathscr{D}^{\alpha_4}] \subset \mathscr{D}^{\alpha_2} + \mathscr{D}^{\alpha_4}$, implying the conclusion. □

Notation 4.60. We construct a graded nilpotent Lie algebra $\mathfrak{m}_- := \bigoplus_{k\geq 1} \mathfrak{m}_{-k}$ as follows:

$$\begin{aligned}
\mathfrak{m}_{-1} &= \bigoplus_{1\leq i\leq 4} \mathbb{C}v_i,\\
\mathfrak{m}_{-2} &= \mathbb{C}v_{23} \oplus \mathbb{C}v_{34},\\
\mathfrak{m}_{-3} &= \mathbb{C}v_{233} \oplus \mathbb{C}v_{234},\\
\mathfrak{m}_{-4} &= \mathbb{C}v_{2334},\\
\mathfrak{m}_{-k} &= 0, \quad k \geq 5,
\end{aligned}$$

where $v_{i_1\ldots i_m} := [v_{i_1\ldots i_{m-1}}, v_{i_m}]$. The Lie algebra structure on $\mathfrak{m}_-$ is defined uniquely by the following rules:

$$[\mathfrak{m}_{-i}, \mathfrak{m}_{-j}] \subset \mathfrak{m}_{-i-j}, \quad [v_1, \mathfrak{m}_-] = 0, \quad [v_{23}, v_{34}] = \frac{1}{2}v_{2334},$$

and there is a table of Lie brackets

(4.32)

	v_{23}	v_{34}	v_{233}	v_{234}
v_2	0	v_{234}	0	0
v_3	$-v_{233}$	0	0	$-\frac{1}{2}v_{2334}$
v_4	$-v_{234}$	0	$-v_{2334}$	0

In the table above, we compute the Lie bracket of left end entry with top end entry. For example, $[v_4, v_{23}] = -v_{234}$ and $[v_3, v_{234}] = -\frac{1}{2}v_{2334}$.

Lemma 4.61. *In Setting 4.53, the symbol algebra of $\mathscr{D}$ at a general point $x \in \mathcal{X}_0$ is isomorphic to $\mathfrak{m}_-$ in Notation 4.60, where we have identifications $\mathcal{N}_x = \mathbb{C}v_1$, $\mathscr{D}_x^{\alpha_2} = \mathbb{C}v_1 + \mathbb{C}v_2$, $\mathscr{D}_x^{\alpha_3} = \mathbb{C}v_3$ and $\mathscr{D}_x^{\alpha_4} = \mathbb{C}v_4$.*

Proof. By Proposition 4.55, Proposition 4.56 and Proposition 4.51 (see also Remark 4.13(*ii*)), we have the description of $\mathfrak{m}(\alpha_i)$ and $\mathfrak{m}(\alpha_i, \alpha_j)$ for $2 \leq i \neq j \leq 4$. In particular, in $\mathfrak{m}_x(\alpha_2, \alpha_3, \alpha_4) := \mathrm{Symb}_x(\mathscr{D})$ we have

$$\begin{aligned}
&[v_1, v_i] = 0, i = 2, 3, 4, \quad (\mathrm{ad}v_2)^2(v_3) = 0, \quad (\mathrm{ad}v_3)^3(v_2) = 0,\\
&[v_2, v_4] = 0, \quad (\mathrm{ad}v_3)^2(v_4) = 0, \quad (\mathrm{ad}v_4)^2(v_3) = 0.
\end{aligned}$$

Then by Proposition 2.15, $\mathrm{Symb}_x(\mathscr{D})$ is a quotient algebra of $\mathfrak{g}_- := \mathfrak{g}_-(C_3) \oplus \mathfrak{g}_-(A_1)$. More precisely, $\mathfrak{g}_- := \bigoplus_{k\geq 1} \mathfrak{g}_{-k}$ as follows:

$$\begin{aligned}
\mathfrak{g}_{-1} &= \bigoplus_{1\leq i\leq 4} \mathbb{C}v_i,\\
\mathfrak{g}_{-2} &= \mathbb{C}v_{23} \oplus \mathbb{C}v_{34},\\
\mathfrak{g}_{-3} &= \mathbb{C}v_{233} \oplus \mathbb{C}v_{234},\\
\mathfrak{g}_{-4} &= \mathbb{C}v_{2334},\\
\mathfrak{g}_{-5} &= \mathbb{C}v_{23344},\\
\mathfrak{g}_{-k} &= 0, \quad k \geq 6.
\end{aligned}$$

Denote by $\mathfrak{q}$ the ideal of $\mathfrak{g}_-$ such that $\mathrm{Symb}_x(\mathscr{D}) = \mathfrak{g}_-/\mathfrak{q}$ as graded nilpotent Lie algebra. By Proposition 3.10, $\dim \mathrm{Symb}_x(\mathscr{D}) = \dim T_x\mathcal{X}_0 = 9$, which implies that $\dim \mathfrak{q} = \dim \mathfrak{g}_- - \dim \mathrm{Symb}_x(\mathscr{D}) = 1$.

To complete the proof of Lemma 4.61, it suffices to show the claim that $\mathfrak{q} = \mathbb{C}v_0$, where $v_0 := v_{23344} + \lambda v_1$ for some $\lambda \in \mathbb{C}$. Note that the graded Lie algebra structure on $\mathfrak{g}_-/\mathbb{C}w_0$ is independent of the choice of $\lambda \in \mathbb{C}$.

Suppose the claim fails. Then there exists $1 \leq k_0 \leq 4$ such that $\mathfrak{q} = \mathbb{C}v_0$ and $v_0 = \lambda v_1 + v_0' + v_0''$, where $v_0'' \in \bigoplus_{k\geq k_0+1} \mathfrak{g}_{-k}$, $0 \neq v_0' \in \mathfrak{g}_{-k_0}$ if $k_0 \geq 2$, and $0 \neq v_0' \in \bigoplus_{2\leq i\leq 4} \mathbb{C}v_i$ if $k_0 = 1$. Then there exists $2 \leq j \leq 4$

such that $[v_j, v_0'] \neq 0$, see table (4.32). Then $0 \neq [v_j, v_0] \in \bigoplus_{k \geq k_0+1} \mathfrak{g}_{-k}$. Since $\mathfrak{q}$ is an ideal of $\mathfrak{g}_-$, we have $0 \neq [v_j, v_0] \in \mathfrak{q} = \mathbb{C}v_0$. In particular, $[v_j, v_0]$ has a nonzero component in $\mathfrak{g}_{-k_0}$. It is a contradiction. Hence the claim holds. □

Lemma 4.62. *In Setting 4.53 the Frobenius bracket of $(T^{\pi^{\alpha_2,\alpha_3}} + T^{\pi^{\alpha_2,\alpha_4}})|_{\mathcal{X}_0}$ induces a homomorphism of meromorphic vector bundles over $\mathcal{X}_0$:*

$$(T^{\pi^{\alpha_2,\alpha_3}}/T^{\pi^{\alpha_2}})|_{\mathcal{X}_0} \otimes (T^{\pi^{\alpha_2,\alpha_4}}/T^{\pi^{\alpha_2}})|_{\mathcal{X}_0} \to T\mathcal{X}_0/(T^{\pi^{\alpha_2,\alpha_3}} + T^{\pi^{\alpha_2,\alpha_3}})|_{\mathcal{X}_0},$$

which is a surjective homomorphism over a nonempty Zariski open subset of $\mathcal{X}_0$.

Proof. It is a direct consequence of Lemma 4.61. More precisely, the weak derivatives of $\mathscr{D}^{\alpha_2}$, $\mathscr{D}^{\alpha_2} + \mathscr{D}^{\alpha_3}$ and $\mathscr{D}^{\alpha_2} + \mathscr{D}^{\alpha_4}$ induces symbol algebras at a general point $x \in \mathcal{X}_0$ as follows:

$$\begin{aligned} \mathrm{gr}(T^{\pi^{\alpha_2}}) &= \mathbb{C}v_1 \oplus \mathbb{C}v_2, \\ \mathrm{gr}(T^{\pi^{\alpha_2,\alpha_3}}) &= \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3 \oplus \mathbb{C}v_{23} \oplus \mathbb{C}v_{233}, \\ \mathrm{gr}(T^{\pi^{\alpha_2,\alpha_4}}) &= \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_4. \end{aligned}$$

Then it is straight-forward to deduce the conclusion from the Lie algebra structure of $\mathfrak{m}_-$ in Notation 4.60. □

Proposition 4.63. *In Setting 4.53 the variety $\mathcal{X}_0^{\alpha_2}$ is biholomorphic to $F(3,4;\mathbb{C}^5)$.*

Proof. By Proposition 4.57, the variety $\mathcal{X}_0^{\alpha_2}$ is smooth. Being the smooth deformation of $F(3,4;\mathbb{C}^5) \cong \mathcal{X}_t^{\alpha_2}$ with $t \neq 0$, $\mathcal{X}_0^{\alpha_2}$ is of Picard number two. The relative Mori contraction $\pi^{\alpha_2,\alpha_k} : \mathcal{X} \to \mathcal{X}^{\alpha_2,\alpha_k}$ induces a relative Mori contraction $\psi^{\alpha_k} : \mathcal{X}^{\alpha_2} \to \mathcal{X}^{\alpha_2,\alpha_k}$ extending $\Psi^{\alpha_k} : A_4/P_{\{\alpha_3,\alpha_4\}} \to A_4/P_{\alpha_i}$, where $i \neq k \in \{3,4\}$. The existence of two elementary contractions of fiber types implies that $\mathcal{X}_0^{\alpha_2}$ is a Fano manifold.

For each $k \in \{3,4\}$, the relative tangent sheaf $T^{\psi^{\alpha_k}}$ is a meromorphic distribution on $\mathcal{X}^{\alpha_2}$, whose singular locus is a proper closed subvariety of $\mathcal{X}_0^{\alpha_2}$. Denote by $\mathcal{E}^{\alpha_k} := T^{\psi^{\alpha_k}}|_{\mathcal{X}_0^{\alpha_2}}$, and $\mathcal{E} := \mathcal{E}^{\alpha_3} + \mathcal{E}^{\alpha_4} \subset T\mathcal{X}_0^{\alpha_2}$. The Frobenius bracket of the meromorphic distribution $\mathcal{E}$ on $\mathcal{X}_0^{\alpha_2}$ induces $F : \mathcal{E}^{\alpha_3} \otimes \mathcal{E}^{\alpha_4} \to T\mathcal{X}_0^{\alpha_2}/\mathcal{E}$, which is a homomorphism of meromorphic vector bundles over $\mathcal{X}_0^{\alpha_2}$.

It is easy to see that $\mathcal{E} = d\pi_0^{\alpha_2}(T^{\pi^{\alpha_2,\alpha_3}} + T^{\pi^{\alpha_2,\alpha_4}})$ and $\mathcal{E}^{\alpha_k} = d\pi_0^{\alpha_2}(T^{\pi^{\alpha_2,\alpha_k}})$ for $k = 3,4$, where $d\pi_0^{\alpha_2}$ is the tangent map of $\pi_0^{\alpha_2}$. By Lemma 4.62, F is surjective at general points of $\mathcal{X}_0^{\alpha_2}$. The conclusion follows from Proposition 3.20. □

Corollary 4.64. *In Setting 4.53 the varieties $\mathcal{X}_0^{\alpha_2,\alpha_3}$ and $\mathcal{X}_0^{\alpha_2,\alpha_4}$ are biholomorphic to A_4/P_{α_4} and A_4/P_{α_3} respectively. The morphisms $\pi_0^{\alpha_2,\alpha_3} : \mathcal{X}_0 \to \mathcal{X}_0^{\alpha_2,\alpha_3}$ and $\pi_0^{\alpha_2,\alpha_4} : \mathcal{X}_0 \to \mathcal{X}_0^{\alpha_2,\alpha_4}$ are $F^d(1,2;\mathbb{C}^4)$-bundle and $(\mathbb{P}^2 \times \mathbb{P}^1)$-bundle respectively.*

Proof. By Proposition 4.63, $\mathcal{X}_0^{\alpha_2} \cong A_4/P_{\{\alpha_3,\alpha_4\}}$. Hence $\mathcal{X}_0^{\alpha_2,\alpha_3} \cong A_4/P_{\alpha_4}$ and $\mathcal{X}_0^{\alpha_2,\alpha_4} \cong A_4/P_{\alpha_3}$. Furthermore, the two elementary Mori contractions $\psi_0^{\alpha_3} : \mathcal{X}_0^{\alpha_2} \to \mathcal{X}_0^{\alpha_2,\alpha_3}$ and $\psi_0^{\alpha_4} : \mathcal{X}_0^{\alpha_2} \to \mathcal{X}_0^{\alpha_2,\alpha_4}$ are $\mathbb{P}^3$-bundle and $\mathbb{P}^1$-bundle respectively. Then by Proposition 4.57 $\pi^{\alpha_2,\alpha_3} : \mathcal{X}_0 \to \mathbb{P}^4$ (resp. $\pi^{\alpha_2,\alpha_4} : \mathcal{X}_0 \to Gr(3,\mathbb{C}^5)$) is a smooth morphism such that each fiber is a Fano manifold admitting a $\mathbb{P}^2$-bundle structure over $\mathbb{P}^3$ (resp. over $\mathbb{P}^1$). By rigidity of projective space and Proposition 3.24, the morphism $\pi_0^{\alpha_2,\alpha_4}$ is a $(\mathbb{P}^2 \times \mathbb{P}^1)$-bundle. By Theorem 1.6, each fiber of $\pi_0^{\alpha_2,\alpha_3}$ is biholomorphic to either $F(2,3;\mathbb{C}^4)$ or $F^d(1,2;\mathbb{C}^4)$. By the local rigidity of $F(2,3;\mathbb{C}^4)$ and Proposition 4.56, the morphism $\pi_0^{\alpha_2,\alpha_3}$ is an $F^d(1,2;\mathbb{C}^4)$-bundle. □

Now we are ready to complete the proof of Proposition 4.52. As a trivial analogue with Construction 1.5, we can define $F^d(2,3;\mathbb{C}^4)$ by using the contact distribution on A_3/P_{α_3} instead of that on A_3/P_{α_1}. Although $F^d(2,3;\mathbb{C}^4) \cong F^(1,2;\mathbb{C}^4)$, we use $F^d(2,3,\mathbb{C}^4)$ in the following to make our discussion compatible with the involved simple roots of A_4.

Proof of Proposition 4.52. We discuss in Setting 4.53. It suffices to deduce a contradiction. In summary of Proposition 4.57, Proposition 4.63 and Corollary 4.64, $\pi^{\alpha_2} : \mathcal{X}_0 \to \mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5)$ is a $\mathbb{P}^2$-bundle and $\pi_0^{\alpha_2,\alpha_3} : \mathcal{X}_0 \to \mathcal{X}_0^{\alpha_2,\alpha_3} = \mathbb{P}^4$ is a $F^d(2,3;\mathbb{C}^4)$-bundle. By Proposition 4.48 there exists a holomorphic section $\sigma : \mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5) \to \mathcal{X}_0$ of $\pi_0^{\alpha_2}$ such that

(i) at any point $x \in \mathcal{X}_0 \setminus \sigma(\mathcal{X}_0^{\alpha_2})$, $\mathcal{K}_x^{\alpha_3}(\mathcal{X}_0)$ consists of a unique element, denoted by $[C_x]$;

(ii) at any point $x \in \mathcal{X}_0 \setminus \sigma(\mathcal{X}_0^{\alpha_2})$, $C_x \cong \mathbb{P}^1$ and $\pi_0^{\alpha_2}$ sends C_x biholomorphically to a line in a fiber of $\psi^{\alpha_3} : \mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5) \to \mathcal{X}_0^{\alpha_2,\alpha_3} = A_4/P_{\alpha_4}$;

(iii) two points $x, y \in \mathbb{P}^2_t \setminus \{\sigma(t)\}$ satisfy $\pi_0^{\alpha_2}(C_x) = \pi_0^{\alpha_2}(C_y)$ if and only if the two lines $\langle x, \sigma(t)\rangle$ and $\langle x, \sigma(t)\rangle$ in $\mathbb{P}^2_t$ coincide, where $t \in \mathcal{X}_0^{\alpha_2}$ is an arbitrary point and $\mathbb{P}^2_t := (\pi_0^{\alpha_2})^{-1}(t)$.

Set $\mathcal{K}^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) := \bigcup_{t \in \mathcal{X}_0^{\alpha_2}} [\pi_0^{\alpha_2}(C_x)] \subset A_4/P_{\{\alpha_2,\alpha_4\}} = \mathcal{K}^{\alpha_3}(\mathcal{X}_0^{\alpha_2})$. Denote by $\mathcal{X}_0^{\alpha_2} \leftarrow U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \to \mathcal{K}^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2})$ the restriction of the universal family $\mathcal{X}_0^{\alpha_2} = A_4/P_{\{\alpha_3,\alpha_4\}} \leftarrow A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}} \to \mathcal{K}^{\alpha_3}(\mathcal{X}_0^{\alpha_2}) = A_4/P_{\{\alpha_2,\alpha_4\}}$.

Since $\pi_0^{\alpha_2,\alpha_3} : \mathcal{X}_0 \to \mathcal{X}_0^{\alpha_2,\alpha_3} = \mathbb{P}^4$ is a $F^d(2,3;\mathbb{C}^4)$-bundle, we can apply Proposition 4.49 to obtain a commutative diagram over $\mathcal{X}_0^{\alpha_2}$ as follows:

(4.33)
$$\begin{array}{ccccc}
\mathcal{X}_0 & \overset{\theta}{\dashrightarrow} & U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) & \hookrightarrow & A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}} \\
& {\scriptstyle \pi_0^{\alpha_2}} \searrow & \downarrow {\scriptstyle \gamma} & \swarrow & \downarrow \\
& & \mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5) & & Gr(2,\mathbb{C}^5),
\end{array}$$

where at any point $t \in \mathcal{X}_0^{\alpha_2}$ the horizontal rational map θ_t is the linear projection from $\mathbb{P}^2_t := (\pi_0^{\alpha_2})^{-1}(t)$ with center $\sigma(t)$.In particular,

(iv) $\gamma : U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \to \mathcal{X}_0^{\alpha_2}$ is a $\mathbb{P}^1$-bundle.

Now we claim that

(v) under the natural surjective morphism $A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}} \to A_4/P_{\alpha_2} = Gr(2,\mathbb{C}^5)$, the variety $U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \subset A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$ is the inverse image of a hyperplane section of $Gr(2,\mathbb{C}^5)$.

To verify the claim (v), it suffices to show that as a divisor on $\mathbf{S} := A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$, $D := U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2})$ satisfies

$$(D \cdot C_i) = \delta_{i2}, \quad [C_i] \in \mathcal{K}^{\alpha_i}(\mathbf{S}), 2 \leq i \leq 4. \tag{4.34}$$

Take a point $[A_4] \in \mathcal{X}_0^{\alpha_2,\alpha_4} = A_4/P_{\alpha_4}$, where A_4 is the corresponding 4-dimensional linear subspace of $\mathbb{C}^5$. The restriction $U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \subset A_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}} \to Gr(2,\mathbb{C}^5)$ on the fiber $(\pi_0^{\alpha_2,\alpha_3})^{-1}([A_4]) \cong F^d(2,3;A_4)$ is $C_2/B \subset A_3/P_{\{\alpha_2,\alpha_3\}} \to Gr(2,\mathbb{C}^4)$. Hence (4.34) holds for $i = 2$ and 3.

Now consider a part of (4.33), which is a commutative diagram as follows:

$$\begin{array}{ccc}
\mathcal{X}_0 & \dashrightarrow & U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \\
\downarrow & \swarrow & \downarrow \\
\mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5) & \longleftarrow & \mathbf{S}
\end{array}$$

Take any $[l_4] \in \mathcal{K}^{\alpha_4}(\mathcal{X}_0^{\alpha_2})$. Restricting on $l_4 \subset \mathcal{X}_0^{\alpha_2}$, we obtain a commutative diagram:

$$\begin{array}{ccc}
\mathbb{P}^2 \times l_4 & \overset{\varphi_1}{\dashrightarrow} & \mathbb{P}^1 \times l_4 \\
\downarrow & \swarrow & \downarrow {\scriptstyle \varphi_2} \\
l_4 & \longleftarrow & \mathbb{P}^2 \times l_4
\end{array}$$

where the horizontal rational map $\varphi_1 : \mathbb{P}^2 \times l_4 \dashrightarrow \mathbb{P}^1 \times l_4$ is the linear projection from $\mathbb{P}^2 \times \{t\}$, $t \in l_4$ with center $\sigma(t) \in \mathbb{P}^2_t := \mathbb{P}^2 \times \{t\}$, and the vertical morphism $\varphi_2 : \mathbb{P}^1 \times l_4 \to \mathbb{P}^2 \times l_4$ is a hyperplane bundle over l_4. By this diagram we can choose $[C_4] \in \mathcal{K}^{\alpha_4}(\mathbf{S})$ such that $C_4 \subset \mathbb{P}^2 \times l_4 \subset \mathbf{S}$ is a section of l_4 and $C_4 \cap U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) = \emptyset$. In particular, $(D \cdot C_4) = 0$, verifying (4.34) and claim (v) too.

Denote by $0 \neq \omega \in \wedge^2(\mathbb{C}^5)^*$ the antisymmetric form on $\mathbb{C}^5$ such that

$$Gr_\omega(2,\mathbb{C}^5) := \{[A] \in Gr(2,\mathbb{C}^5) \mid \omega(A,A) = 0\}$$

is the hyperplane section of $Gr(2,\mathbb{C}^5) \subset \mathbb{P}^9$ mentioned in claim (v). The assertion $\omega \neq 0$ follows from the fact $U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \subsetneq \mathbf{S}$.

Then we can conclude that

(vi) at any point $t = ([A_3], [A_4]) \in \mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5)$ the fiber $U_t^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2})$ is identified with the space $M_t := \{[A_2] \in Gr_\omega(2,\mathbb{C}^5) \mid A_2 \subset A_3\}$.

Denote by $\omega' \in \wedge^2 A_4^*$ the restriction of ω on $A_4 = \mathbb{C}^4 \subset \mathbb{C}^5$. If the point $t = ([A_3], [A_4])$ is general in $\mathcal{X}_0^{\alpha_2} = F(3,4;\mathbb{C}^5)$, then

$$A_3^{\perp_{\omega'}} := \{v \in A_4 \mid \omega'(v, A_3) = 0\} \subset A_4$$

is a linear subspace of dimension one and M_t is exact

$$\{[A_2] \in Gr(2,\mathbb{C}^5) \mid A_3^{\perp_{\omega'}} \subset A_2 \subset A_3\},$$

which is isomorphic to $\mathbb{P}^1$.

However by dimension reason, $\mathrm{Null}(\omega) \neq 0$, where

$$\mathrm{Null}(\omega) := \{v \in \mathbb{C}^5 \mid \omega(v, \mathbb{C}^5) = 0\}.$$

Hence, there exists $[\tilde{A}_3] \in Gr(3,\mathbb{C}^5)$ such that $\mathrm{Null}(\omega) \cap \tilde{A}_3 \neq 0$ and $\tilde{A}_3 \subset \tilde{A}_3^{\perp_\omega} \subset \mathbb{C}^5$, where $\tilde{A}_3^{\perp_\omega} := \{v \in \mathbb{C}^5 \mid \omega(v, \tilde{A}_3) = 0\}$. Choose $\tilde{t} := ([\tilde{A}_3], [\tilde{A}_4]) \in \mathcal{X}_0^{\alpha_2}$. Then by definition we have

$$M_{\tilde{t}} = \{[A_2] \in Gr(2,\mathbb{C}^5) \mid A_2 \subset \tilde{A}_3\} \cong \mathbb{P}^2.$$

It contradicts with the assertion (vi). This completes the proof of Proposition 4.52. □

4.5. Fano deformation of $D_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$.

4.5.1. *Possible degenerations.* The aim in this section is to show the following

Proposition 4.65. *Suppose in Setting 1.11 that $\mathcal{X}_t \cong D_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$ for $t \neq 0$ and $\mathcal{X}_0 \ncong D_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$. Then at a general point $x \in \mathcal{X}_0$, the fibers $F_x^{\alpha_2} \cong \mathbb{P}^1$, $F_x^{\alpha_3} \cong \mathbb{P}^1$, $F_x^{\alpha_4} \cong \mathbb{P}^1$, $F_x^{\alpha_3,\alpha_4} \cong \mathbb{P}^1 \times \mathbb{P}^1$, $F_x^{\alpha_2,\alpha_3} \cong F^d(1,2;\mathbb{C}^4)$ and $F_x^{\alpha_2,\alpha_4} \cong F^d(1,2;\mathbb{C}^4)$.*

Throughout the Subsection 4.5, we discuss in the following setting.

Setting 4.66. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a holomorphic family of connected Fano manifolds such that $\mathcal{X}_t \cong D_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$ for $t \neq 0$.

Firstly we have four possibilities as follows.

Proposition 4.67. *In Setting 4.66, take $x \in \mathcal{X}_0$ general. Then $F_x^{\alpha_2} \cong \mathbb{P}^1$, $F_x^{\alpha_3} \cong \mathbb{P}^1$, $F_x^{\alpha_4} \cong \mathbb{P}^1$ and $F_x^{\alpha_3,\alpha_4} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, one of the following cases occur:*

(A) $F_x^{\alpha_2,\alpha_3} \cong F(2,3;\mathbb{C}^4)$ and $F_x^{\alpha_2,\alpha_4} \cong F(2,3;\mathbb{C}^4)$;
(B) $F_x^{\alpha_2,\alpha_3} \cong F^d(1,2;\mathbb{C}^4)$ and $F_x^{\alpha_2,\alpha_4} \cong F^d(1,2;\mathbb{C}^4)$;
(C) $F_x^{\alpha_2,\alpha_3} \cong F(2,3;\mathbb{C}^4)$ and $F_x^{\alpha_2,\alpha_4} \cong F^d(1,2;\mathbb{C}^4)$;
(D) $F_x^{\alpha_2,\alpha_3} \cong F^d(1,2;\mathbb{C}^4)$ and $F_x^{\alpha_2,\alpha_4} \cong F(2,3;\mathbb{C}^4)$.

Proof. The description of $F_x^{\alpha_i}$ and F^{α_3,α_4} follows from the Fano deformation rigidity of projective spaces and Proposition 3.24. The description of $F_x^{\alpha_2,\alpha_3}$ and $F_x^{\alpha_2,\alpha_4}$ follows from Theorem 1.6. □

Remark 4.68. The positive roots of D_4 are as follows:

$$\begin{aligned}
&\alpha_1, \alpha_2, \alpha_3, \alpha_4; \quad \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_2+\alpha_4;\\
&\alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_4, \alpha_2+\alpha_3+\alpha_4;\\
&\alpha_1+\alpha_2+\alpha_3+\alpha_4; \quad \alpha_1+2\alpha_2+\alpha_3+\alpha_4.
\end{aligned}$$

Take $G = D_4$ and $I = \{\alpha_2, \alpha_3, \alpha_4\}$ in Definition 2.2, then $\mathfrak{g}_-(I) = \bigoplus_{k\geq 1} \mathfrak{g}_{-k}(I)$ is as follows:

$$\begin{aligned}
&\mathfrak{g}_{-1}(I) = \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_3} \oplus \mathfrak{g}_{-\alpha_4},\\
&\mathfrak{g}_{-2}(I) = \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_4},\\
&\mathfrak{g}_{-3}(I) = \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4},\\
&\mathfrak{g}_{-4}(I) = \mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4},\\
&\mathfrak{g}_{-k}(I) = 0 \text{ for } k \geq 5.
\end{aligned} \tag{4.35}$$

Now we fix nonzero vectors $w_1 \in \mathfrak{g}_{-\alpha_1-\alpha_2}$, $w_2 \in \mathfrak{g}_{-\alpha_2}$, $w_3 \in \mathfrak{g}_{-\alpha_3}$, and $w_4 \in \mathfrak{g}_{-\alpha_4}$ respectively. Then (4.35) can be written explicitly as follows:

$$\begin{aligned}
&\mathfrak{g}_{-1}(I) = \mathbb{C}w_1 \oplus \mathbb{C}w_2 \oplus \mathbb{C}w_3 \oplus \mathbb{C}w_4,\\
&\mathfrak{g}_{-2}(I) = \mathbb{C}w_{13} \oplus \mathbb{C}w_{23} \oplus \mathbb{C}w_{14} \oplus \mathbb{C}w_{24},\\
&\mathfrak{g}_{-3}(I) = \mathbb{C}w_{134} \oplus \mathbb{C}w_{234},\\
&\mathfrak{g}_{-4}(I) = \mathbb{C}w_{1342},\\
&\mathfrak{g}_{-k}(I) = 0 \text{ for } k \geq 5,
\end{aligned} \tag{4.36}$$

where $w_{i_1\dots i_m} := [w_{i_1\dots i_{m-1}}, w_{i_m}]$ by inductive definition.

Take $G = D_4$ and $I = \{\alpha_2, \alpha_3\}$ in Definition 2.2, then $\mathfrak{g}_-(I) = \bigoplus_{k\geq 1} \mathfrak{g}_{-k}(I)$ is as follows:

$$\begin{aligned}
&\mathfrak{g}_{-1}(I') = \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_3},\\
&\mathfrak{g}_{-2}(I') = \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4},\\
&\mathfrak{g}_{-3}(I') = \mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4},\\
&\mathfrak{g}_{-k}(I') = 0 \text{ for } k \geq 4.
\end{aligned} \tag{4.37}$$

The choice of w_i is kept unchanged. Then (4.37) can be written explicitly as follows:

$$\begin{aligned}
&\mathfrak{g}_{-1}(I') = \mathbb{C}w_1 \oplus \mathbb{C}w_2 \oplus \mathbb{C}w_{14} \oplus \mathbb{C}w_{24} \oplus \mathbb{C}w_3,\\
&\mathfrak{g}_{-2}(I') = \mathbb{C}w_{13} \oplus \mathbb{C}w_{23} \oplus \mathbb{C}w_{134} \oplus \mathbb{C}w_{234},\\
&\mathfrak{g}_{-3}(I') = \mathbb{C}w_{1342},\\
&\mathfrak{g}_{-k}(I') = 0 \text{ for } k \geq 4.
\end{aligned} \tag{4.38}$$

Convention 4.69. In Subsection 4.5, we denote by $\mathscr{D}^{\alpha_i}$, $\mathscr{D}$ and $\mathscr{D}^{-i}$ the restriction of $\mathcal{D}^{\alpha_i}$, $\mathcal{D}$ and $\mathcal{D}^{-i}$ on $\mathcal{X}_0$ respectively, where the latter is defined in Notation 3.9. For simplicity we write $(\mathfrak{m}_-)_x := \mathfrak{m}_x(\alpha_2, \alpha_3, \alpha_4)$ and $(\mathfrak{m}_{-k})_x := (\mathfrak{m}_{-k}(\alpha_2, \alpha_3, \alpha_4))_x$, where $k \geq 1$ and $x \in \mathcal{X}_0$ is general.

Lemma 4.70. *At $x \in \mathcal{X}_0$ general $\dim(\mathfrak{m}_-)_x = \dim \mathcal{X}_0 = 11$.*

Proof. It is a special case of Proposition 3.10. □

4.5.2. *Exclude possibility of case (C).* Throughout part 4.5.2, we suppose case (C) of Proposition 4.67 occurs, and aim at deducing a contradiction.

Lemma 4.71. *In case (C) of Proposition 4.67, there exists a unique meromorphic line subbundle $\mathcal{N}$ of $\mathscr{D}^{\alpha_2}$ such that $[\mathcal{N}, \mathscr{D}^{\alpha_4}] \subset \mathcal{N} + \mathscr{D}^{\alpha_4}$. Consequently, $[\mathcal{N}, \mathscr{D}^{\alpha_2} + \mathscr{D}^{\alpha_4}] \subset \mathscr{D}^{\alpha_2} + \mathscr{D}^{\alpha_4} \subset \mathscr{D}$.*

Proof. It follows from Proposition 4.51 and the assumption in case (C) directly. □

Construction 4.72. In setting of Lemma 4.71, take $x \in \mathcal{X}_0$ general. Choose a local section $\widetilde{v}_1$ (resp. $\widetilde{v}_3$, $\widetilde{v}_4$) of $\mathcal{N}$ (resp. $\mathscr{D}^{\alpha_3}$, $\mathscr{D}^{\alpha_4}$), which is nonzero in an open neighborhood of x in $\mathcal{X}_0$. Take a local section $\widetilde{v}_2$ of $\mathscr{D}^{\alpha_2}$ such that $(\widetilde{v}_2)_y \notin \mathbb{C}(\widetilde{v}_1)_y$ at any point y in an open neighborhood of x in $\mathcal{X}_0$. Define by induction $k \geq 1$ that $\widetilde{v}_{i_1\cdots i_{k+1}} := [\widetilde{v}_{i_1\cdots i_k}, \widetilde{v}_{i_{k+1}}]$ as local vector field in an open neighborhood of x in $\mathcal{X}_0$. Take a subset $A \subset I := \{\alpha_2, \alpha_3, \alpha_4\}$. When all $\widetilde{v}_{i_j}$ are local sections of $\mathscr{D}^A := \sum_{\beta\in A} \mathscr{D}^\beta$ we denote by $v^A_{i_1\cdots i_k}$ the class of $\widetilde{v}_{i_1\cdots i_k}$ in $\mathrm{Symb}(\mathscr{D}^A)$. When $A = I$ we omit the superscript I, i.e. denote by $v_{i_1\cdots i_k} \in \mathrm{Symb}(\mathscr{D})$ of class of $\widetilde{v}_{i_1\cdots i_k}$. For simplicity we also use $v^A_{i_1\cdots i_k}$ and $v^A_{i_1\cdots i_k}$ to represent the corresponding class in the symbol algebras $\mathrm{Symb}_x(\mathscr{D}^A)$ and $\mathrm{Symb}_x(\mathscr{D})$ at a chosen general point x.

Proposition 4.73. *In setting of Construction 4.72, the symbol algebra of $\mathscr{D}$ at a general point $x \in \mathcal{X}_0$ is a quotient algebra of $\mathfrak{g}_-(B_4)$, denoted by $\mathfrak{g}_-(B_4)/\mathfrak{q}$. More precisely, under the isomorphism $(\mathfrak{m}_-)_x \cong \mathfrak{g}_-(B_4)/\mathfrak{q}$ the elements v_1, v_2, v_3, v_4 have weights $-\beta_1, -\beta_3, -\beta_2, -\beta_4$ respectively, where $\beta_1, \ldots, \beta_3$ are the three long simple roots of B_4, and β_4 is the short one. The ideal $\mathfrak{q}$ is generated by $\mathfrak{g}_{-\beta_1-\beta_2-\beta_3}$ in $\mathfrak{g}_-(B_4)$. We can write explicitly $(\mathfrak{m}_-)_x$ as follows:*

$$\begin{aligned}(\mathfrak{m}_{-1})_x &= \mathbb{C}v_1 \oplus \mathbb{C}v_3 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_4,\\ (\mathfrak{m}_{-2})_x &= \mathbb{C}v_{13} \oplus \mathbb{C}v_{32} \oplus \mathbb{C}v_{24},\\ (\mathfrak{m}_{-3})_x &= \mathbb{C}v_{324} \oplus \mathbb{C}v_{244},\\ (\mathfrak{m}_{-4})_x &= \mathbb{C}v_{3244},\\ (\mathfrak{m}_{-5})_x &= \mathbb{C}v_{32442},\\ (\mathfrak{m}_{-k})_x &= 0 \textit{ for } k \geq 6,\end{aligned} \tag{4.39}$$

where $\dim(\mathfrak{m}_{-k})_x = 4, 3, 2, 1, 1$ *for* $k = 1, \ldots, 5$ *respectively.*

Proof. In case (C) of Proposition 4.67, both $(\mathfrak{m}_-(\alpha_2, \alpha_3))_x$ and $(\mathfrak{m}_-(\alpha_3, \alpha_4))_x$ are standard. Thus by Remark 4.68 we have

$$\mathrm{ad}v_1(v_2) = 0,\ \mathrm{ad}v_3(v_4) = 0,\ (\mathrm{ad}v_i)^2(v_3) = 0,\ (\mathrm{ad}v_3)^2(v_i) = 0 \text{ in } (\mathfrak{m}_-)_x,$$

where $i = 1, 2$. Since $F^{\alpha_2,\alpha_4}_x \cong F^d(1, 2; \mathbb{C}^4)$, we know from Lemma 4.71 and Proposition 4.51 that

$$\mathrm{ad}v_1(v_2) = 0,\ \mathrm{ad}v_1(v_4) = 0,\ (\mathrm{ad}v_2)^2(v_4) = 0,\ (\mathrm{ad}v_4)^3(v_2) = 0 \text{ in } (\mathfrak{m}_-)_x.$$

In summary $(\mathfrak{m}_-)_x$ is a quotient algebra of $\mathfrak{g}_-(B_4)$, where we write the four simple roots of B_4 to be $\beta_1, \ldots, \beta_4$ in order with β_4 being the short simple root, and the elements v_1, v_2, v_3, v_4 have weights $-\beta_1, -\beta_3, -\beta_2, -\beta_4$ respectively. Since $(\mathfrak{m}_{-k}(\alpha_2, \alpha_3))_x = 0$ for all $k \geq 3$, $[v_{13}, v_2] = 0$ in $(\mathfrak{m}_-)_x$. It follows that $(\mathfrak{m}_-)_x$ is a quotient algebra of $\mathfrak{g}_-(B_4)/\mathfrak{q}$, where $\mathfrak{q}$ is the ideal in $\mathfrak{g}_-(B_4)$ generated by $\mathfrak{g}_{-\beta_1-\beta_2-\beta_3}$. It is straightforward to see that $\mathfrak{g}_-(B_4)/\mathfrak{q}$ is isomorphic to the graded Lie algebra described in (4.39). By Lemma 4.70, $\dim(\mathfrak{m}_-)_x = \dim \mathfrak{g}_-(B_4)/\mathfrak{q} = 11$. Hence $(\mathfrak{m}_-)_x \cong \mathfrak{g}_-(B_4)/\mathfrak{q}$. □

Proposition 4.74. *Case (C) of Proposition 4.67 does not occur.*

Proof. Suppose we are in case (C) of Proposition 4.67. Denote by $\mathcal{E}$ the meromorphic distribution on $\mathcal{X}^{\alpha_4}$ such that $\mathcal{E}|_{\mathcal{X}^{\alpha_4}_t}$ coincides with $\mathfrak{g}_{-1}(D_4/P_{\{\alpha_2,\alpha_3\}})$ under the identification $\mathcal{X}^{\alpha_4}_t \cong D_4/P_{\{\alpha_2,\alpha_3\}}$ for each $t \neq 0$. Then the singular locus on $\mathcal{X}^{\alpha_4}$ of $\mathcal{E}$ is a proper closed algebraic subset of $\mathcal{X}^{\alpha_4}_0$. By Remark 4.68 and Proposition 4.73, $\mathcal{E} = d\pi^{\alpha_4}(\mathcal{D} + T^{\pi^{\alpha_2,\alpha_4}})$, where $d\pi^{\alpha_4} : T\mathcal{X} \to T\mathcal{X}^{\alpha_4}$ is the tangent map of $\pi^{\alpha_4} : \mathcal{X} \to \mathcal{X}^{\alpha_4}$.

Take $x \in \mathcal{X}_0$ general. Denote by $\mathscr{E} := \mathcal{E}|_{\mathcal{X}^{\alpha_4}_0}$, and $y := \pi^{\alpha_4}(x) \in \mathcal{X}^{\alpha_4}_0$. We claim that $\mathrm{Symb}_y(\mathscr{E}) \cong \mathfrak{g}_-(\alpha_2, \alpha_3)$, where $\mathfrak{g}_-(\alpha_2, \alpha_3) \subset \mathfrak{g} = \mathrm{Lie}(D_4)$ is as in Definition 2.2. Note that $\mathfrak{g}_-(\alpha_2, \alpha_3)$ has been explicitly described in (4.37) and (4.38).

By abuse of notation, we denote by $v_{i_1\cdots i_k} \in \mathrm{Symb}_y(\mathscr{E})$ the class of the local vector field $d\pi^{\alpha_4}(\widetilde{v}_{i_1\ldots i_k})$ on $\mathcal{X}^{\alpha_4}_0$. Now v_1, v_2, v_3, v_{24}, and v_{244} form a basis of $\mathscr{E}_y$. There is a unique linear isomorphism $\psi : \mathscr{E}_y \to \mathfrak{g}_{-1}(\alpha_2, \alpha_3)$ such that

$$\begin{aligned}&\psi(v_1) = w_1, &&\psi(v_2) = w_2, &&\psi(v_3) = w_3,\\ &\psi(v_{24}) = w_{24}, &&\psi(v_{244}) = w_{14},\end{aligned}$$

where $w_i, w_{ij} \in \mathfrak{g}_{-1}(\alpha_2, \alpha_3)$ are as in Remark 4.68. By direct calculation ψ induces an isomorphism $\Psi : \mathrm{Symb}(\mathscr{E})_y \to (\mathfrak{g}_-(D_4/P_{\{\alpha_2,\alpha_3\}}))_q$ satisfying

$$\begin{aligned}&\Psi(v_{13}) = w_{123}, &&\Psi(v_{32}) = -w_{23}, &&\Psi(v_{324}) = -w_{234},\\ &\Psi(v_{3244}) = -w_{134}, &&\Psi(v_{32332}) = -w_{1342}.\end{aligned}$$

By Proposition 3.19 the variety $\mathcal{X}^{\alpha_4}_0 \cong D_4/P_{\{\alpha_2,\alpha_3\}}$. Thus $\pi^{\alpha_4}_0 : \mathcal{X}_0 \to \mathcal{X}^{\alpha_4}_0$ is a $\mathbb{P}^1$-fibration by Proposition 4.20.

On the other hand, by assumption $F_x^{\alpha_2,\alpha_4} \cong F^d(1,2;\mathbb{C}^4)$. The restriction of $\pi_0^{\alpha_4}$ on $F_x^{\alpha_2,\alpha_4}$ coincides with the morphism $F^d(1,2;\mathbb{C}^4) \to \text{cone}(pt, Q^3)$. In particular, a fiber of $\pi_0^{\alpha_4}$ is biholomorphic to $\mathbb{P}^3$, contracting the assertion that $\pi_0^{\alpha_4}$ is a $\mathbb{P}^1$-fibration. Hence case (C) of Proposition 4.67 does not occur. □

Now we can complete the proof of Proposition 4.65

Proof of Proposition 4.65. By Proposition 4.67, there are four possibilities $(A)-(D)$. By Proposition 4.74, case (C) does not occur. By symmetry of Dynkin diagram, case (D) is also impossible. If case (A) occur, then by Theorem 3.22 the manifold $\mathcal{X}_0 \cong D_4/P_{\{\alpha_2,\alpha_3,\alpha_4\}}$, contradicting to our assumption. Hence only case (B) is possible, verifying the conclusion. □

Acknowledgements. I want to thank Professor Jun-Muk Hwang for substantial discussions on this problem, and thank Professor Baohua Fu for commutations on this problem and encouragement for years. I also want to Professor Jaehyun Hong, and Dr. Yong Hu for communications related with this problem. This work is supported by National Researcher Program of National Research Foundation of Korea (Grant No. 2010-0020413).

References

[1] L. Bonavero, C. Casagrande, S. Druel, On covering and quasi-unsplit families of curves. J. Eur. Math. Soc. (JEMS) 9 (2007), no. 1, 45-57.
[2] A. Čap and H. Schichl, Parabolic geometries and canonical Cartan connections. Hokkaido Math. J. **29** (2000) 453-505.
[3] A. Grothendieck, Sur la classification des fibrs holomorphes sur la sphère de Riemann, Amer. J. Math. 79 (1957) 121-138.
[4] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts Math., Vol. 9, Springer, 1972.
[5] J.-M. Hwang, Rigidity of homogeneous contact manifolds under Fano deformation. J. reine angew. Math. 486 (1997) 153-163.
[6] J.-M. Hwang, Geometry of minimal rational curves on Fano manifolds. School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 335-393, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.
[7] J.-M. Hwang, Q. Li, Deformation of the family of lines on Q^5, preprint.
[8] J.-M. Hwang, N. Mok, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. Invent. Math. 131, 393-418 (1998)
[9] J.-M. Hwang, N. Mok, Deformation rigidity of the rational homogeneous space associated to a long simple root. Ann. scient. Ec. Norm. Sup. 35 (2002) 173-184.
[10] J.-M. Hwang, N. Mok, Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation. Invent. Math. 160, 591-645 (2005)
[11] J. Kollár, Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge, Band 32, Springer Verlag, 1996.
[12] J. M. Landsberg, L. Manivel, On the projective geometry of rational homogeneous varieties. Comment. Math. Helv. 78 (2003), no. 1, 65-100.
[13] Q. Li, Deformation of the product of complex Fano manifolds. C. R. Math. Acad. Sci. Paris 356 (2018), no. 5, 538-541.
[14] B. Pasquier, N. Perrin, Local rigidity of quasi-regular varieties. Math. Z. 265 (2010) 589-600.
[15] J. Tits, Groupes semi-simples complexes et géométrie projective, Séminaire Bourbaki 7 (1954/1955), exposé 112, 11 pages.
[16] A. Weber, J. A. Wiśniewski, On rigidity of flag varieties. Int. Math. Res. Not. IMRN 2018, no. 9, 2967-2979.
[17] J. A. Wiśniewski, On Fano manifolds of large index, Manuscr. Math. 70 (1991), 145-152.
[18] J. A. Wiśniewski, Rigidity of the Mori cone for Fano manifolds. Bulletin of the London Mathematical Society 41, no. 5 (2009): 779-81.
[19] K. Yamaguchi, Differential systems associated with simple graded Lie algebras. Progress in differential geometry, 413-494, Adv. Stud. Pure Math., 22, Math. Soc. Japan, Tokyo, 1993.

Korea Institute for Advanced Study, Seoul, Republic of Korea
qifengli@kias.re.kr