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Harnack Inequality and Gradient Estimate for G-SDEs with Degenerate Noise*

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Abstract

In this paper, the Harnack inequalities for G-SDEs with degenerate noise are derived by method of coupling by change of measure. Moreover, the gradient estimate for the associated nonlinear semigroup $\bar{P}_t$

$$|\nabla \bar{P}_t f| \leq c(p,t)(\bar{P}_t|f|^p)^{\frac{1}{p}}, \quad p>1, t>0$$

is also obtained for bounded and continuous function f. As an application of Harnack inequality, we prove the weak existence of degenerate G-SDEs under some integrable conditions. Finally, an example is presented. All of the above results extends the existed results in the linear expectation setting.

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1 Introduction

Since Peng [10, 11, 12] established the fundamental theory of G-Brownian motion and SDEs driven by it (G-SDEs, in short), the study of G-expectation has received much attention, see a summary paper [13] and references within for details. The G-expectation has been applied in many areas, for instance, stochastic optimization [4, 5], financial markets with volatility uncertainty [2] and the Feyman-Kac formula [6].

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Recently, using method of coupling by change of measure introduced by Wang [15, Chapter 1], Song [14] studied the gradient estimates for nonlinear diffusion semigroups, where the noise is assumed to be non-degenerate. Quite recently, under the nonlinear expectation framework, Yang [20] obtained the dimensional-free Harnack inequality for G-SDEs with non-degenerate noise.

On the other hand, the stochastic Hamiltonian system in the linear probability space, a typical model of degenerate diffusion system, has been investigated in [3, 16, 19].

In this paper, we intend to investigate Harnack inequalities and gradient estimate for G-SDEs with degenerate noise, i.e. the stochastic Hamiltonian system driven by G-Brownian motion. The method is also coupling by change of measure, in which the Girsanov transform in [6] palys a crucial role. Due to the lack of additivity of nonlinear expectation, the Bismut formula [15, (1.8), (1.14)], which is an important technique to get gradient estimate, can not be proved either by coupling by change of measure or Malliavin calculus in the G-SDEs. Instead, we directly estimate the local Lipschitz constant defined below. Moreover, as an application of Harnack inequality, we will prove the existence of weak solution for degenerate G-SDEs perturbed by a drift which only satisfies some integrable condition with respect to a reference nonlinear expectation.

Since the quadratic variation process of G-Brownian motion is a stochastic process, the G-SDEs generally contain two drift terms: $\mathrm{d}t$ and quadratic variation process. Moreover, the Girsanov transform is different from the one in the linear expectation case, see Theorem 2.2 below for more details.

2 Preparations

2.1 G-Expectation and G-Brownian motion

Before moving on, we recall some basic facts on G-expectation and G-Brownian motion. Let $\Omega = C_0([0,\infty);\mathbb{R}^d)$, the $\mathbb{R}^d$-valued and continuous functions on $[0,\infty)$ vanishing at zero, equipped with metric

$$\rho(\omega^1,\omega^2) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left[\max_{t\in[0,n]} |\omega^1_t - \omega^2_t| \wedge 1\right], \quad \omega^1,\omega^2 \in \Omega.$$

For any $T > 0$, set

$$L_{ip}(\Omega_T) = \{\omega \to \varphi(\omega_{t_1},\cdots,\omega_{t_n}) : n \in \mathbb{N}^+, t_1,\cdots,t_n \in [0,T], \varphi \in C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n)\},$$

and

$$L_{ip}(\Omega) = \bigcup_{T>0} L_{ip}(\Omega_T),$$

where $C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n)$ denotes the set of bounded and Lipschitz continuous functions on $\mathbb{R}^d \otimes \mathbb{R}^n$. Let $\mathbb{S}^d$ be the collection of all $d \times d$ symmetric matrices and $\mathbb{S}^d_+ \subset \mathbb{S}^d$ denote all

$d \times d$ positive definite and symmetric matrices. Fix $\underline{\sigma}, \overline{\sigma} \in \mathbb{S}_+^d$ with $\underline{\sigma} < \overline{\sigma}$ and define

$$G(A) := \frac{1}{2} \sup_{\gamma \in [\underline{\sigma}, \bar{\sigma}]} \operatorname{trace}(\gamma^2 A), \ A \in \mathbb{S}^d. \tag{2.1}$$

Then it is not difficult to see

$$G(A) - G(\bar{A}) \geq \frac{\lambda_0(\underline{\sigma}^2)}{2} \operatorname{trace}[A - \bar{A}], \ A \geq \bar{A}, A, \bar{A} \in \mathbb{S}^d, \tag{2.2}$$

where $\lambda_0(\underline{\sigma}^2) > 0$ is the minimal eigenvalue of $\underline{\sigma}^2$.

Let $\bar{\mathbb{E}}^G$ be the nonlinear expectation on Ω such that coordinate process $B = (B_t)_{t \geq 0}$, i.e. $B_t(\omega) = \omega_t, \omega \in \Omega$, is a d-dimensional G-Brownian motion on $(\Omega, L_G^1(\Omega), \bar{\mathbb{E}}^G)$, where $L_G^1(\Omega)$ is the completion of $L_{ip}(\Omega)$ under the norm $(\bar{\mathbb{E}}^G |\cdot|)$. See [14] for details on the construction of $\bar{\mathbb{E}}^G$. For any $p \geq 1$, let $L_G^p(\Omega)$ be the completion of $L_{ip}(\Omega)$ under the norm $(\bar{\mathbb{E}}^G |\cdot|^p)^{\frac{1}{p}}$. Similarly, we can define $L_G^p(\Omega_T)$ for any $T > 0$.

Let

$$M_G^{p,0}([0,T]) = \left\{ \eta_t := \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t);\ \xi_j \in L_G^p(\Omega_{t_j}), N \in \mathbb{N}^+,\ 0 = t_0 < t_1 < \cdots < t_N = T \right\},$$

and $M_G^p([0,T])$ be the completion of $M_G^{p,0}([0,T])$ under the norm

$$\|\eta\|_{M_G^p([0,T])} := \left(\bar{\mathbb{E}}^G \int_0^T |\eta_t|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Moreover, let

$$M_G^2([0,T])^d = \left\{ X = (X^1, X^2, \cdots, X^d), X^i \in M_G^2([0,T]), 1 \leq i \leq d \right\}.$$

Let $\mathcal{M}$ be the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$. According to [1, 8], there exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}$ such that

$$\bar{\mathbb{E}}^G[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X], \ X \in L_G^1(\Omega), \tag{2.3}$$

where $\mathbb{E}_P$ is the linear expectation under probability measure $P \in \mathcal{P}$. $\mathcal{P}$ is called a set that represents $\bar{\mathbb{E}}^G$. In fact, let W^0 be a d-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$, and $\mathbb{H}$ be the set of all progressively measurable stochastic processes valued in $[\underline{\sigma}, \bar{\sigma}]$. For any $\theta \in \mathbb{H}$, define $\mathbb{P}_\theta$ as the law of $\int_0^\cdot \theta_s \mathrm{d}W_s^0$. Then by [1, 8], we can take $\mathcal{P} = \{\mathbb{P}_\theta, \theta \in \mathbb{H}\}$, i.e.

$$\bar{\mathbb{E}}^G[X] = \sup_{\theta \in \mathbb{H}} \mathbb{E}_{\mathbb{P}_\theta}[X], \ X \in L_G^1(\Omega). \tag{2.4}$$

The associated Choquet capacity to $\bar{\mathbb{E}}^G$ is defined by

$$\mathcal{C}(A) = \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).$$

A set $A \subset \Omega$ is called polar if $\mathcal{C}(A) = 0$, and we say that a property holds $\mathcal{C}$-quasi-surely ($\mathcal{C}$-q.s.) if it holds outside a polar set, see [1] for more details on capacity.

Finally, letting $\langle B \rangle$ be the quadratic variation process of B, then by (2.2) and [12, Chapter III, Corollary 5.7], we have $\mathcal{C}$-q.s.

$$\underline{\sigma}^2 < \frac{\mathrm{d}}{\mathrm{d}t}\langle B \rangle_t \leq \bar{\sigma}^2. \tag{2.5}$$

2.2 Girsanov's Transform

The following Girsanov's transform comes from [9, Proposition 5.10].

Theorem 2.1. *Let $\{g_t\}_{t \leq T} \in M_G^2([0,T])^d$. If there exists a constant $\delta > 0$ such that*

$$\bar{\mathbb{E}}^G \exp\left\{\left(\frac{1}{2}+\delta\right)\int_0^T \langle g_u, \mathrm{d}\langle B \rangle_u g_u \rangle\right\} < \infty.$$

Then

$$\bar{B} := B + \int_0^{\cdot} \mathrm{d}\langle B \rangle_u g_u$$

is a G-Brownian motion on $[0,T]$ under $\tilde{\mathbb{E}}[\cdot] = \bar{\mathbb{E}}^G[\tilde{R}_T(\cdot)]$, where

$$\tilde{R}_T = \exp\left[-\int_0^T \langle g_u, \mathrm{d}B_u \rangle - \frac{1}{2}\int_0^T \langle g_u, \mathrm{d}\langle B \rangle_u g_u \rangle\right].$$

According to [6, Remark 5.3], letting $\hat{\Omega} = C_0([0,\infty), \mathbb{R}^{2d})$, we can construct an auxiliary $\hat{G}$-expectation space $(\hat{\Omega}, L^1_{\hat{G}}(\hat{\Omega}), \hat{\mathbb{E}}^{\hat{G}})$ with

$$\hat{G}(A) := \frac{1}{2}\sup_{\gamma \in [\underline{\sigma}, \bar{\sigma}]} \mathrm{trace}\left[A\begin{pmatrix} \gamma^2 & 1 \\ 1 & \gamma^{-2} \end{pmatrix}\right], \ A \in \mathbb{S}^{2d},$$

and a d-dimensional process B' such that $\begin{pmatrix} B \\ B' \end{pmatrix}$ is a $2d$-dimensional $\hat{G}$-Brownian motion and $\langle B, B' \rangle_t = tI_{d\times d}$ under $\hat{\mathbb{E}}^{\hat{G}}$. In addition, the distribution of B under $\bar{\mathbb{E}}^G$ is equal to that of B under $\hat{\mathbb{E}}^{\hat{G}}$. Moreover, letting

$$\tilde{G}(A) = \frac{1}{2}\sup_{\gamma \in [\underline{\sigma}, \bar{\sigma}]} \mathrm{trace}\left[A\gamma^{-2}\right], \quad A \in \mathbb{S}^d, \tag{2.6}$$

then B' is a $\tilde{G}$-Brownian motion under $\hat{\mathbb{E}}^{\hat{G}}$. Letting $\hat{\mathcal{C}}$ be the associated Choquet capacity to $\hat{\mathbb{E}}^{\hat{G}}$, then we have $\hat{\mathcal{C}}$-q.s.

$$\overline{\sigma}^{-2} \leq \frac{\mathrm{d}\langle B' \rangle_t}{\mathrm{d}t} \leq \underline{\sigma}^{-2}. \tag{2.7}$$

As a corollary of Theorem 2.1, we have the following Girsanov's transform, which will be used in the sequel.

Theorem 2.2. *Let* $\{g_t^i\}_{t\le T} \in M_G^2([0,T])^d, i = 1, 2$. *If*

$$\hat{\mathbb{E}}^{\hat{G}} \exp\left\{ \left(\frac{1}{2}+\delta\right) \int_0^T \left(\langle g_s^1, \mathrm{d}\langle B'\rangle_s g_s^1\rangle + \langle g_s^2, \mathrm{d}\langle B\rangle_s g_s^2\rangle + 2\langle g_s^1, g_s^2\rangle \mathrm{d}s\right) \right\} < \infty. \tag{2.8}$$

Then

$$\breve{B} := B + \int_0^\cdot g_u^1 \mathrm{d}u + \int_0^\cdot g_u^2 \mathrm{d}\langle B\rangle_u$$

is a G-Brownian motion on $[0,T]$ *under* $\breve{\mathbb{E}}[\cdot] = \hat{\mathbb{E}}^{\hat{G}}[\breve{R}_T(\cdot)]$, *where*

$$\begin{aligned}\breve{R}_T = \exp\Bigg[&- \int_0^T \left\langle \begin{pmatrix} g_u^1 \\ g_u^2 \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u, \\ B_u \end{pmatrix} \right\rangle \\ &- \frac{1}{2}\int_0^T \left(\langle g_s^1, \mathrm{d}\langle B'\rangle_s g_s^1\rangle + \langle g_s^2, \mathrm{d}\langle B\rangle_s g_s^2\rangle + 2\langle g_s^1, g_s^2\rangle \mathrm{d}s\right) \Bigg].\end{aligned}$$

Proof. Letting $W = \begin{pmatrix} B \\ B' \end{pmatrix}$, we have

$$\langle W\rangle_t = \begin{pmatrix} \langle B\rangle_t & tI_{d\times d} \\ tI_{d\times d} & \langle B'\rangle_t \end{pmatrix}$$

and

$$\begin{aligned}&\int_0^T \left\langle \begin{pmatrix} g^2 \\ g^1 \end{pmatrix}, \mathrm{d}\langle W\rangle \begin{pmatrix} g^2 \\ g^1 \end{pmatrix} \right\rangle \\ &= \int_0^T \left(\langle g_s^1, \mathrm{d}\langle B'\rangle_s g_s^1\rangle + \langle g_s^2, \mathrm{d}\langle B\rangle_s g_s^2\rangle + 2\langle g_s^1, g_s^2\rangle \mathrm{d}s\right).\end{aligned}$$

Let

$$\tilde{W} = W + \int_0^\cdot \mathrm{d}\langle W\rangle \begin{pmatrix} g^2 \\ g^1 \end{pmatrix}.$$

In view of (2.8) and applying Theorem 2.1 for $\left(W, \begin{pmatrix} g^2 \\ g^1 \end{pmatrix}\right)$ replacing (B, g), we conclude that $\tilde{W}$ is a $2d$-dimensional $\hat{G}$-Brownian motion on $[0,T]$ under $\breve{\mathbb{E}}$ defined in Theorem 2.2. In particular,

$$\breve{B} := B + \int_0^\cdot g_u^1 \mathrm{d}u + \int_0^\cdot g_u^2 \mathrm{d}\langle B\rangle_u$$

is a G-Brownian motion on $[0,T]$ under $\breve{\mathbb{E}}$. □

Remark 2.3. *Theorem 2.2 extends the result in [6, Theorem 5.2], where* g^1 *and* g^2 *are assumed to be bounded processes.*

Throughout the paper, the letter C or c will denote a positive constant, and $C(\theta)$ or $c(\theta)$ stands for a constant depending on θ. The value of the constants may change from one appearance to another.

3 Harnack and Gradient Estimate

Consider the following G-SDE on $\mathbb{R}^{m+d}$:

$$\begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = b_1(X_t, Y_t)\mathrm{d}t + \mathrm{d}\langle B\rangle_t b_2(X_t, Y_t) + Q\mathrm{d}B_t, \end{cases} \tag{3.1}$$

where B_t is a d-dimensional G-Brownian motion defined in Section 1, A is an $m \times m$ matrix, M is an $m \times d$ matrix, Q is a $d \times d$ matrix, $b_1, b_2 : \mathbb{R}^{m+d} \to \mathbb{R}^d$.

In this paper, we only consider $m = d = 1$, and the result can be extended to general $m \geq 1$ and $d \geq 1$. In this case, $\underline{\sigma}$ and $\overline{\sigma}$ in (2.1) are two positive constants satisfying $\underline{\sigma} < \overline{\sigma}$, and the corresponding generating function is given by

$$G(a) = \frac{1}{2}\overline{\sigma}^2 a^+ - \frac{1}{2}\underline{\sigma}^2 a^-, \quad a \in \mathbb{R}^1.$$

In this section, we study the Harnack inequalities and gradient estimate for (3.1). To this end, we make the following assumptions:

(A1) $QM \neq 0$.

(A2) There exists $K > 0$ such that

$$|b_1(z) - b_1(\bar{z})| + |b_2(z) - b_2(\bar{z})| \leq K|z - \bar{z}|, \quad z, \bar{z} \in \mathbb{R}^2. \tag{3.2}$$

Remark 3.1. *According to [12, Theorem 1.2],* **(A2)** *implies that* (3.1) *has a unique non-explosive strong solution* (X_t^z, Y_t^z) *in* $M_G^2([0,T])^2$ *for any* $T > 0$ *and* $(X_0, Y_0) = z \in \mathbb{R}^2$.

Denote by $C_b(\mathbb{R}^2)$ ($C_b^+(\mathbb{R}^2)$) the bounded (non-negative bounded) and continuous function on $\mathbb{R}^2$. Let $\bar{P}_t$ be the associated nonlinear semigroup to (X_t^z, Y_t^z), i.e.

$$\bar{P}_t f(z) = \bar{\mathbb{E}}^G f(X_t^z, Y_t^z), \quad f \in C_b(\mathbb{R}^2).$$

For a real-valued functtion f defined on a metric sapce (H, ρ), define

$$|\nabla f(z)| = \limsup_{\rho(x,z) \to 0} \frac{|f(x) - f(z)|}{\rho(x, z)}, \quad z \in H. \tag{3.3}$$

Then $|\nabla f(z)|$ is called the local Lipschitz constant of f at point $z \in H$.

Theorem 3.2. *Assume* **(A1)**-**(A2)** *and let* $T > 0$. *Then there exists some constant* $C > 0$ *depending on* A, K *and* $|Q^{-1}|$ *such that the following assertions hold.*

(1) For any $z = (z_1, z_2), h = (h_1, h_2) \in \mathbb{R}^2$, $p > 1$, *the Harnack inequality*

$$(\bar{P}_T f)^p(z + h) \leq \bar{P}_T f^p(z) \exp\left[C\frac{p}{2(p-1)}\Sigma(T)|h|^2\right], \quad f \in C_b^+(\mathbb{R}^2) \tag{3.4}$$

holds with

$$\Sigma(T) := \underline{\sigma}^{-2} T \left(\frac{1}{T} + \frac{1}{T^2} + 1 + T\right)^2 + \overline{\sigma}^2 T (1 + T)^2. \tag{3.5}$$

(2) The gradient estimate, i.e.

$$\|\nabla \bar{P}_T f\|_\infty \le C\|f\|_\infty \sqrt{\Sigma(T)}, \quad f \in C_b^+(\mathbb{R}^2) \tag{3.6}$$

holds for some constant $C > 0$.

(3) For any $p > 1$, there exists a constant $c(p) > 0$ such that

$$|\nabla \bar{P}_T f(z)| \le c(p) \left(\bar{P}_T |f|^p(z)\right)^{\frac{1}{p}} \sqrt{\Sigma(T)}, \quad z \in \mathbb{R}^2, \quad f \in C_b^+(\mathbb{R}^2). \tag{3.7}$$

Remark 3.3. *With (2.3) in hand, it seems that (3.4) can be derived by taking a supremum in the following Harnack inequality for the linear expectation $\mathbb{E}_P$:*

$$\left(\mathbb{E}_P f(X_t^{z+h}, Y_t^{z+h})\right)^p \le (\mathbb{E}_P f^p(X_t^z, Y_t^z)) \exp\{\Phi(t,h,p)\}. \tag{3.8}$$

However, the method of coupling by change of measure is available for the SDEs driven by Brownian motion. So, it is difficult to get (3.8) since B_t is only a martingale under $\mathbb{E}_P$. Therefore, the results in Theorem 3.2 are non-trivial.

Remark 3.4. *Compared to the SDEs in [14], the SDE (3.1) is allowed to contain drift $\mathrm{d}\langle B\rangle_t b_2$.*

Now, we are in the position to prove Theorem 3.2.

Proof. (1) For any $\eta \in \mathbb{R}^2$, let (X_t^η, Y_t^η) solve (3.1) with $(X_0, Y_0) = \eta$. For $h = (h_1, h_2) \in \mathbb{R}^2$, define

$$\gamma_1(s) = v_1(s)h_2 + \alpha_1(s), \quad s \in [0,T]$$

with

$$v_1(s) = \frac{T-s}{T},$$
$$\alpha_1(s) = -\frac{s(T-s)}{T^2} M \mathrm{e}^{-sA} \Lambda_1(T)^{-1} \left(h_1 + \int_0^T \frac{T-u}{T} \mathrm{e}^{-uA} M h_2 \mathrm{d}u\right), \quad s \in [0,T],$$

where

$$\Lambda_1(T) := \int_0^T \frac{s(T-s)}{T^2} \mathrm{e}^{-2sA} M^2 \mathrm{d}s.$$

It is clear that

$$|\Lambda_1(T)^{-1}| \le cT^{-1} \tag{3.9}$$

holds for some constant $c > 0$.

Let $(\tilde{X}_t, \tilde{Y}_t)$ solve the equation

$$\begin{cases} \mathrm{d}\tilde{X}_t = \{A\tilde{X}_t + M\tilde{Y}_t\}\mathrm{d}t, \\ \mathrm{d}\tilde{Y}_t = b_1(X_t^z, Y_t^z)\mathrm{d}t + b_2(X_t^z, Y_t^z)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}B_t + \gamma_1'(t)\mathrm{d}t \end{cases} \tag{3.10}$$

with $(\tilde{X}_0, \tilde{Y}_0) = z + h$. Then the solution to (3.10) is non-explosive as well.

Set

$$\Theta_1(s) = \left(\mathrm{e}^{As} h_1 + \int_0^s \mathrm{e}^{(s-u)A} M \gamma_1(u) \mathrm{d}u,\ \gamma_1(s) \right), \quad s \in [0, T].$$

Then there exists a constant $C > 0$ such that for any $s \in [0, T]$,

$$\begin{aligned} &|\gamma_1'(s)| \le C \left(\frac{1}{T} + \frac{1}{T^2} \right) |h|, \\ &|\Theta_1(s)| \le C \Big(1 + T \Big) |h|. \end{aligned} \tag{3.11}$$

Note that

$$(\tilde{X}_s, \tilde{Y}_s) = (X_s^z, Y_s^z) + \Theta_1(s), \quad s \in [0, T], \tag{3.12}$$

and in particular, $(\tilde{X}_T, \tilde{Y}_T) = (X_T^z, Y_T^z)$. Let

$$\begin{aligned} \Phi_1(s) &= Q^{-1} \{ b_1(X_s^z, Y_s^z) - b_1(\tilde{X}_s, \tilde{Y}_s) + \gamma_1'(s) \}, \\ \Phi_2(s) &= Q^{-1} \{ b_2(X_s^z, Y_s^z) - b_2(\tilde{X}_s, \tilde{Y}_s) \}, \quad s \in [0, T], \end{aligned}$$

and B' be in Section 2.2. (2.5), (2.7), (3.11) and (3.12) together with **(A1)**-**(A2)** imply $\hat{\mathcal{C}}$-q.s.

$$\begin{aligned} &\int_0^T |\Phi_1(s)|^2 \mathrm{d}\langle B' \rangle_s + \int_0^T |\Phi_2(s)|^2 \mathrm{d}\langle B \rangle_s + 2 \int_0^T \Phi_1(s) \Phi_2(s) \mathrm{d}s \\ &\le \int_0^T \underline{\sigma}^{-2} |\Phi_1(s)|^2 \mathrm{d}s + \int_0^T \overline{\sigma}^2 |\Phi_2(s)|^2 \mathrm{d}s + 2 \int_0^T \Phi_1(s) \Phi_2(s) \mathrm{d}s \\ &\le 2 \int_0^T \underline{\sigma}^{-2} |\Phi_1(s)|^2 \mathrm{d}s + 2 \int_0^T \overline{\sigma}^2 |\Phi_2(s)|^2 \mathrm{d}s \\ &\le C \underline{\sigma}^{-2} \int_0^T \left(|\Theta_1(s)| + |\gamma_1'(s)| \right)^2 \mathrm{d}s + C \overline{\sigma}^2 \int_0^T |\Theta_1(s)|^2 \mathrm{d}s \\ &\le C \underline{\sigma}^{-2} T \left(\frac{1}{T} + \frac{1}{T^2} + 1 + T \right)^2 |h|^2 + C \overline{\sigma}^2 T \left(1 + T \right)^2 |h|^2 \\ &= C \Sigma(T) |h|^2 \end{aligned} \tag{3.13}$$

for some constant $C > 0$ depending on A, K in **(A2)** and $|Q^{-1}|$.

Applying Theorem 2.2, we conclude that

$$\tilde{B} := B + \int_0^{\cdot} \Phi_1(u) \mathrm{d}u + \int_0^{\cdot} \Phi_2(u) \mathrm{d}\langle B \rangle_u$$

is a G-Brownian motion on $[0, T]$ under $\mathbb{E}_1(\cdot) = \hat{\mathbb{E}}^{\hat{G}}(R_1(T)(\cdot))$, where

$$R_1(T) = \exp \left[- \int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d} \begin{pmatrix} B_u' \\ B_u \end{pmatrix} \right\rangle \right.$$

$$-\frac{1}{2}\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\Bigg].$$

Since $\langle \tilde{B}\rangle = \langle B\rangle$, (3.10) reduces to

$$\begin{cases}\mathrm{d}\tilde{X}_t = \{A\tilde{X}_t + M\tilde{Y}_t\}\mathrm{d}t,\\ \mathrm{d}\tilde{Y}_t = b_1(\tilde{X}_t,\tilde{Y}_t)\mathrm{d}t + \mathrm{d}\langle\tilde{B}\rangle_t b_2(\tilde{X}_t,\tilde{Y}_t) + Q\mathrm{d}\tilde{B}_t.\end{cases} \tag{3.14}$$

This means that the distribution of $(\tilde{X}_t,\tilde{Y}_t)$ under $\mathbb{E}_1$ coincides with that of (X_t^{z+h}, Y_t^{z+h}) under $\hat{\mathbb{E}}^{\hat{G}}$ (or $\bar{\mathbb{E}}^{G}$). Thus, Hölder inequality implies for any $f \in C_b^+(\mathbb{R}^2)$ and $p > 1$,

$$\begin{aligned}\bar{P}_T f(z+h) &= \mathbb{E}_1 f(\tilde{X}_T,\tilde{Y}_T)\\ &= \hat{\mathbb{E}}^{\hat{G}}\left[R_1(T)f(X_T^z,Y_T^z)\right]\\ &\le (\bar{P}_T f^p(z))^{\frac{1}{p}}\{\hat{\mathbb{E}}^{\hat{G}}R_1(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}},\end{aligned} \tag{3.15}$$

here we used the fact that the distribution of B under $\bar{\mathbb{E}}^G$ is equal to that of B under $\hat{\mathbb{E}}^{\hat{G}}$. It follows from the definition of $R_1(T)$ and (3.13) that

$$\begin{aligned}&\hat{\mathbb{E}}^{\hat{G}}R_1(T)^{\frac{p}{p-1}}\\ &= \hat{\mathbb{E}}^{\hat{G}}\Bigg\{\exp\Bigg[-\frac{p}{p-1}\int_0^T\left\langle\begin{pmatrix}\Phi_1(u)\\ \Phi_2(u)\end{pmatrix}, \mathrm{d}\begin{pmatrix}B'_u\\ B_u\end{pmatrix}\right\rangle\\ &\quad -\frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^T\left(|\Phi_1(s)|^2\mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\Bigg]\\ &\quad\times\exp\left[\frac{p}{2(p-1)^2}\int_0^T\left(|\Phi_1(s)|^2\mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right]\Bigg\}\\ &\le \exp\left[C\frac{p}{2(p-1)^2}\Sigma(T)|h|^2\right].\end{aligned}$$

Combining this with (3.15), we derive the Harnack inequality (3.4).

(2) Now, we prove the gradient estimate (3.6). Since the distribution of B under $\bar{\mathbb{E}}^G$ is equal to that of B under $\hat{\mathbb{E}}^{\hat{G}}$, we have

$$\bar{P}_T f(z) = \bar{\mathbb{E}}^G f(X_T^z,Y_T^z) = \hat{\mathbb{E}}^{\hat{G}} f(X_T^z,Y_T^z).$$

This and (3.15) yield

$$\begin{aligned}|\bar{P}_T f(z+h) - \bar{P}_T f(z)| &= |\hat{\mathbb{E}}^{\hat{G}}\left[R_1(T)f(X_T^z,Y_T^z)\right] - \hat{\mathbb{E}}^{\hat{G}} f(X_T^z,Y_T^z)|\\ &\le \hat{\mathbb{E}}^{\hat{G}}\left(|f(X_T^z,Y_T^z)||R_1(T)-1|\right).\end{aligned} \tag{3.16}$$

Noting that $|x-1| \le (x+1)|\log x|$ for any $x>0$, we have

$$
\begin{aligned}
&|\bar{P}_T f(z+h) - \bar{P}_T f(z)| \\
&\le \|f\|_\infty \hat{\mathbb{E}}^{\hat{G}} R_1(T)|\log R_1(T)| + \|f\|_\infty \hat{\mathbb{E}}^{\hat{G}} |\log R_1(T)| \\
&= \|f\|_\infty \left(\mathbb{E}_1 |\log R_1(T)| + \hat{\mathbb{E}}^{\hat{G}} |\log R_1(T)| \right).
\end{aligned} \tag{3.17}
$$

Let

$$
\tilde{B}' = B' + \int_0^{\cdot} \Phi_1(u) \mathrm{d}\langle B' \rangle_u + \int_0^{\cdot} \Phi_2(u) \mathrm{d}u.
$$

From Theorem 2.2, we know that $\tilde{B}'$ is a $\tilde{G}$-Brownian motion under $\mathbb{E}_1$. Applying B-D-G inequality and noting $\langle \tilde{B}' \rangle = \langle B' \rangle$ and $\langle \tilde{B} \rangle = \langle B \rangle$, we obtain

$$
\begin{aligned}
&\mathbb{E}_1 |\log R_1(T)| \\
&= \mathbb{E}_1 \bigg| - \int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d} \begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle \\
&- \frac{1}{2} \int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle B' \rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle B \rangle_s + 2\Phi_1(s)\Phi_2(s) \mathrm{d}s \right) \bigg| \\
&\le \mathbb{E}_1 \left| \int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d} \begin{pmatrix} \tilde{B}'_u \\ \tilde{B}_u \end{pmatrix} \right\rangle \right| \\
&+ \frac{1}{2} \mathbb{E}_1 \left| \int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle \tilde{B}' \rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle \tilde{B} \rangle_s + 2\Phi_1(s)\Phi_2(s) \mathrm{d}s \right) \right| \\
&\le \mathbb{E}_1 \left(\int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle \tilde{B}' \rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle \tilde{B} \rangle_s + 2\Phi_1(s)\Phi_2(s) \mathrm{d}s \right) \right)^{\frac{1}{2}} \\
&+ \frac{1}{2} \mathbb{E}_1 \left| \int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle \tilde{B}' \rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle \tilde{B} \rangle_s + 2\Phi_1(s)\Phi_2(s) \mathrm{d}s \right) \right|.
\end{aligned}
$$

This together with (3.13) implies

$$
\mathbb{E}_1 |\log R_1(T)| \le C \left(\Sigma(T)|h|^2 + \sqrt{\Sigma(T)}|h| \right),
$$

here, $\Sigma(T)$ is defined in (3.5). Similarly, we get

$$
\hat{\mathbb{E}}^{\hat{G}} |\log R_1(T)| \le C \left(\Sigma(T)|h|^2 + \sqrt{\Sigma(T)}|h| \right).
$$

Then it follows from (3.17) that

$$
|\bar{P}_T f(z+h) - \bar{P}_T f(z)| \le C\|f\|_\infty \left(\Sigma(T)|h|^2 + \sqrt{\Sigma(T)}|h| \right). \tag{3.18}
$$

This together with (3.3) yields

$$
|\nabla \bar{P}_T f(z)| \le C\|f\|_\infty \sqrt{\Sigma(T)}, \tag{3.19}
$$

which implies (3.6).

(3) In order to get (3.7), let

$$\tilde{R}_1(T) = \exp\Bigg[-\frac{p}{p-1}\int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle - \frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s \right) \Bigg],$$

and

$$\begin{aligned} \hat{B}' &= B' + \int_0^\cdot \frac{p}{p-1}\Phi_1(u)\mathrm{d}\langle B'\rangle_u + \int_0^\cdot \frac{p}{p-1}\Phi_2(u)\mathrm{d}u, \\ \hat{B} &= B + \int_0^\cdot \frac{p}{p-1}\Phi_1(u)\mathrm{d}u + \int_0^\cdot \frac{p}{p-1}\Phi_2(u)\mathrm{d}\langle B\rangle_u. \end{aligned} \tag{3.20}$$

Again by Theorem 2.2, $\hat{B}'$ is also a $\tilde{G}$-Brownian motion under $\tilde{\mathbb{E}}_2(\cdot) = \hat{\mathbb{E}}^{\hat{G}}(\tilde{R}_1(T)(\cdot))$. Using the inequality $|x-1| \le (x+1)|\log x|$ for any $x > 0$ and (3.13), we have

$$\begin{aligned} &\hat{\mathbb{E}}^{\hat{G}}|R_1(T) - 1|^{\frac{p}{p-1}} \\ &\le \hat{\mathbb{E}}^{\hat{G}}|R_1(T)+1|^{\frac{p}{p-1}}|\log R_1(T)|^{\frac{p}{p-1}} \\ &\le c(p)\hat{\mathbb{E}}^{\hat{G}}R_1(T)^{\frac{p}{p-1}}|\log R_1(T)|^{\frac{p}{p-1}} + c(p)\hat{\mathbb{E}}^{\hat{G}}|\log R_1(T)|^{\frac{p}{p-1}} \\ &\le c(p)\exp\left[C\frac{p}{2(p-1)^2}\Sigma(T)|h|^2\right]\hat{\mathbb{E}}^{\hat{G}}\left(\tilde{R}_1(T)|\log R_1(T)|^{\frac{p}{p-1}}\right) \\ &\quad + c(p)\hat{\mathbb{E}}^{\hat{G}}|\log R_1(T)|^{\frac{p}{p-1}} \end{aligned} \tag{3.21}$$

for some constants $C, c(p) > 0$. Combining (3.20) and B-D-G inequality and noting $\langle\hat{B}'\rangle = \langle B'\rangle$ and $\langle\hat{B}\rangle = \langle B\rangle$, we obtain

$$\begin{aligned} &\hat{\mathbb{E}}^{\hat{G}}\left(\tilde{R}_1(T)|\log R_1(T)|^{\frac{p}{p-1}}\right) \\ &= \tilde{\mathbb{E}}_2\Bigg| -\int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle \\ &\qquad - \frac{1}{2}\int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s \right) \Bigg|^{\frac{p}{p-1}} \\ &= \tilde{\mathbb{E}}_2\Bigg| -\int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} \hat{B}'_u \\ \hat{B}_u \end{pmatrix} \right\rangle \\ &\qquad + \left(\frac{p}{p-1} - \frac{1}{2}\right)\int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle \hat{B}'\rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle \hat{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s \right) \Bigg|^{\frac{p}{p-1}} \\ &\le c(p)\tilde{\mathbb{E}}_2\left|\int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} \hat{B}'_u \\ \hat{B}_u \end{pmatrix} \right\rangle\right|^{\frac{p}{p-1}} \\ &+ c(p)\tilde{\mathbb{E}}_2\left|\left(\frac{p}{p-1} - \frac{1}{2}\right)\int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle \hat{B}'\rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle \hat{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s \right)\right|^{\frac{p}{p-1}} \end{aligned}$$

$$\le c(p)\tilde{\mathbb{E}}_2\left|\int_0^T\left(|\Phi_1(s)|^2\mathrm{d}\langle\hat{B}'\rangle_s+|\Phi_2(s)|^2\mathrm{d}\langle\hat{B}\rangle_s+2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right|^{\frac{p}{2(p-1)}}$$
$$+c(p)\tilde{\mathbb{E}}_2\left|\left(\frac{p}{p-1}-\frac{1}{2}\right)\int_0^T\left(|\Phi_1(s)|^2\mathrm{d}\langle\hat{B}'\rangle_s+|\Phi_2(s)|^2\mathrm{d}\langle\hat{B}\rangle_s+2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right|^{\frac{p}{p-1}}$$
$$=: I_1+I_2.$$

Let $\tilde{\mathcal{C}}$ be the Choquet capacity associated to $\tilde{\mathbb{E}}_2$. Noting that $\hat{B}'$ is a $\tilde{G}$-Brownian motion under $\tilde{\mathbb{E}}_2$, then $\tilde{\mathcal{C}}$-q.s. (3.13) holds with (B,B') replacing by $(\hat{B},\hat{B}')$. Thus, we get

$$I_1\le c(p)\left(\Sigma(T)|h|^2\right)^{\frac{p}{2(p-1)}},$$

and

$$I_2\le c(p)\left(\Sigma(T)|h|^2\right)^{\frac{p}{p-1}}.$$

Therefore, we have

$$\left(\hat{\mathbb{E}}^{\hat{G}}\left(\tilde{R}_1(T)|\log R_1(T)|^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{p}}\le c(p)\left(\Sigma(T)|h|^2+\sqrt{\Sigma(T)}|h|\right). \tag{3.22}$$

Similarly by B-D-G inequality and (3.13), we arrive at

$$\left(\hat{\mathbb{E}}^{\hat{G}}|\log R_1(T)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\le c(p)\left(\Sigma(T)|h|^2+\sqrt{\Sigma(T)}|h|\right).$$

This together with (3.16), (3.21), (3.22) and Hölder inequality yields

$$|\nabla\bar{P}_Tf(z)|=\limsup_{h\to0}\frac{|\bar{P}_Tf(z+h)-\bar{P}_Tf(z)|}{|h|}$$
$$\le\left(\bar{P}_T|f|^p(z)\right)^{\frac{1}{p}}\limsup_{h\to0}\frac{\left(\hat{\mathbb{E}}^{\hat{G}}|R_1(T)-1|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}}{|h|}$$
$$\le c(p)\left(\bar{P}_T|f|^p(z)\right)^{\frac{1}{p}}\sqrt{\Sigma(T)},\quad z\in\mathbb{R}^2.$$

This completes the proof.

□

4 Applications of Harnack inequality

As an application of Harnack inequality, in this section, we will prove the weak existence of SDEs perturbed by an integrable drifts with respect to an invariant nonlinear expectation of a regular G-SDE. To this end, we assume that the Harnack inequality holds for the regular G-SDE. The main idea is to prove Novikov's condition by Harnack inequality. We should point out that the following procedure can also be applied for non-degenerate G-SDEs. However, to make the framework consistent, we only consider the stochastic Hamiltonian system. One can refer to [17] and [18] for the linear expectation case. Let A, M, Q, b_1, b_2 and B_t be introduced in Section 3 and $\bar{b}_1, \bar{b}_2 : \mathbb{R}^{m+d} \to \mathbb{R}^d$. For simplicity, we still consider $m = d = 1$. Consider the stochastic Hamiltonian system:

$$
(4.1) \qquad \begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = \bar{b}_1(X_t, Y_t)\mathrm{d}t + \bar{b}_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t \\ \qquad + b_1(X_t, Y_t)\mathrm{d}t + b_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}B_t. \end{cases}
$$

The referenced SDE is

$$
(4.2) \qquad \begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = b_1(X_t, Y_t)\mathrm{d}t + b_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}B_t. \end{cases}
$$

Assume that (4.2) has a unique non-explosive strong solution (X_t^z, Y_t^z) in $M_G^2([0,T])^2$ for any $T > 0$ and $(X_0, Y_0) = z \in \mathbb{R}^2$. Let P_t^0 be the associated nonlinear semigroup to (4.2) defined by

$$
P_t^0 f(z) = \bar{\mathbb{E}}^G f(X_t^z, Y_t^z), \quad f \in C_b(\mathbb{R}^2).
$$

Before moving on, we first introduce the definition of weak solution for the SDE (4.1).

Definition 4.1. $((\tilde{X}, \tilde{Y}), \tilde{B})$ is called a weak solution to (4.1) with initial value (x, y), if $\tilde{B}$ is a G-Brownian motion on some nonlinear space $(\Omega, \tilde{\mathbb{E}})$ and

$$
(4.3) \qquad \begin{cases} \tilde{X}_s = x + \int_0^s \{A\tilde{X}_t + M\tilde{Y}_t\}\mathrm{d}t, \\ \tilde{Y}_s = y + \int_0^s \bar{b}_1(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}t + \int_0^s \bar{b}_2(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}\langle \tilde{B}\rangle_t \\ \qquad + \int_0^s b_1(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}s + \int_0^s b_2(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}\langle \tilde{B}\rangle_t + Q\mathrm{d}\tilde{B}_t, \quad s \geq 0. \end{cases}
$$

In the following, we recall the definition of invariant nonlinear expectation, see [7] for more details.

Definition 4.2. A sublinear expectation $\mathbb{E}_0 : C_b^1(\mathbb{R}^d) \to \mathbb{R}^1$ is said to be an invariant expectation of P_t^0, if

$$
\mathbb{E}_0(P_t^0 f) = \mathbb{E}_0 f, \quad f \in C_b^1(\mathbb{R}^d), t \geq 0.
$$

Theorem 4.1. *Assume that P_t^0 has a unique invariant nonlinear expectation and satisfies the Harnack inequality:*

$$
(4.4) \qquad (P_t^0|f|)^p(z) \leq (P_t^0|f|^p)(\bar{z})\mathrm{e}^{\Phi_p(t,z,\bar{z})}, \quad f \in C_b(\mathbb{R}^2), z, \bar{z} \in \mathbb{R}^2, t > 0
$$

with

$$
(4.5) \qquad \int_0^t \frac{\mathrm{d}s}{\{\mathbb{E}_0 \mathrm{e}^{-\Phi_p(s,z,\cdot)}\}^{\frac{1}{p}}} < \infty, \quad t > 0, z \in \mathbb{R}^2
$$

for each $p > 1$. If $\bar{b}_1$ and $\bar{b}_2$ are continuous and there exists a constant $\varepsilon > 0$ such that

$$
\mathbb{E}_0 \mathrm{e}^{\varepsilon(|\bar{b}_1|^2 + |\bar{b}_2|^2)} < \infty.
$$

Then for any $z \in \mathbb{R}^2$, the stochastic Hamiltonian system (4.1) has a weak solution with initial value z.

Proof. Let (X_t, Y_t) be the solution to (4.2) with initial value z. Define

$$\bar{B}_s = B_s - \int_0^s Q^{-1}[(\bar{b}_1(X_t,Y_t)\mathrm{d}t + \bar{b}_2(X_t,Y_t)\mathrm{d}\langle B\rangle_t)].$$

Then (4.2) can be rewritten as

$$\begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = \bar{b}_1(X_t,Y_t)\mathrm{d}t + \bar{b}_2(X_t,Y_t)\mathrm{d}\langle B\rangle_t \\ \qquad + b_1(X_t,Y_t)\mathrm{d}t + b_2(X_t,Y_t)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}\bar{B}_t. \end{cases} \tag{4.6}$$

By the Markov property, it is sufficient to find out a constant $t_0 > 0$ such that $\{\bar{B}\}_{s\in[0,t_0]}$ is a G-Brownian motion under $\tilde{\mathbb{E}}[\cdot] = \hat{\mathbb{E}}^{\hat{G}}[\tilde{R}(t_0)(\cdot)]$, where

$$\begin{aligned}\tilde{R}(t_0) = \exp\Bigg[&\int_0^{t_0} \left\langle \begin{pmatrix} \bar{b}_1(X_u,Y_u) \\ \bar{b}_2(X_u,Y_u)\end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix}\right\rangle \\ &- \frac{1}{2}\int_0^{t_0} \left(|\bar{b}_1(X_u,Y_u)|^2\mathrm{d}\langle B'\rangle_u + |\bar{b}_2(X_u,Y_u)|^2\mathrm{d}\langle B\rangle_u + 2\bar{b}_1(X_u,Y_u)\bar{b}_2(X_u,Y_u)\mathrm{d}u\right)\Bigg].\end{aligned}$$

According to Theorem 2.2, we only need to prove

$$\begin{aligned}&\hat{\mathbb{E}}^{\hat{G}}\exp\left\{\left(\frac{1}{2}+\delta\right)\left(\int_0^{t_0}\left(|\bar{b}_1(X_t,Y_t)|^2\mathrm{d}\langle B'\rangle_t + |\bar{b}_2(X_t,Y_t)|^2\mathrm{d}\langle B\rangle_t + 2(\bar{b}_1\bar{b}_2)(X_t,Y_t)\mathrm{d}t\right)\right)\right\} \\ &\le \hat{\mathbb{E}}^{\hat{G}}\exp\left\{(1+2\delta)\left(\int_0^{t_0}\underline{\sigma}^{-2}|\bar{b}_1(X_t,Y_t)|^2\mathrm{d}t + \int_0^{t_0}\overline{\sigma}^2|\bar{b}_2(X_t,Y_t)|^2\mathrm{d}t\right)\right\} \\ &\le \bar{\mathbb{E}}^G\exp\left\{(1+2\delta)(\underline{\sigma}^{-2}+\overline{\sigma}^2)\int_0^{t_0}\left(|\bar{b}_1(X_t,Y_t)|^2 + |\bar{b}_2(X_t,Y_t)|^2\right)\mathrm{d}t\right\} < \infty\end{aligned}$$

for some $\delta, t_0 > 0$. Firstly, the Harnack inequality (4.4) implies

$$\mathbb{E}_0\mathrm{e}^{-\Phi_p(t,z,\cdot)}\left(P_t^0\mathrm{e}^{\frac{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}{p}}\right)^p(z) \le \mathbb{E}_0\mathrm{e}^{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}.$$

So for any $s\in(0,1)$ and $\lambda_s = \frac{\varepsilon}{ps}$, by Jensen's inequality and (4.5), we arrive at

$$\begin{aligned}&\bar{\mathbb{E}}^G\exp\left\{\lambda_s\int_0^s\left(|\bar{b}_1(X_t,Y_t)|^2+|\bar{b}_2(X_t,Y_t)|^2\right)\mathrm{d}t\right\} \\ &\le \frac{1}{s}\int_0^s\bar{\mathbb{E}}^G\exp\left\{\frac{\varepsilon}{p}\left(|\bar{b}_1(X_t,Y_t)|^2+|\bar{b}_2(X_t,Y_t)|^2\right)\right\}\mathrm{d}t \\ &= \frac{1}{s}\int_0^s P_t^0\mathrm{e}^{\frac{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}{p}}(z)\mathrm{d}t \\ &\le \frac{1}{s}\int_0^s\frac{\mathrm{d}t}{\{\mathbb{E}_0\mathrm{e}^{-\Phi_p(t,z,\cdot)}\}^{\frac{1}{p}}}\left(\mathbb{E}_0\mathrm{e}^{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}\right)^{\frac{1}{p}} < \infty, \quad z\in\mathbb{R}^2.\end{aligned}$$

Thus, taking t_0 satisfying $\frac{\varepsilon}{pt_0} > (\underline{\sigma}^{-2}+\overline{\sigma}^2)$ and $\delta = \frac{\frac{\varepsilon}{pt_0}}{2(\underline{\sigma}^{-2}+\overline{\sigma}^2)} - \frac{1}{2}$, the proof is completed. □

Next, we give an example in which (4.4) and (4.5) hold.

Example 4.2. *In* (4.1)*, let* $A = 0$*,* $M = Q = 1$*,* $b_2 = 0$ *and* $b_1(x, y) = -x - y$*. Then* (4.2) *reduces to*

$$\begin{cases} \mathrm{d}X_t = Y_t \mathrm{d}t, \\ \mathrm{d}Y_t = (-X_t - Y_t)\mathrm{d}t + \mathrm{d}B_t. \end{cases} \tag{4.7}$$

Firstly, by Theorem 3.2 (1), (4.4) *holds for*

$$\begin{aligned} \Phi_p(T, z, \bar{z}) &= \left[C\frac{p}{p-1} \left(\underline{\sigma}^{-2} T \left(\frac{1}{T} + \frac{1}{T^2} + 1 + T \right)^2 + \overline{\sigma}^2 T \left(1+T\right)^2 \right) \right] |z - \bar{z}|^2 \\ &= c\frac{|z-\bar{z}|^2}{T^3}, \quad T \in (0, 1) \end{aligned}$$

for some constant $c > 0$*.*

Next, by [7, Theorem 3.12], (4.7) *has a unique invariant nonlinear expectation* $\mathbb{E}_0$*. Let* $\theta^0_s = \underline{\sigma}, s \geq 0$ *and* $\mathbb{P}_{\theta^0}$ *be the corresponding probability as represented in* (2.4)*. Then*

$$\bar{\mathbb{E}}^G f(X_t, Y_t) \geq \mathbb{E}_{\mathbb{P}_{\theta^0}} f(X_t, Y_t), \quad f \in C^1_b(\mathbb{R}^2). \tag{4.8}$$

On the other hand, by [16, Theorem 3.1(1)], under the probability $\mathbb{P}_{\theta^0}$*,* (4.7) *has a unique invariant measure* μ_0*:*

$$\mu_0(\mathrm{d}x, \mathrm{d}y) = \frac{1}{2\pi\underline{\sigma}^2} \mathrm{e}^{-\frac{|x|^2+|y|^2}{2\underline{\sigma}^2}} \mathrm{d}x\mathrm{d}y.$$

By [7, Theorem 3.3, Theorem 3.12], letting t *go to infinity in* (4.8)*, we arrive at*

$$\mathbb{E}_0 f \geq \mu_0(f), \quad f \in C^1_b(\mathbb{R}^2).$$

Thus, according to [17, Example 4.3], we have

$$\mathbb{E}_0 \left(\mathrm{e}^{-\Phi_p(t,z,\cdot)} \right) \geq \mu_0 \left(\mathrm{e}^{-\Phi_p(t,z,\cdot)} \right) \geq \mathrm{e}^{-c} \mu_0(B(z, 1 \wedge t^{\frac{3}{2}})) \geq \alpha(z)(1 \wedge t)^{\frac{3}{2}}, \quad t > 0, z \in \mathbb{R}^2$$

for some $\alpha \in C(\mathbb{R}^2)$*. Thus,* (4.5) *holds for* $p > \frac{3}{2}$*.*

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