INITIAL BOUNDS FOR CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY HORADAM POLYNOMIALS

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ABSTRACT. Our present investigation is motivated essentially by the fact that, in Geometric Function Theory, one can find many interesting and fruitful usages of a wide variety of special functions and special polynomials. The main purpose of this article is to make use of the Horadam polynomials $h_n(x)$ and the generating function $\Pi(x,\ z)$, in order to introduce three new subclasses of the bi-univalent function class Σ. For functions belonging to the defined classes, we then derive coefficient inequalities and the FeketeSzegö inequalities. Some interesting observations of the results presented here are also discussed. We also provide relevant connections of our results with those considered in earlier investigations.

keywords and Phrases : Univalent functions, bi-univalent functions, bi-Mocanu-convex functions, bi-$\alpha-$starlike functions, bi-starlike functions, bi-convex functions, Fekete-Szegö problem, Chebyshev polynomials, Horadam polynomials.

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both a function f and it's inverse f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.1).

In 2010, Srivastava et al. [20] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in the recent investigations (see, for example, [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the work of Srivastava et al. [20]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \dots)$ for functions $f \in \Sigma$ is still an open problem.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)) \qquad (z \in \Delta).$$

This subordination will be denoted here by

$$f \prec g \qquad (z \in \Delta)$$

or, conventionally, by

$$f(z) \prec g(z) \qquad (z \in \Delta).$$

In particular, when g is univalent in Δ,

$$f \prec g \qquad (z \in \Delta) \ \Leftrightarrow\ f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The Horadam polynomials $h_n(x,\ a,\ b;\ p,\ q)$, or briefly $h_n(x)$ are given by the following recurrence relation (see [7, 8])):

$$h_n(x) \ = \ pxh_{n-1}(x) + qh_{n-2}(x) \qquad (n \in \mathbb{N}) \tag{1.2}$$

with

$$h_1(x) = a \qquad \text{and} \qquad h_2(x) = bx \tag{1.3}$$

for some real constants a, b, p and q.

The generating function of the Horadam polynomials $h_n(x)$ (see [8]) is given by

$$\Pi(x,\ z) := \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}\ . \tag{1.4}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \Re(z)$.

Note that for particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list few cases here (see, [7, 8] for more details):

(1) For $a = b = p = q = 1$, we have the Fibonacci polynomials $F_n(x)$.
(2) For $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(x)$.
(3) For $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(x)$.

(4) For $a = b = p = 2$ and $q = 1$, we attain the Pell-Lucas polynomials $Q_n(x)$.
(5) For $a = b = 1,\ p = 2$ and $q = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind
(6) For $a = 1,\ b = p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $U_n(x)$ of the second kind.

Recently, in literature, the coefficient estimates are found for functions in the class of univalent and bi-univalent functions associated with certain polynomials like the Faber polynomial [9], the Chebyshev polynomials [5], the Horadam polynomial [16]. Motivated in these lines, estimates on initial coefficients of the Taylor-Maclaurin series expansion (1.1) and Fekete-Szegö inequalities for certain classes of bi-univalent functions defined by means of Horadam polynomials are obtained. The classes introduced in this paper are motivated by the corresponding classes investigated in [1, 11, 16, 15].

2. Coefficient Estimates and Fekete-Szegö Inequalities

A function $f \in \Sigma$ of the form (1.1) belongs to the class $\mathcal{S}^*_\Sigma(\alpha,\ x)$, $\alpha \geq 0$ and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)} \prec \Pi(x,\ z) + 1 - a$$

and for $g(w) = f^{-1}(w)$

$$\frac{wg'(w)}{g(w)} + \alpha\frac{w^2g''(w)}{g(w)} \prec \Pi(x,\ w) + 1 - a,$$

where the real constants a and b are as in (1.3).

Note that $S^*_\Sigma(x) \equiv \mathcal{S}^*_\Sigma(0,\ x)$ was introduced and studied by Srivastava et al. [16].

Remark 2.1. If $a = p = x = 1$, $b = 2$ and $q = 0$, then we have

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)} \prec \frac{1+z}{1-z} \qquad (z \in \Delta).$$

In this case, the function f maps the open unit disk Δ onto the half-plane given by $\Re\left(\frac{+z}{-z}\right) > 0$, since the expression $\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)}$ takes on values in the half-plane. If, on the other hand, we restrict our considerations for a given univalent function $p(z) \in \Delta$, we can investigate the corresponding mapping problems for other regions of the complex $z-$plane instead of the half-plane $\Re(z) > 0$. In this way, one can introduce many other subclasses of the function class $\mathcal{S}^*_\Sigma(\alpha,\ x)$.

Remark 2.2. When $a = 1$, $b = p = 2$, $q = -1$ and $x \to t$, the generating function in (1.4) reduces to that of the Chebyshev polynomial $U_n(t)$ of the second kind, which is given explicitly by

$$U_n(t) = (n+1)_2F_1\left(-n,\ n+2;\ \tfrac{3}{2};\ \tfrac{1-t}{2}\right) = \frac{\sin(n+1)\varphi}{\sin\varphi}, \qquad (t = \cos\varphi)$$

in terms of the hypergeometric function ${}_2F_1$.

In view of Remark 2.2, the bi-univalent function class $\mathcal{S}^*_\Sigma(x)$ reduces to $\mathcal{S}^*_\Sigma(t)$ and this class was studied earlier in [2, 13]. For functions in the class $\mathcal{S}^*_\Sigma(\alpha,\ x)$, the following coefficient estimates and Fekete-Szegö inequality are obtained.

Theorem 2.1. *Let* $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ *be in the class* $\mathcal{S}^*_{\Sigma}(\alpha,\ x)$. *Then*

$$|a_2|\leq\frac{|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(1+4\alpha)\,b-p(1+2\alpha)^2\right]bx^2-qa(1+2\alpha)^2\right|}},\qquad \text{and}\qquad |a_3|\leq\frac{|bx|}{2+6\alpha}+\frac{b^2x^2}{(1+2\alpha)^2}$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\dfrac{|bx|}{2+6\alpha} \\ \qquad \text{if}\qquad |\nu-1|\leq\dfrac{\left|\left[(1+4\alpha)\,b-p(1+2\alpha)^2\right]bx^2-qa(1+2\alpha)^2\right|}{2b^2x^2\,(1+3\alpha)} \\[2ex] \dfrac{|bx|^3\,|\nu-1|}{\left|\left[(1+4\alpha)\,b-p(1+2\alpha)^2\right]bx^2-qa(1+2\alpha)^2\right|} \\ \qquad \text{if}\qquad |\nu-1|\geq\dfrac{\left|\left[(1+4\alpha)\,b-p(1+2\alpha)^2\right]bx^2-qa(1+2\alpha)^2\right|}{2b^2x^2\,(1+3\alpha)}.\end{cases}$$

Proof. Let $f\in\mathcal{S}^*_{\Sigma}(\alpha,\ x)$ be given by Taylor-Maclaurin expansion (1.1). Then, there are analytic functions u and v such that

$$u(0)=0;\quad v(0)=0,\quad |u(z)|<1\quad\text{and}\quad |v(z)|<1\quad(\forall\ z,\ w\in\Delta),$$

we can write

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}=\Pi(x,\ u(z))+1-a \tag{2.1}$$

and

$$\frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}=\Pi(x,\ v(w))+1-a. \tag{2.2}$$

Or, equivalently,

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}=1+h_1(x)-a+h_2(x)u(z)+h_3(x)[u(z)]^2+\ldots \tag{2.3}$$

and

$$\frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}=1+h_1(x)-a+h_2(x)v(w)+h_3(x)[v(w)]^2+\ldots. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}=1+h_2(x)u_1z+[h_2(x)u_2+h_3(x)u_1^2]z^2+\ldots \tag{2.5}$$

and

$$\frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}=1+h_2(x)v_1w+[h_2(x)v_2+h_3(x)v_1^2]w^2+\ldots. \tag{2.6}$$

It is fairly well known that

$$|u(z)|=\left|u_1z+u_2z^2+\ldots\right|<1\qquad\text{and}\qquad |v(z)|=\left|v_1w+v_2w^2+\ldots\right|<1,$$

then

$$|u_k|\leq 1\qquad\text{and}\qquad |v_k|\leq 1\qquad(k\in\mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.5) and (2.6), we have

$$(1+2\alpha)\,a_2 = h_2(x)u_1 \tag{2.7}$$

$$2\,(1+3\alpha)\,a_3 - (1+2\alpha)\,a_2^2 = h_2(x)u_2 + h_3(x)u_1^2 \tag{2.8}$$

$$-\,(1+2\alpha)\,a_2 = h_2(x)v_1 \tag{2.9}$$

and

$$(3+10\alpha)\,a_2^2 - 2\,(1+3\alpha)\,a_3 = h_2(x)v_2 + h_3(x)v_1^2. \tag{2.10}$$

From (2.7) and (2.9), we can easily see that

$$u_1 = -v_1 \tag{2.11}$$

and

$$\begin{aligned} 2(1+2\alpha)^2 a_2^2 &= [h_2(x)]^2(u_1^2+v_1^2) \\ a_2^2 &= \frac{[h_2(x)]^2(u_1^2+v_1^2)}{2(1+2\alpha)^2}\ . \end{aligned} \tag{2.12}$$

If we add (2.8) to (2.10), we get

$$2\,(1+4\alpha)\,a_2^2 = h_2(x)(u_2+v_2) + h_3(x)(u_1^2+v_1^2). \tag{2.13}$$

By substituting (2.12) in (2.13), we reduce that

$$a_2^2 = \frac{[h_2(x)]^3\,(u_2+v_2)}{2\,(1+4\alpha)\,[h_2(x)]^2 - 2h_3(x)(1+2\alpha)^2} \tag{2.14}$$

which yields

$$|a_2| \le \frac{|bx|\,\sqrt{|bx|}}{\sqrt{|[(1+4\alpha)\,b - p(1+2\alpha)^2]bx^2 - qa(1+2\alpha)^2|}}. \tag{2.15}$$

By subtracting (2.10) from (2.8) and in view of (2.11) , we obtain

$$\begin{aligned} 4(1+3\alpha)a_3 - 4(1+3\alpha)a_2^2 &= h_2(x)\,(u_2-v_2) + h_3(x)\left(u_1^2 - v_1^2\right) \\ a_3 &= \frac{h_2(x)\,(u_2-v_2)}{4(1+3\alpha)} + a_2^2. \end{aligned} \tag{2.16}$$

Then in view of (2.12), (2.16) becomes

$$a_3 = \frac{h_2(x)\,(u_2-v_2)}{4(1+3\alpha)} + \frac{[h_2(x)]^2(u_1^2+v_1^2)}{2(1+2\alpha)^2}.$$

Applying (1.3), we deduce that

$$|a_3| \le \frac{|bx|}{2+6\alpha} + \frac{b^2x^2}{(1+2\alpha)^2}.$$

From (2.16), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{h_2(x)\,(u_2-v_2)}{4(1+3\alpha)} + (1-\nu)\,a_2^2. \tag{2.17}$$

By substituting (2.14) in (2.53), we have

$$\begin{aligned} a_3-\nu a_2^2 &= \frac{h_2(x)\,(u_2-v_2)}{4(1+3\alpha)}+\left(\frac{(1-\nu)\,[h_2(x)]^3\,(u_2+v_2)}{2[(1+4\alpha)\,[h_2(x)]^2-h_3(x)(1+2\alpha)^2]}\right)\\ &= h_2(x)\left\{\left(\Omega(\nu,\ x)+\frac{1}{4\,(1+3\alpha)}\right)u_2+\left(\Omega(\nu,\ x)-\frac{1}{4\,(1+3\alpha)}\right)v_2\right\}, \end{aligned} \tag{2.18}$$

where

$$\Omega(\nu,\ x)=\frac{(1-\nu)\,[h_2(x)]^2}{2\,(1+4\alpha)\,[h_2(x)]^2-2h_3(x)(1+2\alpha)^2}.$$

Hence, in view of (1.3), we conclude that

$$\left|a_3-\nu a_2^2\right|\le\begin{cases}\dfrac{|h_2(x)|}{2+6\alpha} & ;0\le|\Omega(\nu,\ x)|\le\dfrac{1}{4\,(1+3\alpha)}\\[2ex] 2\,|h_2(x)|\,|\Omega(\nu,\ x)| & ;|\Omega(\nu,\ x)|\ge\dfrac{1}{4\,(1+3\alpha)}\end{cases}$$

which evidently completes the proof of Theorem 2.2. □

For $\alpha=0$, Theorem 2.1 readily yields the following coefficient estimates for $\mathcal{S}_\Sigma^*(x)$.

Corollary 2.1. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{S}_\Sigma^*(x)$. Then*

$$|a_2|\le\frac{|bx|\sqrt{|bx|}}{\sqrt{|[b-p]bx^2-qa|}},\qquad \textit{and}\qquad |a_3|\le\frac{|bx|}{2}+b^2x^2$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\le\begin{cases}\dfrac{|bx|}{2} & \textit{if } |\nu-1|\le\dfrac{|[b-p]bx^2-qa|}{2b^2x^2}\\[2ex] \dfrac{|bx|^3\,|\nu-1|}{|[b-p]bx^2-qa|} & \textit{if } |\nu-1|\ge\dfrac{|[b-p]bx^2-qa|}{2b^2x^2}.\end{cases}$$

In view of Remark 2.2, Theorem 2.1 can be shown to yield the following result.

Corollary 2.2. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{S}_\Sigma^*(\alpha,\ t)$. Then*

$$|a_2|\le\frac{2t\sqrt{2t}}{\sqrt{|(1+2\alpha)^2-16\alpha^2t^2|}},\qquad \textit{and}\qquad |a_3|\le\frac{t}{1+3\alpha}+\frac{4t^2}{(1+2\alpha)^2}$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\le\begin{cases}\dfrac{2t}{2+6\alpha} & \textit{if } |\nu-1|\le\dfrac{|(1+2\alpha)^2-16\alpha^2t^2|}{8t^2\,(1+3\alpha)}\\[2ex] \dfrac{8t^3\,|\nu-1|}{|(1+2\alpha)^2-16\alpha^2t^2|} & \textit{if } |\nu-1|\ge\dfrac{|(1+2\alpha)^2-16\alpha^2t^2|}{8t^2\,(1+3\alpha)}.\end{cases}$$

Remark 2.3. Results obtained in Corollary 2.1 coincide with results obtained in [16]. For $\alpha = 0$, Corollary 2.2 reduces to the results discussed in [2, 13].

Next, a function $f \in \Sigma$ of the form (1.1) belongs to the class $\mathcal{M}_\Sigma(\alpha,\ x)$, $0 \leqq \alpha \leqq 1$ and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Pi(x,\ z) + 1 - a$$

and for $g(w) = f^{-1}(w)$

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \Pi(x,\ w) + 1 - a,$$

where the real constants a and b are as in (1.3).

Note that the class $\mathcal{M}_\Sigma(\alpha,\ x)$, unifies the classes $S^*_\Sigma(x)$ and $K_\Sigma(x)$ like $\mathcal{M}_\Sigma(0,\ x) \equiv S^*_\Sigma(x)$ and $\mathcal{M}_\Sigma(1,\ x) \equiv K_\Sigma(x)$. In view of Remark 2.1, similarly, one can define many subclasses for the expression $(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$. In view of Remark 2.2, the bi-univalent function classes $\mathcal{M}^*_\Sigma(\alpha,\ x)$ would become the class $\mathcal{M}^*_\Sigma(\alpha,\ t)$ and the class $\mathcal{M}^*_\Sigma(\alpha,\ t)$ introduced and studied by Altınkaya and Yalçin [3]. For functions in the class $\mathcal{M}_\Sigma(\alpha,\ x)$, the following coefficient estimates and Fekete-Szegö inequality are obtained.

Theorem 2.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{M}_\Sigma(\alpha,\ x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(1+\alpha)\,b - p(1+\alpha)^2]bx^2 - qa(1+\alpha)^2|}}, \qquad \text{and} \qquad |a_3| \leq \frac{|bx|}{2+4\alpha} + \frac{b^2x^2}{(1+\alpha)^2}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{2+4\alpha} \\ \qquad \text{if} \qquad |\nu - 1| \leq \dfrac{|[(1+\alpha)\,b - p(1+\alpha)^2]bx^2 - qa(1+\alpha)^2|}{b^2x^2\,(2+4\alpha)} \\ \\ \dfrac{|bx|^3\,|\nu-1|}{|[(1+\alpha)\,b - p(1+\alpha)^2]bx^2 - qa(1+\alpha)^2|} \\ \qquad \text{if} \qquad |\nu - 1| \geq \dfrac{|[(1+\alpha)\,b - p(1+\alpha)^2]bx^2 - qa(1+\alpha)^2|}{b^2x^2\,(2+4\alpha)}. \end{cases}$$

Proof. Let $f \in \mathcal{M}_\Sigma(\alpha,\ x)$ be given by Taylor-Maclaurin expansion (1.1). Then, there are analytic functions u and v such that

$$u(0) = 0; \quad v(0) = 0, \quad |u(z)| < 1 \quad \text{and} \quad |v(z)| < 1 \quad (\forall\ z,\ w \in \Delta),$$

we can write

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \Pi(x,\ u(z)) + 1 - a \tag{2.19}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \Pi(x,\ v(w)) + 1 - a. \tag{2.20}$$

Or, equivalently,

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$=1+h_1(x)-a+h_2(x)u(z)+h_3(x)[u(z)]^2+\ldots \tag{2.21}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)$$
$$=1+h_1(x)-a+h_2(x)v(w)+h_3(x)[v(w)]^2+\ldots. \tag{2.22}$$

From (2.21) and (2.22), we obtain

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$=1+h_2(x)u_1z+[h_2(x)u_2+h_3(x)u_1^2]z^2+\ldots \tag{2.23}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)$$
$$=1+h_2(x)v_1w+[h_2(x)v_2+h_3(x)v_1^2]w^2+\ldots. \tag{2.24}$$

It is fairly well known that

$$|u(z)|=\left|u_1z+u_2z^2+\ldots\right|<1 \qquad \text{and} \qquad |v(z)|=\left|v_1w+v_2w^2+\ldots\right|<1,$$

then

$$|u_k|\leq 1 \qquad \text{and} \qquad |v_k|\leq 1 \qquad (k\in\mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.23) and (2.24), we have

$$(1+\alpha)\,a_2=h_2(x)u_1 \tag{2.25}$$

$$2\,(1+2\alpha)\,a_3-(1+3\alpha)\,a_2^2=h_2(x)u_2+h_3(x)u_1^2 \tag{2.26}$$

$$-\,(1+\alpha)\,a_2=h_2(x)v_1 \tag{2.27}$$

and

$$(3+5\alpha)\,a_2^2-2\,(1+2\alpha)\,a_3=h_2(x)v_2+h_3(x)v_1^2. \tag{2.28}$$

From (2.25) and (2.27), we can easily see that

$$u_1=-v_1 \tag{2.29}$$

and

$$\begin{aligned} 2(1+\alpha)^2a_2^2 &= [h_2(x)]^2(u_1^2+v_1^2)\\ a_2^2 &= \frac{[h_2(x)]^2(u_1^2+v_1^2)}{2(1+\alpha)^2}\ . \end{aligned} \tag{2.30}$$

If we add (2.26) to (2.28), we get

$$2\,(1+\alpha)\,a_2^2=h_2(x)(u_2+v_2)+h_3(x)(u_1^2+v_1^2). \tag{2.31}$$

By substituting (2.30) in (2.31), we reduce that

$$a_2^2 = \frac{[h_2(x)]^3\,(u_2+v_2)}{2\,(1+\alpha)\,[h_2(x)]^2-2h_3(x)(1+\alpha)^2} \tag{2.32}$$

which yields

$$|a_2| \quad \leq \quad \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(1+\alpha)\,b - p(1+\alpha)^2]bx^2 - qa(1+\alpha)^2|}}. \tag{2.33}$$

By subtracting (2.28) from (2.26) and in view of (2.29) , we obtain

$$\begin{aligned} 4(1+2\alpha)a_3 - 4(1+2\alpha)a_2^2 &= h_2(x)\,(u_2 - v_2) + h_3(x)\left(u_1^2 - v_1^2\right) \\ a_3 &= \frac{h_2(x)\,(u_2 - v_2)}{4(1+2\alpha)} + a_2^2. \end{aligned} \tag{2.34}$$

Then in view of (2.30), (2.34) becomes

$$a_3 \quad = \quad \frac{h_2(x)\,(u_2 - v_2)}{4(1+2\alpha)} + \frac{[h_2(x)]^2(u_1^2 + v_1^2)}{2(1+\alpha)^2}.$$

Applying (1.3), we deduce that

$$|a_3| \quad \leq \quad \frac{|bx|}{2+4\alpha} + \frac{b^2x^2}{(1+\alpha)^2}.$$

From (2.34), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{h_2(x)\,(u_2 - v_2)}{4(1+2\alpha)} + (1-\nu)\,a_2^2. \tag{2.35}$$

By substituting (2.32) in (2.53), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{h_2(x)\,(u_2 - v_2)}{4(1+2\alpha)} + \left(\frac{(1-\nu)\,[h_2(x)]^3\,(u_2 + v_2)}{2\,(1+\alpha)\,[h_2(x)]^2 - 2h_3(x)(1+\alpha)^2}\right) \\ &= h_2(x)\left\{\left(\Omega(\nu,\ x) + \frac{1}{4\,(1+2\alpha)}\right)u_2 + \left(\Omega(\nu,\ x) - \frac{1}{4\,(1+2\alpha)}\right)v_2\right\}, \end{aligned} \tag{2.36}$$

where

$$\Omega(\nu,\ x) = \frac{(1-\nu)\,[h_2(x)]^2}{2\,(1+\alpha)\,[h_2(x)]^2 - 2h_3(x)(1+\alpha)^2}.$$

Hence, in view of (1.3), we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|h_2(x)|}{2+4\alpha} & ; 0 \leq |\Omega(\nu,\ x)| \leq \dfrac{1}{4\,(1+2\alpha)} \\ \\ 2\,|h_2(x)|\,|\Omega(\nu,\ x)| & ; |\Omega(\nu,\ x)| \geq \dfrac{1}{4\,(1+2\alpha)} \end{cases}$$

which evidently completes the proof of Theorem 2.2. □

For $\alpha = 1$, Theorem 2.2 readily yields the following coefficient estimates for $\mathcal{K}_\Sigma(x)$.

Corollary 2.3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{K}_\Sigma(x)$. *Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[2b-4p]bx^2 - 4qa|}}, \qquad \text{and} \qquad |a_3| \leq \frac{|bx|}{6} + \frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{6} & \text{if } |\nu - 1| \leq \dfrac{|[2b-4p]bx^2 - 4qa|}{6b^2x^2} \\ \dfrac{|bx|^3 |\nu - 1|}{|[2b-4p]bx^2 - 4qa|} & \text{if } |\nu - 1| \geq \dfrac{|[2b-4p]bx^2 - 4qa|}{6b^2x^2}. \end{cases}$$

In view of Remark 2.2, Theorem 2.2 can be shown to yield the following result.

Corollary 2.4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{M}_\Sigma(\alpha,\ t)$. *Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(1+\alpha)^2 - 4\alpha(1+\alpha)t^2|}}, \qquad \text{and} \qquad |a_3| \leq \frac{t}{1+2\alpha} + \frac{4t^2}{(1+\alpha)^2}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{t}{1+2\alpha} & \text{if } |\nu - 1| \leq \dfrac{|(1+\alpha)^2 - 4\alpha(1+\alpha)t^2|}{8t^2\,(1+2\alpha)} \\ \dfrac{8t^3 |\nu - 1|}{|(1+\alpha)^2 - 4\alpha(1+\alpha)t^2|} & \text{if } |\nu - 1| \geq \dfrac{|(1+\alpha)^2 - 4\alpha(1+\alpha)t^2|}{8t^2\,(1+2\alpha)}. \end{cases}$$

In view of Remark 2.2, Theorem 2.3 can be shown to yield the following result.

Corollary 2.5. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{K}_\Sigma(t)$. *Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-2t^2|}}, \qquad \text{and} \qquad |a_3| \leq \frac{t}{3} + t^2$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{t}{3} & \text{if } |\nu - 1| \leq \dfrac{|1-2t^2|}{6t^2} \\ \dfrac{2t^3 |\nu - 1|}{|1-2t^2|} & \text{if } |\nu - 1| \geq \dfrac{|1-2t^2|}{6t^2}. \end{cases}$$

Remark 2.4. The results obtained in Corollary 2.4 and Corollary 2.5 are coincide with results of Altınkaya and Yalçin [3].

Next, a function $f \in \Sigma$ of the form (1.1) belongs to the class $\mathcal{L}_\Sigma(\alpha,\ x)$, $0 \leqq \lambda \leqq 1$, and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \Pi(x,\ z) + 1 - a$$

and for $g(w) = f^{-1}(w)$

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha} \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\alpha} \prec \Pi(x,\ w) + 1 - a,$$

where the real constants a and b are as in (1.3).

This class also reduces to $\mathcal{S}^*_\Sigma(x)$ and $\mathcal{K}_\Sigma(x)$. Further, as we have discussed in Remark 2.1, we can define many subclasses for the expression $\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$. In view of Remark 2.2, the bi-univalent function class $\mathcal{L}^*_\Sigma(\alpha,\ x)$ would become the class $\mathcal{L}^*_\Sigma(\alpha,\ t)$. For functions in the class $\mathcal{L}_\Sigma(\alpha,\ x)$, the following coefficient estimates are obtained.

Theorem 2.3. *Let $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{L}_\Sigma(\alpha,\ x)$. Then*

$$|a_2|\leq\frac{|bx|\sqrt{2\,|bx|}}{\sqrt{|[(\alpha^2-3\alpha+4)\,b-2p(2-\alpha)^2]bx^2-2qa(2-\alpha)^2|}}\quad \textit{and}\quad |a_3|\leq\frac{|bx|}{6-4\alpha}+\frac{b^2x^2}{(2-\alpha)^2}$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\dfrac{|bx|}{6-4\alpha} \\ \qquad \textit{if}\qquad |\nu-1|\leq\dfrac{|[(\alpha^2-3\alpha+4)\,b-2p(2-\alpha)^2]bx^2-2qa(2-\alpha)^2|}{4b^2x^2\,(3-2\alpha)} \\ \\ \dfrac{2\,|bx|^3\,|\nu-1|}{|[(\alpha^2-3\alpha+4)\,b-2p(2-\alpha)^2]bx^2-2qa(2-\alpha)^2|} \\ \qquad \textit{if}\qquad |\nu-1|\geq\dfrac{|[(\alpha^2-3\alpha+4)\,b-2p(2-\alpha)^2]bx^2-2qa(2-\alpha)^2|}{4b^2x^2\,(3-2\alpha)}.\end{cases}$$

Proof. Let $f\in\mathcal{L}_\Sigma(\alpha,\ x)$ be given by Taylor-Maclaurin expansion (1.1). Then, there are analytic functions u and v such that

$$u(0)=0;\quad v(0)=0,\quad |u(z)|<1\quad\text{and}\quad |v(z)|<1\quad(\forall\ z,\ w\in\Delta),$$

we can write

$$\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}=\Pi(x,\ u(z))+1-a \tag{2.37}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}=\Pi(x,\ v(w))+1-a. \tag{2.38}$$

Or, equivalently,

$$\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$$
$$=1+h_1(x)-a+h_2(x)u(z)+h_3(x)[u(z)]^2+\dots \tag{2.39}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}$$
$$=1+h_1(x)-a+h_2(x)v(w)+h_3(x)[v(w)]^2+\dots. \tag{2.40}$$

From (2.39) and (2.40), we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}=1+h_2(x)u_1z+[h_2(x)u_2+h_3(x)u_1^2]z^2+\dots \tag{2.41}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}=1+h_2(x)v_1w+[h_2(x)v_2+h_3(x)v_1^2]w^2+\ldots. \quad (2.42)$$

It is fairly well known that

$$|u(z)|=\left|u_1z+u_2z^2+\ldots\right|<1 \qquad \text{and} \qquad |v(z)|=\left|v_1w+v_2w^2+\ldots\right|<1,$$

then

$$|u_k|\leq 1 \qquad \text{and} \qquad |v_k|\leq 1 \qquad (k\in\mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.41) and (2.42), we have

$$(2-\alpha)\,a_2=h_2(x)u_1 \quad (2.43)$$

$$2\,(3-2\alpha)\,a_3+\left[(\alpha-2)^2-3\,(4-3\alpha)\right]\frac{a_2^2}{2}=h_2(x)u_2+h_3(x)u_1^2 \quad (2.44)$$

$$-\,(2-\alpha)\,a_2=h_2(x)v_1 \quad (2.45)$$

and

$$\left[8\,(1-\alpha)+\frac{\alpha}{2}\,(\alpha+5)\right]a_2^2-2\,(3-2\alpha)\,a_3=h_2(x)v_2+h_3(x)v_1^2. \quad (2.46)$$

From (2.43) and (2.45), we can easily see that

$$u_1=-v_1 \quad (2.47)$$

and

$$\begin{aligned}2(2-\alpha)^2a_2^2 &= [h_2(x)]^2(u_1^2+v_1^2)\\ a_2^2 &= \frac{[h_2(x)]^2(u_1^2+v_1^2)}{2(2-\alpha)^2}\,.\end{aligned} \quad (2.48)$$

If we add (2.44) to (2.46), we get

$$\left(\alpha^2-3\alpha+4\right)a_2^2=h_2(x)(u_2+v_2)+h_3(x)(u_1^2+v_1^2). \quad (2.49)$$

By substituting (2.48) in (2.49), we reduce that

$$a_2^2 = \frac{[h_2(x)]^3\,(u_2+v_2)}{(\alpha^2-3\alpha+4)\,[h_2(x)]^2-2h_3(x)(2-\alpha)^2} \quad (2.50)$$

which yields

$$|a_2| \leq \frac{|bx|\,\sqrt{2\,|bx|}}{\sqrt{|[(\alpha^2-3\alpha+4)\,b-2p(2-\alpha)^2]bx^2-2qa(2-\alpha)^2|}}. \quad (2.51)$$

By subtracting (2.46) from (2.44) and in view of (2.47) , we obtain

$$\begin{aligned}4(3-2\alpha)a_3-4(3-2\alpha)a_2^2 &= h_2(x)\,(u_2-v_2)+h_3(x)\left(u_1^2-v_1^2\right)\\ a_3 &= \frac{h_2(x)\,(u_2-v_2)}{4(3-2\alpha)}+a_2^2.\end{aligned} \quad (2.52)$$

Then in view of (2.48), (2.52) becomes

$$a_3 = \frac{h_2(x)\,(u_2-v_2)}{4(3-2\alpha)}+\frac{[h_2(x)]^2(u_1^2+v_1^2)}{2(2-\alpha)^2}.$$

Applying (1.3), we deduce that

$$|a_3| \leq \frac{|bx|}{6-4\alpha} + \frac{b^2x^2}{(2-\alpha)^2}.$$

From (2.52), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{h_2(x)\,(u_2 - v_2)}{4(3-2\alpha)} + (1-\nu)\,a_2^2. \tag{2.53}$$

By substituting (2.50) in (2.53), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{h_2(x)\,(u_2 - v_2)}{4(3-2\alpha)} + \left(\frac{(1-\nu)\,[h_2(x)]^3\,(u_2+v_2)}{(\alpha^2-3\alpha+4)\,[h_2(x)]^2 - 2h_3(x)(2-\alpha)^2}\right) \\ &= h_2(x)\left\{\left(\Omega(\nu,\ x) + \frac{1}{4\,(3-2\alpha)}\right)u_2 + \left(\Omega(\nu,\ x) - \frac{1}{4\,(3-2\alpha)}\right)v_2\right\}, \end{aligned} \tag{2.54}$$

where

$$\Omega(\nu,\ x) = \frac{(1-\nu)\,[h_2(x)]^2}{(\alpha^2-3\alpha+4)\,[h_2(x)]^2 - 2h_3(x)(2-\alpha)^2}.$$

Hence, in view of (1.3), we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|h_2(x)|}{6-4\alpha} & ;0 \leq |\Omega(\nu,\ x)| \leq \dfrac{1}{4\,(3-2\alpha)} \\ 2\,|h_2(x)|\,|\Omega(\nu,\ x)| & ;|\Omega(\nu,\ x)| \geq \dfrac{1}{4\,(3-2\alpha)} \end{cases}$$

which evidently completes the proof of Theorem 2.2. □

In view of Remark 2.2, Theorem 2.3 can be shown to yield

Corollary 2.6. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{L}_\Sigma(\alpha,\ t)$*. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(2-\alpha)^2 - (\alpha^2-5\alpha+4)\,t^2|}} \quad \text{and} \quad |a_3| \leq \frac{t}{3-2\alpha} + \frac{4t^2}{(2-\alpha)^2}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{t}{3-2\alpha} & \text{if } |\nu - 1| \leq \dfrac{|(2-\alpha)^2 - (\alpha^2-5\alpha+4)\,t^2|}{8t^2\,(3-2\alpha)} \\ \dfrac{8t^3\,|\nu-1|}{|(2-\alpha)^2 - (\alpha^2-5\alpha+4)\,t^2|} & \text{if } |\nu - 1| \geq \dfrac{|(2-\alpha)^2 - (\alpha^2-5\alpha+4)\,t^2|}{8t^2\,(3-2\alpha)}. \end{cases}$$

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