# SLICE REGULAR FUNCTIONS OF SEVERAL OCTONIONIC VARIABLES.

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Abstract. Octonionic analysis is becoming eminent due to the role of octonions in the theory of  $G_2$  manifold. In this article, a new slice theory is introduced as a generalization of the holomorphic theory of several complex variables to the noncommutative or nonassociative realm. The Bochner-Martinelli formula is established for slice functions of several octonionic variables as well as several quaternionic variables. In this setting, we find the Hartogs phenomena for slice regular functions.

#### 1. INTRODUCTION

Recently, octonionic analysis has been put in the spotlight as the development of the theory of  $G_2$  manifold since  $G_2$  is the automorphism group of the octonion algebra (see [\[22,](#page-17-0)[33\]](#page-18-0)). In this article, we try to establish a new theory related to the octonionic analysis. That is the slice octonionic analysis which generalize the lower dimensional theory to higher dimensional. It reminds us that the slice technique may be helpful in the study of the big problem of Ising model in higher dimensions. Notice that up to now we only know that 2D Ising model is exactly solvable or integrable. For Ising model and the related discrete analysis, we refer to [\[24,](#page-17-1)[29,](#page-18-1)[30\]](#page-18-2).

Now we move on to recall the classical theory on slice analysis. The theory of slice regular functions of one quaternionc variable, initiated by Gentili and Struppa [\[9,](#page-17-2) [10\]](#page-17-3), provides an effective approach to generalize the beautiful theory of holomorphic functions of one complex variable to the non-commutative or even non-associative realm. It turns out to have potential applications in the quantum theory since it demonstrates that the self-adjoint operators of quaternions admits real S-spectrum [\[16\]](#page-17-4). Colombo, Sabadini, and Struppa later extend the theory to the Clifford algebras [\[2,](#page-17-5) [8\]](#page-17-6), and Gentilli and Struppa [\[14\]](#page-17-7) to the octonions. A further extension to real alternative algebras was introduced by Ghiloni and Petrotti in [\[17\]](#page-17-8), and later in [\[21\]](#page-17-9).

The root of this theory lies in its effective approach to construct new functions from the stem function which need not be holomorphic (see [\[18\]](#page-17-10)). This approach goes back to the well-known Futer contruction (see [\[23,](#page-17-11) [31,](#page-18-3) [32\]](#page-18-4)).

Now we recall this construction in more detail (see [\[28\]](#page-18-5)). We consider the holomorphic function  $F$ , defined in a domain D of the complex plan  $\mathbb C$  invariant under the complex conjugate, with values in the complexification of an alternative algebra

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A over R. Then it admits a unique slice regular extension  $f : \Omega_D \to \mathbb{A}$  with

$$
\Omega_D := \{ \alpha + \beta J \mid \alpha + i\beta \in D, \alpha, \beta \in \mathbb{R}, J \in \mathbb{S}_A \} \subseteq \mathcal{Q}_\mathbb{A}
$$

such that the following diagram commutes for every  $J \in \mathbb{S}_{\mathbb{A}}$ :



The construction above depends heavily on the so-called slice complex nature of  $\mathcal{Q}_A$ , the quadratic cone of  $\mathbb{A}$ , i.e.,

$$
\mathcal{Q}_{\mathbb{A}}=\bigcup_{J\in\mathbb{S}_{\mathbb{A}}}\mathbb{C}_{J},
$$

and

$$
\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}
$$

for all  $I, J \in \mathbb{S}_\mathbb{A}$  with  $I \neq \pm J$ . Here  $\mathbb{S}_\mathbb{A}$  denotes the set of square roots of −1 in the algebra A, i.e.,

$$
\mathbb{S}_{\mathbb{A}} := \{ J \in \mathcal{Q}_{\mathbb{A}} \mid J^2 = -1 \},
$$

and the two associated maps are defined respectively by

$$
\Phi_J(a+ib) = a + Jb, \qquad \forall \ a, b \in \mathbb{R},
$$
  

$$
\tilde{\Phi}_J(\alpha + i\beta) = \alpha + J\beta, \qquad \forall \ \alpha, \beta \in \mathbb{A}.
$$

The preceding approach results in the slice theory which provides an effective generalization of the theory of one complex variable to the setting of quaternions or even more general algebras.

It is quite natural to do such an extension so that to generalize the theory of several complex variables to the setting of non-commutative or non-associative realm.

The first attempt was given by Ghiloni and Perotti [\[20\]](#page-17-12). They introduced the class of slice regular functions of several Clifford variables. The definition of the slice functions is based on the concept of stem functions of several variables.

To compare their theory with ours, we need to recall their construction in some details.

Let  $\mathbb{R}_n$  denote the real Clifford algebra of signature  $(0, n)$  generated by  $e_1, \dots, e_n$ . Its elements can be expressed as

$$
x = \sum_{K \in \mathcal{P}(n)} x_K e_K,
$$

where the coefficients  $x_K \in \mathbb{R}$ , the products  $e_K := e_{k_1} e_{k_2} \cdots e_{k_r}$  are the basis elements of the Clifford algebra  $\mathbb{R}_n$ , and the sum runs over the set

$$
\mathcal{P}(n) = \{(k_1, \dots, k_r) \in \mathbb{N}^r \mid r = 0, 1, \dots, n, 1 \le k_1 < \dots < k_r \le n\}.
$$

The unit of the Clifford algebra corresponds to  $K = \emptyset$ , and we set  $e_{\phi} = 1$ .

Consider the stem function

$$
F:D\to\mathbb{R}_m\otimes\mathbb{R}_n,
$$

defined in an open set in  $\mathbb{C}^n$ , invariant w.r.t. complex conjugation in every variable  $z_1, \dots, z_n$ . If we denote it by

(1) 
$$
F = \sum_{K \in \mathcal{P}(n)} e_K F_K, \qquad F_K : D \to \mathbb{R}_m,
$$

then as a stem function, it satisfies the Clifford-intrinsic condition. That is, for each  $K \in \mathcal{P}(n)$ ,  $k \in \{1, \dots, n\}$ , and  $z = (z_1, \dots, z_n) \in D$ , the components  $F_K$ satisfy the compatibility conditions

<span id="page-2-0"></span>
$$
F_K(z_1,\dots, z_{k-1},\overline{z_k},z_{k+1},\dots, z_n) = \begin{cases} F_K(z) & \text{if } k \notin K, \\ -F_K(z) & \text{if } k \in K. \end{cases}
$$

Let  $\Omega(D)$  be the circular subset of  $(\mathcal{Q}_m)^n$  associated to  $D \subset \mathbb{C}^n$ . More precisely,  $\Omega(D)$  consists of all  $x = (x_1, \dots, x_n) \in (\mathcal{Q}_m)^n$  with

$$
x_k = \alpha_k + \beta_k J_k \in \mathbb{C}_{J_k}
$$

for any  $J_k \in \mathbb{S}_m$  and  $\alpha_k, \beta_k \in \mathbb{R}$  provided  $(\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n) \in D$ . Here  $\mathcal{Q}_m$ stands for the quadratic cone in  $\mathbb{R}_m$ , i.e.,

$$
\mathcal{Q}_m := \mathbb{R} \cap \{x \in \mathbb{R}_m \mid t(x) \in \mathbb{R}, \ n(x) \in \mathbb{R}, \ 4n(x) > t(x)^2\},\
$$

where  $t(x) = x + \bar{x}$  denotes the trace of x and  $n(x) = x\bar{x}$  the (squared) norm of a Clifford element x. As usual, we take  $\mathbb{S}_m$  in place of  $\mathbb{S}_\mathbb{A}$  in the case of  $\mathbb{A} = \mathbb{R}_m$ .

With the stem function  $F$  given by  $(1)$  and

$$
x = (x_1, \dots, x_n) = (\alpha_1 + J_1 \beta_1, \dots, \alpha_n + J_n \beta_n) \in \Omega(D),
$$

the slice function  $\mathcal{I}(F)$  is defined as

$$
\mathcal{I}(F)(x) := \sum_{K \in \mathcal{P}(n)} J_K F_K(\alpha_1 + i\beta_1, \cdots, \alpha_n + i\beta_n).
$$

Here

(2) 
$$
J_K := \prod_{k \in K}^{\rightarrow} J_k = J_{k_1} \cdots J_{k_s},
$$

is the odered product.

When the stem function  $F$  is holomorphic, i.e.,

$$
\frac{1}{2} \left( \frac{\partial F}{\partial \alpha_k} + J_k \frac{\partial F}{\partial \beta_k} \right) = 0, \qquad k = 1, \cdots, n.
$$

the slice function  $\mathcal{I}(F)$  is called slice regular on  $\Omega_D$ . Many results for slice regular functions are announced in [\[20\]](#page-17-12).

Observed that the setting considered in [\[20\]](#page-17-12) is too general, Colombo, Sabadini, and Struppa [\[7\]](#page-17-13) chose to move on in some special case about the slice regular functions with  $m = 2$  in which they can provide detail proofs. Moreover, to get rid of the compatibility conditions they restrict their consideration to the case of the upper half space, i.e., they only consider the domain

$$
D \subset (\mathbb{R} \times \mathbb{R}^+)^n.
$$

The purpose of this article is to establish the slice theory of several octonionic variables, which is as a generalization of the theory of several complex variables, instead of the theory of one complex variable.

To overcome the difficulties appearing in [\[7,](#page-17-13)[20\]](#page-17-12), We adopt a new trick by restrict our attention to the same complex structure  $J$ , in contrast to the classical case where the imaginary units may be distinct. Our approach makes many results of several complex variables extended to the non-commutative or non-associative setting with the help of the theory of stem functions. In particular, we establish the Bochner-Martinelli formula for slice functions and Hartogs theorem for slice regular functions in several octonionic variables as well as several quaternionic variables.

2. Slice functions of several octonionic variables

Let  $\mathbb{O}_{\mathbb{C}} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$  be the complexification of the octonions  $\mathbb{O}$ , which can also be expressed as

$$
\mathbb{O}_{\mathbb{C}} = \mathbb{O} + i\mathbb{O} = {\omega = x + iy | x, y \in \mathbb{O}}
$$
  $(i^2 = -1).$ 

It is a complex alternative algebra with a unity w.r.t. the product given by the formula

$$
(x+iy)(u+iv) = xu - yv + i(xv + yu).
$$

For each  $x, y \in \mathbb{O}$ , we define  $(x+iy)^c = \overline{x} + i\overline{y}$  be the complex-linear antiinvolution of  $x + iy$  in  $\mathbb{O}_{\mathbb{C}}$  and  $\overline{x + iy} = x - iy$  be the complex conjugation of  $x + iy$  in  $\mathbb{O}_{\mathbb{C}}$ .

<span id="page-3-0"></span>**Definition 2.1.** Let D be an open subset in  $\mathbb{C}^n$ . A function

$$
F:D\to\mathbb{O}_\mathbb{C}
$$

is called an  $\mathbb{O}\text{-}stem\ function\ on\ D,\ if\ F\ is\ complex\ intrinsic,\ i.e.$ 

$$
F(\overline{z}) = \overline{F(z)}, \qquad \forall \ z \in D.
$$

Notice that by complex intrinsincity,  $F$  can be extended to the axially symmetric set generated by D, i.e.,  $D \cup conj(D)$ , where

$$
conj(D) := \{ z \in \mathbb{C}^n | \overline{z} \in D \}.
$$

However, the extended set may be non-connected.

<span id="page-3-1"></span>**Remark 2.2.** (1) In the case of  $n = 1$ , Definition [2.1](#page-3-0) was introduced by Ghiloni and Petrotti in [\[18\]](#page-17-10) even in any alternative algebra instead of octonions. Here, we initiate the study to the case of higher dimensions with  $n > 1$ .

(2)A function  $F = F_1 + iF_2$  is a  $\mathbb{O}$ -stem function if and only if the  $\mathbb{O}$ -valued components  $F_1, F_2$  constitute an even-odd pair, i.e.,

$$
F_1(\overline{z}) = F_1(z), \qquad F_2(\overline{z}) = -F_2(z), \qquad \forall \ z \in D.
$$

(3) By means of a basis  $\mathcal{B} = \{e_0, \dots, e_7\}$  of  $\mathbb{O}$  as a 8-dimensional real vector space. F can be identity with a complex intrinsic surface in  $\mathbb{C}^8$ . Let

$$
F(z) = F_1(z) + iF_2(z) = \sum_{k=0}^{7} F^k(z)e_k
$$

with  $F^k(z) \in \mathbb{C}$ . Then

$$
\tilde{F} = (F^0, \cdots, F^7) : D \to \mathbb{C}^8
$$

satisfies

$$
\tilde{F}(\overline{z}) = \overline{\tilde{F}(z)}.
$$

Giving  $\mathbb O$  the unique manifold structure as a real vector space, we get that a stem function F is of class  $C^k (k = 0, \dots, \infty)$  or real-analytic if and only if the same property for  $\tilde{F}$ . This notion is clearly independent of the choice of the basis of  $\mathbb{O}$ .

In several octonionic variables  $\mathbb{O}^n$ , we define

$$
\mathbb{O}^n_s:=\bigcup_{J\in\mathbb{S}}\mathbb{C}^n_J,
$$

where  $S$  denotes the unit sphere of the imaginary octonions, *i.e.*,

$$
\mathbb{S} = \{ a \in \mathbb{O} \mid a^2 = -1 \},
$$

and

$$
\mathbb{C}^n_J:=(\mathbb{C}_J)^n
$$

with

$$
\mathbb{C}_J := \mathbb{R} + J\mathbb{R} \subset \mathbb{O}, \qquad \forall \ J \in \mathbb{S}.
$$

Given an open subset D of  $\mathbb{C}^n$ , let  $\Omega_D$  be the subset of  $\mathbb{O}_s^n$  generated by D:

 $\Omega_D := \{ x = \alpha + \beta J \in \mathbb{C}_J^n \mid \forall J \in \mathbb{S}, \ \alpha, \beta \in \mathbb{R}^n \text{ with } \alpha + i\beta \in D \} \subset \mathbb{O}_s^n,$ 

and

$$
D_J := \Omega_D \cap \mathbb{C}_J^n, \qquad J \in \mathbb{S}.
$$

Sets of this type as  $\Omega_D$  will be called circular sets in  $\mathbb{O}^n_s$ , which is an open subset of  $\mathbb{O}_s^n$ .

<span id="page-4-0"></span>**Definition 2.3.** A stem function  $F: D \to \mathbb{O}_{\mathbb{C}}$  induces a (left) slice function

<span id="page-4-1"></span>
$$
f = \mathcal{I}(F) : \Omega_D \to \mathbb{O}.
$$

For any  $x = \alpha + J\beta \in D_J$ ,  $\forall J \in \mathbb{S}$ , we set

(3)  $f(x) := F_1(z) + JF_2(z) \qquad (z = \alpha + i\beta).$ 

The slice function is well defined, since  $(F_1, F_2)$  is an even-odd pair w.r.t.  $\beta$  and then

$$
f(\alpha + (-\beta)(-J)) = F_1(\bar{z}) + (-J)F_2(\bar{z}) = F_1(z) + JF_2(z) = f(\alpha + \beta J).
$$

There is an analogous definition for right slice functions when the element  $J \in \mathbb{S}$ is placed on the right of  $F_2(z)$ . From now on, the term slice functions will always mean left slice function.

We denote the set of  $\mathbb O$ -stem functions on  $D$  as

$$
\mathfrak{S}(D,\Omega) := \{ F : D \to \mathbb{O}_{\mathbb{C}} \mid F : D \to \mathbb{O}_{\mathbb{C}} \text{ is a } \mathbb{O}\text{-stem function}\}
$$

and denote the set of (left) slice function on  $\Omega_D$  by

$$
\mathcal{S}(\Omega_D, \mathbb{O}) := \{ f : \Omega_D \to \mathbb{O} \mid f = \mathcal{I}(F), \ F : D \to \mathbb{O}_{\mathbb{C}} \text{ is a } \mathbb{O}\text{-stem function}\}.
$$

Therefore, the lift map  $\mathcal I$  is a bijection

$$
\mathcal{I}: \mathfrak{S}(D,\Omega) \longrightarrow \mathcal{S}(D,\Omega).
$$

Remark 2.4.  $\mathcal{S}(\Omega_D, \mathbb{O})$  is a real vector space, since

$$
\mathcal{I}(F+G) = \mathcal{I}(F) + \mathcal{I}(G),
$$

and

$$
\mathcal{I}(aF) = a\mathcal{I}(F)
$$

for every complex intrinsic function F, G on D and  $a \in \mathbb{R}$ .

From Definition [2.3,](#page-4-0) we obtain the following representation formulas for slice functions.

<span id="page-5-0"></span>**Proposition 2.5.** Let  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$ , and  $J, K \in \mathbb{S}$  with  $J \neq K$ . Then

 $f(\alpha + \beta I) = (I - K) ((J - K)^{-1} f(\alpha + \beta J)) - (I - J) ((J - K)^{-1} f(\alpha + \beta K))$ for each  $I \in \mathbb{S}$ ,  $\alpha, \beta \in \mathbb{R}^n$  with  $\alpha + \beta I \in D_I$ .

*Proof.* For any  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$ , by [\(3\)](#page-4-1),

$$
f(\alpha + \beta J) - f(\alpha - \beta K) = (J - K)F_2(\alpha + \beta i)
$$

Hence

$$
F_2(\alpha + \beta i) = (J - K)^{-1}(f(\alpha + \beta J) - f(\alpha + \beta K))
$$

and

$$
F_1(\alpha + \beta i) = f(\alpha + \beta J) - JF_2(\alpha + \beta i)
$$
  
=  $f(\alpha + \beta J) - J((J - K)^{-1}(f(\alpha + \beta J) - f(\alpha + \beta K))).$ 

Therefore,

$$
f(\alpha + \beta I) = F_1(\alpha + \beta i) + IF_2(\alpha + \beta i)
$$
  
=  $f(\alpha + \beta J) + (I - J)((J - K)^{-1}(f(\alpha + \beta J) - f(\alpha + \beta K)))$   
=  $(J - K + I - J)((J - K)^{-1}f(\alpha + \beta J)) + (I - J)((J - K)^{-1}f(\alpha + \beta K))$   
=  $(I - K)((J - K)^{-1}f(\alpha + \beta J)) + (I - J)((J - K)^{-1}f(\alpha + \beta K)).$ 

By settling  $K = -J$  in Proposition [2.5,](#page-5-0) we obtain the following result.

Corollary 2.6. Let  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$  and  $J \in \mathbb{S}$ . Then

$$
f(\alpha + \beta I) = \frac{1}{2}(f(\alpha + \beta J) + f(\alpha - \beta J)) - \frac{I}{2}(J(f(\alpha + \beta J) - f(\alpha - \beta J)))
$$
  
each  $I \in \mathbb{S}$ ,  $\alpha, \beta \in \mathbb{R}^n$  with  $\alpha + \beta I \in D$ .

for each  $I \in \mathbb{S}$ ,  $\alpha, \beta \in \mathbb{R}^n$  with  $\alpha + \beta I \in D_I$ .

When  $n = 1$ , Ghiloni and Petrotti [\[18\]](#page-17-10) introduced the useful concepts of the spherical derivative and the spherical value. Now we generalize them to the slice functions of several octonionic variables.

**Definition 2.7.** Let  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$ . The spherical value of f at  $x \in \Omega_D$  is the element of O

$$
v_s f(x) := \frac{1}{2}(f(x) + f(\overline{x}))
$$

and the spherical derivative of f at  $x \in \Omega_D \backslash \mathbb{R}^n$  is the element of  $\mathbb O$ 

$$
\partial_s f(x) := \frac{1}{2} Im(x)^{-1} (f(x) - f(\overline{x})).
$$

In this way, we get two slice functions associated with  $f$ , given by  $(3)$ . Namely, vsf is induced on  $\Omega_D$  by the stem function  $F_1(z)$  and  $\partial_s f$  is induced on  $\Omega_D \backslash \mathbb{R}^n$  by  $F<sub>2</sub>(z)$ .

Since these stem functions are  $\mathbb O$  valued,  $v_s f$  and  $\partial_s f$  are constant on every "sphere"

$$
\mathbb{S}_x := \{ y = \alpha + \beta I \mid x = \alpha + \beta J, \alpha, \beta \in \mathbb{R}^n, I \in \mathbb{S} \}.
$$

Therefore

$$
\partial_s(\partial_s f) = 0, \qquad \partial_s(v_s f) = 0
$$

for every f. Moreover,  $\partial_s f(x) = 0$  if and only if f is constant on  $\mathbb{S}_x$ . In this case, f has value  $v_s f(x)$  on  $\mathbb{S}_x$ .

If  $\Omega_D \cap \mathbb{R}^n \neq \emptyset$ , under mild reguarity conditions on F, we get that  $\partial_s f$  can be contiously extended as a slice function on  $\Omega_D$ . For example, it is sufficient to assume that  $F_2(z)$  is of class  $C^1$ . By definition, the following identity holds for every  $x \in \Omega_D$ :

$$
f(x) = v_s f(x) + Im(x) \partial_s f(x).
$$

We will consider slice functions of several octonionic variables induced by stem functions of class  $C^1$ . They consist of the real vector space

$$
\mathcal{S}^1(\Omega_D,\mathbb{O}) := \{ f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D,\mathbb{O}) | F \in C^1(D,\mathbb{O}_{\mathbb{C}}) \}.
$$

Let  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D, \mathbb{O})$  and  $z = \alpha + i\beta \in D$ . Then the partial derivatives  $\partial F/\partial \alpha_t$  and  $i\partial F/\partial \beta_t$  are continous  $\mathbb{O}-$ stem functions on D for any  $t=1,2,\cdots,n$ . The same property holds for their linear combinations

$$
\frac{\partial F}{\partial z_t} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha_t} - i \frac{\partial F}{\partial \beta_t} \right) \quad \text{and} \quad \frac{\partial F}{\partial \overline{z}_t} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha_t} + i \frac{\partial F}{\partial \beta_t} \right),
$$

where  $z = (z_1, \dots, z_n), \ \alpha = (\alpha_1, \dots, \alpha_n), \ \beta = (\beta_1, \dots, \beta_n), \ t = 1, 2, \dots, n.$ 

<span id="page-6-0"></span>**Definition 2.8.** Let  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D, \mathbb{O})$ . We set

$$
\frac{\partial f}{\partial x} := \mathcal{I}(\frac{\partial F}{\partial z}) = \left(\mathcal{I}(\frac{\partial F}{\partial z_1}), \cdots, \mathcal{I}(\frac{\partial F}{\partial z_n})\right),
$$

and

$$
\frac{\partial f}{\partial \bar{x}} := \mathcal{I}(\frac{\partial F}{\partial \bar{z}}) = \left(\mathcal{I}(\frac{\partial F}{\partial \bar{z}_1}), \cdots, \mathcal{I}(\frac{\partial F}{\partial \bar{z}_n})\right)
$$

with

$$
\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \bar{x}} = \left(\frac{\partial f}{\partial \bar{x}_1}, \cdots, \frac{\partial f}{\partial \bar{x}_n}\right).
$$

These maps are continous slice maps on  $\Omega_D$ ,  $t = 1, 2, \cdots, n$ .

#### 3. Slice regular functions of several octonionic variables

Left multiplication by i defines a complex structure on  $\mathbb{O}_{\mathbb{C}}$ . With respect to this structure, a  $C^1$  function

$$
F = F_1 + iF_2 : D \to \mathbb{O}_{\mathbb{C}}
$$

is holomorphic if and only if its components  $F_1$ ,  $F_2$  satisfy the Cauchy-Riemann equations:

$$
\frac{\partial F_1}{\partial \alpha_t} = \frac{\partial F_2}{\partial \beta_t}, \qquad \frac{\partial F_1}{\partial \beta_t} = -\frac{\partial F_2}{\partial \alpha_t}, \qquad (z = \alpha + i\beta \in D)
$$

i.e.,

$$
\frac{\partial F}{\partial \overline{z}_t} = 0,
$$

where  $z = (z_1, \dots, z_n), \ \alpha = (\alpha_1, \dots, \alpha_n), \ \beta = (\beta_1, \dots, \beta_n), \ t = 1, 2, \dots, n.$ 

This condition is equivalent to require that, for any basis  $B$ , the complex surface  $\tilde{F}$  (see Remark [2.2\)](#page-3-1) is holomorphic. Set

$$
\frac{\partial F}{\partial z} := \left( \frac{\partial F}{\partial z_1}, \frac{\partial F}{\partial z_2}, \cdots, \frac{\partial F}{\partial z_n} \right), \n\frac{\partial F}{\partial \bar{z}} := \left( \frac{\partial F}{\partial \bar{z}_1}, \frac{\partial F}{\partial \bar{z}_2}, \cdots, \frac{\partial F}{\partial \bar{z}_n} \right).
$$

The set of all holomorphic O-stem functions is denoted by

$$
\mathcal{H}(D,\mathbb{O}_{\mathbb{C}}):=\{F\in C^{1}(D,\mathbb{O}_{\mathbb{C}}): \frac{\partial F}{\partial \overline{z}}(z)=0, \ \forall z\in D\}.
$$

<span id="page-7-0"></span>**Definition 3.1.** A (left) slice function  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D, \mathbb{O})$  is (left) slice regular if its associated stem function  $F$  is holomorphic. We will denote the vector space of slice regular functions on  $\Omega_D$  by

$$
\mathcal{SR}(\Omega_D, \mathbb{O}) := \{ f \in \mathcal{S}^1(\Omega_D, \mathbb{O}) \mid f = \mathcal{I}(F), F : D \to \mathbb{O}_{\mathbb{C}} \text{ is holomorphic} \}.
$$

**Remark 3.2.** A function  $f \in S^1(\Omega_D, \mathbb{O})$  is slice ragular if and only if the slice map

$$
\frac{\partial f}{\partial \overline{x}} = \left(\frac{\partial f}{\partial \overline{x}_1}, \cdots, \frac{\partial f}{\partial \overline{x}_n}\right)
$$

(cf. Definition [2.8](#page-6-0) in Section 2) vanishies identically. Moreover, if f is slice regular, then also

$$
\frac{\partial f}{\partial x} = \left(\mathcal{I}(\frac{\partial F}{\partial z_1}), \cdots, \mathcal{I}(\frac{\partial F}{\partial z_n})\right)
$$

is slice regular on  $\Omega_D$ .

**Proposition 3.3.** Let  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D, \mathbb{O})$ . Then f is slice regular on  $\Omega_D$  if and only if the restriction

$$
f_J = f|_{D_J}: D_J \to \mathbb{O}
$$

is holomorphic for every  $J \in \mathbb{S}$  with respect to the complex structures on  $D_J$  and  $\mathbb{O}$ defined by left multiplication by J.

Proof. Notice that

$$
f_J(\alpha + \beta J) = F_1(\alpha + i\beta) + JF_2(\alpha + i\beta).
$$

If  $F$  is holomorphic, then

$$
\frac{\partial f_J}{\partial \alpha} + J \frac{\partial f_J}{\partial \beta} = \frac{\partial F_1}{\partial \alpha} + J \frac{\partial F_2}{\partial \alpha} + J \left( \frac{\partial F_1}{\partial \beta} + J \frac{\partial F_2}{\partial \beta} \right) = 0
$$

at every point  $x = \alpha + J\beta \in D_J$ .

Conversely, assume that  $f_J$  is holomorphic at every  $J \in \mathbb{S}$ . Then

$$
0=\frac{\partial f_J}{\partial \alpha}+J\frac{\partial f_J}{\partial \beta}=\frac{\partial F_1}{\partial \alpha}-\frac{\partial F_2}{\partial \beta}+J(\frac{\partial F_2}{\partial \alpha}+\frac{\partial F_1}{\partial \beta})
$$

at every point  $z = \alpha + i\beta \in D$ . From the arbitrariness of J it follows that  $F_1, F_2$ satisfy the Cauchy-Riemann equations.

**Remark 3.4.** The even-odd character of the pair  $(F_1, F_2)$  and the proof of preceding proposition show that, in oder to get slice regularity of  $f = \mathcal{I}(F)$  with  $F \in C^1$ , it is sufficient to assume that two functions  $f_J$ ,  $f_K$  with  $J \neq K$  are holomorphic on domains  $D_J$  and  $D_K$  respectively (cf. Proposition [2.5](#page-5-0)). The possibility  $K = -J$  is not excluded which means that the single function  $f_J$  must be holomorphic on  $D_J$ .

4. Products of slice functions of several octonionic variables

In general, the pointwise product of two slice functions is not a slice function. However, pointwise product in the algebra  $\mathbb{O}_{\mathbb{C}}$  of  $\mathbb{O}$ -stem functions induces a natural product on slice functions, which is similar to the case of slice function of one variable.

**Definition 4.1.** Let  $f = \mathcal{I}(F)$ ,  $g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D, \mathbb{O})$ . The product of f and g is the slice function

$$
f \cdot g := \mathcal{I}(FG) \in \mathcal{S}(\Omega_D, \mathbb{O}).
$$

The preceding definition is well-posed, since the pointwise product

$$
FG = (F_1 + iF_2)(G_1 + iG_2) = F_1G_1 - F_2G_2 + i(F_1G_2 + F_2G_1)
$$

of complex intrinsic functions is still complex intrinsic. It follows directly from the definition that the product is distributive. The spherical derivative satisfies a Leibniz-type product rule, where evaluation is replaced by spherical value:

$$
\partial_s(f \cdot g) = (\partial_s f)(v_s g) + (v_s f)(\partial_s g).
$$

Remark 4.2. In general,

$$
(f \cdot g)(x) \neq f(x)g(x).
$$

If  $x = \alpha + \beta J$  belongs to  $D_J$  and  $z = \alpha + i\beta$ , then

$$
(f \cdot g)(x) = F_1(z)G_1(z) - F_2(z)G_2(z) + J(F_1(z)G_2(z) + F_2(z)G_1(z),
$$

while

$$
f(x)g(x) = F_1(z)G_1(z) + (JF_2(z))(JG_2(z)) + F_1(z)(JG_2(z)) + (JF_2(z))G_1(z).
$$

If the components  $F_1, F_2$  of the first stem function F are real-valued, or if F and G are both O-valued, then

$$
(f \cdot g)(x) = f(x)g(x), \qquad \forall \ x \in \Omega_D.
$$

In this case, we will use also the notation  $fg$  in place of  $f \cdot g$ .

**Definition 4.3.** A slice function  $f = \mathcal{I}(F)$  is called real if the  $\mathbb{O}\text{-}valued$  components  $F_1, F_2$  of its stem function are real valued. Equivalently, f is real if the spherical value  $v_s f$  and the spherical derivative  $\partial_s f$  are real valued.

A real slice function f has the characteristic property that for every  $J \in \mathbb{S}$ , the image  $f_J$  is contained in  $\mathbb{C}_J$ .

**Definition 4.4.** A slice function  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$  is real if and only if  $f_J(D_J) \subseteq \mathbb{C}_J$ for every  $J \in \mathbb{S}$ .

*Proof.* Assume that  $f(\mathbb{C}^n_J \cap \Omega_D) \subseteq \mathbb{C}_J$  for every  $J \in \mathbb{S}$ . Let  $f = \mathcal{I}(F)$ . If  $x =$  $\alpha + J\beta \in \Omega_D$  and  $z = \alpha + i\beta$ , then

$$
f(x) = F_1(z) + JF_2(z) \in \mathbb{C}_J
$$

and

$$
f(\bar{x}) = F_1(\bar{z}) + JF_2(\bar{z}) = F_1(z) - JF_2(z) \in \mathbb{C}_J.
$$

This implies that

$$
F_1(z), F_2(z) \in \bigcap_{J \in \mathbb{S}} \mathbb{C}_J = \mathbb{R}.
$$

 $\Box$ 

**Proposition 4.5.** If f, g are slice regular on  $\Omega_D$ , then the product f  $\cdot$  g is slice regular on  $\Omega_D$ .

*Proof.* Let  $f = \mathcal{I}(F)$ ,  $g = \mathcal{I}(G)$ ,  $H = FG$ . If F and G satisfy the Cauchy-Riemann equations, the same holds for  $H$ . This follows from the validity of the Leibniz product rule, that can be checked using a basis representation of F and G.  $\Box$ 

We consider two polynomials or convergent power series

$$
f(x) = \sum_{\mu} x^{\mu} a_{\mu}, \qquad g(x) = \sum_{\nu} x^{\nu} b_{\nu},
$$

where

 $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{N}^n$ ,  $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$ ,  $x^{\mu} := x_1^{\mu_1} \cdots x_n^{\mu_n}$ for  $x = (x_1, \dots, x_n) \in \mathbb{O}_s^n$ , and

$$
a_{\mu} := a_{\mu_1 \mu_2 \cdots \mu_n} \mathbb{O}, \qquad b_{\nu} := b_{\nu_1, \cdots \nu_n} \in \mathbb{O}.
$$

The star product  $f * g$  of f and g is the convergent power series, defined as

$$
f * g := \sum_{\gamma} x^{\gamma} (\sum_{\mu + \nu = \gamma} a_{\mu} b_{\nu}).
$$

**Proposition 4.6.** Let  $f(x) = \sum_{\mu} x^{\mu} a_{\mu}$  and  $g(x) = \sum_{\nu} x^{\nu} b_{\nu}$  be polynomials or convergent power series, where  $a_{\mu}, b_{\nu} \in \mathbb{O}$ ,  $\mu, \nu \in \mathbb{N}^n$ . Then the product of f and g, viewed as slice regular functions, coincides with the star product

$$
f * g = \mathcal{I}(FG) = \mathcal{I}(F) * \mathcal{I}(G).
$$

*Proof.* Let  $f = \mathcal{I}(F)$ ,  $q = \mathcal{I}(G)$ ,  $H = FG$ , and

$$
F(z) = \sum_{\mu} z^{\mu} a_{\mu}, \qquad G(z) = \sum_{\mu} z^{\mu} b_{\mu}.
$$

Since  $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}$  is contained in the commutative and associative center of  $\mathbb{O}_{\mathbb{C}}$ , we have

$$
H(z) = F(z)G(z)
$$
  
= $(\sum z^{\mu} a_{\mu})(\sum z^{\nu} b_{\nu})$   
= $\sum_{\mu,\nu} z^{\mu} z^{\nu} (a_{\mu} b_{\nu})$ 

Denote by  $A_{\gamma}(z)$  and  $B_{\gamma}(z)$  the real components of the complex power

$$
z^{\gamma} = (\alpha + i\beta)^{\gamma}, \qquad \alpha, \ \beta \in \mathbb{R}^n, \quad \gamma \in \mathbb{N}^n.
$$

Let  $c_{\gamma} = \sum_{\mu+\nu=\gamma} a_{\mu} b_{\nu}$  for each  $\gamma$ . Therefore, we have

$$
H(z) = \sum_{\gamma} z^{\gamma} c_{\gamma}
$$
  
= 
$$
\sum_{\gamma} (A_{\gamma}(z) + iB_{\gamma}(z))c_{\gamma}
$$
  
= 
$$
\sum_{\gamma} A_{\gamma}(z)c_{\gamma} + i(\sum_{\gamma} B_{\gamma}(z)c_{\gamma})
$$
  
=: 
$$
H_1(z) + iH_2(z)
$$

and then, if  $x = \alpha + \beta J$ ,  $z = \alpha + i\beta$ ,

$$
\mathcal{I}(H)(x) = H_1(z) + JH_2(z) = \sum_{\gamma} A_{\gamma}(z)c_{\gamma} + J(\sum_{\gamma} B_{\gamma}(z)c_{\gamma}).
$$

On the other hand,

$$
f * g(x) = \sum_{\gamma} (\alpha + \beta J)^{\gamma} c_{\gamma}
$$
  
= 
$$
\sum_{\gamma} (A_{\gamma}(z) + JB_{\gamma}(z)) c_{\gamma}
$$

since  $A_{\gamma}$  and  $B_{\gamma}$  are all real. From these, the result follows.

## 5. Zeros of slice functions of several octonionic variables

The zero sets of slice functions exhibits many interesting algebraic and topological properties. Some relevant theories, concerning the zeros of slice functions of quternionic and octonionic variable, have been studied deeply in [\[11,](#page-17-14) [13,](#page-17-15) [19,](#page-17-16) [27\]](#page-18-6).

The zero set

$$
V(f)=\{x\in\mathbb{O}^n_s\ |\ f(x)=0\}
$$

of a slice function  $f \in \mathcal{S}(\Omega_D, \mathbb{Q})$  has a particular structure. We will see that, for every fixed  $x = \alpha + J\beta \in \mathcal{S}(\Omega_D)$ , the "sphere"

$$
\mathbb{S}_x = \{ \alpha + I\beta \mid I \in \mathbb{S} \}
$$

is entirely contained in  $V(f)$  or  $\mathbb{S}_x$  contains at most one zero of f.

**Proposition 5.1.** Let  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$ . For  $x \in \Omega_D \backslash \mathbb{R}^n$ , the restriction of f to  $\mathbb{S}_x$  is injective or constant.

*Proof.* Given  $x, x' \in \mathbb{S}_x$ , if  $f(x) = f(x')$ , then

$$
(x - x')\partial_s f(x) = (Im(x) - Im(x'))\partial_s f(x) = 0.
$$

If  $\partial_s f(x) \neq 0$ , this implies  $x = x'$ .

If  $\partial_s f(x) = 0$ , then  $f|_{\mathbb{S}_x}$  is a constant due to the representation formula.  $\Box$ 

This result leads to a structure theorem of the zero of  $f$  restricted to the sphere  $\mathbb{S}_x.$ 

**Theorem 5.2.** (Structure of  $V(f)$ ) Let  $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D, \mathbb{O})$ . Let  $x = \alpha + J\beta \in \mathbb{C}$  $\Omega_D$  and  $z = \alpha + i\beta \in D$ . Then one of the following mutually exclusive statements holds:

 $(1)$   $\mathbb{S}_x \cap V(f) = \emptyset$ .

(2)  $\mathbb{S}_x \subset V(f)$ . In this case x is called a real (if  $x \in \mathbb{R}^n$ ) or spherical (if  $x \notin \mathbb{R}^n$ ) zereo of f.

(3)  $\mathbb{S}_x \cap V(f)$  consists of a single, non-real point. In this case x is called a non-real zero of f in  $\mathbb{S}_x$ .

**Remark 5.3.** We remark that the preceding theorem shows that when restricted to any sphere  $\mathbb{S}_x$ , the zeros of f have the same behavior either  $n = 1$  or  $n > 1$ .

## 6. Bochner-Martinelli formula and Hartogs theorem

The Bochner-Martinelli formula is an important formmula in several complex variables (see Theorem 1.1.4 [\[26\]](#page-18-7)). We now extend it to slice functions of several octonionic variables. As an application, we shall see that there appear the Hartogs phenomena when  $n > 1$  in our setting.

On  $\mathbb{C}^n_J$  with any  $J\in\mathbb{S},$  we consider the Bochner-Matinalli kernel

$$
\omega_x(\xi) := \frac{(n-1)!}{(2\pi J)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\overline{\xi}_j - \overline{x}_j}{|\xi - x|^{2n}} \overline{d\xi}_1 \wedge \cdots \wedge \overline{d\xi}_{j-1} \wedge \overline{d\xi}_{j+1} \wedge \cdots \wedge \overline{d\xi}_n \wedge d\xi.
$$

Here  $\xi = (x_1, \dots, \xi_n) \in \mathbb{C}^n$  and  $d\xi := d\xi_1 \wedge \dots \wedge d\xi_n$ .

<span id="page-11-0"></span>**Theorem 6.1.** Assume that  $f \in S^1(\overline{\Omega}_D, \mathbb{O})$ ,  $J \in \mathbb{S}$ , and  $D_J$  is a bounded domain with  $C^1$  boundary in  $\mathbb{C}^n_J$ . Then for any  $x \in D_J$ ,

$$
f(x) = \int_{\partial D_J} \omega_x(\xi) f(\xi) - (-1)^n \int_{D_J} \omega(\xi) \wedge \overline{\partial} f(\xi).
$$

Moreover, for any  $q \in \Omega_D$  there exists  $I \in \mathbb{S}$  such that  $q \in \mathbb{C}^n_I$  and

$$
f(q) = \int_{\partial D_J} \frac{1}{2} (\omega_x(\xi) + \omega_{\overline{x}}(\xi)) f(\xi) - \frac{I}{2} (J(\omega_x(\xi) + \omega_{\overline{x}}(\xi)) f(\xi))
$$
  
-  $(-1)^n \int_{D_J} \frac{1}{2} (\omega_x(\xi) + \omega_{\overline{x}}(\xi)) \wedge \overline{\partial} f(\xi) - \frac{I}{2} (J(\omega_x(\xi) + \omega_{\overline{x}}(\xi)) \wedge \overline{\partial} f(\xi)).$ 

*Proof.* By defintion, for any  $f \in S^1(\overline{\Omega}_D, \mathbb{O})$  there exists  $F \in C^1(D, \mathbb{O}_\mathbb{C})$  such that  $f = \mathcal{I}(F)$ .

Foe any  $z = \alpha + \beta i \in D$ , we write

$$
F(z) = F_1(z) + iF_2(z)
$$
  
= 
$$
\sum_{k=0}^{7} F_1^k(z)e_k + iF_2^k(z)e_k
$$
  
= 
$$
\sum_{k=0}^{7} (F_1^k(z) + iF_2^k(z))e_k
$$
  
= 
$$
\sum_{k=0}^{7} F^k(z)e_k
$$

where  $\{e_0, \dots e_7\}$  is a basis of  $\mathbb{O}$  and

$$
F^k = F_1^k + iF_2^k \in C^1(\overline{D}, \mathbb{C}).
$$

We abuse of notation by denoting  $\phi_J$  either the isomorphism

$$
\phi_J : \mathbb{C}^n \to \mathbb{C}^n_J
$$

or the isomorphism

$$
\phi_j:\mathbb{C}\to\mathbb{C}_J
$$

which sends  $i$  to  $J$ .

Define

$$
F_J^k := \phi_J \circ F^k \circ \phi_J^{-1} \in C^1(\overline{D}_J, \mathbb{C}_J).
$$

Then

$$
F_J^k(x) = \phi_J \circ F^k(z) = F_1^k(z) + JF_2^k(z),
$$

where

 $x = \phi_J(z)$ .

From this it follows that

$$
f(x) = \mathcal{I}(F)(x)
$$
  
=  $F_1(z) + JF_2(z)$   
=  $\sum_{k=0}^{7} (F_1^k(z) + JF_2^k(z))e_k$   
=  $\sum_{k=0}^{7} F_J^k(x)e_k.$ 

Notice that  $F_J^k \in C^1(\overline{D}_J, \mathbb{C}_J)$ , and  $D_J$  is a bounded domain with  $C^1$  boundary in  $C_J^n$ . By the Bochner-Martinelli formula in the function theory of several complex variables, we obtain

$$
F_J^k(x) = \int_{\partial D_J} F_J^k(\xi) \omega_x(\xi) - \int_{D_J} \overline{\partial} F_J^k(\xi) \wedge \omega_x(\xi).
$$

A straight calculation shows that

$$
F_J^k(x) = \int_{\partial D_J} \omega_x(\xi) F_J^k(\xi) - (-1)^n \int_{D_J} \omega_x(\xi) \wedge \overline{\partial} F_J^k(\xi).
$$

Now we have

$$
f(x) = \sum_{k=0}^{7} F_j^k(x)e_k
$$
  
= 
$$
\int_{\partial D_j} (\omega_x(\xi)\sum_{k=0}^{7} F_j^k(\xi))e_k - (-1)^n \int_{D_j} (\omega_x(\xi) \wedge \sum_{k=0}^{7} \overline{\partial} F_j^k(\xi))e_k
$$
  
= 
$$
\int_{\partial D_j} \omega_x(\xi)(\sum_{k=0}^{7} F_j^k(\xi)e_k) - (-1)^n \int_{D_j} \omega_x(\xi) \wedge (\sum_{k=0}^{7} \overline{\partial} F_j^k(\xi)e_k)
$$
  
= 
$$
\int_{\partial D_j} \omega_x(\xi)f(\xi) - (-1)^n \int_{D_j} \omega_x(\xi) \wedge \overline{\partial} f(\xi).
$$

In the third equation above, we used the alternativity of octonions . Apply the octonionic representation formula with the function f on the domain  $\Omega_D$  (cf. Propo-sition [2.5\)](#page-5-0), we have the other formula.  $\square$ 

As a direct corollary, we get the Cauchy formula for slice regular funtions of sevral octonionic variables.

**Corollary 6.2.** Let  $f \in S^1(\overline{\Omega}_D, \mathbb{O}) \cap SR(\Omega_D, \mathbb{O}), J \in \mathbb{S}$ , and  $D_J$  is a bounded domain with  $C^1$  boundary in  $\mathbb{C}^n_J$ . Then for any  $x \in D_J$ ,

$$
f(x) = \int_{\partial D_J} \omega_x(\xi) f(\xi).
$$

Moreover, for any  $q \in \Omega_D$ ,

$$
f(q) = \int_{\partial D_J} \frac{1}{2} (\omega_x(\xi) + \omega_{\overline{x}}(\xi)) f(\xi) - \frac{I}{2} (J(\omega_x(\xi) + \omega_{\overline{x}}(\xi)) f(\xi)),
$$

where  $q \in \mathbb{C}^n_I$ ,  $\forall I \in \mathbb{S}$ .

By setting  $n = 1$  in Theorem [6.1,](#page-11-0) we obtain the Cauchy integral formula for slice functions of octonionic variable of class  $C<sup>1</sup>$ , which was obtained by Ghiloni and Perotti [\[18\]](#page-17-10).

Hartogs's theorem [\[25\]](#page-17-17) is a fundemental result in the theory of several complex variables. Now we generalize the Hartogs Theorem to the case of several octonionic variables.

Let D be a domain in  $\mathbb{C}^n$ ,  $f \in \mathcal{S}(\Omega_D, \mathbb{O})$ . For any  $a \in \mathbb{C}^n$ , we denote

$$
D_{j,a} = \{ z_j \in \mathbb{C} : (a_1, \cdots, a_{j-1}, z_j, a_{j+1}, \cdots, a_n) \in D \}
$$

and

$$
\Omega_{D_{j,a}} = \bigcup_{J \in \mathbb{S}} \phi_J(D_{j,a}).
$$

We consider the functions on  $D_{j,a}$ 

$$
F_{j,a}(z) = F(a_1, \cdots, a_{j-1}, z_j, a_{j+1}, \cdots, a_n)
$$

and its lift

$$
f_{j,a} := \mathcal{I}(F_{j,a})
$$

for any  $a \in \mathbb{C}^n$ ,  $j = 1, \dots, n$ ,

**Theorem 6.3.** If for any  $a \in \mathbb{C}^n$ ,  $j = 1, \dots, n$ ,

$$
f_{j,a} := \mathcal{I}(F_{j,a}) \in \mathcal{SR}(\Omega_{D_{j,a}}, \mathbb{O}),
$$

then  $f \in \mathcal{SR}(\Omega_D)$ .

*Proof.* If  $f_{j,a} = \mathcal{I}(F_{j,a}) \in \mathcal{SR}(\Omega_{D_j,a}, \mathbb{O})$ , which is well-defined (see remark [2.2\)](#page-3-1). From Definition [3.1,](#page-7-0) we have

$$
F_{j,a} \in \mathcal{H}(D_{j,a},\mathbb{O}_{\mathbb{C}}).
$$

This means that

$$
F\in\mathcal{H}(D,\mathbb{O}_{\mathbb{C}})
$$

by the Hartogs Theorem for the holomorphic functions of several complex variables. Hence  $f \in \mathcal{SR}(\Omega_D, \mathbb{O})$  by definition.

**Theorem 6.4.** (Hartogs) Assume that  $n \geq 2$ . Let  $D \subset \mathbb{C}^n$  be a domain,  $K \subset D$ be a compact set such that  $D\backslash K$  is connected in  $\mathbb{C}^n$ . If  $f \in \mathcal{SR}(\Omega_{D\backslash K}, \mathbb{O})$ , then there is a function  $g \in \mathcal{SR}(\Omega_D, \mathbb{O})$ , such that  $g|_{\Omega_{D\setminus K}} = f$ .

*Proof.* For any  $f \in \mathcal{SR}(\Omega_{D\setminus K}, \mathbb{O})$ , by definition there exists a function  $F \in \mathcal{H}(D\setminus K, \mathbb{O}_\mathbb{C})$ , such that  $f = \mathcal{I}(F)$ . Notice that

$$
F(z) = \sum_{k=0}^{7} F^k(z)e_k,
$$

here  $\{e_0, \dots, e_7\}$  is a basis of  $\mathbb{O}$  and

$$
F^k\in \mathcal{H}(D\backslash K,\mathbb{C}).
$$

By the classical Hartogs theorem for several complex variables, there is a function  $G^k \in \mathcal{H}(D, \mathbb{C})$  such that  $G^k|_{D \setminus K} = F^k$ . Now set

$$
G(z) := \sum_{k=0}^{7} G^k(z)e_k,
$$

then  $G \in \mathcal{H}(D, \mathbb{O}_{\mathbb{C}})$  so that  $g = \mathcal{I}(G)$ , is the desired function.

#### 7. Slice functions of several quaternionic variables

In the previous sections, we consider the slice theory of several octonionic variables. The similar theory holds with the octonions replaced by the quaternions. In this section, the results are stated without proof.

Let  $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$  be the complexification of  $\mathbb{H}$ , i.e.,

$$
\mathbb{H}_{\mathbb{C}} = \{ \omega = x + iy \mid x, y \in \mathbb{H} \} \qquad (i^2 = -1).
$$

 $\mathbb{H}_{\mathbb{C}}$  is a complex alternative algebra with a unity w.r.t. the product given by the formula

$$
(x+iy)(u+iv) = xu - yv + i(xv + yu).
$$

Since  $\mathbb{H}_{\mathbb{C}} = \mathbb{H} + i\mathbb{H}$ , in  $\mathbb{H}_{\mathbb{C}}$  two commuting operators are defined: the complexlinear antiinvolution

$$
\omega \mapsto \omega^c = (x + iy)^c = \overline{x} + i\overline{y}
$$

and the complex conjungation defined by

$$
\overline{\omega} = \overline{x + iy} = x - iy.
$$

**Definition 7.1.** Let  $D \subset \mathbb{C}^n$  be an open subset. A function  $F: D \to \mathbb{H}_{\mathbb{C}}$  is called a H-stem function, if F is complex intrinsic, i.e.  $F(\bar{z}) = \overline{F(z)}$ , for each  $z \in D$ .

**Remark 7.2.** (1)  $F = F_1 + iF_2$  is a  $H$ -stem function if and only if  $F_1, F_2$  form an even-odd pair, i.e. for each  $z \in D$ , we have  $F_1(\overline{z}) = F_1(z)$ , and  $F_2(\overline{z}) = -F_2(z)$ .

(2) Consider  $\mathbb H$  as a 4-dimensional real vector space. By means of a basis  $\mathcal B =$  ${e_0, e_1, e_2, e_3}$  of H, F can be identity with a complex intrinsic surface in  $\mathbb{C}^n$ .

Let  $F(z) = F_1(z) + iF_2(z) = \sum_{k=0}^{3} F^k(z)e_k$  with  $F^k(z) \in \mathbb{C}$ . Then

$$
\tilde{F} = (F^0, F^1, F^2, F^3) : D \to \mathbb{C}^4
$$

satisfies  $\tilde{F}(\bar{z}) = \tilde{F}(z)$ . Giving  $\mathbb H$  the unique manifold structure as a real vector space, we get that a stem function F is of class  $C^k$ ( $k = 0, \dots, \infty$ ) or real-analytic if and only if the same property for  $\tilde{F}$ . This notion is clearly independent of the choice of the basis of H.

Given an open subset  $D$  of  $\mathbb{C}^n$ , let  $\Omega_D$  be the subset of  $\mathbb{H}^n_s$  obtained by the action on D of the square roots of  $-1$ :

$$
\Omega_D:=\{x=\alpha+\beta J\in\mathbb{C}^n_J\mid \alpha+i\beta\in D, \alpha,\beta\in\mathbb{R}^n,\ J\in\mathbb{S}_{\mathbb{H}}\}\subset\mathbb{H}^n_s
$$

and

$$
D_J=D\cap\mathbb{C}^n_J,
$$

where

$$
\mathbb{S}_{\mathbb{H}} = \{a \in \mathbb{H} \mid a^2 = -1\}, \qquad \mathbb{H}^n_s = \bigcup_{J \in \mathbb{S}_{\mathbb{H}}} \mathbb{C}^n_J.
$$

Sets of this type will be called circular sets in  $\mathbb{H}_s^n$ , which is an open subset of  $\mathbb{H}_s^n$ .

**Definition 7.3.** Any stem function  $F : D \to \mathbb{H}_{\mathbb{C}}$  induces a (left) slice function  $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ . If  $x = \alpha + J\beta \in D_J$ , we set

$$
f(x) := F_1(z) + JF_2(z) \qquad (z = \alpha + i\beta)
$$

We will denote the set of (left) slice function on  $\Omega_D$  by

 $\mathcal{S}(\Omega_D, \mathbb{H}) := \{f : \Omega_D \to \mathbb{H} \mid f = \mathcal{I}(F), F : D \to \mathbb{H}_{\mathbb{C}} \text{ is a } \mathbb{H} \text{-stem function}\},\$ and

$$
\mathcal{S}^1(\Omega_D, \mathbb{H}) := \{ f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D, \mathbb{H}) \mid F \in C^1(D, \mathbb{H}_{\mathbb{C}}),\
$$

 $\mathcal{SR}(\Omega_D, \mathbb{H}) := \{f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D, \mathbb{H}) \mid F : D \to \mathbb{H}_{\mathbb{C}} \text{ is holomorphic}\}.$ 

As in the section above, we let

$$
\omega_x(\xi) := \frac{(n-1)!}{(2\pi J)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\overline{\xi}_j - \overline{x}_j}{|\xi - x|^{2n}} \overline{d\xi}_1 \wedge \cdots \wedge \overline{d\xi}_{j-1} \wedge \overline{d\xi}_{j+1} \wedge \cdots \wedge \overline{d\xi}_n \wedge d\xi
$$

on  $\mathbb{C}^n_J$ .

**Theorem 7.4.** Let  $f \in S^1(\overline{\Omega}_D, \mathbb{H})$ ,  $J \in \mathbb{S}_{\mathbb{H}}$ . Assume that  $D_J$  be a bounded domain with  $C^1$  boundary in  $\mathbb{C}^n_J$ , then for any  $x \in D_J$ ,

$$
f(x) = \int_{\partial D_J} \omega_x(\xi) f(\xi) - (-1)^n \int_{D_J} \omega_x(\xi) \wedge \overline{\partial} f(\xi).
$$

Moreover, for any  $q \in \Omega_D$ ,

$$
f(q) = \int_{\partial D_J} \omega_q(\xi) f(\xi) - (-1)^n \int_{D_J} \omega_q(\xi) \wedge \overline{\partial} f(\xi)
$$

where

$$
\omega_q(\xi) = \frac{1}{2} (\omega_x(\xi) + \omega_{\overline{x}}(\xi)) - \frac{I}{2} (J(\omega_x(\xi) + \omega_{\overline{x}}(\xi)), \qquad q \in \mathbb{C}_I^n, \quad I \in \mathbb{S}_{\mathbb{H}}.
$$

**Theorem 7.5.** Let  $f \in S^1(\overline{\Omega}_D, \mathbb{H}) \cap \mathcal{SR}(\Omega_D, \mathbb{H})$ ,  $J \in \mathbb{S}$ . Assume that  $D_J$  be a bounded domain with  $C^1$  boundary in  $\mathbb{C}^n_J$ , then for any  $x \in D_J$ ,

$$
f(x) = \int_{\partial D_J} \omega_x(\xi) f(\xi).
$$

Moreover, for any  $q \in \Omega_D$ ,

$$
f(q) = \int_{\partial D_J} \omega_q(\xi) f(\xi),
$$

where

$$
\omega_q(\xi) = \frac{1}{2} (\omega_x(\xi) + \omega_{\overline{x}}(\xi)) - \frac{I}{2} (J(\omega_x(\xi) + \omega_{\overline{x}}(\xi)), \qquad \forall q \in \mathbb{C}_I^n, \quad I \in \mathbb{S}_{\mathbb{H}}.
$$

Let D be a domain in  $\mathbb{C}^n$ . For any  $a = (a_1, a_2, \dots, a_n) \in D$  and  $j = 1, \dots, n$ , we consider the slices of  $D$  defined by

$$
D_{j,a} = \{u \in \mathbb{C} : (a_1, \cdots, a_{j-1}, u, a_{j+1}, \cdots, a_n) \in D\} \subset \mathbb{C}
$$

and the related sets

$$
\Omega_{D_{j,a}} = \{ \alpha_j + \beta_j J \in \mathbb{H} : \alpha + i\beta \in D, \alpha, \beta \in \mathbb{R}^n, J \in \mathbb{S}_{\mathbb{H}} \} \subset \mathbb{H}
$$
  
Here we denote  $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n).$ 

Associated with the functions of  $f \in \mathcal{S}(\Omega_D, \mathbb{H})$  and  $F \in \mathfrak{S}(\Omega_D, \mathbb{H}_{\mathbb{C}})$ , we consider their slices

$$
F_{j,a}(u) = F(a_1, \dots, a_{j-1}, u, a_{j+1}, \dots, a_n), \forall u \in D_{j,a};
$$
  
\n
$$
f_{j,a}(x_j) = f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n), \forall x_j \in \Omega_{D_{j,a}} \subset \mathbb{H}.
$$

It is easy to see that if  $f = \mathcal{I}(F)$  then

$$
f_{j,a} := \mathcal{I}(F_{j,a}).
$$

Moreover, if  $f_{j,a} := \mathcal{I}(F_{j,a}) \in \mathcal{SR}(\Omega_{D_{j,a}}, \mathbb{H})$  for any  $a \in D \subset \mathbb{C}^n$ ,  $j = 1, \dots, n$ , then

$$
f\in\mathcal{SR}(\Omega_D,\mathbb{H}).
$$

**Theorem 7.6.** Let D be a domain in  $\mathbb{C}^n$  and  $f \in \mathcal{S}(\Omega_D, \mathbb{H})$ . If for any  $a \in D \subset \mathbb{C}^n$ and  $j = 1, \cdots, n$ ,

$$
f_{j,a}\in\mathcal{SR}(\Omega_{D_{j,a}},\mathbb{H}),
$$

then

$$
f\in\mathcal{SR}(\Omega_D,\mathbb{H}).
$$

Now we can state the Hartogs theorem in the version of several quaternionic variables.

**Theorem 7.7.** Assume that  $n \geq 2$ . Let  $D \subset \mathbb{C}^n$  be a domain,  $K \subset D$  be a compact set such that  $D\backslash K$  is connected in  $\mathbb{C}^n$ . If  $f \in \mathcal{SR}(\Omega_{D\backslash K}, \mathbb{H})$ , then there is a function  $g \in \mathcal{SR}(\Omega_D, \mathbb{H})$ , such that  $g|_{\Omega_{D\setminus K}} = f$ .

## 8. Conclusions

We initiate the study of the theory of slice regular functions of sevevral octonionic variables as well as several quaternionic variables. The related Bochner-Martinelli formula and the Hartogs theorem are established. It deserves to consider further extensions of the classical theory of several complex variables to these new settings.

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