

**INFINITESIMALLY SMALL SPHERES
AND
CONFORMALLY INVARIANT METRICS**

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ABSTRACT. The *modulus metric* (also called the *capacity metric*) on a domain $D \subset \mathbb{R}^n$ can be defined as $\mu_D(x, y) = \inf\{\text{cap}(D, \gamma)\}$, where $\text{cap}(D, \gamma)$ stands for the capacity of the condenser (D, γ) and the infimum is taken over all continua $\gamma \subset D$ containing the points x and y . It was conjectured by J. Ferrand, G. Martin and M. Vuorinen in 1991 that every isometry in the modulus metric is a conformal mapping. In this note, we confirm this conjecture and prove new geometric properties of surfaces that are spheres in the metric space (D, μ_D) .

1. CONFORMAL MAPPINGS AND ISOMETRIES OF THE MODULUS METRIC

A continuous one-to-one mapping $f : D \rightarrow \Omega$ from a domain $D \subset \mathbb{R}^n$, $n \geq 2$, onto a domain $\Omega \subset \mathbb{R}^n$ is conformal if it maps smooth curves in D onto smooth curves in Ω preserving oriented angles between intersecting curves. The class of conformal mappings, which is rich in planar domains (thanks to the Riemann mapping theorem!), becomes very restrictive in dimensions ≥ 3 . Precisely, by the classical Liouville's theorem (see, [7, p. 388], [18, p. 19], and references therein), in dimensions $n \geq 3$, every C^4 conformal mapping $f : D \rightarrow \Omega$ is a restriction to D of a Möbius self-map of $\bar{\mathbb{R}}^n$, where $\bar{\mathbb{R}}^n$ is the one point compactification of \mathbb{R}^n .

An important characterization of conformal mappings $f : D \rightarrow \Omega$, as well as a characterization of their generalization, quasiconformal mappings, can be given in terms of their dilatation $H_f(x)$. Throughout the text we use the following standard notations. By $|x|$ we denote the Euclidean norm in \mathbb{R}^n and by $S(x, r) = \{y \in \mathbb{R}^n : |y - x| = r\}$ and $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ we denote, respectively, the sphere and the open ball in \mathbb{R}^n centered at x with radius $r > 0$. We also use the following shorter notations $\mathbb{S} = S(\bar{0}, 1)$ and $\mathbb{B} = B(\bar{0}, 1)$ for the unit sphere and unit ball centered at $\bar{0} = (0, \dots, 0)$. Then the dilatation $H_f(x)$ can be defined as

$$H_f(x) = \limsup_{r \rightarrow 0} \frac{\max_{y \in S(x, r)} |f(y) - f(x)|}{\min_{y \in S(x, r)} |f(y) - f(x)|}. \quad (1.1)$$

According to a celebrated theorem proved by Yu. G. Reshetnyak in [10] (see also Theorem 5.10 in [11, Chapter II]) and by F. W. Gehring, see Theorem 16 in [7], a sense preserving homeomorphism $f : D \rightarrow \Omega$ is conformal if and only if its dilatation $H_f(x)$ is 1 a.e. on D and $H_f(x) < \infty$ on D .

Another geometric characterization of conformal mappings can be given in terms of modules of families of curves. Precisely, a sense preserving homeomorphism $f : D \rightarrow \Omega$ satisfies equation (1.1) and therefore it is conformal if and only if it preserves the modulus of every family Γ of curves in D . To define the modulus of a family Γ of curves $\gamma \subset D$, we consider a class $\mathcal{A}(\Gamma)$ of metrics $\rho \geq 0$ admissible for Γ in the following sense: $\rho \in \mathcal{A}(\Gamma)$ if and only if ρ is a non-negative Borel measurable function satisfying $\int_\gamma ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. Now the modulus $\text{mod}(\Gamma)$ is defined as

$$\text{mod}(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_D \rho^n dm. \quad (1.2)$$

The fact that the modulus $\text{mod}(\Gamma)$ defined by (1.2) is conformally invariant is classical, see [7] and [18, p. 54]. But verification of invariance of modules of all families of curves under a mapping $f : D \rightarrow \Omega$ is impractical and there is no need to verify invariance of modules of every family of curves. It was shown by Gehring [7] that a mapping will be conformal if it preserves modules of curves connecting boundary components of the so-called ring domains. Later, J. Ferrand, G. Martin and M. Vuorinen suggested in [6] that it might be sufficient to verify invariance of modules for some other specific families of curves.

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On this way, these authors studied in [6] a conformal invariant μ_D , which can be defined as follows. If D is a domain in \mathbb{R}^n and $x, y \in D$, then

$$\mu_D(x, y) = \inf\{\text{mod}(\Gamma(D, l)) : l \in C_{xy}\}, \quad (1.3)$$

where C_{xy} is the family of all Jordan arcs l joining x to y in D and $\Gamma(D, l)$ is the family of all curves γ (not necessarily Jordan) in D joining l and ∂D . It was mentioned in [18, p. 103] that the function $\mu_D(x, y)$ defines a metric on D , which is called the *modulus metric*, if and only if ∂D is of positive conformal capacity. Thus, a domain D supplied with the modulus metric μ_D becomes a metric space (D, μ_D) .

The modulus metric is conformally invariant and quasi-invariant under quasiconformal mappings which makes it very useful in the theory of quasiconformal mappings; see, for instance, [17], [18]. It was conjectured by Ferrand, Martin and Vuorinen (see [6, p. 195]) that every mapping $f : D \rightarrow \Omega$, which is an *isometry* with respect to the metrics μ_D and μ_Ω , is conformal. These authors have shown in [6] that this is indeed the case when D is a ball in \mathbb{R}^n . An essential progress towards the solution of this conjecture was made recently in [2], where the authors proved that f is conformal if $n = 2$, thus settling the conjecture in this case, and that f is quasiconformal in the case $n \geq 3$. The main goal of this paper is to prove the following.

Theorem 1. *Let D and Ω be domains in \mathbb{R}^n , $n \geq 2$, such that ∂D and $\partial\Omega$ have positive conformal capacities. Suppose that f is an isometry of the metric space (D, μ_D) onto the metric space (Ω, μ_Ω) . Then f is a conformal mapping.*

Thus, Theorem 1 proves Ferrand-Martin-Vuorinen conjecture in all dimensions. The method used to prove this result is purely geometrical. It is based on application of polarization transformation in the spirit of papers [13] and [16], where polarization was used to solve Pólya-Szegő problem on continuous symmetrization.

One more goal of this work is to study geometric properties of the μ_D -spheres, that are level surfaces with respect to the modulus metric, defined by $S_{\mu_D}(x, r) = \{y \in D : \mu_D(x, y) = r\}$ and the μ_D -balls defined by $B_{\mu_D}(x, r) = \{y \in D : \mu_D(x, y) < r\}$. In this direction we prove the following result.

Theorem 2. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, such that ∂D has positive conformal capacity and let $x_0 \in D$. Then there is $r_0 > 0$ such that for all $0 < r < r_0$, the μ_D -sphere $S_{\mu_D}(x_0, r)$ is a topological sphere in \mathbb{R}^n which satisfies the interior and exterior cone conditions and the ball $B_{\mu_D}(x_0, r)$ is starlike with respect to x_0 .*

The rest of the paper is organized as follows. Section 2 contains necessary background from Potential Theory. In Section 3, a geometric transformation called *polarization* will be used to establish some properties of the modulus metric and geometric properties of μ_D -balls stated in Theorem 2. In Section 4, we use Reshetnyak's characterization of conformality by an invariance property of collections of infinitesimally small spheres and the lemma on the three spheres from elementary geometry to prove our Theorem 1. Finally, in Section 5, we discuss some related open problems.

2. CONDENSER CAPACITY AND MODULUS METRICS

A condenser in $\overline{\mathbb{R}}^n$, $n \geq 2$, is a pair (D, K) , where D is a domain in $\overline{\mathbb{R}}^n$ and K is a non-empty compact subset of D . For every condenser (D, K) , we denote by $H(D, K)$ the class of functions admissible for (D, K) ; i.e. $H(D, K)$ consists of all C^1 functions with compact support in D satisfying $u(x) \geq 1$ for $x \in K$. If $D \subset \mathbb{R}^n$, the conformal capacity of the condenser (D, K) is defined by

$$\text{cap}(D, K) = \inf_u \int_{D \setminus K} |\nabla u|^n dm, \quad (2.1)$$

where m is the n -dimensional Lebesgue measure and the infimum is taken over the class $H(D, K)$. If D contains the point ∞ , $\text{cap}(D, K)$ is defined by means of an auxiliary Möbius transformation. Since we do not use other capacities in this paper, everywhere below we use a shorter term "capacity" instead of "conformal capacity".

It is instrumental for us, that the capacity of a condenser (D, K) is conformally invariant, see [9], and, due to Ziemer's theorem [21, Theorem 3.8] (see also [7, Theorem 1] and Proposition 10.2 in [12]), it coincides with the modulus of the family $\Gamma(D, K)$ of curves γ joining K and ∂D in D ; i.e.,

$$\text{cap}(D, K) = \text{mod}(\Gamma(D, K)).$$

Therefore, the modulus metric μ_D can be alternatively defined as

$$\mu_D(x, y) = \inf\{\text{cap}(D, \gamma) : \gamma \in C_{xy}\}. \quad (2.2)$$

Thus, the modulus metric is, in a certain sense, the “*capacity metric*”. The latter definition has some advantages when studying properties of the metric. Next, we introduce necessary terminology and recall several known or “semi-known” properties of capacities. It would be convenient to list these properties as a series of lemmas.

First, we recall that a compact set $E \subset \overline{\mathbb{R}^n}$ is said to be of *zero capacity* if there exists a domain D with $E \subset D$ such that $\text{cap}(D, E) = 0$. Otherwise, E is said to be a set of positive capacity (or a set of positive conformal capacity as we state it in Theorem 1).

It is well known that the infimum in the definition of the capacity can be taken, with the same result, over different classes of functions. Indeed, we recall first that every closed set $E \subset \mathbb{R}^n$ may contain points which are regular for the Dirichlet problem for the n -Laplacian and points which are irregular for this problem, see, for instance, Section 9.5 in [9] for the definition and the properties of irregular boundary points. One but not both of these sets can be empty. Let $I(E)$ denote the set of irregular points of E for the problem under consideration. Let $C(D)$ be the space of continuous functions in D , let $C^\infty(D)$ be the space of infinitely differentiable functions in D and let $C_0^\infty(D)$ be the subspace of $C^\infty(D)$ of functions with compact support in D . Let $H^{1,n}(D)$ be the completion of $C^\infty(D)$ and let $H_0^{1,n}(D)$ be the completion of $C_0^\infty(D)$ with respect to the norm

$$\|v\|_{1,n} = \left(\int_D |v|^n dm \right)^{1/n} + \left(\int_D |\nabla v|^n dm \right)^{1/n}, \quad v \in C^\infty(D).$$

Also, let $W^{1,n}(D) = H^{1,n}(D) \cap C(D)$ and $W_0^{1,n}(D) = H_0^{1,n}(D) \cap C(D)$.

Lemma 1 (see [9, pp. 28-29]). *Let (D, K) be a condenser in \mathbb{R}^n . Then*

$$\text{cap}(D, K) = \inf_{u \in H_1(D, K)} \int_{D \setminus K} |\nabla u|^n dm, \quad (2.3)$$

where $H_1(D, K)$ is the family of functions u in the space $W_0^{1,n}(D)$ satisfying $u \geq 1$ on K .

We can restrict the classes of functions over which the infima in (2.1) and (2.3) are taken to the subclasses of so-called *monotone functions*. A continuous function f on a domain $D \subset \mathbb{R}^n$ is called monotone on D if for any relatively compact domain Ω in D ,

$$\sup_{x \in \partial\Omega} f(x) = \sup_{x \in \Omega} f(x) \quad \text{and} \quad \inf_{x \in \partial\Omega} f(x) = \inf_{x \in \Omega} f(x). \quad (2.4)$$

Given a domain D and a compact set $K \subset D$, by $H^*(D, K)$ we denote the family of monotone on $D \setminus K$ functions u in the space $W_0^{1,n}(D)$ satisfying $u \geq 1$ on K .

Lemma 2 (cf. [12, p. 54]). *Let (D, K) be a condenser in \mathbb{R}^n . Then*

$$\text{cap}(D, K) = \inf_{u \in H^*(D, K)} \int_{D \setminus K} |\nabla u|^n dm.$$

It is well known that every condenser (D, K) has a unique potential function $\omega_{D, K}$. In our next lemma, we summarize some well-known properties of this function.

Lemma 3 (see, [9, pp. 194, 211, 212], [20, p. 104]). *For every condenser (D, K) with positive capacity there is a function $\omega_{D, K}$, called the potential function, which minimizes the integral in (2.3). The potential function $\omega_{D, K}$ is a solution to the n -Laplace equation in $D \setminus K$ and satisfies the following boundary conditions*

$$\omega_{D, K}(x) \rightarrow 1 \quad \text{as } x \in D \setminus K \text{ approaches regular points of } K$$

and

$$\omega_{D, K}(x) \rightarrow 0 \quad \text{as } x \in D \setminus K \text{ approaches regular points of } \partial D.$$

Furthermore, if D is bounded, then $\omega_{D, K} \in H_0^{1,n}(D)$.

The potential function $\omega_{D, K}$ possessing properties described in Lemma 3 is, in fact, the n -harmonic measure of the compact set K with respect to the domain $D \setminus K$. In particular, $\omega_{D, K}$ is monotone on $D \setminus K$. For further properties of the n -harmonic measure the reader may consult [9, Chapter 11]. The following convergence lemma is a useful tool often used to prove existence of condensers with special properties.

Lemma 4 ([9, Theorem 2.2]). *Let (D_k, K_k) , $k \in \mathbb{N}$, be a sequence of condensers in $\overline{\mathbb{R}^n}$ such that $D_k \subset D_{k+1}$, $K_{k+1} \subset K_k$ for all $k \in \mathbb{N}$, $\cup_{k=1}^\infty D_k = D$, $\cap_{k=1}^\infty K_k = K$, and (D, K) is a condenser of positive capacity. Then*

$$\text{cap}(D_k, K_k) \rightarrow \text{cap}(D, K), \quad \text{as } k \rightarrow \infty.$$

We will need the following monotonicity property of the capacity.

Lemma 5. *Let (D_1, K_1) and (D_2, K_2) be condensers in \mathbb{R}^n such that $D_2 \subset D_1$ and $K_1 \subset K_2$. Then*

$$\text{cap}(D_1, K_1) \leq \text{cap}(D_2, K_2). \quad (2.5)$$

Furthermore, if $I(K_1) = \emptyset$ and $K_2 \setminus K_1$ contains a compact set K_3 such that $I(K_3) = \emptyset$ and if, in addition, $D_1 \setminus K_1$ is connected and contains K_3 and $D_1 \setminus K_3$ is connected and contains K_1 then (2.5) holds with the sign of strict inequality.

Proof. The non-strict inequality (2.5) is well known, see, for instance, [9, Theorem 2.2], and follows immediately from the definition (2.1). Thus, we have to prove only the statement about the cases of equality. Also, since the capacity of a condenser defined by (2.1) is conformally invariant we may assume without loss of generality that $\infty \notin D_1$.

For $\delta > 0$, let $K_3(\delta) = \{x \in \mathbb{R}^n : \text{dist}(x, K_3) < \delta\}$. If $\delta < \text{dist}(K_3, \partial(D_1 \setminus K_1))$, then $\overline{K_3(\delta)}$ and $\partial(K_3(\delta))$ are compact subsets of $D_1 \setminus K_1$. Let ω be the potential function of the condenser $(D_1, K_1 \cup K_3)$ and let

$$t_m = \max\{\omega(x) : x \in \partial(K_3(\delta))\}.$$

It follows from the maximum principle for solutions of the n -Laplace equation (see, for instance, [9, p. 115]) that $0 < t_m < 1$. Let $t_m < t_\delta < 1$. Our assumptions imply that ω is continuous on the set $\overline{K_3(\delta)}$ and therefore the set $V = \{x \in \overline{K_3(\delta)} : \omega(x) > t_\delta\}$ is open in \mathbb{R}^n and its complement $E = \overline{\mathbb{R}^n} \setminus V$ is compact in $\overline{\mathbb{R}^n}$.

Since $\Omega = \overline{\mathbb{R}^n} \setminus K_3$ is open and connected and $E \subset \Omega$, the pair (Ω, E) is a condenser with the potential function ω_E given by

$$\omega_E(x) = \begin{cases} \frac{1-\omega(x)}{1-t_\delta} & \text{if } x \in \Omega \setminus E \\ 1 & \text{if } x \in E. \end{cases}$$

Let $\varepsilon > 0$ be as small as we will need it later. It follows from the convergence Lemma 4 that there exists a domain D_ε such that $K_2 \subset D_\varepsilon \subset \overline{D_\varepsilon} \subset D_1$, the boundary ∂D_ε is regular for the n -Laplace equation, and

$$\text{cap}(D_\varepsilon, K_1 \cup K_3) < \text{cap}(D_1, K_1 \cup K_3) + \varepsilon. \quad (2.6)$$

Let ω_ε denote the potential function of the condenser $(D_\varepsilon, K_1 \cup K_3)$. It follows from the maximum principle for solutions of the n -Laplace equation that

$$\omega_\varepsilon(x) \leq \omega(x) \quad \text{for all } x \in D_\varepsilon.$$

The latter equation implies that the set $V_{\varepsilon, \delta} = \{x \in \overline{K_3(\delta)} : \omega_\varepsilon(x) > t_\delta\}$ is an open subset of V , the set $E_{\varepsilon, \delta} = \overline{\mathbb{R}^n} \setminus V_{\varepsilon, \delta}$ is compact in $\overline{\mathbb{R}^n}$, and $E_{\varepsilon, \delta} \supset E$. Furthermore, the pair $(\Omega, E_{\varepsilon, \delta})$ can be considered as a condenser and the function

$$\omega_{\varepsilon, \delta}(x) = \begin{cases} \frac{1-\omega_\varepsilon(x)}{1-t_\delta} & \text{if } x \in \Omega \setminus E_{\varepsilon, \delta} \\ 1 & \text{if } x \in E_{\varepsilon, \delta} \end{cases}$$

is the potential function of $(\Omega, E_{\varepsilon, \delta})$. This yields the following inequality:

$$\int_{V_{\varepsilon, \delta} \setminus K_3} |\nabla \omega_\varepsilon|^n dm = (1-t_\delta)^n \text{cap}(\Omega, E_{\varepsilon, \delta}) \geq (1-t_\delta)^n \text{cap}(\Omega, E) = \int_{\Omega \setminus E} |\nabla \omega|^n dm. \quad (2.7)$$

Notice that the function

$$u(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } x \in D_\varepsilon \setminus V_{\varepsilon, \delta} \\ t_\delta & \text{if } x \in V_{\varepsilon, \delta} \end{cases}$$

is admissible for the condenser (D_ε, K_1) . This together with relations (2.6) and (2.7) implies

$$\begin{aligned} \text{cap}(D_1, K_1) &\leq \text{cap}(D_\varepsilon, K_1) \leq \int_{D_\varepsilon \setminus (K_1 \cup K_3)} |\nabla \omega_\varepsilon|^n dm - \int_{V_{\varepsilon, \delta} \setminus K_3} |\nabla \omega_\varepsilon|^n dm \\ &= \text{cap}(D_\varepsilon, K_1 \cup K_3) - (1-t_\delta)^n \text{cap}(\Omega, E_{\varepsilon, \delta}) \\ &\leq \text{cap}(D_1, K_1 \cup K_3) + \varepsilon - (1-t_\delta)^n \text{cap}(\Omega, E). \end{aligned} \quad (2.8)$$

Finally, assuming that $\varepsilon < (1-t_\delta)^n \text{cap}(\Omega, E)$ and using the non-strict monotonicity property of capacity of condensers, we conclude from (2.8) that

$$\text{cap}(D_1, K_1) < \text{cap}(D_1, K_1 \cup K_3) \leq \text{cap}(D_1, K_2) \leq \text{cap}(D_2, K_2),$$

which is the required strict monotonicity property. \square

We note here that the value of the modulus metric $\mu_D(x, y)$ does not change if we replace C_{xy} with the family K_{xy} of all continua (connected compact sets) in D containing x, y . Precisely, we have the following result.

Lemma 6 (see [6, p. 191]). *The modulus metric μ_D can be alternatively defined as*

$$\mu_D(x, y) = \inf\{\text{cap}(D, K) : K \in K_{xy}\}. \quad (2.9)$$

Proof. Let $\nu_D(x, y)$ denote the infimum in (2.9). Since $C_{xy} \subset K_{xy}$ it is immediate from (2.2) and (2.9) that $\nu_D(x, y) \leq \mu_D(x, y)$.

To prove the reverse inequality we consider a sequence of continua K_k , $k \in \mathbb{N}$, in K_{xy} such that $\text{cap}(D, K_k) \rightarrow \nu_D(x, y)$ as $k \rightarrow \infty$ and a sequence of $\varepsilon_k > 0$ such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. It follows from the convergence Lemma 4 that for every $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that the set $K(\delta_k) = \{x \in \mathbb{R}^n : \text{dist}(x, K_k) \leq \delta_k\}$ is a compact subset of D and

$$\text{cap}(D, K(\delta_k)) \leq \text{cap}(D, K_k) + \varepsilon_k. \quad (2.10)$$

Furthermore, the interior of $K(\delta_k)$ is a non-empty connected open set containing points x and y . Therefore, for every $k \in \mathbb{N}$, there exists a Jordan arc $\gamma_k \subset K(\delta_k)$ (one may assume that it is analytic if necessary) joining points x and y . Now, by (2.5) and (2.10),

$$\mu_D(x, y) \leq \text{cap}(D, \gamma_k) \leq \text{cap}(D, K_k) + \varepsilon_k.$$

Taking the limit in the last inequality we obtain $\mu_D(x, y) \leq \nu_D(x, y)$, which combined with the reverse inequality mentioned above gives (2.9) \square

An advantage of the definition of $\mu_D(x, y)$ given by (2.9) is that it is easier to establish existence of a continuum minimizing the capacity in the right-hand side of (2.9) than to prove that this extremal continuum is a Jordan arc. For instance, Lemma 6 below guarantees, in most cases, existence of a continuum extremal for problem (2.2) but does not provide enough information to conclude that this continuum is a Jordan arc extremal for problem (1.3). Similar existence results are known for some other problems (see [5]) but for the problem under consideration it was not recorded in the literature available for us. Thus, we provide its proof here.

Lemma 7. *Let D be a domain in \mathbb{R}^n such that ∂D has positive capacity, let $E \subset D$ be a connected compact set and let $x, y \in E$. Suppose that there is a sequence of Jordan arcs $\gamma_k \in C_{xy}$ such that $\gamma_k \subset E$ for all $k \in \mathbb{N}$ and $\text{cap}(D, \gamma_k) \rightarrow \mu_D(x, y)$ as $k \rightarrow \infty$. Then there exists a continuum $\beta \subset E$ containing x and y such that*

$$\mu_D(x, y) = \text{cap}(D, \beta). \quad (2.11)$$

Proof. Let $\omega_k = \omega_{D, \gamma_k}$ be the potential function of the condenser (D, γ_k) that is also the n -harmonic measure of γ_k with respect to $D \setminus \gamma_k$. From the maximum and minimum principles for n -harmonic functions and Corollary 2.5 in [5] we conclude that for every $k \in \mathbb{N}$ the function ω_k is monotone (in the sense of definition (2.4)) on the domain $D \setminus \{x\}$. Using this fact, Proposition 1.6 in [5] and passing to a subsequence if necessary, we may assume that the sequence of functions ω_k converges locally uniformly on $D \setminus \{x\}$ to a continuous function ω which has generalized partial derivatives on $D \setminus \{x\}$, satisfying

$$\int_D |\nabla \omega|^n dm \leq \lim_{k \rightarrow \infty} \int_{D \setminus \gamma_k} |\nabla \omega_k|^n dm = \lim_{k \rightarrow \infty} \text{cap}(D, \gamma_k) = \mu_D(x, y). \quad (2.12)$$

Let ω_E be the n -harmonic measure of E with respect to $D \setminus E$. Since $\gamma_k \subset E$, the Carleman's principle for n -harmonic measures (see Theorem 11.3 in [9]) implies that $0 \leq \omega_k \leq \omega_E$ on $D \setminus E$, for every $k \in \mathbb{N}$. Therefore, letting $k \rightarrow \infty$, we conclude that $0 \leq \omega \leq \omega_E$ on $D \setminus E$. For every regular boundary point $\zeta \in \partial D$, we have

$$0 \leq \liminf_{z \rightarrow \zeta} \omega(z) \leq \limsup_{z \rightarrow \zeta} \omega(z) \leq \lim_{z \rightarrow \zeta} \omega_E(z) = 0.$$

Therefore,

$$\omega(z) \rightarrow 0, \text{ as } z \rightarrow \partial D \setminus I(\partial D).$$

Let $\beta = \omega^{-1}(1) \subset E$. Suppose that β is not connected. Let $S \subset D \setminus \beta$ be a topological sphere such that both connected components Ω_1 and Ω_2 of $\mathbb{R}^n \setminus S$ intersect β . Note that, since S is compact,

$$\max_{z \in S} \omega(z) < 1. \quad (2.13)$$

Suppose that there is a subsequence γ_{k_m} of γ_k such that $\gamma_{k_m} \cap S = \emptyset$, for all $m \in \mathbb{N}$. We may assume, passing to a subsequence of γ_{k_m} if needed, that γ_{k_m} , $m \in \mathbb{N}$, lies in the same component of $\mathbb{R}^n \setminus S$,

say Ω_1 . Then ω_{k_m} are n -harmonic functions on Ω_2 which converge locally uniformly to ω on Ω_2 . From Theorem 6.13 in [9, p. 117], ω is n -harmonic on Ω_2 . Let $w_0 \in \Omega_2 \cap \beta \neq \emptyset$. Then

$$\omega(w_0) = 1 \geq \omega(z),$$

for all $z \in \Omega_2$. From the maximum principle for n -harmonic functions (see, for instance, [9, p. 115]), $\omega = 1$ on Ω_2 . Since ω is continuous on $\Omega_2 \cup S$, we obtain that $\omega = 1$ on S , which contradicts (2.13). We conclude that there exists $p \in \mathbb{N}$ such that $\gamma_k \cap S \neq \emptyset$ for every $k \geq p$. Let $z_k \in \gamma_k \cap S$, $k \geq p$. Since S is compact we may assume that $z_k \rightarrow z_0 \in S$. Since $\omega_k \rightarrow \omega$ uniformly on S ,

$$\omega(z_0) = \lim_{k \rightarrow \infty} \omega_k(z_k) = 1,$$

contradicting (2.13). Therefore β is connected. Since obviously β is closed, $\beta \in K_{xy}$.

Let $z \in D \setminus \beta$. Since $\omega(z) < 1$ and $\omega_k = 1$ on γ_k , there exists $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that $B(z, \varepsilon) \cap \gamma_k = \emptyset$ for every $k \geq k_0$. Therefore, for every $k \geq k_0$, ω_k is n -harmonic on $B(z, \varepsilon)$ and $\omega_k \rightarrow \omega$ locally uniformly on $B(z, \varepsilon)$. From Theorem 6.13 in [9, p. 117], ω is n -harmonic on $B(z, \varepsilon)$. Since $z \in D \setminus \beta$ was arbitrary, ω is n -harmonic on $D \setminus \beta$. We conclude that ω is n -harmonic on $D \setminus \beta$ with boundary values 1 on β and 0 on every regular boundary point of D . From Theorem 11.1(c) [9, p. 209]) we get that ω is equal to the potential function $\omega_{D, \beta}$ of the condenser (D, β) and therefore

$$\text{cap}(D, \beta) = \int_{D \setminus \beta} |\nabla \omega|^n dm. \quad (2.14)$$

Finally, (2.11) follows from (2.12), (2.14) and Lemma 6. \square

Every continuum K such that $\mu_D(x, y) = \text{cap}(D, K)$ for some points $x, y \in K$ will be called μ_D -minimizer. Simple examples show that a μ_D -minimizer may not exist for some domains D and some pairs of points x and y and, if exist, it is not unique, in general. We conjecture that every μ_D -minimizer $\gamma_D(x, y)$ is a smooth Jordan arc joining x and y .

In the last lemma of this section, we recall well-known properties of the function $\mu_D(x, y)$, which, in particular, show that $\mu_D(x, y)$ is indeed a metric.

Lemma 8. *Let D be a domain in \mathbb{R}^n such that ∂D has positive capacity. Then the following holds.*

- (1) $\mu_D(x, y)$ is a continuous function of x and y .
- (2) $\mu_D(x, y) = 0$ if and only if $x = y$.
- (3) If ∂D contains a continuum E , then $\mu_D(x, y) \rightarrow \infty$ if $x \in D$ is fixed and $y \rightarrow E$.
- (4) For every triple x, y, z of distinct points in D the triangle inequality holds, i.e.,

$$\mu_D(x, z) \leq \mu_D(x, y) + \mu_D(y, z).$$

For the proof of properties (1) and (2) we refer to [18] and [5, p. 115]. Property (3) follows from the monotonicity Lemma 5 and Lemma 7.35 in [18]. For the triangle inequality see, for instance, [18, p. 103].

3. MODULUS METRIC AND POLARIZATION

A geometric transformation called *polarization* was introduced by V. Wolontis [19]. Two modern approaches to this transformation are popular now. The first one was developed by V. N. Dubinin who also suggested the term ‘‘polarization’’ for this transformation, see his book [4], and the other approach first appeared in [14] and then was developed in full generality in [3]. In this paper we use polarization with respect to spheres in \mathbb{R}^n , which continuously depend on some geometric parameters. The latter approach was inspired by papers [13] and [16], where polarization was used to solve Pólya-Szegő problem on continuous symmetrization.

Polarization of a set $E \subset \overline{\mathbb{R}^n}$ with respect to a sphere $S(x_0, r)$ can be defined as follows. Given $x \in \mathbb{R}^n \setminus \{x_0\}$, by x^* we denote the point in \mathbb{R}^n symmetric to x with respect to $S(x_0, r)$, i.e.

$$x^* = x_0 + r^2 \frac{x - x_0}{|x - x_0|^2}.$$

The points x_0 and $x = \infty$ are considered symmetric to each other with respect to every sphere centered at x_0 . Let $E^* = \{x \in \overline{\mathbb{R}^n} : x^* \in E\}$. Thus, E^* consists of all points that are symmetric to the points of E with respect to $S(x_0, r)$. In other words, E^* is a reflection of E with respect to $S(x_0, r)$.

Definition 1. Let E and (D, K) be a set and a condenser in \mathbb{R}^n , respectively.

- (a) Polarization of E with respect to $S(x_0, r)$ is defined as

$$E_p = \left((E \cup E^*) \cap \overline{B_r(x_0)} \right) \cup \left((E \cap E^*) \setminus \overline{B_r(x_0)} \right). \quad (3.1)$$

(b) Polarization of a condenser (D, K) with respect to $S(x_0, r)$ is defined as (D_p, E_p) .

It is well-known that E_p defined by (3.1) is a compact set if E is compact and that D_p is an open set if D is open. On the other side, polarization does not preserve connectivity. Simple examples, well known to the experts, show that there are simply connected domains D the polarization of which consists of infinitely many connected components and some of these connected components are infinitely connected domains. Thus, the polarization (D_p, K_p) of a condenser (D, K) is not, in general, a condenser as it was defined in Section 2. However, everywhere below, we polarize condensers (D, K) with respect to the spheres $S(x_0, r)$ such that $B(x_0, r) \subset D$. In this case, $D_p = D$ and the resulting pair $(D_p, K_p) = (D, K_p)$ is again a condenser in the sense of our definition in Section 2. The following theorem describes the effect of polarization on the capacity of a condenser.

Theorem 3 ([4]). *Let (D, K) be a condenser in \mathbb{R}^n and (D_p, K_p) be the polarization of (D, K) with respect to a sphere $S(x_0, r)$. Suppose further that D_p is connected. Then (D_p, K_p) is a condenser and*

$$\text{cap}(D_p, K_p) \leq \text{cap}(D, K). \quad (3.2)$$

Remark 1. In dimension $n = 2$, the cases when equality occurs in (3.2) were discussed under a variety of assumptions in [4], [13], and [1]. Also, in dimensions $n \geq 3$, the cases of equality in polarization inequalities for the Newtonian capacity were discussed in [13] and [1]. In dimensions $n \geq 3$, the question on the cases when equality sign occurs in (3.2), i.e. in polarization inequality for the conformal capacity, remains open. Resolving this question would be an important advance in the theory of symmetrization that also will lead to simpler proofs of some of our results presented below.

Remark 2. If K_p is the polarization of a connected compact set K with respect to a sphere $S(x_0, r)$, then K_p is compact but not necessarily connected. But one can easily see that the restriction $\tilde{K}_p = K_p \cap \overline{B(x_0, r)}$ of K_p to the closed ball $\overline{B(x_0, r)}$ is always compact and connected.

Combining Theorem 3 with properties of condenser capacity discussed in Section 2, we obtain new useful properties of the μ_D -metric presented in Lemma 9 and Lemma 10 below.

Lemma 9. *Let $D \subset \mathbb{R}^n$ be a domain such that ∂D has positive capacity.*

- (1) *Suppose that $\overline{B(x_0, r)} \subset D$. Then for every pair of points x and y in $\overline{B(x_0, r)}$ there is a μ_D -minimizer $\gamma_{\mu_D}(x, y)$ which lies in $\overline{B(x_0, r)}$.*
- (2) *For $x_0 \in D$, let $R_0 = \text{dist}(x_0, \partial D)$. Let $x_1 \in B(x_0, R_0)$ and $R_1 = |x_1 - x_0|$. Then, for every $x \in \mathbb{S}$, the μ_D -distance $\mu_D(x_1, x_0 + tx)$, considered as a function of t , is non-decreasing on $R_1 \leq t < R_0$.*
- (3) *If $B_{\mu_D}(x_0, s) \subset B(x_0, R_0)$, then $B_{\mu_D}(x_0, s)$ is starlike with respect to x_0 .*

Proof. (1) Let $x, y \in \overline{B(x_0, r)}$ and let $\gamma_k, k \in \mathbb{N}$, be a sequence of continua in K_{xy} such that

$$\text{cap}(D, \gamma_k) \rightarrow \mu_D(x, y), \quad \text{as } k \rightarrow \infty. \quad (3.3)$$

Let γ_k^p denote the polarization of γ_k with respect to the sphere $S(x_0, r)$ and let $\tilde{\gamma}_k = \gamma_k^p \cap \overline{B(x_0, r)}$. As we mentioned above in Remark 2, $\tilde{\gamma}_k$ is a connected compact set in $\overline{B(x_0, r)}$ and $x, y \in \tilde{\gamma}_k$. Hence, $\tilde{\gamma}_k \in K_{xy}$. Now, it follows from Theorem 3 and Lemma 5 that

$$\mu_D(x, y) \leq \text{cap}(D, \tilde{\gamma}_k) \leq \text{cap}(D, \gamma_k^p) \leq \text{cap}(D, \gamma_k), \quad \text{for all } k \in \mathbb{N}. \quad (3.4)$$

Taking the limit in (3.4) and taking into account (3.3), we conclude that

$$\text{cap}(D, \tilde{\gamma}_k) \rightarrow \mu_D(x, y), \quad \text{as } k \rightarrow \infty.$$

Now, an existence of the required μ_D -minimizer $\gamma_{\mu_D}(x, y) \subset \overline{B(x_0, r)}$ follows from Lemma 7.

(2) For $x \in \mathbb{S}$ and t_1, t_2 such that $R_1 < t_1 < t_2 < R_0$, let $y_1 = x_0 + t_1x, y_2 = x_0 + t_2x$. Suppose that $\gamma_2 = \gamma_{\mu_D}(x_1, y_2)$ is a μ_D -minimizer for the points x_1, y_2 . Let $\gamma_{2,p}$ denote the polarization of γ_2 with respect to the sphere $S(x_0, r)$ with $r = \sqrt{t_1 t_2}$ and let $\tilde{\gamma}_2 = \gamma_{2,p} \cap \overline{B(x_0, r)}$. Since the points y_1 and y_2 are symmetric with respect to $S(x_0, r)$ it follows that $y_1 \in \tilde{\gamma}_2$. Hence, same argument as in part (1) of this proof shows that $\tilde{\gamma}_2 \in K_{x_1 y_1}$. Therefore, applying Theorem 3 and Lemma 5 as above, we conclude that

$$\mu_D(x_1, y_2) = \text{cap}(D, \gamma_{\mu_D}(x_1, y_2)) \geq \text{cap}(D, \gamma_{2,p}) \geq \text{cap}(D, \tilde{\gamma}_2) \geq \mu_D(x_1, y_1),$$

which proves the required monotonicity property.

(3) Now, if $B_{\mu_D}(x_0, s) \subset B(x_0, R_0)$ and $y \in B_{\mu_D}(x_0, s)$, then $\mu_D(x_0, x_0 + t(y - x_0)) \leq \mu_D(x_0, y) \leq s$ for all $t, 0 \leq t \leq 1$, by the monotonicity property proved above. Hence, $x_0 + t(y - x_0) \in B_{\mu_D}(x_0, s)$ for $0 \leq t \leq 1$, which proves that $B_{\mu_D}(x_0, s)$ is starlike with respect to x_0 . \square

Lemma 10. *Let D be a domain in \mathbb{R}^n such that ∂D has positive capacity. For $x_0 \in D$, let $R_0 = \text{dist}(x_0, \partial D)$. Then the function $\mu(x) = \mu_D(x_0, x)$ does not have relative extrema in the ball $B(x_0, R_0)$ except for the absolute minimum at x_0 .*

Proof. (1) Suppose that there exist $x_* \neq x_0$ and $r > 0$ such that $2r < d = |x_* - x_0|$, $\overline{B(x_*, 2r)} \subset B(x_0, R_0)$, and $\mu(x_*) \leq \mu(x)$ for all $x \in \overline{B(x_*, 2r)}$. By part (1) of Lemma 9, there is a continuum $\gamma(x_*) \subset \overline{B(x_0, d)}$ such that

$$\mu(x_*) = \text{cap}(D, \gamma(x_*)). \quad (3.5)$$

Since $\gamma(x_*)$ is closed and connected there are closed and connected sets $\gamma_1 \subset \gamma(x_*)$ and $\gamma_2 \subset \gamma(x_*)$ satisfying the following conditions: (a) $\gamma_1 \subset \overline{B(x_0, d)} \setminus B(x_*, 2r)$ and contains the point x_0 and some point $x_1 \in S(x_*, 2r) \cap \overline{B(x_0, d)}$, (b) $\gamma_2 \subset \overline{B(x_*, r)}$ and contains the point x_* and some point $x_2 \in S(x_*, r)$.

Conditions (a) and (b) show that the continua γ_* , γ_1 and γ_2 satisfy assumptions of Lemma 5 concerning the cases of equality in this lemma and therefore

$$\text{cap}(D, \gamma_1) < \text{cap}(D, \gamma_1 \cup \gamma_2) \leq \text{cap}(D, \gamma(x_*)). \quad (3.6)$$

Since $\gamma_1 \in K_{x_0 x_1}$ we have $\mu(x_1) \leq \text{cap}(D, \gamma_1)$. Since $x_1 \in S(x_*, 2r)$ the latter inequality combined with relations (3.5) and (3.6) contradicts our assumption that $\mu(x_*) \leq \mu(x)$ for all $x \in \overline{B(x_*, 2r)}$. Therefore, $\mu(x)$ can not have relative minimum in $B(x_0, R_0)$ except for the absolute minimum at x_0 .

(2) Suppose that there exist x^* and $r > 0$ such that $\overline{B(x^*, 2r)} \subset B(x_0, R_0)$ and $\mu(x) \leq \mu(x^*)$ for all $x \in B(x^*, r)$. Let H be the hyperplane passing through x_0 and orthogonal to $x^* - x_0$. Below we use the following notations: $d = |x^* - x_0|$, $\rho = |x - x_0|$, and $L = \sqrt{\rho^2 + d^2} + r$; see Figure 1, which illustrates notations used in the proofs of this section.

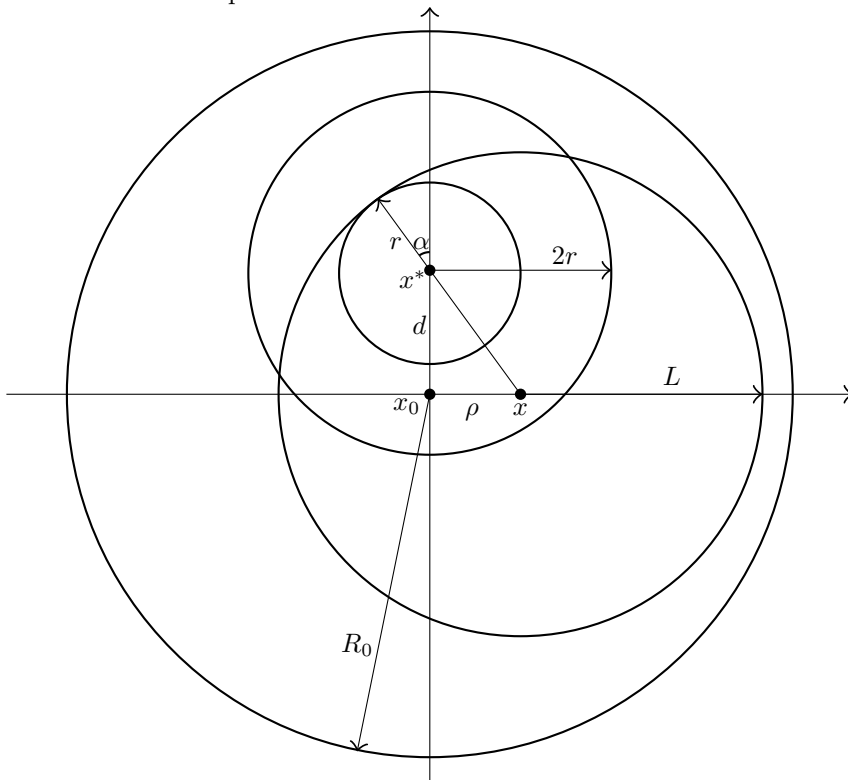


Figure 1. Spheres of Lemma 10.

An elementary geometric calculation shows that if $x \in H$ is such that

$$\rho = |x - x_0| < \frac{r(2d + r)}{2(d + r)}, \quad (3.7)$$

then

$$x_0 \in B(x, L) \quad \text{and} \quad \overline{B(x^*, r)} \subset \overline{B(x, L)} \subset B(x_0, R_0). \quad (3.8)$$

Let

$$y_t = x^* + t \frac{x^* - x}{|x^* - x|}, \quad 0 \leq t \leq r. \quad (3.9)$$

Since conditions (3.8) are satisfied it follows from Lemma 8 that $\mu_D(x_0, y_t)$ considered as a function of t is a non-decreasing function on $0 \leq t \leq r$. Furthermore, since $\mu_D(x_0, x)$ attained its relative maximum at $x = x^*$ it follows that $\mu_D(x_0, x)$ is constant on every radial segment of the ball $B(x^*, r)$ of the form (3.9) if x satisfies condition (3.7). Let $\Phi = \Phi(x^*, x_0, r, \alpha)$ denote the spherical cone, which has a vertex at x^* , radius r , and forms a central angle of opening $\alpha = \arctan \frac{r(2d+r)}{2d(d+r)}$ with the segment $\{y = x^* + t \frac{x^* - x_0}{|x^* - x_0|} : 0 \leq t \leq r\}$. The latter segment is a radius of the ball $B(x^*, r)$. Since every end point $x \in S(x^*, r)$ of the radial segment from x^* to x , which is in the spherical cone Φ , satisfies condition (3.7) it follows that $\mu_D(x_0, x)$ is constant on Φ . Obviously, Φ has interior points and the latter conclusion contradicts the fact established in part (1) of this proof that $\mu(x)$ can not have relative minimum in $B(x_0, R_0) \setminus \{x_0\}$. Thus, our assumption was wrong and $\mu_D(x_0, x)$ does not have relative maxima in $B(x_0, R_0)$. \square

Remark 3. We conjecture that the modulus metric $\mu(x) = \mu_D(x_0, x)$ considered as a function of $x \in D$ can not have relative minima or relative maxima at any point $x \in D$, $x \neq x_0$. We want to stress here that our proof of Lemma 10 is based on polarization and therefore it can not be applied to all points $x \in D$ because polarization changes the domain D , in general.

In the proof below, we will use the following notations. Let x_0 and x be points in D , let $\mu_D(x_0, x) = \mu$, and let $0 < |x - x_0| = r < R$, where $R = \text{dist}(x_0, \partial D)$. Also, let

$$\alpha_0 = \arctan \frac{\sqrt{R^2 - r^2}}{r} \quad (3.10)$$

and

$$\rho_{ext}(\alpha) = R - r \sec \alpha \quad \text{and} \quad \rho_{int}(\alpha) = \frac{r(R \cos \alpha - r)}{R - r \cos \alpha}. \quad (3.11)$$

For $0 < \alpha < \alpha_0$, by $\Phi_{ext}(\alpha)$ we denote the spherical cone with the vertex at x and radius ρ_{ext} that forms the central angle of opening α with the vector $v = x - x_0$. Similarly, by $\Phi_{int}(\alpha)$ we denote the spherical cone with the vertex at x and radius ρ_{int} that forms the central angle of opening α with the vector $-v_1 = x_0 - x$. We will call $\Phi_{ext}(\alpha)$ and $\Phi_{int}(\alpha)$ the exterior cone and the interior cone of $S_{\mu_D}(x_0, \mu)$, respectively. Now we are ready to prove the cone property of the μ_D -spheres stated in Theorem 2 in the Introduction.

Proof of Theorem 2. (1) We claim that, for every α , $0 < \alpha < \alpha_0$, $\Phi_{ext}(\alpha) \subset D \setminus \overline{B_{\mu_D}(x_0, \mu)}$. Let $H_{1/2}$ be a hyperplane passing through the point $x_{1/2} = \frac{1}{2}(x + x_0)$ and orthogonal to the vector $v = x - x_0$. Let $x_{1/2} \in H_{1/2}$ be such that $|x_{1/2} - x_0| = R/2$. Then the angle formed by the vectors v and $v_{1/2} = x_{1/2} - x_0$ equals α_0 ; this is how the value of α_0 in the formula (3.10) was calculated.

Suppose now that for some angle $\alpha < \alpha_0$ there is a point $y \in \Phi_{ext}(\alpha)$ that is in $\overline{B_{\mu_D}(x_0, \mu)}$. By Lemma 9, the function $\mu_D(x_0, x)$ is continuous and can not have relative minimum. Therefore, there is a point $y^* \in \Phi_{ext}(\alpha)$ such that $\mu_D(x_0, y^*) < \mu$.

Let l be the line passing through the points y^* and x and let l intersects $H_{1/2}$ at the point x^* . Then the angle α^* formed by the vectors $v^* = x - x^*$ and $v = x - x_0$ is less than α ; i.e., $0 \leq \alpha^* < \alpha < \alpha_0$. Consider the sphere $S(x^*, \rho)$ with $\rho = \sqrt{|y^* - x^*||x - x^*|}$. The points x and y^* are symmetric with respect to $S(x^*, \rho)$. Notice also that $x_0 \in B(x^*, \rho)$ and $B(x^*, \rho) \subset B(x_0, R_0)$. The latter inclusion follows, after simple calculations, from our definition of the radius $\rho_{ext}(\alpha)$ defined in (3.11).

Let $\gamma_{\mu_D}(x_0, y^*)$ be a μ_D -minimizer for the points x_0 and y^* and let $\gamma_p \subset \overline{B(x^*, \rho)}$ denote the connected component of the polarization of $\gamma_{\mu_D}(x_0, y^*)$ with respect to the sphere $S(x^*, \rho)$. Then γ_p is a continuum in D containing the points x_0 and x and therefore $\gamma_p \subset K_{x_0 x}$. Now, using Theorem 3, we obtain the following

$$\mu_D(x_0, y^*) = \text{cap}(D, \gamma_{\mu_D}(x_0, y^*)) \geq \text{cap}(D, \gamma_p) \geq \mu_D(x_0, x) = \mu.$$

The latter inequalities contradicts our assumption that $\mu_D(x_0, y^*) < \mu$, which proves our claim on the exterior spherical cone.

(2) Now, we prove that $\Phi_{int}(\alpha) \subset B_{\mu_D}(x_0, \mu)$. The proof is similar to the proof given in part (1). We assume that there is a point y in $\Phi_{int}(\alpha)$ that is not in $B_{\mu_D}(x_0, \mu)$. Since, by Lemma 11, $\mu_D(x_0, x)$ does not have relative maxima in $B(x_0, R_0)$ it follows that there is a point y^* in $\Phi_{int}(\alpha)$ such that $\mu_D(x_0, y^*) > \mu$. As in part (1), we consider the line l passing through the points y^* and x . Let α^* denote the angle formed by the vectors $v^* = y^* - x$ and $-v = x_0 - x$. Then $0 \leq \alpha^* < \alpha < \alpha_0$. Let x^* be the point on l such that $|x^* - x| = R_0 - |x^* - x_0|$. Consider the sphere $S(x^*, \rho)$ with the radius $\rho = \sqrt{|x - x^*||y^* - x^*|}$. Then the points y^* and x are symmetric with respect to $S(x^*, \rho)$. Furthermore, $x_0 \in B(x^*, \rho) \subset B(x_0, R_0)$.

Let $\gamma_{\mu_D}(x_0, x)$ be a μ_D -minimizer for points x_0 and x and let $\gamma_p \subset \overline{B(x^*, \rho)}$ denote the connected component of the polarization of $\gamma_{\mu_D}(x_0, x)$ with respect to the sphere $S(x^*, \rho)$. Then γ_p is a continuum in D containing the points x_0 and y^* and therefore $\gamma_p \subset K_{x_0 y^*}$. Now, using Theorem 3, we obtain the following

$$\mu = \mu_D(x_0, x) = \text{cap}(D, \gamma_{\mu_D}(x_0, x)) \geq \text{cap}(D, \gamma_p) \geq \mu_D(x_0, y^*).$$

The latter inequalities contradicts our assumption that $\mu_D(x_0, y^*) > \mu$, which proves our claim on the interior spherical cone. The proof of Theorem 2 is complete. \square

It follows from Theorem 2 and its proof above that stronger versions of statements of Lemma 9 hold true. We present these stronger versions in the following corollary.

Corollary 1. *Under the assumptions of Theorem 2, the following statements hold true.*

- (1) *Let $L(\alpha_0)$ denote the spherical lune which is the intersection of all balls $B(y, R_0/2)$ having centers at $y \in H_{1/2}$ such that $|y - x_0| = R_0/2$. Then there is a μ_D -minimizer $\gamma_{\mu_D}(x_0, x)$ contained in $L(\alpha_0)$.*
- (2) *Let $y \in \mathbb{S}$. The function $\mu_D(t) = \mu_D(x_0, x_0 + ty)$ is strictly increasing for $0 \leq t < R_0$. In particular, $S_{\mu_D}(x_0, \tau)$ can not contain intervals of a line passing through the point x_0 .*
- (3) *If $\overline{B_{\mu_D}(x_0, \tau)} \subset B(x_0, R_0)$, then $B_{\mu_D}(x_0, \tau)$ is strictly starlike, i.e. every ray l_+ in \mathbb{R}^n with the initial point at x_0 intersect the μ_D -sphere $S_{\mu_D}(x_0, \tau)$ at one point.*

Proof. Part (1) follows from the fact used in the proof of Theorem 2 that every ball $B(y, R_0/2)$ with the center $y \in H_{1/2}$ such that $|y - x_0| = R_0/2$ contains a μ_D -minimizer. Parts (2) and (3) follow immediately from the cone properties of the μ_D -spheres. \square

4. INFINITESIMALLY SMALL SPHERES AND CONFORMALITY

Everyone who studied Complex Analysis remembers that conformal mapping transforms small circles to “*infinitesimally small circles*”. However, it was not easy for us to find a precise definition of this term, especially in the n -dimensional setting, in the accessible literature. For our purposes, we adapt the definition introduced by Yu. G. Reshetnyak [10].

Definition 2. Let D be a domain in \mathbb{R}^n and $x_0 \in D$.

- (1) A parameterized family $\mathcal{U}(x_0) = \{U_t(x_0) : 0 < t \leq t_0\}$, of neighborhoods $U_t(x_0) \subset D$ of x_0 is called *almost spherical* if the following holds:
 - (a) $U_t(x_0) \subset U_{t_0}(x_0)$ for all t and there is a homeomorphism φ from $U_{t_0}(x_0)$ to \mathbb{R}^n such that $\varphi(U_t(x_0)) = S(\bar{0}, t)$ for $0 < t \leq t_0$.
 - (b) $\max_{x \in \partial U_t(x_0)} |x - x_0| / \min_{x \in \partial U_t(x_0)} |x - x_0| \rightarrow 1$ as $t \rightarrow 0$.
- (2) If $\mathcal{U}(x_0) = \{U_t(x_0) : 0 < t \leq t_0\}$ is an almost spherical family of neighborhoods $U_t(x_0)$ in D , then the family $\Sigma(x_0) = \{\partial U_t(x_0) : 0 < t \leq t_0\}$ consisting of the boundary surfaces of $U_t(x_0)$ will be called an *infinitesimally small sphere* centered at x_0 .

With this terminology, the main result of Reshetnyak’s paper [10] can be stated in the following form.

Theorem 4 ([10]). *Let D be a domain in \mathbb{R}^n and let $\mathcal{S} = \{\Sigma(x_0) : x_0 \in D\}$ be a collection of infinitesimally small spheres $\Sigma(x_0)$ centered at x_0 such that one such sphere is assigned to each point $x_0 \in D$. Then a homeomorphism f from D onto a domain $\Omega \in \mathbb{R}^n$ is conformal if and only if for every $x_0 \in D$ the image $f(\Sigma(x_0))$ is an infinitesimally small sphere in Ω centered at $f(x_0)$.*

In view of this Reshetnyak’s theorem, to prove Theorem 1 we have to show that for every domain $D \subset \mathbb{R}^n$ and every point $x_0 \in D$ an appropriate truncation $\Sigma'(x_0) = \{S_{\mu_D}(x_0, t) : 0 < t \leq t_0(x_0)\}$ of the family of level sets of the modulus metric μ_D is an infinitesimally small sphere in D centered at x_0 . This will be established in Lemma 12 below. To prove this lemma, we will use polarization with respect to appropriate spheres. An existence of such spheres follows from our next lemma that can be seen as an exercise in elementary geometry.

Lemma 11 (Lemma on 3 spheres). *Let S_1 and S_2 be two concentric spheres centered at $x_0 \in \mathbb{R}^n$ of radii $R_1 = R$ and $R_2 = kR$, respectively, with $R > 0$ and $0 < k < 1$. Then for every pair of points $x_1 \in S_1$ and $x_2 \in S_2$ there is a sphere S_3 of radius $R_3 = R_3(x_1, x_2)$ such that:*

- (1) x_1 and x_2 are symmetric with respect to S_3 .
- (2) x_0 and x_2 belong to the closed ball bounded by S_3 .
- (3) $\frac{k}{\sqrt{1+k^2}}R \leq R_3(x_1, x_2) \leq \frac{k}{1-k}R$.

Proof. Let $x_1 \in S_1$ and $x_2 \in S_2$. Using translation and scaling, if necessary, we may assume without loss of generality that S_1 is the sphere of radius 1 centered at $x_0 = \bar{0}$, then S_2 is the sphere of radius k centered at $x_0 = \bar{0}$. Furthermore, using rotation and reflection, again if necessary, we may assume that x_1 and x_2 belong to a two-dimensional plane P , embedded in \mathbb{R}^n , and that in the plane P the points x_1 and x_2 have the following two-dimensional coordinates: $x_1 = (-\sqrt{1 - k^2 \sin^2 \theta}, k \sin \theta)$ and $x_2 = (k \cos \theta, k \sin \theta)$, $0 \leq \theta \leq \pi$. Thus, under these assumptions, the points x_1 and x_2 lie on the same horizontal line $L \subset P$. See Figure 2, which illustrates notations used in this proof.

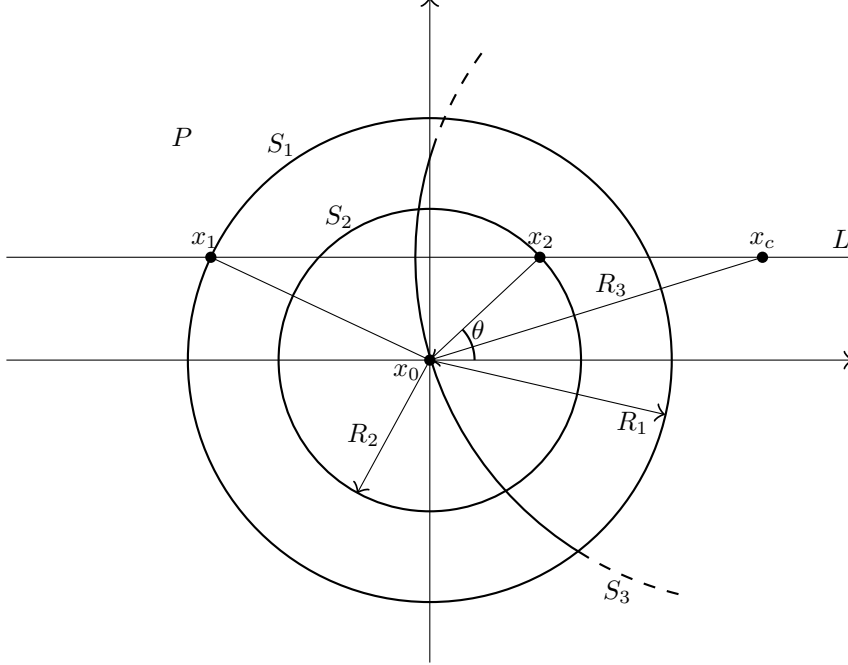


Figure 2. Three spheres lemma.

1) First, we assume that $\pi - \operatorname{arccot} k \leq \theta \leq \pi$. In this case, we define S_3 to be a sphere in \mathbb{R}^n centered at the point $x_c \in P$ with coordinates $(0, k \sin \theta)$ in the plane P and radius

$$R_3 = \sqrt{-k \cos \theta \sqrt{1 - k^2 \sin^2 \theta}}. \quad (4.1)$$

The latter equation shows that the points x_1 and x_2 are symmetric with respect to S_3 . Furthermore, an elementary calculation shows that for all θ , $\pi - \operatorname{arccot} k \leq \theta \leq \pi$, the following equation holds:

$$R_3^2 = -k \cos \theta \sqrt{1 - k^2 \sin^2 \theta} \geq k^2 \sin^2 \theta = (\operatorname{dist}(x_c, x_0))^2. \quad (4.2)$$

Moreover, equality occurs in (4.2) only for $\theta = \pi - \operatorname{arccot} k$. Thus, inequality (4.2) implies that the points x_2 and $x_0 = \bar{0}$ belong to a closed ball in \mathbb{R}^n bounded by S_3 . In particular, if $\theta = \pi - \operatorname{arccot} k$, then the sphere S_3 passes through the origin $x_0 = \bar{0}$. Therefore the sphere S_3 satisfies conditions (1) and (2) for the values of θ under consideration. In addition, it is immediate from (4.1) that the radius R_3 is an increasing function of θ . Hence,

$$\frac{k}{\sqrt{1 + k^2}} \leq R_3 = R_3(x_1, x_2) \leq \sqrt{k}, \quad \text{when } \pi - \operatorname{arccot} k \leq \theta \leq \pi. \quad (4.3)$$

2) Now we turn to the case when $0 \leq \theta \leq \pi - \operatorname{arccot} k$. We claim that in this case there is a unique point $x_c \in L$, with coordinates $(\lambda_c, k \sin \theta)$, $\lambda_c > 0$, in the plane P , such that the points x_1 and x_2 are symmetric with respect to the sphere $S_3 = S_3(x_1, x_2)$ centered at x_c with radius $R_3 = \sqrt{\lambda_c^2 + k^2 \sin^2 \theta}$. Notice that under these conditions, the sphere S_3 passes through the origin $x_0 = \bar{0}$.

First we introduce necessary notations. We fix θ , $0 \leq \theta < \pi - \operatorname{arccot} k$, and consider a point $x(\rho)$ on the line L , which is uniquely determined by the following conditions:

- (a) $x(\rho)$ lies further to the right on L than x_2 and the point $x_i = (0, k \sin \theta)$.
- (b) The distance from $x(\rho)$ to the origin x_0 equals ρ .

Let $S(\rho)$ denote the sphere centered at $x(\rho)$ such that the points x_1 and x_2 are symmetric with respect to $S(\rho)$. Then the radius $\tau = \tau(\rho)$ of this sphere can be found from the equation

$$\tau^2 = (\sqrt{\rho^2 - k^2 \sin^2 \theta} - k \cos \theta) \left(\sqrt{\rho^2 - k^2 \sin^2 \theta} + \sqrt{1 - k^2 \sin^2 \theta} \right). \quad (4.4)$$

To prove the existence part of our claim, it is enough to show that the equation $\tau(\rho) = \rho$ has at least one solution in the case under consideration. The existence of such solution follows from the continuity of function $\tau(\rho)$ given by (4.4) and the following ‘‘boundary’’ relations:

- (α) If $0 \leq \theta \leq \pi/2$ and $\rho = k$, then $\tau(\rho) = \tau(k) = 0 < k = \rho$.
- (α') If $\pi/2 < \theta < \pi - \operatorname{arccot} k$ and $\rho = k \sin \theta$, then it follows from our argument in part (1) of this proof that $\tau(\rho) = \tau(k \sin \theta) < k \sin \theta = \rho$.
- (β) Using equation (4.4) one can easily show that the function $\tau^2(\rho)$ has the following asymptotic expansion:

$$\tau^2(\rho) = \rho^2 + (\sqrt{1 - k^2 \sin^2 \theta} - k \cos \theta) + o(\rho),$$

where $o(\rho) \rightarrow 0$ when $\rho \rightarrow \infty$.

Relations (α) and (β) show that the difference $\tau(\rho) - \rho$ changes its sign when ρ runs from k to ∞ ; similarly, relations (α') and (β) show that $\tau(\rho) - \rho$ changes its sign when ρ runs from $\pi - \operatorname{arccot} k$ to ∞ . Therefore, in each of these cases equation

$$\rho^2 = (\sqrt{\rho^2 - k^2 \sin^2 \theta} - k \cos \theta)(\sqrt{\rho^2 - k^2 \sin^2 \theta} + \sqrt{1 - k^2 \sin^2 \theta}) \quad (4.5)$$

has at least one solution in the corresponding interval. In fact, equation (4.5) can be easily solved and its unique solution $R_3 = R_3(x_1, x_2)$ is

$$R_3^2 = \left(\frac{k^2}{\sqrt{1 - k^2 \sin^2 \theta} - k \cos \theta} + k \cos \theta \right)^2 + k^2 \sin^2 \theta. \quad (4.6)$$

Differentiating both sides of equation (4.6) with respect to θ and then simplifying the output, we obtain:

$$2R_3 \frac{dR_3}{d\theta} = - \frac{2k^3 \sin \theta}{\sqrt{1 - k^2 \sin^2 \theta} (\sqrt{1 - k^2 \sin^2 \theta} - k \cos \theta)^2} < 0. \quad (4.7)$$

Since the derivative $\frac{dR_3}{d\theta}$ is negative, the radius $R_3 = R_3(\theta)$, considered as a function of θ , strictly decreases from $R_3(0) = \frac{k}{1-k}$ to $R_3(\pi - \operatorname{arccot} k) = \frac{k}{\sqrt{1+k^2}}$ when θ varies from 0 to $\pi - \operatorname{arccot} k$. The latter together with the inequality (4.2) proves part (3) of the lemma. Now the proof is complete. \square

Remark 4. In notations of Lemma 11, suppose that $x_1 = (-\sqrt{1 - k^2 \sin^2 \theta}, k \sin \theta)$, $x_2 = (k \cos \theta, k \sin \theta)$ and that $0 \leq \theta \leq \pi/2$. In this case, the monotonicity property of the radius $R_3(x_1, x_2)$ established in the proof of Lemma 11 implies the following bounds for this radius:

$$\frac{k}{\sqrt{1 - k^2}} R \leq R_3(x_1, x_2) \leq \frac{k}{1 - k} R. \quad (4.8)$$

Thus, $R_3(x_1, x_2) \rightarrow \infty$ uniformly on $0 \leq \theta \leq \pi/2$ as $k \rightarrow 1$.

Lemma 12. *Let D be a domain in \mathbb{R}^n . For every $x_0 \in D$, the family $\Sigma(x_0) = \{S_{\mu_D}(x_0, t) : t \in (0, \infty)\}$ of the level surfaces of the modulus metric $\mu_D(x_0, y)$ considered as a function of $y \in D$ has a truncation $\Sigma'(x_0) = \{S_{\mu_D}(x_0, t) : 0 < t \leq t_0(x_0)\}$ which is an infinitesimally small sphere centered at x_0 .*

Proof. Fix $x_0 \in D$. We have to show that an appropriate truncation of $\Sigma(x_0)$ satisfies conditions (a) and (b) of part (1) of Definition 2.

Since, by Lemma 8, the function $\mu_D(x_0, x)$ is continuous and $\mu_D(x_0, x) \rightarrow 0$ as $x \rightarrow x_0$, there is $t_0 > 0$ such that $S_{\mu_D}(x_0, t) \subset S(x_0, R_0)$ for all t , $0 < t \leq t_0$. Here $R_0 = \operatorname{dist}(x_0, \partial D)$. We claim that $\Sigma'(x_0) = \{S_{\mu_D}(x_0, t) : 0 < t \leq t_0\}$ is an infinitesimally small sphere.

Indeed, consider the mapping φ defined by

$$\varphi(x) = \frac{x}{|x|} \mu_D(x_0, x), \quad x \in B_{\mu_D}(x_0, t_0). \quad (4.9)$$

Since $\mu_D(x_0, x)$ is continuous by part (1) of Lemma 8 and it is strictly increasing by part (2) of Corollary 1, it follows from (4.9) that φ maps $B_{\mu_D}(x_0, t_0)$ continuously and one-to-one onto the ball $B(\bar{0}, t_0)$ and such that $\varphi(S_{\mu_D}(x_0, t)) = S(\bar{0}, t)$ for all t , $0 < t \leq t_0$. Therefore, the family $\mathcal{U}(x_0) =$

$\{B_{\mu_D}(x_0, t) : 0 < t \leq t_0\}$ of neighborhoods of x_0 and the mapping φ satisfy condition (a) of part (1) of Definition 2.

It remains to show that the family $\Sigma'(x_0)$ satisfies condition (b) of part (1) of Definition 2. Suppose that this condition is not satisfied for a sequence of the μ_D -spheres $S_i = S_{\mu_D}(x_0, t_i)$, $i \in \mathbb{N}$, where $t_i \rightarrow 0$ as $i \rightarrow \infty$. Then there are an index $i_0 \in \mathbb{N}$ and k , $0 < k < 1$, such that for every $i \geq i_0$ there are points $x_i, y_i \in S_{\mu_D}(x_0, t_i)$ such that

$$\frac{|x_0 - x_i|}{|x_0 - y_i|} = k \quad \text{for all } i \geq i_0. \quad (4.10)$$

Furthermore, since, by Lemma 10, $\mu_D(x_0, x)$ does not have relative maxima in $B(x_0, R_0)$ it follows that for every $i \geq i_0$ there is $z_i \in B(x_0, R_0)$ such that

$$\frac{|x_i - z_i|}{|x_0 - y_i|} < \frac{1}{i} \quad \text{and} \quad \mu_D(x_0, z_i) > t_i. \quad (4.11)$$

Let $k_i = |x_0 - z_i|/|x_0 - y_i|$. From (4.10) and (4.11), we conclude that the inequalities

$$\frac{k}{2} \leq k_i \leq \frac{1+k}{2} < 1 \quad (4.12)$$

hold for all $i \geq i_1$, if $i_1 \geq \max\{i_0, \frac{2}{k}, \frac{2}{1-k}\}$.

It follows from Lemma 11 that for every $i \geq i_1$ there is a sphere \overline{S}_i such that z_i and y_i are symmetric with respect to S_i , the points x_0 and z_i belong to the closed ball \overline{B}_i bounded by S_i and such that the radius R_i of S_i satisfies the inequalities

$$R_i \leq \frac{k_i}{1-k_i}|y_i - x_0| \leq \frac{1+k}{1-k}|y_i - x_0|, \quad (4.13)$$

where the second inequality follows from (4.12). Since $y_i \rightarrow x_0$ as $i \rightarrow \infty$ it follows from (4.13) that there is $i_* \geq i_1$ such that $R_{i_*} < R_0/2$. Since $x_0 \in \overline{B}_{i_*}$, the latter implies that $\overline{B}_{i_*} \subset B(x_0, R_0)$.

Let $\gamma_{\mu_D}(x_0, y_{i_*})$ be a μ_D -minimizer for the points x_0 and y_{i_*} and let $\gamma_p \subset \overline{B}_{i_*}$ denote the connected component of the polarization of $\gamma_{\mu_D}(x_0, y_{i_*})$ with respect to the sphere S_{i_*} . Then γ_p is a continuum in D containing the points x_0 and z_i and therefore $\gamma_p \subset K_{x_0 z_i}$. Now, using Theorem 3 and (4.11), we obtain the following

$$\mu_D(x_0, y_{i_*}) = \text{cap}(D, \gamma_{\mu_D}(x_0, y_{i_*})) \geq \text{cap}(D, \gamma_p) \geq \mu_D(x_0, z_i) > t_{i_*}.$$

The latter inequality contradicts our assumption that $\mu_D(x_0, y_{i_*}) = t_{i_*}$.

Thus, our assumption that there is a sequence $S_i = S_{\mu_D}(x_0, t_i)$, $i \in \mathbb{N}$, of μ_D -spheres, which do not satisfy condition (b) of part (1) of Definition 2 leads to a contradiction. Therefore, the family $\Sigma'(x_0)$ satisfies this condition, which completes our proof of Lemma 12. \square

Now we are ready to prove our main result.

Proof of Theorem 1. Let $f : D \rightarrow \Omega$ be an isometry of the metric space (D, μ_D) onto the metric space (Ω, μ_Ω) . For $x_0 \in D$, let $R_D(x_0) = \text{dist}(x_0, \partial D)$ and $R_\omega(x_0) = \text{dist}(f(x_0), \partial \Omega)$. To each $x_0 \in D$ we assign a family $\Sigma'(x_0) = \{S_{\mu_D}(x_0, t) : 0 < t \leq t_0\}$ of μ_D -spheres $S_{\mu_D}(x_0, t)$ such that $S_{\mu_D}(x_0, t) \subset B(x_0, R_D(x_0))$ and $f(S_{\mu_D}(x_0, t)) \subset B(f(x_0), R_\omega(f(x_0)))$ for all t , $0 < t \leq t_0$. It follows from Lemma 12 that, for each x_0 , $\Sigma'(x_0)$ is an infinitesimally small sphere in D centered at x_0 . Also, since f is an isometry from (D, μ_D) to (Ω, μ_Ω) it follows that $f(S_{\mu_D}(x_0, t)) = S_{\mu_\Omega}(f(x_0), t)$. Since $S_{\mu_\Omega}(f(x_0), t) = f(S_{\mu_D}(x_0, t)) \subset B(f(x_0), R_\omega(f(x_0)))$ for all t , $0 < t \leq t_0$, it follows from Lemma 12 that $\Sigma'_f(f(x_0)) = \{f(S_{\mu_D}(x_0, t)) : 0 < t \leq t_0\}$ is an infinitesimally small sphere in Ω . Thus, for every point x_0 in D there is an infinitesimally small sphere $\Sigma'(x_0)$ that is mapped by f onto an infinitesimally small sphere in Ω which is centered at $f(x_0)$. Therefore, by Theorem 4, f is a conformal mapping from D to Ω . \square

Remark 5. It is tempting to use the polarization technique alone, without referencing to the rather deep Reshetnyak's Theorem 4, to prove conformality of isometries f between metric spaces (D, μ_D) and (Ω, μ_Ω) . At a first glance it looks possible since, if the image $f(S(x_0, r))$ of a sphere $S(x_0, r) \subset D$ is not a round sphere, then it is squeezed between two spheres $S_1 = S(f(x_0), r)$ and $S_2 = S(f(x_0), R)$ such that $0 < r/R = k < 1$. Then, by Lemma 11, we may find the third sphere S_3 and then use polarization with respect to S_3 as in the proof of Lemma 12 to get a contradiction to the assumption of non-roundness of $f(S(x_0, r))$. The only obstacle for this "proof" is the inequality (4.8) of Remark 4. Precisely, this inequality shows that the radius R_3 of the sphere S_3 may grow without bounds as $k \rightarrow 1$ and therefore polarization with respect to S_3 will eventually destroy the domain D if it is not the whole space \mathbb{R}^n .

5. OPEN QUESTIONS AND FURTHER RESEARCH

Our polarization approach is essentially geometrical and thus can be adapted to prove similar results for some other metrics. What is needed is a few basic properties of the metric and polarization inequality akin to (3.2). But polarization (or symmetrization) alone does not provide enough information to study, for instance, delicate properties of the μ_D -spheres and μ_D -minimizers while other tools are not available. This is why many questions about their structure remain open. Below, we mention three of them.

Problem 1. Prove that the μ_D -spheres in $D \subset \mathbb{R}^n$, $n \geq 3$, generically are smooth topological spheres or finite collections of disjoint smooth topological spheres.

Describe the structure of $S_{\mu_D}(x_0, r)$ near its critical points; i.e. near the points $x \in D$, where $S_{\mu_D}(x_0, r)$ is not smooth.

Problem 2. Prove that every μ_D -minimizer $\gamma_{\mu_D}(x, y)$ in $D \subset \mathbb{R}^n$, $n \geq 3$, is a smooth Jordan arc. Currently it is not known if $\gamma_{\mu_D}(x, y)$ is an irreducible continuum or even whether or not $\gamma_{\mu_D}(x, y)$ may have interior points.

To state our next problem we need some terminology. If D is a domain in $\overline{\mathbb{R}^n}$, $x, y \in D$, and $\gamma \in K_{xy}$ is such that $\mu_D(x, y) = \text{cap}(D, \gamma)$, then we say that γ is a μ_D -minimizer with endpoints x, y . We say that a family $\Gamma = \{\gamma\}$ of μ_D -minimizers $\gamma \subset D$ foliates D if: (a) $\cup_{\gamma \in \Gamma} \gamma = D$ and (b) if $\gamma_1, \gamma_2 \in \Gamma$ and there is $x \in \gamma_1 \cap \gamma_2$, which is not an endpoint for at least one of these μ_D -minimizers, then either $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$.

Problem 3. Let D be a domain in \mathbb{R}^n , $n \geq 3$, supplied with the μ_D -metric. Then D has a family of μ_D -minimizers foliating D if and only if D is a topological ball or D is a topological spherical shell.

Remark 6. In all three problems stated above we assume that $n \geq 3$. In the planar case, when $n = 2$, these problems are easier and can be resolved within the frame of the Jenkins' theory on extremal partitioning, see [15]. But this is already a topic for another paper.

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