

A BOUNDEDNESS THEOREM FOR CONE SINGULARITIES

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ABSTRACT. A cone singularity is a normal affine variety X with an effective one-dimensional torus action with a unique fixed point $x \in X$ which lies in the closure of any orbit of the k^* -action. In this article, we prove a boundedness theorem for cone singularities in terms of their dimension, singularities, and isotropies. Given d and N two positive integers and ϵ a positive real number, we prove that the class of d -dimensional ϵ -log canonical cone singularities with isotropies bounded by N forms a bounded family.

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1. INTRODUCTION

In algebraic geometry we are mostly interested on smooth varieties. However, singular varieties appear naturally when studying smooth objects. For instance, in Mori's theory singularities appear when running a minimal model program (see, e.g. [KM98, HK10]), and singularities also appear on the Gromov-Hausdorff limit of a sequence of Kähler-Einstein manifolds (see, e.g. [DS14, DS16]). Since the introduction of the minimal model program its been clear that certain classes of singularities are of special importance for birational geometers [Kol13]. The development of projective geometry has been tangled with the understanding of the theory of singularities [Xu17]. Indeed, Kawamata log terminal singularities, which are the main class of singularities on the MMP, are a local analogue of Fano varieties. Unfortunately, a complete characterization of klt singularities in dimensions greater or equal than four seems to be unfeasible [Kol16]. However, it is expected that the boundedness of Fano varieties due to Birkar [Bir16a, Bir16b] will have a vast number of applications to the understanding of klt singularities. In this article, we investigate an application of such boundedness result to the study of the so-called cone singularities (see, e.g. [LS13]). Cone singularities appear naturally in many contexts of algebraic geometry: Toric geometry [CLS11, Ful93], \mathbb{T} -varieties [AH06, AHS08, AIP⁺12], terminal 3-fold singularities [Hay05a, Hay05b, Tzi05], stability theory of klt singularities [LX16, LX17, Li17],

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the graded ring of a valuation [Tei03], Gromov-Hausdorff limits [DS14, DS16], and Cox rings of Fano varieties [Bro13, GOST15], among many others. A cone over a projective variety Y is a local version of such variety, and the global geometry of Y is reflected on the singularity at the vertex of the cone. For instance, the cone singularity is Kawamata log terminal if and only if the corresponding normalized Chow quotient Y is of Fano type. Henceforth, it is expected that the boundedness of Fano varieties implies the boundedness of some klt cone singularities, and more generally the boundedness of certain klt singularities. In this article, we give a first step in this direction proving that some natural class of klt cone singularities forms a bounded family.

A cone singularity is a normal affine variety X with an effective one-dimensional torus action with a unique fixed point $x \in X$ which lies in the closure of any orbit of the k^* -action. The fixed point for the torus action is often called the vertex of the cone singularity. We say that the cone singularity has isotropies bounded by N if for every point of the cone singularity the corresponding isotropy group is either k^* or a finite group of order less than or equal to N . By definition, the only point at which the isotropy group is k^* is the vertex for the torus action.

In order to obtain bounded families of Fano varieties [Bir16b, Theorem 1.1], it is necessary to impose that such projective varieties have mild singularities [Bir17]. It does not suffice to assume that the Fano varieties are Kawamata log terminal; it is indeed necessary to bound the log discrepancies away from zero. This forces us to work with the class of ϵ -log canonical singularities for some positive real number ϵ . Analogously, in order to show that a class of cone singularities is bounded, it is necessary to impose that they have ϵ -log canonical singularities around the vertex. We prove that a bound on the dimension, singularities, and isotropies are necessary and sufficient to obtain a bounded class of cone singularities:

Theorem 1. *Let d and N be positive integers and ϵ a positive real number. The class of d -dimensional ϵ -log canonical cone singularities with isotropies bounded by N forms a bounded family.*

In subsection 2.4, we will give examples where the statement of the theorem fails if we weaken the conditions on d , N or ϵ . It is then expected that many algebraic invariants take finitely many possible values on the class of singularities considered in the main theorem. We remark two invariants which may be of particular interest.

The minimal log discrepancy is an invariant defined to measure the singularities of an algebraic variety (see, e.g. [Amb99, Mor18]). Its importance relies on its connection with termination of flips [Sho04]. In this direction, we prove the finiteness of minimal log discrepancies of the aforementioned cone singularities.

Corollary 1. *Let d and N be positive integers and ϵ a positive real number. The set of minimal log discrepancies of d -dimensional ϵ -log canonical cone singularities with isotropies bounded by N forms a finite set.*

On the other hand, Chenyang Xu proved the finiteness of the algebraic fundamental group of a klt singularity [Xu14]. This result is related to the finiteness of the fundamental group of the smooth locus of log Fano pairs [TX17]. In this direction, we prove the existence of a bound for the order of such groups for certain cone singularities.

Corollary 2. *Let d and N be positive integers and ϵ a positive real number. The possible orders of the algebraic fundamental group of d -dimensional ϵ -log canonical cone singularities with isotropies bounded by N form a finite set.*

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2. PRELIMINARIES

All varieties in this paper are quasi-projective and normal over a fixed algebraically closed field k of characteristic zero unless stated otherwise. In this section we collect some definitions and preliminary results which will be used in the proof of the main theorem.

2.1. Cone singularities. In this subsection, we will introduce the definition of cone singularities which will be used in this paper and we will prove some basic properties of these singularities.

Definition 2.1. A point $x \in X$ is said to be a *cone singularity* if X is a normal affine variety which admits an effective k^* -action such that $x \in X$ is the unique fixed point which is contained in the closure of any orbit for the action. In the above situation, we say that k^* gives X the structure of a *cone singularity*. The point $x \in X$ which is invariant under the k^* -action is called the *vertex*. We will often say that $x \in X$ is a cone singularity to precise that x is the vertex for the k^* -action.

Definition 2.2. Let X be a quasi-projective variety with a k^* -action embedded in a projective space \mathbb{P}^N . There exists an open set of the variety X on which all the orbits of the k^* -action have the same dimension d and the same degree k . The *Chow quotient* of X is the closure of the set of points which correspond to such orbits on $\text{Chow}_{d,k}(\mathbb{P}^N)$, the Chow variety which parametrizes cycles of dimension d and degree k on the projective space \mathbb{P}^N . The isomorphism class of the Chow quotient is independent from the embedding of X in a projective space. The *normalized Chow quotient* of X is the normalization of the Chow quotient of X .

We start recalling a classic theorem which characterizes affine normal varieties with effective k^* -actions due to Demazure (see, e.g. [Dem88, 3.5]).

Theorem 2.3. *Let X be a normal affine variety with an effective k^* -action. Then, we can write*

$$X \simeq \text{Spec} \left(\bigoplus_{n \geq 0} H^0(Y, \mathcal{O}_Y(nD)) \right)$$

where Y is a quasi-projective variety and D is a \mathbb{Q} -Cartier \mathbb{Q} -divisor on Y .

Notation 2.4. Given a normal affine variety X with an effective k^* -action, we will denote by Y a quasi-projective variety and by D a \mathbb{Q} -Cartier \mathbb{Q} -divisor on Y realizing the isomorphism in Theorem 2.3. We will write

$$D = \sum_{i=1}^k \frac{p_i}{q_i} D_i,$$

where the D_i 's are pairwise different prime divisors on Y and p_i and q_i are coprime integers. We denote by \tilde{X} the relative spectrum of the divisorial sheaf

$$\mathcal{A}(D) := \bigoplus_{n \geq 0} \mathcal{O}_Y(nD)$$

on Y . Observe that the natural inclusion of sheaves $\mathcal{O}_Y \hookrightarrow \mathcal{A}(D)$ induces a good quotient $\pi: \tilde{X} \rightarrow Y$ for the k^* -action on \tilde{X} . The k^* -action on \tilde{X} is induced by the \mathbb{N} -grading of $\mathcal{A}(D)$. We have a birational contraction $r: \tilde{X} \rightarrow X$ which contracts a divisor E_0 on \tilde{X} , moreover this divisor is fixed by the k^* -action and dominates Y . Hence, we have an induced rational map $X \dashrightarrow Y$ which by abuse of notation we may denote by π as well.

The following proposition gives a characterization of cone singularities, it follows from [LS13, §4].

Proposition 2.5. *A normal affine variety with an effective k^* -action is a cone singularity if and only if Y is projective and D is a semiample and big \mathbb{Q} -Cartier \mathbb{Q} -divisor. In particular, the birational morphism $r: \tilde{X} \rightarrow X$ contracts the divisor E_0 to the vertex, and we have a good quotient $\pi: X - \{x\} \rightarrow Y$.*

Remark 2.6. We know that every cone singularity corresponds to a semiample and big \mathbb{Q} -Cartier \mathbb{Q} -divisor on a projective variety Y . Furthermore, we may replace the variety Y with the image of the morphism induced by a sufficiently large and divisible multiple of D to assume that Y is projective and D is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. This latter model is said to be minimal in the sense of [AH06, Definition 8.7]. Indeed, this variety is the normalized Chow quotient of Y , which in this case coincides with the GIT quotient since the GIT decomposition has a unique maximal dimensional chamber (see, e.g. [HK00]). Observe that pulling-back D to a higher model of Y does not change the cone singularity $x \in X$; however, it changes the model \tilde{X} introduced in 2.4.

In what follows, we may say that the cone singularity $x \in X$ corresponds to the ample \mathbb{Q} -Cartier \mathbb{Q} -divisor D on the normalized Chow quotient Y , or simply, corresponds to the couple (Y, D) . Observe that our definition of couples differs from the classic one in which D is assumed to be reduced (see Definition 2.9).

The following lemma gives a description of the canonical divisor of X in terms of the couple (Y, D) (see, e.g. [PS11, Theorem 3.21] or [Wat81, Theorem 2.8]).

Lemma 2.7. *Let $x \in X$ be a cone singularity corresponding to the couple (Y, D) . Then the canonical divisor of X is given by*

$$K_X = \pi^*(K_Y) + \sum_{i=1}^k (q_i - 1) \pi_*^{-1} D_i,$$

and the canonical divisor of \tilde{X} is given by

$$K_{\tilde{X}} = \pi^*(K_Y) + \sum_{i=1}^k (q_i - 1) \pi_*^{-1} D_i - E_0.$$

The following lemma is proved in [ADHL15, Proposition 1.3.5.7] in the context of Cox rings.

Lemma 2.8. *Let $x \in X$ be a cone singularity corresponding to the couple (Y, D) . Let $x_0 \in X$ any point which is not the vertex. The order of the isotropy group of k^* at x_0 equals the Cartier index of D at $\pi(x_0)$.*

2.2. Log discrepancies. In this subsection, we will introduce the definition of log discrepancies. We will prove a formula relating the log discrepancies of a cone singularity with the log discrepancies with respect to certain pair structure on the normalized Chow quotient. This formula is implicit in the proof of [LS13, Theorem 4.7].

Definition 2.9. A couple (Y, B) consists of a normal quasi-projective algebraic variety Y and a \mathbb{Q} -divisor B on Y . A couple (Y, B) is said to be a *sub-pair* if the \mathbb{Q} -divisor $K_Y + B$ is \mathbb{Q} -Cartier. A sub-pair (Y, B) is said to be a *pair* if B is an effective \mathbb{Q} -divisor.

Definition 2.10. Consider a pair (Y, B) , a projective birational morphism $f: Y' \rightarrow Y$ from a quasi-projective normal variety Y' and a prime divisor E on Y' . We define the *log discrepancy* of (Y, B) with respect to E to be

$$a_E(K_Y + B) = 1 + \text{coeff}_E(K_{Y'} - f^*(K_Y + B)).$$

We say that a pair (Y, B) is ϵ -log canonical if the log discrepancies with respect to any prime divisor over E are greater or equal to ϵ . We say that a pair (Y, B) is *Kawamata log terminal*, or simply *klt*, if the log discrepancies with respect to any prime divisor over Y are greater than zero.

Remark 2.11. A pair (Y, B) is ϵ -log canonical if and only if there exists a projective birational morphism $f: Y' \rightarrow Y$ from a smooth quasi-projective variety Y' such that the exceptional locus E of f is divisorial, the divisor $E \cup f_*^{-1}(B)$ has simple normal crossing support on Y' , and the \mathbb{Q} -divisor

$$K_{Y'} - f^*(K_Y + B),$$

has coefficients greater or equal than $\epsilon - 1$.

Proposition 2.12. *Let $x \in X$ be a cone singularity corresponding to the couple (Y, D) . The fields of fractions of X is isomorphic to $k(Y)[M]$ where M is the lattice of characters of the torus. Hence, every divisorial valuation on Y induces a divisorial valuation on X .*

Notation 2.13. Given a divisorial valuation E on Y we will denote by E_X the corresponding divisorial valuation on X . Moreover, we will denote by χ^u the *character* of the torus corresponding to $u \in M$. Hence, every rational function on X has the form $f\chi^u$ where f is a rational function on Y and $u \in M$.

Definition 2.14. Given a projective variety Y , a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on Y , a projective birational morphism $f: Y' \rightarrow Y$, and a prime divisor E on Y , we define the *Weil index* of D at E to be the smallest positive integer μ such that $\mu f^*(D)$ is a Weil divisor at the generic point of E , i.e. the coefficient of $\mu f^*(D)$ at E is an integer. If E is non-exceptional over Y then the Weil index of D with respect to E is just q_E where $\frac{p_E}{q_E}$ is the coefficient of D at E with p_E and q_E coprime integers. Observe that the Weil index of D at E does not depend on $f: Y' \rightarrow Y$, it only depends on the divisorial valuation corresponding to E . We will denote the Weil index of D at E by $W_E(D)$.

The following proposition is straightforward from the definition of Weil index.

Proposition 2.15. *Let D be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on a projective variety Y . The Weil index at any exceptional divisor over Y is less than or equal to the Cartier index of D .*

The following proposition is well-known (see e.g. [PS11, Proposition 3.11]).

Proposition 2.16. *Any k^* -invariant Cartier divisor is principal on a cone singularity $x \in X$, i.e. for any k^* -invariant Cartier divisor D on X we may find a rational function $f \in k(Y)$ and $u \in M$ such that $D = \text{div}_X(f\chi^u)$.*

Remark 2.17. The ring associated to the cone singularity $x \in X$ has a natural M -grading (see, e.g. [AH06]). However, the weighted monoid of such grading is isomorphic to \mathbb{N} . We say that an element $u \in M$ is *positive* if it lies in the weighted monoid, and we say it is *negative* if its additive inverse is positive.

Since any k^* -invariant Cartier divisor D on a cone singularity is principal, we are interested on the principal divisor corresponding to the rational function $f\chi^u$ on $x \in X$. The following proposition gives us the corresponding principal divisor on X (see, e.g. [PS11, Proposition 3.14]).

Proposition 2.18. *Let $x \in X$ be a cone singularity corresponding to the couple (Y, D) and $f\chi^u$ a rational function on X . Then we can write*

$$\text{div}_X(f\chi^u) = \sum_{i=1}^k q_i \left(u \frac{p_i}{q_i} + \text{ord}_{D_i}(f) \right) \pi_*^{-1} D_i,$$

and

$$\text{div}_{\tilde{X}}(f\chi^u) = uE_0 + \sum_{i=1}^k q_i \left(u \frac{p_i}{q_i} + \text{ord}_{D_i}(f) \right) \pi_*^{-1} D_i.$$

Here $\text{div}_X(f)$ denotes the principal divisor on X associated to the rational function $f \in k(X)$.

Proposition 2.19. *Let $x \in X$ be a Kawamata log terminal cone singularity corresponding to the ample \mathbb{Q} -Cartier \mathbb{Q} -divisor D on the projective variety Y . There exist a boundary divisor B on Y such that for each divisorial valuation E over Y we have*

$$a_{E_X}(K_X) = W_E(D)a_E(K_Y + B).$$

Moreover, the divisor $-(K_Y + B)$ is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor.

Proof. Since the cone singularity $x \in X$ is Kawamata log terminal then K_X is \mathbb{Q} -Cartier and k^* -invariant. Hence, by Proposition 2.16, we know that we can write

$$(2.1) \quad mK_X = \operatorname{div}_X(f\chi^u),$$

where f is a rational function on Y and $u \in M$. Pushing-forward the divisor mK_X to Y via π , and considering equation (2.1), Lemma 2.7, and Proposition 2.18, we obtain

$$(2.2) \quad m(K_Y + B) = \operatorname{div}_Y(f) + uD,$$

where $B = \sum_{i=1}^k \frac{q_i-1}{q_i} D_i$ is an effective \mathbb{Q} -divisor. Moreover, since $H := \operatorname{div}_Y(f)$ is principal we deduce that (Y, B) is a pair and $-(K_Y + B)$ is an ample \mathbb{Q} -divisor since u and m are integers of opposite sign. Hence, it suffices to prove the equality relating the log discrepancies of (Y, B) with those of X . Let $f: Y' \rightarrow Y$ a projective birational morphism from a normal projective variety Y' and E a prime divisor on Y' . Observe that the cone singularity corresponding to the couple (Y, D) is equal to the cone singularity corresponding to the couple $(Y', f^*(D))$ (see Remark 2.6). Let \tilde{X}' be the relative spectrum on Y' of the divisorial sheaf

$$\bigoplus_{m \geq 0} \mathcal{O}_{Y'}(mf^*(D)).$$

Observe that the center of E_X on \tilde{X}' is just the strict transform of E on \tilde{X}' . Thus, we have a commutative diagram as follows:

$$\begin{array}{ccccc} \tilde{X}' & \xrightarrow{\tilde{f}} & \tilde{X} & \xrightarrow{r} & X \\ \pi' \downarrow & & \downarrow \pi & \swarrow \pi & \\ Y' & \xrightarrow{f} & Y & & \end{array}$$

By Proposition 2.18, we have that

$$\operatorname{coeff}_{E_X}(\tilde{f}^*(r^*(K_X))) = \frac{\operatorname{div}_{\tilde{X}'}(f\chi^u)}{m} = \frac{W_E(D)}{m} (u\operatorname{coeff}_E(f^*(D)) + \operatorname{coeff}_E(f^*(H))).$$

On the other hand, by Lemma 2.7 we have that

$$\operatorname{coeff}_{E_X}(K_{\tilde{X}'}) = W_E(D)\operatorname{coeff}_E(K_Y) + W_E(D) - 1.$$

Hence, the log discrepancy of K_X is given by

$$a_{E_X}(K_X) = W_E(D) \left(\operatorname{coeff}_E(K_Y) - \frac{u}{m}\operatorname{coeff}_E(f^*(D)) - \frac{1}{m}\operatorname{coeff}_E(f^*(H)) + 1 \right).$$

From equation (2.2), we deduce that

$$\frac{u}{m}\operatorname{coeff}_E(f^*(D)) - \frac{1}{m}\operatorname{coeff}_E(f^*(H)) = -\operatorname{coeff}_E(f^*(K_Y + B)),$$

so we can write

$$a_{E_X}(K_X) = W_E(D) (\operatorname{coeff}_E(K_Y) - \operatorname{coeff}_E(f^*(K_Y + B)) + 1) = W_E(D)a_E(Y, D).$$

Thus, for any divisor E over Y we get the relation

$$a_{E_X}(K_X) = W_E(D)a_E(K_Y + B).$$

□

Definition 2.20. A log pair (Y, B) is said to be *log Fano* if it is klt and $-(K_Y + B)$ is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. In what follows, we may call (Y, B) the *log Fano quotient* of the cone singularity. Observe that the log Fano quotient (Y, B) of a cone singularity $x \in X$ may not be equal to the corresponding couple (Y, D) .

The following remark relates the log Fano quotient and the corresponding couple of a klt cone singularity.

Remark 2.21. Let $x \in X$ be a cone singularity corresponding to the ample \mathbb{Q} -Cartier \mathbb{Q} -divisor D on the projective variety Y . The boundary divisor on Y associated to the log Fano quotient of the cone singularity is

$$B = \sum_{i=1}^k \left(1 - \frac{1}{q_i}\right) D_i,$$

where the q_i 's are as in Notation 2.4. This means that the log Fano quotient of a cone singularity has a boundary with standard coefficients in the sense of [Mor18, Definition 2.4]. Observe that the log Fano quotient of a cone singularity is uniquely determined by the pair (Y, D) . On the other hand, the isomorphism class of $x \in X$ is determined by the couple (Y, D) up to isomorphisms in the first component and linear equivalence on the second component, i.e. if D_1 and D_2 are two \mathbb{Q} -Cartier \mathbb{Q} -divisors such that $D_1 \sim D_2$, then the couples (Y, D_1) and (Y, D_2) determine isomorphic cone singularities (see, e.g. [AH06, Proposition 8.6]). Here, we say that two \mathbb{Q} -divisors D_1 and D_2 are linearly equivalent if $D_1 - D_2$ is a principal divisor on Y .

Corollary 2.22. *Let ϵ be a positive real number and N be a positive integer. Let $x \in X$ be an ϵ -log canonical cone singularity with isotropies bounded by N , then its log Fano quotient (Y, B) is $\frac{\epsilon}{N}$ -log canonical.*

Proof. Let (Y, D) the couple corresponding to the cone singularity $x \in X$, and let E be a prime divisor over Y . By Proposition 2.15, we know that the Weil index of D at E is at most the Cartier index of D . On the other hand, by Proposition 2.8, we know that the Cartier index of D is bounded by N . Therefore, we have the inequality $W_E(D) \leq N$ for any prime divisor E over Y . By Proposition 2.19, we obtain

$$a_E(K_Y + B) = \frac{a_{E_X}(K_X)}{W_E(D)} \geq \frac{\epsilon}{N}.$$

□

2.3. Bounded families of Fano varieties. In this subsection, we recall the boundedness of Fano varieties due to Birkar [Bir16a, Bir16b] and a result about the Neron-Severi space on families of Fano varieties due to Hacon and Xu (see [HX15, Proposition 2.8]).

Definition 2.23. We say that a class of schemes \mathcal{C} is *bounded* if there exists a morphism $\phi: \mathcal{X} \rightarrow T$ between two schemes of finite type such that every scheme on the class \mathcal{C} appears as a geometric fiber of ϕ .

If the class of schemes \mathcal{C} is a class of projective varieties we will also require that ϕ is a projective morphism between possibly reducible quasi-projective varieties.

If \mathcal{C} is a class of couples (Y, B) we say that it is *log bounded* if there exists a projective morphism $\phi: \mathcal{X} \rightarrow T$ between quasi-projective varieties and a \mathbb{Q} -divisor $\mathcal{B} \subset \mathcal{X}$ such that for each $(Y, B) \in \mathcal{C}$ there exists a closed point $t \in T$ and an isomorphism $Y \simeq \mathcal{X}_t$ so that the support of B is contained in the support of \mathcal{B}_t under this isomorphism. Moreover, we say that the class of couples (Y, B) is *log bounded with coefficients* or *strictly log bounded* if the isomorphism $Y \simeq \mathcal{X}_t$ induces an isomorphism of B and \mathcal{B}_t with their corresponding coefficients. We will call $\phi: \mathcal{X} \rightarrow T$ the *bounding family* and $\mathcal{B} \subset \mathcal{X}$ the *bounding divisor*.

We say that a class \mathcal{C} of couples (Y, D) is *log bounded with coefficients up to linear equivalence* or *strictly log bounded up to linear equivalence*, if there exists a class of couples \mathcal{C}' which is log bounded with coefficients, such that for each $(Y, D) \in \mathcal{C}$ there exists $(Y', D') \in \mathcal{C}'$ and an isomorphism $f: Y \rightarrow Y'$ for which $f^*(D') \sim D$, or equivalently, $D' \sim f_*(D)$.

Finally, we say that a class of singularities \mathcal{C} is *bounded*, if for each $x \in X$ belonging to \mathcal{C} we can find an affine neighborhood of $x \in U \subset X$, so that the schemes U are bounded in the above sense.

The following proposition is a consequence of the functoriality of polyhedral divisors (see, e.g. [AH06, Proposition 8.6]).

Proposition 2.24. *A class of cone singularities is bounded if the class of corresponding couples is strictly log bounded up to linear equivalence.*

Proof. Let \mathcal{C}_X be a class of cone singularities and \mathcal{C}_Y be the corresponding class of couples. By [Ale94, Theorem 3.10], it suffices to prove that for every sequence $x_i \in X_i$ of cone singularities there exists an infinite sub-sequence which is bounded. Let (Y_i, D_i) be the corresponding sequence of couples, where Y_i is projective and D_i is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on Y_i . Let $\phi: \mathcal{X} \rightarrow T$ and $\mathcal{D} \subset \mathcal{X}$ be a family and a divisor realizing the strictly log boundedness of (Y_i, D_i) up to linear equivalence. Passing to a subsequence, we may assume that the points t_i corresponding to the couples (Y_i, D_i) are dense on T . Now, we consider the variety

$$\mathcal{X}_X := \text{Spec} \left(\bigoplus_{m \geq 0} H^0(\mathcal{X}/T, m\mathcal{D}) \right),$$

which has an structure morphism $\phi_X: \mathcal{X}_X \rightarrow T$. By construction we have isomorphisms

$$\mathcal{X}_{X, t_i} \simeq \text{Spec} \left(\bigoplus_{m \geq 0} H^0(\mathcal{X}_{t_i}, m\mathcal{D}_{t_i}) \right) \simeq \text{Spec} \left(\bigoplus_{m \geq 0} H^0(Y_i, mD_i) \right) \simeq X_{t_i}.$$

Therefore, we conclude that the morphism $\phi_X: \mathcal{X}_X \rightarrow T$ is a bounding family for the cone singularities $x_i \in X_i$ which belong to the class \mathcal{C}_X . \square

Theorem 2.25. *Let d be a positive integer and ϵ a positive real number. The set of varieties Y for which the following conditions hold:*

- *Y is a projective variety of dimension d ,*
- *There exists a boundary divisor B on Y such that (Y, B) is an ϵ -log canonical pair, and*
- *the \mathbb{Q} -Cartier \mathbb{Q} -divisor $-(K_Y + B)$ is ample,*

forms a bounded family.

We are also interested to bound the pairs (Y, B) . In order to do so, we also need to impose a condition on the coefficients of B as the following corollary shows.

Corollary 2.26. *Let d be a positive integer, ϵ a positive real number and \mathcal{R} a set of rational numbers satisfying the descending chain condition. The set of pairs (Y, B) for which the following conditions hold:*

- *Y is a projective variety of dimension d ,*
- *the pair (Y, B) is ϵ -log canonical,*
- *the \mathbb{Q} -Cartier \mathbb{Q} -divisor $-(K_Y + B)$ is ample, and*
- *the coefficients of B belong to \mathcal{R} .*

forms a log bounded family. Moreover, if \mathcal{R} is finite, then the pairs (Y, B) forms a strictly log bounded family.

Proof. By Theorem 2.25, we may find a positive real number C and for each Y as in the statement we may choose a very ample Cartier divisor A_Y on Y so that $A_Y^d \leq C$. We may further assume that $(-K_Y) \cdot A^{d-1} \leq C$. Let δ be a positive rational number which is smaller than any element of \mathcal{R} , then we have that

$$\frac{1}{\delta} \text{red}(B) \cdot A^{d-1} \leq B \cdot A^{d-1} \leq -K_Y \cdot A^{d-1} \leq C.$$

Thus, by [Ale94, Lemma 3.7.(2)] we conclude that the pairs (Y, B) are log bounded. If \mathcal{R} is finite, then the log boundedness of (Y, B) with coefficients follows from the log boundedness of (Y, B) by taking all the possible combinations for the coefficients of the bounding divisor on \mathcal{R} . \square

Definition 2.27. Given a projective morphism $\phi: \mathcal{X} \rightarrow T$ we say that \mathcal{X} is of *Fano type over T* if there exists a big boundary \mathcal{B} over T on \mathcal{X} such that $(\mathcal{X}, \mathcal{B})$ is klt and $K_{\mathcal{X}} + \mathcal{B} \sim_{\mathbb{Q}, T} 0$.

Proposition 2.28. *Let $\phi: \mathcal{X} \rightarrow T$ be a projective morphism such that \mathcal{X} is of Fano type over T . Up to a base change, for every $t \in T$ the following three conditions hold:*

- *The restriction morphism $\rho_t: N^1(\mathcal{X}/T) \rightarrow N^1(\mathcal{X}_t)$ is an isomorphism,*
- *the restriction morphism induce isomorphisms $\text{Cox}(\mathcal{X}/T) \simeq \text{Cox}(\mathcal{X}_t)$,*
- *we have that $\rho_t(\text{Mov}(\mathcal{X}/T)) = \text{Mov}(\mathcal{X}_t)$, and*
- *there is a one-to-one correspondence between the two Mori chamber decompositions.*

Proof. The first, third and fourth claims are proved in [HX15, Proposition 2.8]. Since the Cox ring $\text{Cox}(\mathcal{X}/T)$ is finitely generated (see, e.g. [BCHM10, Corollary 1.3.2]), after shrinking T we may assume that the restriction of the generating sections of the Cox ring $\text{Cox}(\mathcal{X}/T)$ to \mathcal{X}_t are injective for every $t \in T$. On the other hand, by [HX15, Proposition 2.7] or [dFH11, Theorem 1.1], we know that they are also surjective, proving the second claim. \square

2.4. Examples. In this subsection, we give some examples in which the statement of the main theorem does not hold if we weaken the assumptions on d, ϵ or N .

Example 2.29. Let (x_0, \dots, x_d) be the coordinates of the $(d+1)$ -dimensional affine space k^{d+1} . Consider the cone singularity structure on k^{d+1} given by the diagonal action of k^* . In this case, the action has trivial isotropies, e.g. the isotropy at every closed point which is not the origin is trivial while the isotropy at the origin is k^* . Moreover, smooth points are 1-log canonical and form an unbounded family whenever the dimension is not bounded. Hence, the statement of the main theorem fails if we drop the condition on the dimension.

Example 2.30. In this example, we show that the statement of the main theorem fails if we drop the condition on ϵ -log canonical singularities. Indeed, consider the cone over a rational curve of degree m , i.e. the cone singularity given by

$$X_m := \text{Spec} \left(\bigoplus_{n \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(nmH)) \right)$$

where H is the class of a point on \mathbb{P}^1 . It is well-known that the log discrepancy at the exceptional divisor obtained by blowing-up the maximal ideal of the vertex of X_m is $\frac{2}{n}$. Hence, the cone singularities X_m are not ϵ -log canonical for some fixed positive real number ϵ , even if all of them are log canonical. However, the k^* -action given by the grading has trivial isotropies and all these are 2-dimensional singularities. This sequence of surfaces singularities give an example in which the main theorem fails if we drop the condition on ϵ . Indeed, the Cartier index of K_{X_m} is m . Thus, for m unbounded the above sequence of 2-dimensional log canonical singularities with isotropies bounded by 1 does not form a bounded family.

Example 2.31. In this example, we show that the statement of the main theorem fails if we drop the condition on the isotropies bounded by N . Consider the well-known A_n -singularities:

$$A_n := \{(x, y, z) \mid x^2 y^2 - z^n = 0\} \subset k^3.$$

We claim that any k^* -action on A_n which give it the structure of a cone singularity has isotropy greater or equal than n along either the curve $x = 0$ or the curve $y = 0$. Indeed, any k^* -action on A_n is induced by a sub-torus of the torus action $(k^*)^2$ on A_n given by

$$(t_1, t_2) \cdot (x, y, z) = (t_1 t_2^n x, t_1^{-1} t_2^n y, t_2^2 z).$$

A subtorus embedding $k^* \hookrightarrow (k^*)^2$ has the form $t \mapsto (t^a, t^b)$ for certain integers a and b . Hence, any k^* -action on A_n is given by

$$(2.3) \quad t \cdot (x, y, z) = (t^{a+bn} x, t^{-a+bn} y, t^{2b} z),$$

where a and b are integers. We check that the above action gives A_n a cone singularity structure if and only if $b \neq 0$. If $b = 0$ the action (2.3) is given by $t \cdot (x, y, z) = (t^a x, t^{-a} y, z)$ and the curve $z = 1$ is an orbit which does not contain the origin in its closure. Hence, we may assume that $b \neq 0$ and therefore the action (2.3) has isotropy $-a + bn$ on the curve $x = 0$ and isotropy $a + bn$ on the curve $y = 0$. Observe that we have

$$2|b|n = |2bn| = |(a + bn) + (-a + bn)| \leq |a + bn| + |-a + bn|.$$

Therefore, for n large enough, either the isotropy at $x = 0$ or the isotropy at $y = 0$ is getting arbitrarily large. Observe that the A_n singularities are canonical surface singularities which don't form a bounded family since their algebraic fundamental groups have arbitrarily large order.

3. PROOF OF BOUNDEDNESS

Proof of Theorem 1. Let d and N be positive integers and ϵ a positive real number. Denote by $\mathcal{C}_{d,\epsilon,N}$ the class of d -dimensional ϵ -log canonical cone singularities with isotropies bounded by N . Denote by $\mathcal{C}_{d,\epsilon,N}^{\text{quot}}$ the class of corresponding couples (Y, D) associated to the cone singularities in $\mathcal{C}_{d,\epsilon,N}$. By Proposition 2.24 and [Ale94, Theorem 3.1], it suffices to prove that for every sequence $(Y_i, D_i) \in \mathcal{C}_{d,\epsilon,N}^{\text{quot}}$ we can find an infinite sub-sequence which is log bounded with coefficients up to linear equivalence. We denote by $x_i \in X_i$ the corresponding sequence of cone singularities.

Step 1: In this step, we prove that the log Fano quotients (Y_i, B_i) of $x_i \in X_i$ belong to a strictly log bounded family which only depends on d, ϵ and N . Indeed, by Lemma 2.22, we know that the log Fano quotient (Y_i, B_i) has $\frac{\epsilon}{N}$ -log canonical singularities and $-(K_{Y_i} + B_i)$ is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. Moreover, since the coefficients of B_i have the form $1 - \frac{1}{n}$ for some positive integer n and the pairs (Y_i, B_i) are $\frac{\epsilon}{N}$ -log canonical, we conclude that n is at most $\frac{N}{\epsilon}$. Thus, the coefficients of B_i belong to a finite set which only depends on ϵ and N . Hence, we can apply Corollary 2.26 to deduce that the pairs (Y_i, B_i) belong to a strictly log bounded family which only depends on $d - 1, \epsilon$ and N . We denote by $\phi: \mathcal{X} \rightarrow T$ the bounding family for the Y_i 's and $\mathcal{B} \subset \mathcal{X}$ the bounding divisor for the B_i 's. We denote by \mathcal{D}_i the \mathbb{Q} -Cartier \mathbb{Q} -divisor on \mathcal{X} such that $(\mathcal{D}_i)|_{t_i} = D_i$. As in the proof of [HX15, Proposition 2.8], we may assume that all fibers of $\mathcal{X} \rightarrow T$ are \mathbb{Q} -factorial. Observe that this last assumption does not change the isomorphism class of the cone singularities, however it may change the models \tilde{X}_i .

Step 2: We denote by $t_i \in T$ the points on the base of the bounding family for which $(\mathcal{X}_{t_i}, \mathcal{B}_{t_i}) \simeq (Y_i, B_i)$ holds. Up to a base change and shrinking T we may assume that the t_i 's are dense on T and that $\text{Cox}(\mathcal{X}/T) \simeq \text{Cox}(\mathcal{X}_{t_i})$ for every $t \in T$ (see Proposition 2.28). Shrinking more if needed, we may assume that the points of $N^1(\mathcal{X}/T)$ corresponding to classes of Weil divisors (resp. Cartier divisors) are identified via ρ_t

with the points of $N^1(\mathcal{X}_i)$ corresponding to classes of Weil divisors (resp. Cartier divisors).

Step 3: In this step we will compute the log discrepancy of the divisor E_i contracted by $\tilde{X}_i \rightarrow X_i$ with respect to the canonical divisor K_{X_i} . This computation will be expressed in terms of the log Fano quotient (Y_i, B_i) of the cone singularity. By the proof of Proposition 2.19, we know that we can write

$$D_i = \frac{m_i}{u_i}(K_{Y_i} + B_i) + \frac{1}{u_i}H_i,$$

where m_i and u_i are integers of opposite sign, and H_i is a principal divisor on Y_i . By Lemma 2.7 and Proposition 2.18, we conclude that there is an equality

$$a_{E_i}(K_{X_i}) = \frac{u_i}{m_i}.$$

Hence, by the assumption on X_i being ϵ -log canonical we deduce that $\frac{m_i}{u_i} \leq \frac{1}{\epsilon}$.

Step 4: In this step we prove that the possible classes of \mathbb{Q} -linearly equivalence of \mathcal{D}_i on $N^1(\mathcal{X}/T)$ belong to a finite set. Observe that we have the relation

$$(3.1) \quad \mathcal{D}_i \sim_{\mathbb{Q},T} \frac{m_i}{u_i}(K_{\mathcal{X}/T} + \mathcal{B}_i),$$

for each i . Observe that the set

$$\left\{ \mathcal{F} \in N^1(\mathcal{X}/T) \mid \mathcal{F} \sim_{\mathbb{Q},T} r(K_{\mathcal{X}/T} + \mathcal{B}_i) \text{ and } 0 \leq r \leq \frac{1}{\epsilon} \right\}$$

is a compact subspace of $N^1(\mathcal{X}/T)$. On the other hand, since $N\mathcal{D}_i$ is a Cartier divisor, the divisors \mathcal{D}_i belong to a lattice inside $N^1(\mathcal{X}/T)$. From the equation (3.1) we conclude that there are finitely many possible \mathbb{Q} -linearly classes $\mathcal{F}_1, \dots, \mathcal{F}_r$ on $N^1(\mathcal{X}/T)$ for which any \mathcal{D}_i is \mathbb{Q} -linearly equivalent to some \mathcal{F}_j . Passing to a subsequence we may assume that all the \mathcal{D}_i are \mathbb{Q} -linearly equivalent to each other.

Step 5: We prove that the sequence $(\mathcal{X}, \mathcal{D}_i)$ is log bounded up to linear equivalence over T . We can write

$$\mathcal{D}_i = \mathcal{D}_{i,f} + \mathcal{D}_{i,W}^+ - \mathcal{D}_{i,W}^-,$$

where the three divisors are effective, $\mathcal{D}_{i,f}$ has coefficients on the interval $(0, 1) \cap \mathbb{Z}[\frac{1}{N}]$, and the two latter divisors are integral. Since the pairs $(\mathcal{X}, \mathcal{B}_i)$ are strictly log bounded, we conclude that the couples $(\mathcal{X}, \mathcal{D}_{i,f})$ are strictly log bounded as well. Thus, passing to a subsequence we may assume that for each i and j we have that

$$\mathcal{D}_{i,W}^+ - \mathcal{D}_{i,W}^- = \mathcal{D}_{j,W}^+ - \mathcal{D}_{j,W}^-.$$

Since the Cox ring of \mathcal{X} relative to T is finitely generated, we may find a finite basis E_1, \dots, E_r for the effective Weil divisors on $N^1(\mathcal{X}/T)$ up to linear equivalence. We will denote by k the smallest positive integer such that for every Weil divisor E on \mathcal{X} the multiple kE is Cartier. For each i we can write

$$\mathcal{D}_{i,W}^+ \sim_T \sum_{l=1}^r a_{i,l}^+ E_l + \sum_{l=1}^r k b_{i,l}^+ E_l,$$

and

$$\mathcal{D}_{i,W}^- \sim_T \sum_{l=1}^r a_{i,l}^- E_l + \sum_{l=1}^r k b_{i,l}^- E_l,$$

where the $a_{i,l}^+$'s and the $a_{i,l}^-$'s are positive integers in the interval $(0, k-1]$, while the $b_{i,l}^+$'s and $b_{i,l}^-$'s are positive integers. Observe that there are finitely many possible Weil divisors

$$\sum_{l=1}^r a_{i,l}^+ E_l \quad \text{and} \quad \sum_{l=1}^r a_{i,l}^- E_l,$$

Hence, passing to a subsequence we may assume that for every i and j we have

$$\sum_{l=1}^r kb_{i,l}^+ E_l - \sum_{l=1}^r kb_{i,l}^- E_l \sim_{\mathbb{Q}, T} \sum_{l=1}^r kb_{j,l}^+ E_l - \sum_{l=1}^r kb_{j,l}^- E_l$$

are two Cartier divisors which are \mathbb{Q} -linearly equivalent over T , therefore for every i and j we have

$$\sum_{l=1}^r kb_{i,l}^+ E_l - \sum_{l=1}^r kb_{i,l}^- E_l \sim_T \sum_{l=1}^r kb_{j,l}^+ E_l - \sum_{l=1}^r kb_{j,l}^- E_l.$$

Thus, for each i we can write

$$\mathcal{D}_i \sim_T \mathcal{D}_{i,f} + \left(\sum_{l=1}^r a_{i,l}^+ E_l - \sum_{l=1}^r a_{i,l}^- E_l \right) + \left(\sum_{l=1}^r kb_{i,l}^+ E_l - \sum_{l=1}^r kb_{i,l}^- E_l \right)$$

where there are finitely many possible \mathbb{Q} -divisors

$$\mathcal{D}_{i,f} + \left(\sum_{l=1}^r a_{i,l}^+ E_l - \sum_{l=1}^r a_{i,l}^- E_l \right)$$

and all the Weil divisors

$$\sum_{l=1}^r kb_{i,l}^+ E_l - \sum_{l=1}^r kb_{i,l}^- E_l$$

are linearly equivalent over T to a fixed integral divisor. Thus, we deduce that the sequence $(\mathcal{X}, \mathcal{D}_i)$ is strictly log bounded up to linear equivalence over T .

Step 6: In this step we complete the proof; we prove that the couples (Y_i, D_i) are strictly log bounded up to linear equivalence. Indeed, by the fifth step we may pass to a subsequence in which

$$K_{\mathcal{X}} + \mathcal{D}_i \sim_T K_{\mathcal{X}} + \mathcal{D}_j$$

for every i and j . Moreover, since the t_i 's are dense in T we have that the linear equivalence

$$(K_{\mathcal{X}} + \mathcal{D}_i)|_{t_l} \sim (K_{\mathcal{X}} + \mathcal{D}_j)|_{t_l},$$

holds for all but finitely many t_l 's. Thus, we conclude that for all but finitely many t_l 's the linear equivalence

$$K_{Y_l} + D_l \sim K_{Y_l} + D_{l,1}$$

holds, where $D_{l,1} = (\mathcal{D}_1)|_{t_l}$. Since the pairs $(Y_l, D_{l,1})$ are strictly log bounded by the family $\phi: \mathcal{X} \rightarrow T$ and the divisor \mathcal{D}_1 , we conclude that the pairs (Y_l, D_l) are strictly log bounded up to linear equivalence. \square

Proof of Corollary 1. This follows from Theorem 1, and the fact that minimal log discrepancies take finitely many values on bounded families [Amb99, §2]. \square

Proof of Corollary 2. This follows from Theorem 1, and the upper semi-continuity of the order of algebraic fundamental groups [BKS03, Corollary 17]. \square

REFERENCES

- [Ale94] Valery Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810, DOI 10.1142/S0129167X94000395. MR1298994 ↑8, 9, 10
- [Amb99] Florin Ambro, *On minimal log discrepancies*, Math. Res. Lett. **6** (1999), no. 5-6, 573–580, DOI 10.4310/MRL.1999.v6.n5.a10. MR1739216 ↑2, 12
- [AIP⁺12] Klaus Altmann, Nathan Owen Itten, Lars Petersen, Hendrik Süß, and Robert Vollmert, *The geometry of T -varieties*, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17–69, DOI 10.4171/114-1/2. MR2975658 ↑1
- [ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR3307753 ↑4
- [AH06] Klaus Altmann and Jürgen Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. **334** (2006), no. 3, 557–607, DOI 10.1007/s00208-005-0705-8. MR2207875 ↑1, 4, 5, 7, 8
- [AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süß, *Gluing affine torus actions via divisorial fans*, Transform. Groups **13** (2008), no. 2, 215–242, DOI 10.1007/s00031-008-9011-3. MR2426131 ↑1
- [Bir16a] Caucher Birkar, *Anti-pluricanonical systems on Fano varieties*, 2016. <https://arxiv.org/abs/1603.05765v3>. ↑1, 7
- [Bir16b] ———, *Singularities of linear systems and boundedness of Fano varieties*, 2016. <https://arxiv.org/abs/1609.05543v1>. ↑1, 2, 7
- [Bir17] ———, *Birational geometry of algebraic varieties*, 2017. <https://arxiv.org/abs/1801.00013>. ↑2
- [Bro13] Morgan Brown, *Singularities of Cox rings of Fano varieties*, J. Math. Pures Appl. (9) **99** (2013), no. 6, 655–667, DOI 10.1016/j.matpur.2012.10.003 (English, with English and French summaries). MR3055212 ↑2
- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468, DOI 10.1090/S0894-0347-09-00649-3. MR2601039 ↑9
- [BKS03] Károly Böröczky Jr., János Kollár, and Tamás Szamuely (eds.), *Higher dimensional varieties and rational points*, Bolyai Society Mathematical Studies, vol. 12, Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2003. Papers from the Summer School and Conference held in Budapest, September 3–21, 2001. MR2011742 ↑12
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322 ↑1
- [Dem88] Michel Demazure, *Anneaux gradués normaux*, Introduction à la théorie des singularités, II, Travaux en Cours, vol. 37, Hermann, Paris, 1988, pp. 35–68 (French). MR1074589 ↑3
- [dFH11] Tommaso de Fernex and Christopher D. Hacon, *Deformations of canonical pairs and Fano varieties*, J. Reine Angew. Math. **651** (2011), 97–126, DOI 10.1515/CRELLE.2011.010. MR2774312 ↑9
- [DS14] Simon Donaldson and Song Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math. **213** (2014), no. 1, 63–106, DOI 10.1007/s11511-014-0116-3. MR3261011 ↑1, 2
- [DS16] Ved Datar and Gábor Székelyhidi, *Kähler-Einstein metrics along the smooth continuity method*, Geom. Funct. Anal. **26** (2016), no. 4, 975–1010, DOI 10.1007/s00039-016-0377-4. MR3558304 ↑1, 2
- [Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR1234037 ↑1
- [GOST15] Yoshinori Gongyo, Shinnosuke Okawa, Akiyoshi Sannai, and Shunsuke Takagi, *Characterization of varieties of Fano type via singularities of Cox rings*, J. Algebraic Geom. **24** (2015), no. 1, 159–182, DOI 10.1090/S1056-3911-2014-00641-X. MR3275656 ↑2
- [Hay05a] Takayuki Hayakawa, *Divisorial contractions to 3-dimensional terminal singularities with discrepancy one*, J. Math. Soc. Japan **57** (2005), no. 3, 651–668. MR2139726 ↑1
- [Hay05b] ———, *Gorenstein resolutions of 3-dimensional terminal singularities*, Nagoya Math. J. **178** (2005), 63–115, DOI 10.1017/S0027763000009120. MR2145316 ↑1
- [HK00] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, DOI 10.1307/mmj/1030132722. Dedicated to William Fulton on the occasion of his 60th birthday. MR1786494 ↑4
- [HK10] Christopher D. Hacon and Sándor J. Kovács, *Classification of higher dimensional algebraic varieties*, Oberwolfach Seminars, vol. 41, Birkhäuser Verlag, Basel, 2010. MR2675555 ↑1
- [HX15] Christopher D. Hacon and Chenyang Xu, *Boundedness of log Calabi-Yau pairs of Fano type*, Math. Res. Lett. **22** (2015), no. 6, 1699–1716, DOI 10.4310/MRL.2015.v22.n6.a8. MR3507257 ↑7, 9, 10
- [Kol16] János Kollár, *New examples of terminal and log canonical singularities*, 2016. <https://arxiv.org/abs/1107.2864>. ↑1
- [Kol13] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács. MR3057950 ↑1

- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original. MR1658959 ↑1
- [Li17] Chi Li, *K-semistability is equivariant volume minimization*, *Duke Math. J.* **166** (2017), no. 16, 3147–3218, DOI 10.1215/00127094-2017-0026. MR3715806 ↑1
- [LS13] Alvaro Liendo and Hendrik Süß, *Normal singularities with torus actions*, *Tohoku Math. J. (2)* **65** (2013), no. 1, 105–130, DOI 10.2748/tmj/1365452628. MR3049643 ↑1, 3, 4
- [LX16] Chi Li and Chenyang Xu, *Stability of Valuations and Kollr Components*, 2016. <https://arxiv.org/abs/1604.05398>. ↑1
- [LX17] ———, *Stability of Valuations: Higher Rational Rank*, 2017. <https://arxiv.org/abs/1707.05561>. ↑1
- [Mor18] Joaquín Moraga, *On minimal log discrepancies and Kollár components*, 2018. <https://arxiv.org/abs/1810.10137>. ↑2, 7
- [PS11] Lars Petersen and Hendrik Süß, *Torus invariant divisors*, *Israel J. Math.* **182** (2011), 481–504, DOI 10.1007/s11856-011-0039-z. MR2783981 ↑4, 5
- [Sho04] V. V. Shokurov, *Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips*, *Tr. Mat. Inst. Steklova* **246** (2004), no. *Algebr. Geom. Metody, Svyazi i Prilozh.*, 328–351 (Russian, with Russian summary); English transl., *Proc. Steklov Inst. Math.* **3(246)** (2004), 315–336. MR2101303 ↑2
- [Tei03] Bernard Teissier, *Valuations, deformations, and toric geometry*, *Valuation theory and its applications*, Vol. II (Saskatoon, SK, 1999), *Fields Inst. Commun.*, vol. 33, Amer. Math. Soc., Providence, RI, 2003, pp. 361–459. MR2018565 ↑2
- [Tzi05] Nikolaos Tziolas, *Three dimensional divisorial extremal neighborhoods*, *Math. Ann.* **333** (2005), no. 2, 315–354, DOI 10.1007/s00208-005-0676-9. MR2195118 ↑1
- [TX17] Zhiyu Tian and Chenyang Xu, *Finiteness of fundamental groups*, *Compos. Math.* **153** (2017), no. 2, 257–273, DOI 10.1112/S0010437X16007867. MR3604863 ↑2
- [Wat81] Keiichi Watanabe, *Some remarks concerning Demazure’s construction of normal graded rings*, *Nagoya Math. J.* **83** (1981), 203–211. MR632654 ↑4
- [Xu14] Chenyang Xu, *Finiteness of algebraic fundamental groups*, *Compos. Math.* **150** (2014), no. 3, 409–414, DOI 10.1112/S0010437X13007562. MR3187625 ↑2
- [Xu17] ———, *Interaction Between Singularity Theory and the Minimal Model Program*, 2017. <https://arxiv.org/abs/1712.01041>. ↑1

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