EKEDAHL-OORT STRATA ON THE MODULI SPACE OF CURVES OF GENUS FOUR

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Abstract. We study the induced Ekedahl-Oort stratification on the moduli of curves of genus 4 in positive characteristic.

1. Introduction

Let k be an algebraically closed field with $\mathrm{char}(k) = p > 0$. Let $\mathcal{A}_g \otimes k$ be the moduli space (stack) of principally polarized abelian varieties of dimension g defined over k and let $\mathcal{M}_g \otimes k$ be the moduli space of (smooth projective) curves of genus g defined over k. Ekedahl and Oort introduced a stratification on $\mathcal{A}_g \otimes k$ consisting of 2^g strata, cf. [13, 2]. These strata are indexed by n-tuples $\mu = [\mu_1, \dots, \mu_n]$ with $0 \leq n \leq g$ and $\mu_1 > \mu_2 > \cdots > \mu_n > 0$. The largest stratum is the locus of ordinary abelian varieties corresponding to the empty n-tuple $\mu = \emptyset$. Their cycle classes have been studied by [16].

Via the Torelli map $\tau : \mathcal{M}_g \otimes k \to \mathcal{A}_g \otimes k$ we can pull back this stratification to $\mathcal{M}_g \otimes k$ and it is natural to ask what stratification this provides. Similarly, we can ask for the induced stratification on the hyperelliptic locus $\mathcal{H}_g \otimes k$. We denote the induced strata on $\mathcal{M}_g \otimes k$ by Z_μ. We say a (smooth) curve has Ekedahl-Oort type μ if the corresponding point in $\mathcal{M}_g \otimes k$ lies in Z_μ.

Here we are interested in the existence of curves of genus 4 with given Ekedahl-Oort type. By a curve we mean a smooth irreducible projective curve defined over k. For $g \leq 3$, we know the situation for the induced Ekedahl-Oort stratification on $\mathcal{M}_g \otimes k$. But for $g \geq 4$ much less is known. Elkin and Pries [4] give a complete classification for hyperelliptic curves when $p = 2$. Our first result describes this stratification on $\mathcal{H}_4 \otimes k$ with $p = 3$. In the following we write simply $\mathcal{A}_g$ (resp. $\mathcal{M}_g$, $\mathcal{H}_g$) for $\mathcal{A}_g \otimes k$ (resp. $\mathcal{M}_g \otimes k$, $\mathcal{H}_g \otimes k$). Recall that the indices μ of the Ekedahl-Oort strata are partially ordered by

$$\mu = [\mu_1, \dots, \mu_n] \preceq \upsilon = [\upsilon_1, \dots, \upsilon_m]$$

if $n \leq m$ and $\mu_i \leq \upsilon_i$ for $i = 1, \dots, n$.

Theorem 1.1. *Let k be an algebraically closed field with* $\mathrm{char}(k) = 3$. *A smooth hyperelliptic curve of genus 4 over k has a-number ≤ 2. In particular, $Z_\mu \cap \mathcal{H}_4$ is* *empty if* $\mu \succeq [3,2,1]$*. If* $\mu \not\succeq [3,2,1]$*, the codimension of* $Z_\mu \cap \mathcal{H}_4$ *in* $\mathcal{H}_4$ *equals the expected codimension* $\sum_{i=1}^n \mu_i$*. Moreover, in the cases* $\mu = [4,1], [3,1], [3,2], [2,1]$ *and* $[1]$ *the intersection* $Z_\mu \cap \mathcal{H}_4$ *is irreducible.*

Part of Theorem 1.1 was known. Frei [5] proved that hyperelliptic curves in odd characteristic cannot have a-number $g-1$. Glass and Pries ([6, Theorem 1]) showed that the intersection of $\mathcal{H}_g$ with the locus V_l of p-rank $\leq l$ has codimension $g-l$ in characteristic $p > 0$. Pries ([14, Theorem 4.2]) showed that $Z_{[2]} \cap \mathcal{H}_4$ has dimension 5 for $p \geq 3$.

The following result shows that certain Ekedahl-Oort strata in $\mathcal{M}_4$ are not empty.

Theorem 1.2. *Let* k *be an algebraically closed field of characteristic* p*. For any odd prime* p *with* $p \equiv \pm 2 \pmod 5$*, the loci* $Z_{[4,2]}$ *and* $Z_{[4,3]}$ *in* $\mathcal{M}_4$ *are non-empty. For any prime* $p \equiv -1 \pmod 5$*, there exist superspecial curves of genus* 4 *in characteristic* p*.*

To prove Theorem 1.2, we use cyclic covers of the projective line in positive characteristic. Furthermore, we give an alternative but much shorter proof of a result of Kudo [8] showing that there exists a superspecial curve of genus 4 in characteristic p for all p with $p \equiv 2 \pmod 3$. Related results on Newton polygons of cyclic covers of the projective line and on the existence of curves with given Newton polygon can be found in [9, 10].

2. Proof of Theorem 1.1

Let X be a hyperelliptic curve of genus 4 defined over k with $p = 3$. Before giving the proof of Theorem 1.1, we prove several propositions needed for Theorem 1.1 and give a basis of the de Rham cohomology of a hyperelliptic curve of genus 4 defined over k.

We first show that any smooth hyperelliptic curve of genus 4 has a-number at most 2.

Proposition 2.1. *A hyperelliptic curve of genus* 4 *in characteristic* 3 *has* a*-number at most* 2*.*

Proof. Any smooth hyperelliptic curve X can be written as $y^2 = f(x)$ with $f(x) = \sum_{i=0}^9 a_i x^i \in k[x]$ and $\mathrm{disc}(f) \neq 0$. By putting a branch point at 0 and by scaling we may assume that $a_1 = a_9 = 1$ and

$$f(x) = x^9 + a_8 x^8 + \cdots + a_2 x^2 + x \tag{1}$$

with $a_i \in k$ for $i = 2, \ldots, 8$. As a basis of $H^0(X, \Omega^1_X)$ we choose $\omega_i = x^i/y\, \mathrm{d}x$ for $i = 0, \ldots, 3$. Then the Cartier-Manin matrix H, i.e. the matrix of the Cartier operator acting on the holomorphic differentials with respect to a basis, of curve X is

$$H = \begin{pmatrix} a_2 & 1 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ a_8 & a_7 & a_6 & a_5 \\ 0 & 0 & 1 & a_8 \end{pmatrix}^{1/3}, \tag{2}$$

where $H^{1/3} = (h_{ij}^{1/3})$ if $H = (h_{ij})$. Since $\mathrm{rank}(H) \geq 2$, we have for the a-number $a = 4 - \mathrm{rank}(H) \leq 2$. □

Remark 2.2. Note that the map from the parameter space of the a_i $(i = 2, \ldots, 8)$ to the hyperelliptic locus has finite fibres. Indeed, if ϕ is an isomorphism between two smooth hyperelliptic curves given by $f_1(x) = \sum_{i=1}^{9} a_i x^i$ and $f_2(x) = \sum_{i=1}^{9} b_i x^i$ as in (1) that induces an isomorphism of $\mathbb{P}^1$ fixing 0 and ∞, then ϕ is given by scaling $x \mapsto \alpha x$ and $y \mapsto \beta y$. We obtain $\alpha^9/\beta^2 = \alpha/\beta^2 = 1$ and hence $\alpha^8 = 1, \beta^2 = \alpha$.

We let Y be the open subset of affine space with coordinates $(a_2, \ldots, a_8)$ such that $\mathrm{disc}(f) \neq 0$. Denote by T_a the locus of curves of genus g with a-number $\geq a$ in $\mathcal{M}_4$ and by X_f the smooth projective hyperelliptic curve defined by the equation $y^2 = f(x)$ as in (1). Let H_f be the Cartier-Manin matrix of the curve X_f. In the following we simply write X (resp. H) for X_f (resp. H_f). Now we give a result about the intersection $\mathcal{H}_4 \cap T_a$ with $a \leq 2$.

Proposition 2.3. *The locus of $\mathcal{H}_4 \cap T_a$ with $a \leq 2$ is irreducible of codimension $a(a+1)/2$.*

Proof. For $a = 0$, we consider the curve with equation $y^2 = f(x) = x^9 + tx^5 + x$ defined over k where t is a primitive element in $\mathbb{F}_9$. Then $\mathrm{disc}(f) = 2 \neq 0$ and by (2) we have $\mathrm{rank}(H) = 4$. Hence there is a curve with $a = 0$ and note that $\mathcal{H}_4$ is irreducible of dimension 7. Then by semicontinuity the generic hyperelliptic curve is ordinary and $T_0 \cap \mathcal{H}_4$ is irreducible of dimension 7.

The condition $a = 1$ means $\mathrm{rank}(H) = 3$. We show that the locus in Y with $\mathrm{rank}(H) = 3$ is given by

$$(a_8 a_6 - a_5)(a_2 a_4 - a_5) + (a_2 - a_3 a_8)(a_2 a_7 - a_8) = 0\,.$$

Indeed if $a_2 = a_8 = 0$ and $\mathrm{disc}(f) \neq 0$, then by Gauss reduction the rank of H is equal to the rank of

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{1/3}.$$

Since we want $\mathrm{rank}(H) > 2$ we must have $a_5 \neq 0$. Then this implies $\mathrm{rank}(H) = 4$ and the curve is ordinary.

Suppose one of a_2, a_8 is not zero; by symmetry we can assume $a_2 \neq 0$ and the rank of H is equal to the rank of

$$\begin{pmatrix} a_2 & 1 & 0 & 0 \\ 0 & a_4 - a_5/a_2 & a_3 & a_2 \\ 0 & a_7 - a_8/a_2 & a_6 & a_5 \\ 0 & 0 & 1 & a_8 \end{pmatrix}^{1/3} .$$

We have $\det(H) = 0$ as $\mathrm{rank}(H) = 3$ and hence

$$(a_8a_6 - a_5)(a_2a_4 - a_5) + (a_2 - a_3a_8)(a_2a_7 - a_8) = 0 . \tag{3}$$

Note that equation (3) can be rewritten as

$$a_7a_2^2 + 2(a_3a_7a_8 + a_4a_5 + 2a_2a_4a_6a_8 + a_2a_8)a_2 + a_3a_8^2 + a_5^2 + 2a_5a_6a_8 = 0 .$$

This is a 6-dimensional subspace of Y, which is irreducible. Also if we take $a_2 = a_7 = a_8 = 1$ and $a_i = 0$ for $i \neq 2, 7, 8$, then $\mathrm{disc}(f) = 2 \neq 0$ and $\mathrm{rank}(H) = 3$. Hence there is a curve with $a = 1$ and by semicontinuity $T_1 \cap \mathcal{H}_4$ is irreducible of codimension 1.

For $a = 2$, we want to show that the locus in Y with $a = 2$ is given by $a_2 = a_5 = a_8 = 0$. Since we want $\mathrm{rank}(H) = 2$ and the first and fourth row of H are linearly independent, we have several situations to deal with: *i*) $a_2 = 0$, *ii*) $a_8 = 0$ and *iii*) $a_2a_8 \neq 0$.

For the first two cases, if the $\mathrm{rank}(H) = 2$, then the second and third rows of H are linear combinations of the first and fourth rows. Therefore we have $a_2 = a_5 = a_8 = 0$. For the third case, if $a_2a_8 \neq 0$, let e_i to be the i-th row of H, then with some $b, c, s, t \in k$ we have

$$be_1 + ce_4 = e_2, \; se_1 + te_4 = e_3 .$$

This implies $a_3 = a_2/a_8, a_4 = a_5/a_2, a_6 = a_5/a_8, a_7 = a_8/a_2$ and hence

$$f(x) = (x^2 + (a_5/a_8)^{1/3}x + (a_2/a_8)^{1/3})^3(x^3 + a_8x^2 + (a_8/a_2)x) ,$$

which does not have distinct roots, a contradiction. Then we have $a_2 = a_5 = a_8 = 0$, which defines an irreducible sublocus in Y. Indeed, if we take $a_3 = 1, a_7 = 2$ and $a_i = 0$ for $i \neq 3, 7$, then $\mathrm{disc}(f) = 1 \neq 0$ and $\mathrm{rank}(H) = 2$. So we find a curve with $\mathrm{rank}(H) = 2$. Hence by semicontinuity $T_2 \cap \mathcal{H}_4$ is irreducible of codimension 3. □

We have seen that any hyperelliptic curve over k with a-number 2 is given by an equation $y^2 = f(x)$ as in (1) with $(a_2, \ldots, a_8) \in Y$ and $a_2 = a_5 = a_8 = 0$. We will now use the de Rham cohomology $H^1_{dR}(X)$ for a curve X of genus g. Recall that this is a vector space of dimension $2g$ provided with a non-degenerate pairing, cf. [13, Section 12]. Now we consider the action of Verschiebung V on the de Rham cohomology of a curve X given by equation (1) with a-number 2. First we give a basis of the de Rham cohomology of a hyperelliptic curve with a-number 2. Let X be a smooth irreducible complete curve over k with equation

$$y^2 = f = x^9 + a_7x^7 + a_6x^6 + a_4x^4 + a_3x^3 + x,\ a_i \in k\,, \tag{4}$$

where the discriminant of f is non-zero. Let $\pi : X \to \mathbb{P}^1$ be the hyperelliptic map. Take an open affine cover $\mathcal{U} = \{U_1, U_2\}$ with $U_1 = \pi^{-1}(\mathbb{P}^1 - \{0\})$ and $U_2 = \pi^{-1}(\mathbb{P}^1 - \{\infty\})$. By Section [12, Section 5], the de Rham cohomology $H^1_{dR}(X)$ can be described as

$$H^1_{dR}(X) = Z^1_{dR}(\mathcal{U})/B^1_{dR}(\mathcal{U})$$

with $Z^1_{dR}(\mathcal{U}) = \{(t, \omega_1, \omega_2)|\ t \in \mathcal{O}_X(U_1 \cap U_2), \omega_i \in \Omega^1_X(U_i), \mathrm{d}t = \omega_1 - \omega_2\}$ and $B^1_{dR}(\mathcal{U}) = \{(t_1 - t_2, \mathrm{d}t_1, \mathrm{d}t_2)|\ t_i \in \mathcal{O}_X(U_i)\}$. Then $V(H^1_{dR}(X)) = H^0(X, \Omega^1_X)$ and V coincides with the Cartier operator on $H^0(X, \Omega^1_X)$.

For $1 \le i \le 4$, put $s_i(x) = xf'(x) - 2if(x)$ with $f'(x)$ the formal derivative of $f(x)$ and write $s_i(x) = s_i^{\le i}(x) + s_i^{>i}(x)$ with $s_i^{\le i}(x)$ the sum of monomials of degree $\le i$. By Köck and Tait [7], $H^1_{dR}(X)$ has a basis with respect to $\{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(X)$:

$$\gamma_i = [(\frac{y}{x^i}, \frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x, -\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)],\ i = 1, \dots, 4, \tag{5}$$

$$\lambda_j = [(0, \frac{x^j}{y}\mathrm{d}x, \frac{x^j}{y}\mathrm{d}x)],\ j = 0, \dots, 3, \tag{6}$$

where $\psi_i(x) = s_i^{\le i}(x)$ and $\phi_i(x) = s_i^{>i}(x)$. Also we have $\langle \gamma_i, \lambda_j \rangle \ne 0$ if $j = i - 1$ and $\langle \gamma_i, \lambda_j \rangle = 0$ otherwise. Now we give the action of Verschiebung.

Lemma 2.4. *Let X be a smooth hyperelliptic curve over k with equation (4). Let $\{U_1, U_2\}$ be a covering of X as above. Then for the basis of $H^1_{dR}(X)$ given by (5) and (6) , we have $V(\lambda_0) = V(\lambda_3) = V(\gamma_2) = V(\gamma_3) = 0$ and*

$$V(\lambda_1) = a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1 + \lambda_0\,, V(\lambda_2) = \lambda_3 + a_6^{1/3}\lambda_2 + a_3^{1/3}\lambda_1\,,$$
$$V(\gamma_1) = \lambda_2 + a_6^{1/3}\lambda_1 + a_3^{1/3}\lambda_0, V(\gamma_4) = a_4^{1/3}\lambda_2 + (1 - (a_3a_7)^{1/3} + (a_4a_6)^{1/3})\lambda_1 + a_6^{1/3}\lambda_0\,.$$

Proof. Since V coincides with the Cartier operator on $H^0(X, \Omega^1_X)$, we have $V(h\mathrm{d}x) = (-\mathrm{d}^2h/\mathrm{d}x^2)^{1/3}\mathrm{d}x$ with $h \in k(x)$. We will compute $V(\gamma_1)$ and the rest of the lemma will follow easily by using a similar argument. Note that we always have for $1 \le i \le 4$

$$V(\frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x) = V(-\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)$$

as $0 = V(\mathrm{d}(y/x^i)) = V(\frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x) - V(-\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)$. So it suffices to compute $V(\frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x)$ instead of computing $V(\gamma_i)$. For $i = 1$, we have

$$V(\frac{\psi_1(x)}{2x^{1+1}y}\mathrm{d}x) = V(\frac{1}{xy}\mathrm{d}x) = (-\frac{\mathrm{d}(\mathrm{d}(\frac{1}{xy})/\mathrm{d}x)}{dx})^{1/3}\,\mathrm{d}x\,.$$

Note that $\mathrm{d}f'(x)/\mathrm{d}x = 0$ and by a calculation we have

$$-\frac{\mathrm{d}(\mathrm{d}(\frac{1}{xy})/\mathrm{d}x)}{\mathrm{d}x} = \frac{x^9 + a_6x^6 + a_3x^3}{x^3y^3}\,.$$

Hence $V(\psi_1(x)/(2x^{1+1}y)\mathrm{d}x) = (x^2/y + a_6^{1/3}x/y + a_3^{1/3}/y)\,\mathrm{d}x$ and we have $V(\gamma_1) = \lambda_2 + a_6^{1/3}\lambda_1 + a_3^{1/3}\lambda_0$. □

Now we give a proof of Theorem 1.1.

Proof of Theorem 1.1. The theorem holds for cases $\mu = [0]$ and $[1]$ by Proposition 2.3 where we showed that $T_a \cap \mathcal{H}_4$ is irreducible with codimension $a(a+1)/2$ for $a \leq 2$. Also $T_3 \cap \mathcal{H}_4$ is empty by Proposition 2.1, hence $Z_\mu \cap \mathcal{H}_4 = \varnothing$ for any $\mu \succeq [3,2,1]$.

We only need to prove that the theorem is true for the remaining nine Ekedahl-Oort strata, that is six strata consisting of curves with a-number 2 and three strata consisting of curves with a-number 1.

As an outline of the proof, we first show that for $\mu = [2,1], [3,1], [3,2], [4,1], [4,2]$ and $[4,3]$ the codimension of $Z_\mu \cap \mathcal{H}_4$ in $\mathcal{H}_4$ equals the expected codimension $\sum_{i=1}^n \mu_i$ with $\mu = [\mu_1, \ldots, \mu_n]$. For $\mu = [2], [3]$ and $[4]$, combined with the fact $V_l \cap \mathcal{H}_4$ is non-empty of codimension $4 - l$ in $\mathcal{H}_4$ for $l = 0, 1, 2$ by Glass and Pries [6, Theorem 1], the intersection $Z_\mu \cap \mathcal{H}_4$ also has the expected codimension. In the cases $\mu = [2,1], [3,1], [3,2], [4,1], [4,2]$ and $[4,3]$, a curve with Ekedahl-Oort type μ can be written as equation (4) by the proof of Proposition 2.3.

Throughout the proof, denote by X a smooth hyperelliptic curve given by equation $y^2 = f(x)$ as in (4) with Ekedahl-Oort type μ. Denote

$$Y_2 := V(H^0(X, \Omega_X^1)) = V(\langle \lambda_0, \ldots, \lambda_3 \rangle) \text{ and } Y_6 := Y_2^\perp$$

with respect to the paring $\langle\,,\,\rangle$ on $H^1_{dR}(X)$. Put $v : \{0, 1, \ldots, 8\} \to \{0, 1, \ldots, 4\}$ the final type of X. From Lemma 2.4 above, we know that Y_2 is a 2-dimensional subspace of $H^0(X, \Omega_X^1)$ generated by $V(\lambda_1)$ and $V(\lambda_2)$.

Let $\mu = [2,1]$. By Proposition 2.3 the intersection $T_2 \cap \mathcal{H}_4$ is irreducible of dimension 4. For any curve X corresponding to a point in $\mathcal{H}_4 \cap T_2$, we have by Lemma 2.4

$$\begin{aligned} V(Y_2) &= \langle V^2(\lambda_1), V^2(\lambda_2) \rangle = \langle V(a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1 + \lambda_0), V(\lambda_3 + a_6^{1/3}\lambda_2 + a_3^{1/3}\lambda_1) \rangle \\ &= \langle a_7^{1/9}V(\lambda_2) + a_4^{1/9}V(\lambda_1), a_6^{1/9}V(\lambda_2) + a_3^{1/9}V(\lambda_1) \rangle\,. \end{aligned} \tag{7}$$

We consider the curve associated to $(a_3, a_4, a_6, a_7) = (1, 0, 0, 2)$. Then $\mathrm{disc}(f) = 1 \neq 0$. Moreover, $V^n(Y_2) = Y_2$ for any $n \in \mathbb{Z}_{>0}$. Hence the semi-simple rank of V acting on $H^0(X, \Omega^1_X)$ is 2 and the Ekedahl-Oort type of this curve is $[2, 1]$. Since the p-rank can only decrease under specialization, the generic point of $\mathcal{H}_4 \cap T_2$ has Ekedahl-Oort type $[2, 1]$ and $Z_{[2,1]} \cap \mathcal{H}_4$ is irreducible of dimension 4.

Now we move to the case $\mu = [3, 1]$. We show that a curve with Ekedahl-Oort type $[3, 1]$ has equation (4) with $a_7a_3 = a_6a_4$ and $\mathrm{disc}(f) = a_3a_4^2 + a_6a_7 + 1 \neq 0$. Then the irreducibility and dimension will follow naturally.

Suppose X has Ekedahl-Oort type $[3, 1]$, then X is given by equation (4) with $\dim(V(Y_2)) = 1$. Then by Lemma 2.4 and relation (7), we have $a_3a_7 = a_4a_6$.

Put $Y_3 := V(Y_6)$ then we have

$$\dim Y_3 = v(6) = v(2) + 4 - 2 = 3$$

by the properties of the final type v. If we take $(a_3, a_4, a_6, a_7) = (0, 1, 0, 0)$, then $\mathrm{disc}(f) = 1 \neq 0$. Note that $V(\gamma_1) = \lambda_2$ and $V(\gamma_4) = \lambda_2 + \lambda_1$, hence $Y_2 = \langle \lambda_3, \lambda_1 + \lambda_0 \rangle$ by the Lemma 2.4. Furthermore, it is easy to check that $V^2(\lambda_2) = 0$ and $V^n(\lambda_1) = \lambda_1$. Then we get $v(1) = 1$. Also there exists an element $\gamma = \sum_{i=1}^3 b_i\gamma_i$ with $b_i \in k$ in Y_6 such that $b_1 \neq 0$, otherwise it contradicts that $\langle \gamma, \lambda_0 + \lambda_1 \rangle = 0$. Thus $b_1^{1/3}\lambda_2 = V(\gamma) \in Y_3$ and by simple computation we have $\dim V(Y_3) = 2$. Then there is a curve with Ekedahl-Oort type $[3, 1]$ and by semicontinuity we have the $Z_{[3,1]} \cap \mathcal{H}_4$ is irreducible of dimension 3.

Let $\mu = [3, 2]$, we show that the smooth hyperelliptic curve X with Ekedahl-Oort type $[3, 2]$ can be written as

$$y^2 = f(x) = x^9 + a_7x^7 + \alpha^3a_7x^6 + a_4x^4 + \alpha^3a_4x^3 + x \tag{8}$$

with $a_4, a_7, \alpha \in k^*$ satisfying $\alpha^3a_7^2 + \alpha a_7 = a_4 + \alpha a_4^2$ and $\mathrm{disc}(f) = (a_4\alpha + a_7\alpha^2 + 1)^9 \neq 0$.

Indeed, if the curve X is given by equation (4) with Ekedahl-Oort type $[3, 2]$, then we have $v(2) = 1$ and $v(1) = 1$. By Lemma 2.4 and relation (7), the condition $v(2) = 1$ implies $a_3a_7 = a_4a_6$. Also Y_6 is generated by λ_i for $i = 0, \ldots, 3$ and $\sum_{j=1}^4 b_j\gamma_j$ with $b_j \in k$ and $\langle \sum_{j=1}^4 b_j\gamma_j, Y_2 \rangle = 0$.

If $a_7 = 0$, by $a_3a_7 = a_4a_6$ we have (i) $a_6 = 0$ or (ii) $a_4 = 0$.

If we suppose $a_7 = a_6 = 0$, then $Y_2 = \langle a_4^{1/3}\lambda_1 + \lambda_0, \lambda_3 + a_3^{1/3}\lambda_1 \rangle$ and Y_6 is generated by λ_i and $\sum_{j=1}^4 b_j\gamma_j$ with

$$b_1 + b_2a_4^{1/3} = b_2a_3^{1/3} + b_4 = 0,\ b_1, \ldots, b_4 \in k\,. \tag{9}$$

Write $Y_3 = V(Y_6)$. If $a_4 = 0$, then we have $V^2(Y_2) = \langle\, 0\, \rangle$, a contradiction since X has Ekedahl-Oort type $[3, 2]$. Now suppose $a_4 \neq 0$, then for all b_1, b_4 satisfying (9) we have

$$V(Y_6) = Y_3 = \langle Y_2, (1 + a_4^{2/9}a_3^{1/9})\lambda_2 + (\frac{a_3}{a_4})^{1/9}\lambda_1 + a_3^{1/3}\lambda_0) \rangle\,.$$

Since $v(3) = 1$, we have $1 + a_4^{2/9} a_3^{1/9} = 0$, which implies $a_3^1 a_4^2 = -1$. In this case we have $\mathrm{disc}(f) = a_3^3 a_4^6 + 1 = 0$, a contradiction.

Now if $a_7 = a_4 = 0$, Y_6 is generated by λ_i and $\sum_{j=1}^4 b_j\gamma_j$ with $b_1 = b_2 a_3^{1/3} + a_6^{1/3} b_3 + b_4 = 0$. By Lemma 2.4 we have $V(\gamma_4) = \lambda_1 + a_6^{1/3}\lambda_0$, hence

$$\begin{aligned} Y_3 = V(Y_6) &= \langle Y_2, V(b_2\gamma_2 + \cdots + b_4\gamma_4)\rangle = \langle Y_2, V(\gamma_4)\rangle \\ &= \langle \lambda_0, \lambda_3 + a_6^{1/3}\lambda_2 + a_3^{1/3}\lambda_1, \lambda_1 + a_6^{1/3}\lambda_0\rangle\,. \end{aligned}$$

Therefore we have $V(Y_3) = Y_2$, a contradiction with $v(3) = 1$.

Now assume $a_7 \neq 0$ and put $\alpha = (a_6/a_7)^{1/3}$. Then we have $a_3 = \alpha^3 a_4$ by relation $a_7 a_3 = a_6 a_4$, and

$$Y_2 = \langle a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1 + \lambda_0, \lambda_3 + \alpha a_7^{1/3}\lambda_2 + \alpha a_4^{1/3}\lambda_1\rangle\,. \tag{10}$$

By a similar argument to the above, Y_6 is generated by λ_i and $\sum_{j=1}^4 b_j\gamma_j$ with $\langle \sum_{j=1}^4 b_j\gamma_j, Y_2\rangle = 0$, this implies

$$b_4 - \alpha b_1 = b_3 a_7^{1/3} + b_2 a_4^{1/3} + b_1 = 0, b_i \in k\,.$$

Then $Y_3 = V(Y_6) = \langle Y_2, V(b_1\gamma_1 + b_4\gamma_4)\rangle$ and this equals $\langle Y_2, \xi\rangle$ with

$$\xi = (1 + \alpha^{1/3} a_4^{1/3})\lambda_2 + (\alpha a_7^{1/3} + \alpha^{1/3})\lambda_1 + (\alpha^{1/3} a_4^{1/3} + \alpha^{4/3} a_7^{1/3})\lambda_0\rangle\,.$$

Since X has Ekedahl-Oort type $\mu = [3, 2]$, we have $v(3) = 1$. Then $V(\langle\,\xi\,\rangle) = V(Y_2) = V(a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1)$ by relation (10) and Lemma 2.4. Thus we have

$$\alpha a_7^{2/3} + (\alpha a_7)^{1/3} = a_4^{1/3} + \alpha^{1/3} a_4^{2/3}$$

and hence

$$\alpha^3 a_7^2 + \alpha a_7 = a_4 + \alpha a_4^2\,. \tag{11}$$

If $\alpha = 0$, by equality (11) we have $a_4 = 0$ and in equation (8) we have $f = x^9 + a_7x^7 + x$ and one can easily show that $v(1) = 0$, a contradiction as X has Ekedahl-Oort type $\mu = [3, 2]$. By a similar argument we can prove $a_4 \neq 0$. If we take $(a_7, \alpha, a_4) = (2, 2, 1)$ in equation (8), we have $\mathrm{disc}(f) = 2 \neq 0$. Then there is a curve with Ekedahl-Oort type $[3, 2]$ and by semicontinuity we have $Z_{[3,2]} \cap \mathcal{H}_4$ is irreducible of dimension 2.

Let $\mu = [4, 1]$. We show that any smooth hyperelliptic curve with Ekedahl-Oort type $[4, 1]$ can be written as

$$y^2 = f(x) = x^9 + a_7x^7 + \alpha^3 a_7 x^6 - \alpha^9 a_7 x^4 - \alpha^{12} a_7 x^3 + x \tag{12}$$

with $a_7, \alpha \in k^*$ and $\mathrm{disc}(f) = 2\alpha^{10} a_7 + \alpha^2 a_7 + 1 \neq 0$. Then it will follow that $Z_{[4,1]} \cap \mathcal{H}_4$ is irreducible of dimension 2. Indeed, if X is given by equation (4) with Ekedahl-Oort type [4,1], then $v(2) = 1, v(1) = 0$ and by Lemma 2.4 and relation (7) we have $a_3 a_7 = a_4 a_6$.

$a)$: If $a_7 = 0$, we have $a_6 = 0$ or $a_4 = 0$. Assume $a_6 = a_7 = 0$, then by relation (7) we have $V(Y_2) = \langle V(\lambda_1)\rangle$. By Lemma 2.4, we have $a_4 = 0$ since the p-rank of X is zero. But then X has Ekedahl-Oort type $[4,2]$ by a similar argument with $Y_6 = Y_2^{\perp}$ and $Y_3 = V(Y_6)$ as in case $\mu = [3,2]$. Now suppose $a_7 = a_4 = 0$. We have $a_6 = 0$ since X has p-rank 0. Then again X has Ekedahl-Oort type $[4,2]$.

$b)$: Now assume $a_7 \neq 0$. Put $\alpha = (a_6/a_7)^{1/3}$ and we have $a_3 = \alpha^3 a_4$ by equation $a_7a_3 = a_6a_4$. Write $Y_1 = V(Y_2) = \langle a_7^{1/9}V(\lambda_2) + a_4^{1/9}V(\lambda_1)\rangle$. Suppose we have $V^m(Y_1) = 0$ and $V^{m-1}(Y_1) \neq 0$ for some $m \in \mathbb{Z}_{>0}$. For $0 \leq i \leq m$, put $V^i(Y_1) = \langle g_i(\lambda_0, \lambda_3) + c_i\lambda_2 + d_i\lambda_1\rangle$ with $g_i(x,y) \in k[x,y]$. Then we have

$$c_i = (\alpha c_{i-1}^{1/3} + d_{i-1}^{1/3})a_7^{1/3}, d_i = (\alpha c_{i-1}^{1/3} + d_{i-1}^{1/3})a_4^{1/3} \tag{13}$$

for $1 \leq i \leq m$. Now $V(V^{m-1}(Y_1)) = 0$. Therefore by Lemma 2.4 we have

$$V^m(Y_1) = V(\langle c_{m-1}\lambda_2 + d_{m-1}\lambda_1 + g_{m-1}(\lambda_0,\lambda_3)\rangle) = 0\,.$$

Hence we have $c_m = d_m = 0$ as $V(\langle\lambda_0,\lambda_3\rangle) = 0$ by Lemma 2.4. Thus we obtain $(\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3})a_7^{1/3} = (\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3})a_4^{1/3} = 0$, which implies $\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3} = 0$ as $a_7 \neq 0$. Using the inductive assumption (13) again, we have

$$\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3} = ((\alpha^3 a_7^{1/3} + a_4^{1/3})(\alpha^{1/3}c_{m-2} + d_{m-2}))^{1/3} = 0\,.$$

Since $V^{m-1}(Y_1) \neq 0$, we have $\alpha^{1/3}c_{m-2} + d_{m-2} \neq 0$ and hence $(\alpha^3 a_7^{1/3} + a_4^{1/3}) = 0$. This implies $a_4 = -\alpha^9 a_7$ and $a_3 = \alpha^3 a_4 = -\alpha^{12}a_7$. Now we compute $Y_3 = V(Y_6)$ and this equals

$$\langle Y_2, (1 - \alpha^{10/3}a_7^{1/3})\lambda_2 + (\alpha a_7^{1/3} + \alpha^{1/3})\lambda_1 + g(\lambda_0,\lambda_3)\rangle$$

for some $g(x,y) \in k[x,y]$. Combined with

$$\begin{aligned} Y_2 &= \langle a_7^{1/3}(\lambda_2 - \alpha^3\lambda_1) + \lambda_0, \lambda_3 + \alpha a_7^{1/3}(\lambda_2 - \alpha^3\lambda_1)\rangle \\ &= \langle \lambda_3 - \alpha\lambda_0, a_7^{1/3}(\lambda_2 - \alpha^3\lambda_1) + \lambda_0\rangle\,, \end{aligned}$$

we have $v(3) = 1$ if

$$\alpha^3(-1 + \alpha^{10/3}a_7^{1/3}) = (\alpha a_7^{1/3} + \alpha^{1/3})\,,$$

this is equivalent to $\alpha^3(\alpha^{16}-1)a_7 = \alpha^9 + \alpha$. Otherwise X has Ekedahl-Oort type $[4,1]$ for general pair $(a_7,\alpha) \in \mathbb{A}_k^2$. Hence we have the desired conclusion for $\mu = [4,1]$. Moreover if in equation (12) we take $(a_7,\alpha) = (v^{10}, v^9)$ with v a primitive element in $\mathbb{F}_{27}$, then $\mathrm{disc}(f) = v^{21} \neq 0$ and there is a curve with equation (12) has Ekedahl-Oort type associated to $\mu = [4,1]$. Hence by semicontinuity we have proved the theorem is true for $\mu = [4,1]$.

For $\mu = [4,2]$, from the discussion in the case $[4,1]$ above, a hyperelliptic curve X with Ekedahl-Oort type $[4,2]$ is either given by equation (4) with $a_7 = a_6 = a_4 = 0$, or it can be written as

$$y^2 = f(x) = x^9 + a_7x^7 + \alpha^3 a_7 x^6 - \alpha^9 a_7 x^4 - \alpha^{12} a_7 x^3 + x$$

with $a_7, \alpha \in k, a_7 \neq 0$ satisfying $\alpha^3(\alpha^{16}-1)a_7 = \alpha^9+\alpha$ and $\mathrm{disc}(f) \neq 0$. Moreover,the curve with equation $y^2 = x^9 + x^7 + x$ has $\mathrm{disc}(f) = 1 \neq 0$ and Ekedahl-Oort type $\mu = [4,2]$. Hence $Z_{[4,2]} \cap \mathcal{H}_4$ is non-empty of dimension 1.

For $\mu = [4,3]$, a curve X with Ekedahl-Oort type $[4,3]$ is given by (4) with $V(Y_2) = \langle 0 \rangle$. Then by Lemma 2.4 we have $a_7 = a_6 = a_4 = a_3 = 0$. This implies X is isomorphic to the curve with equation $y^2 = x^9 + x$. Now we have proved the theorem for $\mu = [2,1], [3,1], [3,2], [4,1], [4,2]$ and $[4,3]$. Also for $\mu = [2], [3]$ and $[4]$, by Glass and Pries [6, Theorem 1] the intersection $V_l \cap \mathcal{H}_4$ has codimension $4 - l$ in $\mathcal{H}_4$ for $l = 0, 1, 2$. Since we have showed that $Z_{[2,1]}$ (resp. $Z_{[3,1]}$ and $Z_{[4,1]}$) intersects $\mathcal{H}_4$ with codimension 3 (resp. 4 and 5), it follows that $Z_\mu \cap \mathcal{H}_4$ has the expected codimension for $\mu = [2], [3]$ and $[4]$. □

3. Proof of Theorem 1.2

We prove Theorem 1.2 using cyclic covers of the projective line in characteristic $p > 0$. First we introduce some general facts on cyclic covers of the projective line and give a basis of the first de Rham cohomology for them.

Let k be an algebraically closed field of characteristic $p > 0$. We fix an integer $m \geq 2$ with $p \nmid m$. Write $a = (a_1, \dots, a_N)$ for an N-tuple of positive integers with $N \geq 3$. We say a is a monodromy vector of length N if

$$\sum_{i=1}^{N} a_i \equiv 0 \pmod{m}, \qquad \gcd(a_i, m) = 1,\ i = 1, \dots, N. \tag{14}$$

There is an action of $(\mathbb{Z}/m\mathbb{Z})^* \times \mathfrak{S}_N$ on the set of monodromy vectors of length N, where the symmetric group $\mathfrak{S}_N$ acts by permutation of indices and $(\mathbb{Z}/m\mathbb{Z})^*$ acts by multiplication on vectors. Two monodromy vectors a and a' are called equivalent if they are in the same orbit.

Let $P_1, \dots, P_N$ be the distinct points in $\mathbb{P}^1$ and x be a coordinate on $\mathbb{P}^1$. By a change of coordinates, we may assume $P_1 = 0$ and $P_N = \infty$. Denote by $x - \xi_i$ with $\xi_i \in k$ the local parameter of P_i ($\xi_1 = 0$) for $1 \leq i \leq N-1$. We consider a smooth projective curve X given by equation

$$y^m = f_a(x) = \Pi_{i=1}^{N-1}(x - \xi_i)^{a_i}\,. \tag{15}$$

Note that the isomorphism class of the curve depends only on the orbit of monodromy vector a. For $N = 3$, the supersingularity of cyclic covers of the projective line has been studied and examples of supersingular curves was given for $4 \leq g \leq 11$, see [10]. In [3], Elkin gave a bound for the a-number of X for $m \geq 2$ and $N \geq 3$.

A curve defined by equation (15) is equipped with a $\mathbb{Z}/m\mathbb{Z}$ action generated by $\epsilon : (x, y) \mapsto (x, \zeta^{-1} y)$ with $\zeta \in k$ a primitive m-th root of unity. This ϵ also induces an automorphism on $H^0(X, \Omega^1_X)$. Then we can decompose

$$H^0(X, \Omega^1_X) = \bigoplus_{n=1}^{m-1} H^0(X, \Omega^1_X)_{(n)}\,, \tag{16}$$

where $H^0(X, \Omega^1_X)_{(n)} := \{\omega \in H^0(X, \Omega^1_X) \mid \epsilon^*(\omega) = \zeta^n \omega\}$ is the ζ^n-eigenspace of $H^0(X, \Omega^1_X)$. Denote by $\langle z \rangle := z - \lfloor z \rfloor$ the fractional part of z for any $z \in \mathbb{R}$. Put

$$b(i, n) := \lfloor (na_i)/m \rfloor,\ \omega_{n,l} := y^{-n} x^l \Pi_{i=1}^{N-1} (x - \xi_i)^{b(i,n)}\, \mathrm{d}x\,.$$

Then (see for example [11]) the space $H^0(X, \Omega^1_X)_{(n)}$ is generated by $\omega_{n,l}$ with $0 \leq l \leq -2 + \sum_{i=1}^N \langle na_i/m \rangle$ and

$$d_n = \dim H^0(X, \Omega^1_X)_{(n)} = -1 + \sum_{i=1}^N \langle na_i/m \rangle\,.$$

By the Hurwitz formula, the curve has genus $g = 1 + ((N-2)m - N)/2$.

Now we introduce a basis of de Rham cohomology of a curve X given by equation (15). Fix a monodromy vector a together with an ordered N-tuple $(\xi_1, \ldots, \xi_N)$. Let $\pi : X \to \mathbb{P}^1$ be the $\mathbb{Z}/m\mathbb{Z}$-cover. Put $U_1 = \pi^{-1}(\mathbb{P}^1 - \{0\})$ and $U_2 = \pi^{-1}(\mathbb{P}^1 - \{\infty\})$. For the open affine cover $\mathcal{U} = \{U_1, U_2\}$, we consider the de Rham cohomology $H^1_{dR}(X)$ similar to the hyperelliptic case in Section 2 above, i.e.

$$H^1_{dR}(X) \cong Z^1_{dR}(\mathcal{U})/B^1_{dR}(\mathcal{U})$$

with $Z^1_{dR}(\mathcal{U}) = \{(t, \omega_1, \omega_2) | t \in \mathcal{O}_X(U_1 \cap U_2), \omega_i \in \Omega^1_X(U_i), \mathrm{d}t = \omega_1 - \omega_2\}$ and $B^1_{dR}(\mathcal{U}) = \{(t_1 - t_2, \mathrm{d}t_1, \mathrm{d}t_2) | t_i \in \mathcal{O}_X(U_i)\}$.

By (16), the vector space $H^0(X, \Omega^1_X)$ is generated by $\omega_{n,l}$ with $1 \leq n \leq m-1$ and $0 \leq l \leq -2 + \sum_{i=1}^N \langle na_i/m \rangle$. Moreover for the basis of $H^1(X, \mathcal{O}_X)$, we have the following result.

Lemma 3.1. *Let $K(X)$ be the function field of X. For all integers n, l such that $1 \leq n \leq m-1$ and $0 \leq l \leq -2 + \sum_{i=1}^N \langle na_i/m \rangle$, the elements $f_{n,l} := y^n x^{-l-1} \Pi_{i=1}^{N-1} (x - \xi_i)^{-b(i,n)} \in K(X)$ are regular on $U_1 \cap U_2$ and their residue classes $[f_{n,l}]$ form a basis of $H^1(X, \mathcal{O}_X)$ with respect to $\{U_1, U_2\}$.*

Proof. It suffices to show that the $f_{n,l}$ are regular on $U_1 \cap U_2$ for all integers n, l such that $\omega_{n,l} \in H^0(X, \Omega^1_X)$. One can check the linear independence by checking the order of $f_{n,l}$ at ∞.

Note that for P_i with $i = 2, \ldots, N-1$, we have

$$\begin{aligned}\mathrm{ord}_{P_i}(\frac{y^n}{x^{l+1}\Pi_{i=1}^{N-1}(x-\xi_i)^{b(i,n)}}) &= n\,\mathrm{ord}_{P_i}(y) - b(i,n)\,\mathrm{ord}_{P_i}(x-\xi_i)\\ &= na_i - mb(i,n) = na_i - m\lfloor\frac{na_i}{m}\rfloor \geq 0\,.\end{aligned}$$

Then $f_{n,l}$ is regular on $U_1 \cap U_2$ for $1 \leq n \leq m-1$ and $0 \leq l \leq -2 + \sum_{i=1}^{N}\langle na_i/m\rangle$. $\square$

Put $s_a(x) := \Pi_{i=1}^{N-1}(x-\xi_i)^{b(i,n)}$. Denote by $h_a(x)$ the polynomial in $k[x]$ such that

$$\frac{nxs_a(x)f_a'(x) + ((l+1)s_a(x) + xs_a'(x))f_a(x)}{s_a^2(x)} = \Pi_{i=1}^{N-1}(x-\xi_i)^{a_i-b(i,n)-1}h_a(x)\,,$$

where $f_a'(x)$ (resp. $s_a'(x)$) is the formal derivative of $f_a(x)$ (resp. $s_a(x)$). In the following we write simply $s(x)$ (resp. $f(x)$, $h(x)$) for $s_a(x)$ (resp. $f_a(x)$, $h_a(x)$).

Then we have the following result.

Proposition 3.2. *Let X be a smooth projective curve over k given by equation (15). Then $H^1_{dR}(X)$ has a basis with respect to $\mathcal{U} = \{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(\mathcal{U})$:*

$$\alpha_{n,l} = [(0, \omega_{n,l}, \omega_{n,l})],\ 1 \leq n \leq m-1,\ 0 \leq l \leq -2 + \sum_{i=1}^{N}\langle na_i/m\rangle\,, \tag{17}$$

$$\beta_{n,l} = [(f_{n,l}, \frac{\psi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\mathrm{d}x, -\frac{\phi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\mathrm{d}x)]\,, \tag{18}$$

where $t(x) = \Pi_{i=1}^{N-1}(x-\xi_i)^{a_i-b(i,n)-1}$ and $\psi(x) + \phi(x) = h(x)$ with $\psi(x)$ the sum of monomials of degree $\leq l+1$.

Proof. We use the exact sequence

$$0 \to H^0(X, \Omega^1_X) \to H^1_{dR}(X) \to H^1(X, \mathcal{O}_X) \to 0\,.$$

The elements $\alpha_{n,l}$ are images of $\omega_{n,l}$ under the canonical map.

By Lemma 3.1, we have $[f_{n,l}] \in H^1(X, \mathcal{O}_X)$ for any n, l such that $\omega_{n,l} \in H^0(X, \Omega^1_X)$. To prove the theorem, we need to show that the elements $\beta_{n,l}$ are well defined and are mapped to the element $[f_{n,l}]$ in $H^1(X, \mathcal{O}_X)$. We first show that $\psi_{n,l}(x)/(x^{l+2}y^{m-n}) \in \mathcal{O}(U_1)$ and $\phi_{n,l}(x)/(x^{l+2}y^{m-n}) \in \mathcal{O}(U_2)$. Next we show

$$\mathrm{d}f_{n,l} = \frac{\psi_{n,l}(x)}{x^{l+2}y^{m-n}} - (-\frac{\phi_{n,l}(x)}{x^{l+2}y^{m-n}})$$

and then we will have the desired conclusion.

Note that for any P_i with $i = 1, \ldots, N-1$, we have

$$\begin{aligned}\mathrm{ord}_{P_i}(\frac{t(x)}{y^{m-n}}\,\mathrm{d}x) &= \mathrm{ord}_{P_i}(\frac{\Pi_{i=1}^{N-1}(x-\xi_i)^{a_i-b(i,n)-1}}{y^{m-n}}) + \mathrm{ord}_{P_i}(\mathrm{d}x)\\ &= m(a_i - \lfloor\frac{na_i}{m}\rfloor - 1) - a_i(m-n) + m - 1 = m\langle\frac{na_i}{m}\rangle - 1 \geq 0\,,\end{aligned}$$

since $\gcd(a_i, m) = 1$. Hence

$$\mathrm{ord}_{P_i}(\frac{\phi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\mathrm{d}x)) \geq \mathrm{ord}_{P_i}(\frac{t(x)}{y^{m-n}}\,\mathrm{d}x) \geq 0$$

and $\phi_{n,l}(x)t(x)/(x^{l+2}y^{m-n})\mathrm{d}x$ is regular at P_i for $i = 2, \ldots, N-1$. By a similar argument, $\psi_{n,l}(x)t(x)/(x^{l+2}y^{m-n})\mathrm{d}x$ is also regular at P_i for $i = 2, \ldots, N-1$. For $P_1 = 0$, we have

$$\mathrm{ord}_{P_1}(\frac{\phi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x) \geq \mathrm{ord}_{P_1}(\frac{t(x)}{y^{m-n}}\,\mathrm{d}x) \geq 0\,,$$

since all the monomials of $\phi_{n,l}(x)$ has degree $\geq l+2$. Then the residue class of the element $\phi_{n,l}(x)t(x)/(x^{l+2}y^{m-n})\mathrm{d}x$ is regular on U_2. For $P_N = \infty$, by a similar calculation we have

$$\mathrm{ord}_{P_N}(\frac{\psi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x) \geq \mathrm{ord}_{P_N}(\frac{x^{l+1}t(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x) \geq 0\,.$$

Let n, l be integers such that $\omega_{n,l} \in H^0(X, \Omega^1_X)$. Then

$$\begin{aligned}\mathrm{d}(f_{n,l}) = \mathrm{d}(\frac{y^n}{x^{l+1}s(x)}) &= \frac{nxs(x)f'(x) + ((l+1)s(x) + xs'(x))f(x)}{x^{l+2}s^2(x)y^{m-n}}\,\mathrm{d}x\\ &= \frac{t(x)h(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x = \frac{t(x)\psi_{n,l}(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x - \frac{-t(x)\phi_{n,l}(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x\,.\end{aligned}$$

□

Remark 3.3. The pairing $\langle\ ,\ \rangle$ for this basis is as follows: $\langle\alpha_{i_1,j_1}, \beta_{i_2,j_2}\rangle \neq 0$ if $(i_1, j_1) = (i_2, j_2)$ and $\langle\alpha_{i_1,j_1}, \beta_{i_2,j_2}\rangle = 0$ otherwise. Indeed, for $(i_1, j_1) = (i_2, j_2)$ we have $\mathrm{ord}_\infty(1/x\,\mathrm{d}x) = -1$ and hence $\langle\alpha_{i_1,j_1}, \beta_{i_2,j_2}\rangle \neq 0$. For the other cases, the proof is similar to the proof of [15, Theorem 4.2.1].

Now for $p = 2$ and $N = 4$, we have the following

Corollary 3.4. *Let k be an algebraically closed field of characteristic p and a be a monodromy vector satisfying relation (14) with $a = (1, 1, 1, m-3)$. Let X be a curve of genus $m-1$ over k given by equation*

$$y^m = x(x-1)(x-\xi)\,,$$

where $\xi \neq 0, 1 \in k$. Then $H^1_{dR}(X)$ has a basis with respect to $\mathcal{U} = \{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(\mathcal{U})$:

$$\alpha_{i,0} = [(0, \frac{1}{y^i}\,\mathrm{d}x, \frac{1}{y^i}\,\mathrm{d}x)],\ \frac{m}{3} < i \leq m-1\,, \alpha_{j,1} = [(0, \frac{x}{y^j}\,\mathrm{d}x, \frac{x}{y^j}\,\mathrm{d}x)],\ \frac{2m}{3} < j \leq m-1\,,$$

$$\beta_{i,0} = [(\frac{y^i}{x}, \frac{\xi}{xy^{m-i}}\mathrm{d}x, -\frac{x+(\xi+1)}{y^{m-i}}\mathrm{d}x)],\ \ i \text{ even and } \frac{m}{3} < i \leq m-1,$$

$$\beta_{i,0} = [(\frac{y^i}{x}, 0, -\frac{(\xi+1)}{y^{m-i}}\mathrm{d}x)],\ \ i \text{ odd and } \frac{m}{3} < i \leq m-1,$$

$$\beta_{j,1} = [(\frac{y^j}{x^2}, 0, 0)],\ \ j \text{ even and } \frac{2m}{3} < j \leq m-1,$$

$$\beta_{j,1} = [(\frac{y^j}{x^2}, \frac{\xi}{x^2y^{m-j}}\mathrm{d}x, -\frac{1}{y^{m-j}}\mathrm{d}x)],\ \ j \text{ odd and } \frac{2m}{3} < j \leq m-1,$$

Proof. Note that $a = (a_1, a_2, a_3, a_4) = (1, 1, 1, m-3)$. Then by definition $b(i, n) = \langle na_i/m\rangle = 0$ for any $1 \leq n \leq m-1$ and $1 \leq i \leq 3$. Moreover, the differential form $\omega_{n,l} = y^{-n}x^l\,\mathrm{d}x$ is holomorphic for $1 \leq n \leq m-1$ if and only if $0 \leq l \leq -2 + \sum_{i=1}^4 \langle na_i/m\rangle \leq -2 + 3 = 1$. If $0 \leq n \leq m/3$, then $\sum_{i=1}^4 \langle na_i/m\rangle = 1$ and $H^0(X, \Omega^1_X)_{(n)} = \langle 0\rangle$. The rest of the corollary follows from Proposition 3.2. ☐

We can now give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let X be a curve of genus 4 with equation

$$y^5 = x(x-1)(x-\xi)\,,$$

where $\xi \in k\backslash\mathbb{F}_p$ and $p = 2\,(\bmod 5)$. We first show that X has Ekedahl-Oort type $[4, 2]$. Then the case $p = -2\,(\bmod 5)$ is similar and hence we omit it. Write $p = 5r+2$ with $r \in \mathbb{Z}_{\geq 0}$. Since p is a prime, either $r = 0$ or r is an odd positive integer.

Let $Y_8 := H^1_{dR}(X)$ and $Y_4 := H^0(X, \Omega^1_X)$. By Corollary 3.4, we have $V(Y_8) = Y_4$ and $Y_4 = \langle \alpha_{2,0}, \alpha_{3,0}, \alpha_{4,0}, \alpha_{4,1}\rangle$.

Note that

$$\mathcal{C}(\frac{1}{y^2}\,\mathrm{d}x) = \mathcal{C}(\frac{y^{5r}}{y^{5r+2}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}((x(x-1)(x-\xi))^r\,\mathrm{d}x) = 0\,.$$

Then similarly $\mathcal{C}(1/y^3\,\mathrm{d}x) = \mathcal{C}((x(x-1)(x-\xi))^{4r+1}\,\mathrm{d}x)/y^4\,, \mathcal{C}(1/y^4\,\mathrm{d}x) = \mathcal{C}((x(x-1)(x-\xi))^{2r}\,\mathrm{d}x)/y^2$ and $\mathcal{C}(x/y^4\,\mathrm{d}x) = \mathcal{C}(x^{2r+1}((x-1)(x-\xi))^{2r}\,\mathrm{d}x)/y^2$. One can show that the coefficient of $x^{p-1} = x^{5r+1}$ in $x^{2r+i}((x-1)(x-\xi))^{2r}$ cannot be simultaneously zero for $i = 0, 1$ and $\xi \in k\backslash\mathbb{F}_p$. Similarly, the coefficients of x^{p-1} and x^{2p-1} in $(x(x-1)(x-\xi))^{4r+1}$ are both not zero. Then $Y_2 := V(Y_4) = \langle \alpha_{2,0},\, \gamma\alpha_{4,0} + \eta\alpha_{4,1}\rangle$ with $\gamma, \eta \in k^*$. Denote by $Y_6 = Y_2^{\perp}$ the orthogonal complement with respect to the pairing on $H^1_{dR}(X)$. Hence by a calculation using Corollary 3.4, we have

$$Y_6 = \langle \alpha_{2,0}, \alpha_{3,0}, \alpha_{4,0}, \alpha_{4,1}, \beta_{3,0}, \lambda_0\beta_{4,0} + \lambda_1\beta_{4,1} \rangle \,,$$

where $\lambda_i \in k^*$. This implies $Y_3 := V(Y_6) = \langle \alpha_{2,0}, \alpha_{4,0}, \alpha_{4,1} \rangle$ and $V(Y_3) = \langle \alpha_{2,0} \rangle$. We obtain that X has Ekedahl-Oort type $[4, 2]$.

Now we show that $Z_{[4,3]}$ is non-empty in $\mathcal{M}_4$ for any odd prime p with $p \equiv \pm 2(\bmod 5)$.

Take now $m = 5$ and monodromy vector $a = (1, 1, 1, 2)$ in equation (15) and consider a curve X given by equation

$$y^5 = x(x - \xi)(x + \xi)\,, \tag{19}$$

where $\xi \in k^*$. For $p \neq 2, 5$, the curve is of genus 4. Moreover by Lemma 3.1, the vector space $H^0(X, \Omega^1_X)$ has a basis given by forms $y^2\,\mathrm{d}x, y^3\,\mathrm{d}x, y^4\,\mathrm{d}x$ and $xy^4\,\mathrm{d}x$. Now if $p > 2$ and $p \equiv 2(\bmod 5)$, then write $p = 5r + 2$ with r an odd positive integer and we consider the action of the Cartier operator $\mathcal{C}$. By a similar calculation as in the case $Z_{[4,2]}$, we have

$$\mathcal{C}(\frac{\mathrm{d}x}{y^2}) = \mathcal{C}(\frac{x\,\mathrm{d}x}{y^4}) = 0,\, \mathcal{C}(\frac{\mathrm{d}x}{y^3}) = \eta_1\frac{x\,\mathrm{d}x}{y^4},\, \mathcal{C}(\frac{\mathrm{d}x}{y^4}) = \eta_2\frac{\mathrm{d}x}{y^2}$$

with some $\eta_1, \eta_2 \in k^*$. Then X has Ekedahl-Oort type $[4, 3]$ and $v(2) = 0$ with v the final type, cf. [13, 16]. This implies X is supersingular, see [1, page 1379]. By a similar argument, one can show that for $p \equiv 3(\bmod 5)$, the curve has Ekedahl-Oort type $[4, 3]$ and hence is supersingular.

Now we show the existence of superspecial curves of genus 4 in characteristic $p \equiv -1(\bmod 5)$. Again let X be the same curve given by equation (19) and write $p = 5r + 4$ with r some positive integer. Then

$$\mathcal{C}(\frac{\mathrm{d}x}{y^2}) = \mathcal{C}(\frac{y^{15r+10}}{y^{15r+10+2}}\,\mathrm{d}x) = \frac{1}{y^3}\mathcal{C}(x^{3r+2}(x^2 - v^2)^{3r+2}\,\mathrm{d}x) = 0.$$

By a similar calculation, we have $\mathcal{C}(\mathrm{d}x/y^3) = \mathcal{C}(x\,\mathrm{d}x/y^4) = \mathcal{C}(\mathrm{d}x/y^4) = 0$. Then $\mathcal{C}(H^0(X, \Omega^1_X)) = 0$ and X is superspecial. □

4. An alternative proof of Kudo's result

In [8], Kudo showed that there is a superspecial non-hyperelliptic curve of genus 4 over k for any odd prime $p \equiv 2\,(\bmod 3)$ by viewing such curves as an intersection of a quadric and a cubic in $\mathbb{P}^3$. Using our approach, we can give an alternative proof of Kudo's result.

Proposition 4.1. *[8, Theorem 3.1] There exists a superspecial curve of genus 4 in characteristic $p \equiv 2\,(\bmod 3)$.*

Proof. Consider the monodromy vector $a = (1,1,1,1,1,1)$. Let X be the smooth projective curve of genus 4 with equation

$$y^3 = x(x-\xi)(x-\xi^3)(x-\xi^5)(x-\xi^7) = x^5 + x\,,$$

where $\xi \in k$ is a primitive 8-th root of unity.

By Lemma 3.1, the vector space $H^0(X, \Omega^1_X)$ has a basis consisting of forms $1/y\,\mathrm{d}x$, $1/y^2\,\mathrm{d}x, x/y^2\,\mathrm{d}x$ and $x^2/y^2\,\mathrm{d}x$. Write $p = 3r+2$ with r an odd positive integer. Then

$$\mathcal{C}(\frac{1}{y}\,\mathrm{d}x) = \mathcal{C}(\frac{y^{6r+3}}{y^{6r+3+1}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}(x^{2r+1}(x^4+1)^{2r+1}\,\mathrm{d}x)\,.$$

Since r is an odd integer, the coefficient of x^{np-1} in $x^{2r+1}(x^4+1)^{2r+1}$ for any $n \in \mathbb{Z}_{>0}$ is zero. Similarly, we have $\mathcal{C}(x^i\,\mathrm{d}x/y^2) = 0$ for $i = 0,1,2$. ☐

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