Review on rationality problems of algebraic k-tori

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Abstract

Rationality problems of algebraic $k-tori$ are closely related to rationality problems of the invariant field, also known as Noether's Problem. We describe how a function field of algebraic $k-tori$ can be identified as an invariant field under a group action and that a $k-tori$ is rational if and only if its function field is rational over k. We also introduce character group of $k-tori$ and numerical approach to determine rationality of $k-tori$.

Contents

1 Introduction

Let k be a field and K is a finitely generated field extension of k. K is called *rational over* k or *k-rational* if K is isomorphic to $k(x_1, ..., x_n)$ where x_i are transcendental over k and algebraically independent. There are also relaxed notions of rationality. K is called *stably k-rational* if $K(y_1, ..., y_m)$ is $k-rational$ for some transcendental and algebraically independent y_i. K is called $k - unirational$ if $k \subset K \subset k(x_1, ..., x_n)$ for some pure transcendental extension $k(x_1, ..., x_n)/k$.

The Noether's Problem is the question of rationality of the invariant field under finite group action. For example, if $K = \mathbb{Q}(x_1, x_2)$ and $G = \{1, \sigma\} \cong C_2$ and G acts on K as permutation of variables x_1, x_2 (i.e. σ fixes $\mathbb{Q}$, $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$), then the invariant field K^G is $\mathbb{Q} - rational$.

Example 1.1 *$K = \mathbb{Q}(x, y)$ and $G \cong C_2$, acting on K as permutation of variables. Let $\frac{f}{g} \in K^G$, f, g are coprime. We have*

$$\frac{f(x,y)}{g(x,y)} = \sigma(\frac{f(x,y)}{g(x,y)}) = \frac{f(y,x)}{g(y,x)}$$

By observing that $gcd(f(x, y), g(x, y)) = gcd(f(y, x), g(y, x)) = 1$, we have $f(x, y) = f(y, x)$ and $g(x, y) = g(y, x)$.
Therefore, $K^G = \{\frac{f(x,y)}{g(x,y)} | f, g \text{ are symmetric}\}$, field of fractions (quotient field) of $S = \{f \in \mathbb{Q}[x, y] | f(x, y) = f(y, x)\}$. It is easy to see that $\psi : S \to \mathbb{Q}[s, t]$ is isomorphism, where

$$\psi(x + y) = s, \quad \psi(xy) = t$$

Therefore, $S \cong \mathbb{Q}[x, y]$ and $K^G \cong \mathbb{Q}(x, y)$, $\mathbb{Q} - rational$.

We can also consider case of G acting on both of coefficients and variables.

Example 1.2 *$K = \mathbb{C}(x, y)$ and $G = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\} \cong C_2$. Suppose G acts on K by permuting x, y and as complex conjugation on coefficients.*
For example, $\sigma(ix + (1-i)xy + y^2) = -iy + (1+i)yx + x^2$. Then, $K^G \cong \mathbb{R}(x, y)$, is $\mathbb{R} - rational$.

Proof. *For* $\frac{f(z,w)}{g(z,w)} \in K^G$*, where* f, g *are coprime,* $\sigma(f)$ *and* $\sigma(g)$ *are also coprime. From* $\frac{f}{g} = \frac{\sigma(f)}{\sigma(g)}$*, we have* $f = \sigma(f)$ *and* $g = \sigma(g)$*. Thus,* K^G *is quotient field of* S *where* $S := \{f(z,w) \in \mathbb{C}[z,w] | f = \sigma(f)\}$*.*

Define a map $\psi : S \to \mathbb{R}[x,y]$ *as*

$$z = x + yi, w = x - yi$$

and

$$\psi(f)(x,y) = f(z,w)$$

The coefficients of $\psi(f)$ *are real numbers. This is because, if we let* $f(z,w) = \sum_{n,m} a_{n,m} z^n w^m$*, we have that*

$$\psi(f)(x,y) = f(z,w) = \sigma(f(z,w)) = \sigma(\sum_{n,m} a_{n,m} z^n w^m) = \sum_{n,m} \overline{a_{n,m}} w^n z^m$$

$$= \sum_{n,m} \overline{a_{n,m}(x+iy)^n (x-iy)^m} = \overline{\psi(f)(x,y)}.$$

Therefore, $\psi(f) = \overline{\psi(f)}$*,* $\psi(f) \in \mathbb{R}[x,y]$*. It is easy to see that* ψ *is actually isomorphism,* $S \cong \mathbb{R}[x,y]$*, and* $K^G \cong \mathbb{R}(x,y)$*.*

Another perspective to view this *change of variables* is identifying the field with rational function field of algebraic $k - tori$. (see **Example 2.5** and **Example 2.6**)

2 Algebraic $k - tori$

Let k be a field. Then $\mathbb{A}_k^n$ is *n-dimension affine space* over the field k, simply k^n with usual vector space structure on it. A subset X of $\mathbb{A}_k^n$ is an *algebraic k-variety* (*k-variety* in short) if it is a set of zeros of a system of equations with n variables $x_1, ... x_n$ over k. The ideal of polynomials that vanish on every points of X will be denoted by $I(X)$. The *coordinate ring* of a variety X is defined to be the quotient

$$A(X) := k[x_1, ..., x_n]/I(X)$$

Projective varieties can be similarly defined as the set of zeros of a system of homogeneous equations. *Projective* $n-space$ $\mathbb{P}_k^n$ is defined as set of lines passing the origin in $\mathbb{A}_k^{n+1}$.

If X, Y are varieties, a map $f : X \to Y$ is called *regular* if it can be presented as fraction of polynomials p/q, where q does not vanishes in X. A map $f : X \to Y$ is called *rational* if it is regular on Zariski open dense set. (Formally, a regular map is defined as an equivalence class of pairs $< U, f_U >$ where U is Zariski open subset of U. See [1]) Let X be a variety, $K(X)$ is the *rational function field*, or *function field* in short, the set of rational maps $f : X \to \mathbb{A}_k$. For example, if X is an affine variety over algebraically closed field k, $K(X)$ is quotient field of $A(X)$.

Example 2.1 *Let* $X = \{(x, y) \in \mathbb{A}_{\mathbb{C}}^2 | xy = 1\}$ *be a variety over* $\mathbb{C}$.
Then, $A(X) = \mathbb{C}[x, y]/(xy - 1) \cong \mathbb{C}[x, \frac{1}{x}]$ *and* $K(X) \cong \mathbb{C}(x)$.

Two varieties X, Y are *isomorphic* (resp. *birationally isomorphic*) if there is a bijective regular map (resp. rational map) $f : X \to Y$ and its inverse is also regular (resp. rational).

A variety X in $\mathbb{A}_k^n$ is an *algebraic group* if it has a group structure on it, where the group operation and inversions are regular maps. (i.e. $* : X \times X \to X$ and $^{-1} : X \to X$ are regular)

Algebraic $k-tori$, or algebraic $k-torus$, is a special type of algebraic group over k. We call an algebraic group as $k - tori$ when it is isomorphic to some power of multiplicative group over $\overline{k}$, the algebraic closure of k.

Definition 2.1 (Multiplicative Group) *Let k be a field, the multiplicative group* $\mathbb{G}_m(k)$ *is algebraic group in* $\mathbb{A}_k^2$, *defined as* $\{(x, y) \in \mathbb{A}_k^2 | xy = 1\}$, *with operation* $\cdot : \mathbb{G}_m(k) \times \mathbb{G}_m(k) \to \mathbb{G}_m(k)$ *of* $(x, \frac{1}{x}) \cdot (y, \frac{1}{y}) = (xy, \frac{1}{xy})$

Example 2.2 $\mathbb{G}_m(\mathbb{R})$ *is the curve* $xy = 1$ *on the real affine plane. It is isomorphic to* $\mathbb{R}^\times$ *as a group.* ($(x, y) \to x$ *is group isomorphism.*)

As field changes, same system of equations can define different varieties. For instance, the equation $xy = 1$ in previous example defines $\mathbb{G}_m(\mathbb{C})$ in $\mathbb{A}_{\mathbb{C}}^2$, which is different from $\mathbb{G}_m(\mathbb{R})$. If E is a field and F is its algebraic closure, an irreducible variety V over F entails the ring of equations, I. If I happens to be in $E[\mathbf{x}]$ (ring of polynomials over E), we can define $V(E)$, a variety over E defined by equations in I. This can be viewed as *restriction* of scalar. Extension of scalar can be defined similarly.

Definition 2.2 (Algebraic k-tori) *Let k be a field with algebraic closure $\overline{k}$. If T is an algebraic group over k, it is $k-torus$ if and only if*

$$T(\overline{k}) \cong (\mathbb{G}_m(\overline{k}))^r$$

for some r. The r is called dimension of T.

Example 2.3 $T = \mathbb{G}_m(\mathbb{R})$ *is one dimensional* $\mathbb{R}-tori$. *This is because* $T(\mathbb{C}) = \mathbb{G}_m(\mathbb{C})$.

From now, let $k^\times = \mathbb{G}_m(k)$ be the one dimensional torus over k. There are two one-dimensional $\mathbb{R}$-tori, one can be recognized as $\mathbb{R}^\times$, the other one can be recognized as $SO(2)$ as a group.

Example 2.4 *The norm one torus N is a real algebraic group in $\mathbb{A}^2_{\mathbb{R}}$, defined by equation $x_1^2 + x_2^2 = 1$ (i.e. $N = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{R}} | x_1^2 + x_2^2 = 1\}$), and operation $\cdot : N \times N \to N$ such that*

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$$

Indeed, N is isomorphic to $SO(2)$ as a group.
Also, $N(\mathbb{C}) = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} | x_1^2 + x_2^2 = 1\}$ is isomorphic to $C^\times$ as algebraic group. The map $\psi : N(\mathbb{C}) \to \mathbb{C}^\times$

$$\psi(x_1, x_2) = x_1 + ix_2$$

is isomorphism. Therefore, N is one dimensional real torus.

If T is a $k-torus$, T is called *split over* K if it satisfies $T(K) \cong (K^\times)^s$ for some extension K/k and some s. For instance, $\mathbb{R}^\times$ is split over $\mathbb{R}$, N is not.

It is easy to find split torus such as $(\mathbb{R}^{\times})^2$ or $(\mathbb{R}^{\times})^3$, being another torus. Also, for any integer r, N^r is r-dimensional $\mathbb{R}-tori$. Meanwhile, there are also some non-trivial(not a product of low-dimensional torus) torus.

Example 2.5 *Let P be a real algebraic group in $\mathbb{A}^4_{\mathbb{R}}$, defined as*

$$P = \{(x_1, x_2, x_3, x_4) \in \mathbb{A}^4_{\mathbb{R}} | x_1x_3 - x_2x_4 = 1, x_1x_4 + x_2x_3 = 0\}$$

Alternatively,

$$P = \{A \in M_{2\times 2}(\mathbb{R}) \mid AA^t = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \quad s \in \mathbb{R}\backslash\{0\}\}$$

and operation $\cdot : P \times P \to P$ such that

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1, x_3y_3 - x_4y_4, x_3y_4 + x_4y_3)$$

Which is compatible with complex multiplication of

$$(x_1 + x_2i, x_3 + x_4i) \cdot (y_1 + y_2i, y_3 + y_4i)$$

Moreover, $P(\mathbb{C})$ is isomorphic to $(\mathbb{C}^{\times})^2$, by sending

$$(x_1, x_2, x_3, x_4) \to ((x_1 + x_2i, x_3 + x_4i), (x_1 - x_2i, x_3 - x_4i)) = ((z, \frac{1}{z}), (w, \frac{1}{w}))$$

Therefore, P is 2-dimensional $\mathbb{R}-tori$.

By tracking the function fields of $P(\mathbb{R})$ and $P(\mathbb{C})$, we have the same trick of change of variables as in **Example 1.2**.

Example 2.6 *In the previous example, the coordinate ring of $P(\mathbb{C})$ is*

$$A(P(\mathbb{C})) = \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1x_3 - x_2x_4 - 1, x_1x_4 + x_2x_3) \cong \mathbb{C}[z, \frac{1}{z}, w, \frac{1}{w}]$$

where $z = x_1 + x_2i$ and $w = x_1 - x_2i$. The function field of $P(\mathbb{C})$ is

$$K(P(\mathbb{C})) \cong \mathbb{C}(z, w)$$

Let $G = Gal(\mathbb{C}/\mathbb{R})$ acts on $K(P(\mathbb{C}))$ as in ***Example 1.2****. Observe that the coordinate ring of $P(\mathbb{R})$ is $A(P(\mathbb{R})) = A(P(\mathbb{C}))^G$ and the function field of $P(\mathbb{R})$ is $K(P(\mathbb{R})) = K(P(\mathbb{C}))^G \cong \mathbb{C}(z, w)^G$ (note that G actions on $K(P(\mathbb{C}))$ and $\mathbb{C}(z, w)$ are equivalent through the isomorphism). In short, we have that*

$$K(P(\mathbb{R})) \cong \mathbb{C}(z, w)^G$$

Therefore, when $G = Gal(\mathbb{C}/\mathbb{R})$ action on $C(z, w)$ is given, we can convert the rationality problem to the rationality problem of $K(P(\mathbb{R}))$, the function field of $P(\mathbb{R})$. In this sense, the following definition and theorem are natural.

Definition 2.3 (Rationality of $k - variety$) *We say that a variety X over k is rational if, equivalently,*

(1) X is birationally isomorphic to $\mathbb{P}^n_k$ for some n.
(2) $K(X) \cong k(x_1, .., x_n)$

If K/k is Galois extension, a $k - tori$ T is $K - rational$ if it is rational as a K-variety $T(K)$. If k is algebraically closed, there is unique n-dimension tori $T_n = (k^\times)^n$. Since the function field of T_n is $k(x_1, ..., x_n)$, thus T_n is k-rational.

Theorem 2.1 *The following two problems are equivalent.*

(1) The rationality problem of n dimensional $k - tori$ T
(2) The rationality problem of invariant field K^G

where $G = Gal(\overline{k}/k)$ and $K = k(x_1, ..., x_n)$.

There is a connection between the G action on K and $k - tori$ T, connecting the two rationality problems given in the previous theorem. To be specific, the character group of T determines both the G action and T uniquely.

3 Character group of $k - tori$

Definition 3.1 (Character group of $k - tori$) *Let T be $k-tori$. Then $\mathbb{X}(T)$, the character group of T is the set of algebraic group homomorphisms(a regular map preserving the group structure) from T to $\overline{k}^{\times}$, denoted by $Hom(T, \mathbb{G}_m)$ or $Hom(T, \overline{k}^{\times})$.*

The character group $\mathbb{X}(T)$ of T has a group structure defined by component-wise multiplication. Also, if T is split over L for finite Galois extension of base field k, $G = Gal(L/k)$ acts on $\mathbb{X}(T)$. Moreover, it is known that $\mathbb{X}(T)$ is torsion-free $\mathbb{Z}$-module(i.e. isomorphic to $\mathbb{Z}^n$ for some n). Therefore, $\mathbb{X}(T)$ is a $G-lattice$ (a free $\mathbb{Z} - module$ with G-action).

Example 3.1 *If $T = \mathbb{C}^{\times}$ is multiplicative group of $\mathbb{C}$, then $\mathbb{X}(T)$ is set of regular functions $f : \mathbb{C}^{\times} \to \mathbb{C}^{\times}$ such that $f(xy) = f(x)f(y)$ for $x, y \in \mathbb{C}^{\times}$. Since f is a rational function, it is a meromorphic function over $\mathbb{C}$. Also, we have $f(\mathbb{C}^{\times}) \subset \mathbb{C}^{\times}$, which implies 0 is the only point where f can have zeros or poles. Therefore, $f(t) = t^n$ for some $n \in \mathbb{Z}$. If we write a function $t \to t^n$ as t^n, we have*

$$\mathbb{X}(T) = \{t^n | n \in \mathbb{Z}\} \cong \mathbb{Z}^1$$

as a group. $G = Gal(\mathbb{C}/\mathbb{C}) = \{id\}$ acts trivially on $\mathbb{X}(T)$.

In general, if k is algebraically closed, the character group of $(k^{\times})^n = \mathbb{G}_m^n$ is

$\mathbb{X}(\mathbb{G}_m^n) = \{f_{t_1,...t_n} : \mathbb{G}_m^n \to \mathbb{G}_m | f_{t_1,...t_n}(x_1, ...x_n) = \prod_i x_i^{t_i}, t_i \in \mathbb{Z}\}$

$= \prod_{i=1}^n \{f_t : \mathbb{G}_m \to \mathbb{G}_m | f_t(x_i) = x_i^t, t \in \mathbb{Z}\} \cong \mathbb{Z}^n$

Example 3.2 *Let P be the 2-dimension $\mathbb{R} - tori$ in **Example 2.5**. Then, the character group of P is*

$$\mathbb{X}(P) = \{f_{t_1,t_2} : P \to \mathbb{C}^{\times} | f_{t_1,t_2}(x_1, x_2, x_3, x_4) = (x_1 + x_2 i)^{t_1}(x_1 - x_2 i)^{t_2}\}$$

Let $z = x_1 + x_2 i$, $w = x_1 - x_2 i$, then we have the natural extension of $\mathbb{X}(P)$ to $\mathbb{X}(P(\mathbb{C}))$

$$\mathbb{X}(P(\mathbb{C})) = \{f_{t_1,t_2} : P(\mathbb{C}) \to \mathbb{C}^\times | f_{t_1,t_2}((z, \frac{1}{z}), (w, \frac{1}{w})) = z^{t_1} w^{t_2}\} \cong \mathbb{Z}^2$$

Observe that the complex conjugation $\sigma \in G$, exchanges z and w, thus acting on $\mathbb{Z}^2$ as 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

It is known that when a $G = Gal(K/k)$ action (as $\mathbb{Z}$-linear function) on $\mathbb{Z}^n$ is given, there exists unique n-dimensional $k - tori$ which has the given $G - lattice$ as its character group. Furthermore, there are conditions of $G - lattice$ corresponding to the rationality conditions of $k - tori$ and of invariant fields.

4 Flabby resolution and numerical approach

This section contains many results in [2]. Let G be a group and M be a $G - lattice$ ($M \cong \mathbb{Z}^n$ as group and has G-linear action on it). M is called a *permutation G-lattice* if $M \cong \bigoplus_{1 \le i \le m} \mathbb{Z}[G/H_i]$ for some subgroups $H_1, ..., H_m$ of G (equivalently, there exists a $\mathbb{Z}$-basis of M such that G acts on M as permutation of the basis). M is called *stably permutation G-lattice* if $M \bigoplus P \cong Q$ for some permutation $G - lattices$ P and Q. M is called *invertible* if it is a direct summand of a permutation G-lattice, i.e. $P \cong M \bigoplus M'$ for some permutation G-lattice P and M'.

Definition 4.1 (1st Group Cohomology) *Let G be a group and M be a G-lattice. For $g \in G$ and $m \in M$, let $g.m = m^g$ be g acting on m. The first group cohomology $H^1(G, M)$ is a group defined as*

$$H^1(G, M) = Z^1(G, M)/B^1(G, M)$$

where $Z^1(G, M) = \{f : G \to M | f(gh) = f(g)^h f(h)\}$ and $B^1(G, M) = \{f : G \to M | f(g) = m_f^g m_f^{-1}$ for some $m_f \in M\}$

$H^1(G, M) = 0$ simply implies that if $f : G \to M$ satisfies $f(gh) = f(g)^h f(h)$, then there exists $m \in M$ such that $f(g) = m^g m^{-1}$. M is called *coflabby* if $H^1(G, M) = 0$.

Definition 4.2 (-1st Tate Cohomology) *Let G be finite group of order n and M be a G-lattice. The -1st group cohomology $\hat{H}^{-1}(G, M)$ is a group defined as*

$$\hat{H}^{-1}(G, M) = Z^{-1}(G, M)/B^{-1}(G, M)$$

where

$$Z^{-1}(G, M) = \{m \in M | \sum_{g \in G} m^g = 0\}$$

,

$$B^{-1}(G, M) = \{\sum_{g \in G} m_g^{g-id} | m_g \in M\}$$

Similarly, M is called *flabby* if $\hat{H}^{-1}(G, M) = 0$. It is clear that a $k - tori$ is rational if and only if $\mathbb{X}(T)$ is permutation G-lattice. Thus, the rationality problems of $k - tori$ and invariant fields can be reduced into problem of finding permutation G-lattice(equivalent to find finite subgroup of $GL(n, \mathbb{Z})$. However, this problem is not solved yet, even though there are many results in weakened problems.

Let $C(G)$ be the category of all G-lattices and $S(G)$ be the category of all permutation G-lattices. Define equivalence relation on $C(G)$ by M_1 M_2 if and only if there exist $P_1, P_2 \in S(G)$ such that $M_1 \bigoplus P_1 \cong M_2 \bigoplus P_2$. Let $[M]$ be equivalence class containing M under this relation.

Theorem 4.1 (Endo and Miyata [3, Lemma 1.1], Colliot-Thélène and Sansuc [4, Lemma 3]) *For any G-lattice M, there is a short exact sequence of G-lattices $0 \to M \to P \to F \to 0$ where P is permutation and F is flabby.*

In the previous theorem, $[F]$ is called the *flabby class* of M, denoted by $[M]^{fl}$.

Theorem 4.2 (Akinari and Aiichi [2, 17pp]) *If M is stably permutation, then $[M]^{fl}$. If M is invertible, $[M]^{fl}$ is invertible.*

It is not difficult to see that

$$M \ is \ permutation \ \Rightarrow \ M \ is \ stably \ permutation$$

Furthermore, it is true that

$$M \ is \ stably \ permutation \ \Rightarrow \ M \ is \ invertible \ \Rightarrow M \ is \ flabby \ and \ coflabby$$

In [2], they gave the complete list of stably permutation lattices for dimension 4 and 5 by computing $[M]^{fl}$ for finite subgroup of $GL(n, \mathbb{Z})$, which is equivalent to classifying stably rational tori. Thus, the rationality problems for low dimensional $k - tori$ can be resolved by finding conditions which can determine a stably permutation M is permutation or not.

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