

Averaging principle for equation driven by a stochastic measure¹

Vadym Radchenko

2010 Mathematics Subject Classification: 60H05; 60H10; 60G57

Keywords: Averaging principle; symmetric integral; stochastic measure; stochastic ordinary differential equations; Doss–Sussmann transformation

Abstract

Equation with the symmetric integral with respect to stochastic measure is considered. For the integrator, we assume only σ -additivity in probability and continuity of the paths. It is proved that the averaging principle holds for this case, the rate of convergence to the solution of the averaged equation is estimated.

1 Introduction

Averaging is an important method to describe the main part of the behavior of dynamical systems. It allows to avoid the detailed analysis of fast-changing variables and consider the simplified equations. This approach is well developed for deterministic and stochastic systems.

Averaging principle for non-random equations is considered in details, for example, in [20]. The stochastic case was studied mainly for equations driven by Wiener process. Averaging of equations was considered in [18], slow-fast systems – in [2], [4, Section 7.9], [25].

Other stochastic integrators were also considered. Averaging of the system with α -stable noises was studied in [1], fractional Brownian motion – in [12], Poisson process – in [8], [11].

In these papers the strong convergence to the solutions of averaged equations was studied, a similar result is obtained in the given paper. The weak convergence in averaging scheme was considered in [2], [3], [5], [21, Section II.3].

We will consider averaging of equation driven by general stochastic measure μ . For μ we assume only σ -additivity in probability and continuity of the paths. This integrator includes many classes of processes, see examples in Section 2.1. In the previous papers, the scaling invariance of the driving processes was very important in the proofs, we do not assume such property for μ .

In the given paper, the following equation is considered

$$\circ dX_t^\varepsilon = \sigma(X_t^\varepsilon) \circ d\mu_t + b(X_t^\varepsilon, t/\varepsilon) dt, \quad 0 \leq t \leq T, \quad X_0^\varepsilon = X_0,$$

where \circ denotes the symmetric integral, defined in [16] (see Section 2.2). We prove that

$$\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t| \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

for \bar{X}_t that is the solution to the equation

$$\circ d\bar{X}_t = \sigma(\bar{X}_t) \circ d\mu_t + \bar{b}(\bar{X}_t) dt, \quad 0 \leq t \leq T, \quad \bar{X}_0 = X_0.$$

Under some additional assumptions, we obtain the rate of convergence.

The rest of the paper is organized as follows. Section 2 contains the basic facts concerning SMs and symmetric integral. In Section 3 we formulate and prove the main result of the paper (Theorem 3.1).

By C and $C(\omega)$ we will denote positive finite constant and random constant respectively whose exact values are not important.

¹The final version will be published in "Stochastics"

2 Preliminaries

2.1 Stochastic measures

Let $L_0 = L_0(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (more precisely, the set of equivalence classes). Convergence in L_0 means the convergence in probability. Let X be an arbitrary set and \mathcal{B} a σ -algebra of subsets of X .

Definition 2.1. A σ -additive mapping $\mu : \mathcal{B} \rightarrow L_0$ is called *stochastic measure* (SM).

We do not assume the moment existence or martingale properties for SM. In other words, μ is L_0 -valued vector measure.

For deterministic measurable functions $f : X \rightarrow \mathbb{R}$ the integral $\int_X f d\mu$ is defined. Construction of the integral and basic facts concerning general SMs may be found in [6, Chapter 7], [13, Chapter 1]. In particular, every bounded measurable f is integrable with respect to any μ . An analog of the Lebesgue dominated convergence theorem holds for this integral, see [6, Proposition 7.1.1]. Some additional facts and review of results about equations driven by SMs are given in [14].

Important examples of SMs are orthogonal stochastic measures, α -stable random measures defined on a σ -algebra for $\alpha \in (0, 1) \cup (1, 2]$ (see [19, Chapter 3]).

Many examples of the SMs on the Borel subsets of $[0, T]$ may be given by the Wiener-type integral

$$\mu(A) = \int_{[0, T]} \mathbf{1}_A(t) dX_t. \quad (2.1)$$

We note the following cases of processes X_t in (2.1) that generate SM.

1. X_t – any square integrable continuous martingale.
2. $X_t = W_t^H$ – the fractional Brownian motion with Hurst index $H > 1/2$, see Theorem 1.1 [10].
3. $X_t = S_t^k$ – the sub-fractional Brownian motion for $k = H - 1/2$, $1/2 < H < 1$, see Theorem 3.2 (ii) and Remark 3.3 c) in [24].
4. $X_t = Z_H^k(t)$ – the Hermite process, $1/2 < H < 1$, $k \geq 1$, see [9], [22, Section 3.1.3]. $Z_H^2(t)$ is known as the Rosenblatt process, see also [23, Section 3].

Main result of this paper will be obtained under the following assumption on μ .

Assumption A1. μ is a SM on Borel subsets of $[0, T]$, and the process $\mu_t = \mu((0, t])$ has continuous paths on $[0, T]$.

Processes X_t in examples 1–4 are continuous, therefore A1 holds in these cases.

Give an another example. Let η be an arbitrary SM defined on Borel subsets of $[a, b] \subset \mathbb{R}$, function $f : [0, T] \times [a, b] \rightarrow \mathbb{R}$ be such that $f(0, x) = 0$, and

$$|f(t, x) - f(s, y)| \leq L(|t - s| + |x - y|^\gamma), \quad \gamma > 1/2, \quad L \in \mathbb{R}.$$

Then $f(\cdot, x)$ is absolutely continuous for each x , $\left| \frac{\partial f(t, x)}{\partial t} \right| \leq L$ a. e., and we can define SM

$$\mu(A) = \int_{[a, b]} d\eta(x) \int_A \frac{\partial f(t, x)}{\partial t} dt, \quad A \in \mathcal{B}([0, T]). \quad (2.2)$$

The σ -additivity of μ follows from the analog of the Lebesgue dominated convergence theorem, see details in [15, Section 3]. Theorem 1 of [15] implies that the process

$$\mu_t = \mu((0, t]) = \int_{[a, b]} f(t, x) d\eta(x), \quad t \in [0, T],$$

has a continuous version. Thus, in this case the process $X_t = \mu_t$ in (2.1) defines an SM that satisfies A1.

In some propositions we will impose the following condition.

Assumption A2. There exists a real-valued finite measure \mathbf{m} on (X, \mathcal{B}) with the following property: if a measurable function $h : X \rightarrow \mathbb{R}$ is such that $\int_X h^2 d\mathbf{m} < +\infty$ then h is integrable with respect to μ on X .

This assumption holds in examples 2, 3 for the Lebesgue measure \mathbf{m} (see [10], [24]), for α -stable random measure and the control measure \mathbf{m} (see (3.4.1) [19]). If martingale X_t in example 1 has the deterministic characteristic then A2 is fulfilled for $\mathbf{m}(A) = \int_A d\langle X_t \rangle$.

If A2 holds for SM η in (2.2) then it holds for μ . This follows from the boundedness of $\frac{\partial f(t, x)}{\partial t}$.

This assumption is used in the following statement.

Lemma 2.2. (*Corollary 3.3 [16]*) *If A2 holds then the set of random variables*

$$\left\{ \sum_{k=1}^j \left(\int_X f_k d\mu \right)^2 \mid f_k : X \rightarrow \mathbb{R} \text{ are measurable, } \sum_{k=1}^j f_k^2(x) \leq 1, \quad j \geq 1 \right\}$$

is bounded in probability.

Recall that set of random variables ξ_α , $\alpha \in \mathcal{A}$ is bounded in probability if

$$\sup_{\alpha \in \mathcal{A}} \mathbb{P}(|\xi_\alpha| \geq c) \rightarrow 0, \quad c \rightarrow \infty.$$

2.2 Symmetric integral

The symmetric integral of random functions with respect to stochastic measures was considered in [16]. We review the basic facts and definitions.

Definition 2.3. Let ξ_t and η_t be random processes on $[0, T]$, $0 = t_0^n < t_1^n < \dots < t_{j_n}^n = T$ be a sequence of partitions such that $\max_k |t_k^n - t_{k-1}^n| \rightarrow 0$, $n \rightarrow \infty$. We define

$$\int_{(0, T]} \xi_t \circ d\eta_t := \mathbb{p} \lim_{n \rightarrow \infty} \sum_{k=1}^{j_n} \frac{\xi_{t_{k-1}^n} + \xi_{t_k^n}}{2} (\eta_{t_k^n} - \eta_{t_{k-1}^n}) \quad (2.3)$$

provided that this limit in probability exists.

For Wiener process η_t and adapted ξ_t we obtain the classical Stratonovich integral. If η_t and ξ_t are Hölder continuous with exponents γ_η and γ_ξ , $\gamma_\eta + \gamma_\xi > 1$, then value of (2.3) equals to the integral defined in [26].

The following theorem describes the class of processes for which the integral exists.

Assumption A3. V_t is a continuous process of bounded variation on $[0, T]$.

Theorem 2.4. (*Theorem 4.6 [16]*) *Let A1 and A3 hold, $f \in \mathbb{C}^{1,1}(\mathbb{R}^2)$. Then integral (2.3) of $f(\mu_t, V_t)$ with respect to μ_t is well defined, and*

$$\int_{(0, T]} f(\mu_t, V_t) \circ d\mu_t = G(\mu_t, V_t) - G(\mu_0, V_0) - \int_{(0, T]} G'_2(\mu_t, V_t) dV_t,$$

where $G(x, v) = \int_0^x f(y, v) dy$.

We will consider a stochastic equation of the form

$$\circ dX_t = \sigma(X_t) \circ d\mu_t + b(X_t, t) dt, \quad 0 \leq t \leq T, \quad (2.4)$$

X_0 is a given random variable.

Definition 2.5. A process X_t , $0 \leq t \leq T$ is a *solution* to (2.4) if:

- 1) $X_t = f(\mu_t, Y_t)$, $f \in \mathbb{C}^{2,1}(\mathbb{R}^2)$, Y_t is a continuous process of bounded variation;
- 2) for any process $Z_s = \psi(\mu_s, X_s)$, $\psi \in \mathbb{C}^{1,1}(\mathbb{R}^2)$, we have

$$\int_{(0,t]} Z_s \circ dX_s = \int_{(0,t]} Z_s \sigma(X_s) \circ d\mu_s + \int_{(0,t]} Z_s b(X_s, s) ds, \quad t \in [0, T].$$

For $Z_s \equiv 1$ we get the usual integral form of the stochastic equation. This form of Definition 2.5 2) was important for the proof of uniqueness of the solution.

Solution to (2.4) was obtained in [16] using the Doss–Sussmann transformation.

Assumption A4. 1) $\sigma \in \mathbb{C}^2(\mathbb{R})$ and the derivatives σ' , σ'' are bounded;

2) $b \in \mathbb{C}(\mathbb{R}^2)$;

3) for each $c > 0$ there exists a $L(c)$ such that

$$|b(x, t) - b(y, t)| \leq L(c)|x - y|, \quad |x|, |y| \leq c;$$

4) b is bounded.

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the solution of the equation

$$\frac{\partial F}{\partial r}(r, x) = \sigma(F(r, x)), \quad F(0, x) = x, \quad (2.5)$$

which exists globally because of our assumptions. Set $H(r, x) = F^{-1}(r, x)$, where the inverse is taken with respect to x . We have that $F, H \in \mathbb{C}^{2,2}(\mathbb{R}^2)$ and

$$\frac{\partial F}{\partial x}(r, x) = \exp\left(\int_0^r \sigma'(F(s, x)) ds\right) \quad (2.6)$$

(see calculations in (5.5)–(5.11) [17]).

Theorem 2.6. (*Theorem 5.3 [16]*) Let A1 and A4 hold, X_0 be an arbitrary random variable. Then equation (2.4) has a unique solution $X_t = F(\mu_t, Y_t)$, where Y_t is the solution of the random equation

$$Y_t = H(0, X_0) + \int_0^t \frac{\partial H}{\partial x}(\mu_s, F(\mu_s, Y_s)) b(F(\mu_s, Y_s), s) ds. \quad (2.7)$$

Further, we will need Lipschitz properties of functions in (2.7).

Using (2.5), we obtain

$$\begin{aligned} F(r, x) = x + \int_0^r \sigma(F(s, x)) ds &\Rightarrow \frac{\partial F(r, x)}{\partial x} = 1 + \int_0^r \sigma'(F(s, x)) \frac{\partial F(s, x)}{\partial x} ds \\ &\stackrel{A4.1}{\Rightarrow} \left| \frac{\partial F(r, x)}{\partial x} \right| \leq 1 + C \int_0^r \left| \frac{\partial F(s, x)}{\partial x} \right| ds. \end{aligned}$$

The Gronwall inequality implies that

$$\left| \frac{\partial F(r, x)}{\partial x} \right| \leq \exp\{Cr\}.$$

Therefore, for μ_s with continuous paths we obtain that

$$\left| \frac{\partial F(\mu_s, x)}{\partial x} \right| \leq C(\omega).$$

Also from A4.1) and (2.6) we obtain that

$$\left| \frac{\partial F(\mu_s, x)}{\partial x} \right| \geq C(\omega) > 0.$$

Therefore,

$$\left| \frac{\partial H(\mu_s, x)}{\partial x} \right| \leq C(\omega). \quad (2.8)$$

Here $C(\omega)$ do not depend of $x \in \mathbb{R}$, $s \in [0, T]$.

Note that we use continuous version of process μ_s , solve (2.7) for each fixed ω . Therefore, our estimates hold for all $\omega \in \Omega$.

3 Averaging principle

For each $\varepsilon > 0$ consider the equation

$$\circ dX_t^\varepsilon = \sigma(X_t^\varepsilon) \circ d\mu_t + b(X_t^\varepsilon, t/\varepsilon) dt, \quad 0 \leq t \leq T, \quad X_0^\varepsilon = X_0, \quad (3.9)$$

X_0 is an arbitrary random variable.

From Theorem 2.6 it follows that $X_t^\varepsilon = F(\mu_t, Y_t^\varepsilon)$, where

$$Y_t^\varepsilon = H(0, X_0) + \int_0^t \frac{\partial H}{\partial x}(\mu_s, F(\mu_s, Y_s^\varepsilon)) b(F(\mu_s, Y_s^\varepsilon), s/\varepsilon) ds. \quad (3.10)$$

Assumption A4.4) and (2.8) imply that

$$|Y_t^\varepsilon| \leq C(\omega), \quad |X_t^\varepsilon| \leq C(\omega), \quad (3.11)$$

where $C(\omega)$ does not depend of $t \in [0, T]$, $\varepsilon > 0$.

Assume that there exist the following limit

$$\bar{b}(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t b(y, s) ds.$$

Obviously, \bar{b} satisfies A4.2)-4) (as function of one variable).

Assumption A5. Function $G(y, r) = \int_0^r (b(y, s) - \bar{b}(y)) ds$, $r \in \mathbb{R}_+$, $y \in \mathbb{R}$ is bounded.

This holds, for example, if $b(y, s)$ is periodic in s for each fixed y .

Averaged form of (3.9) is the following

$$\circ d\bar{X}_t = \sigma(\bar{X}_t) \circ d\mu_t + \bar{b}(\bar{X}_t) dt, \quad 0 \leq t \leq T, \quad \bar{X}_0^\varepsilon = X_0. \quad (3.12)$$

From Theorem 2.6 it follows that $\bar{X}_t = F(\mu_t, \bar{Y}_t)$, where

$$\bar{Y}_t = H(0, X_0) + \int_0^t \frac{\partial H}{\partial x}(\mu_s, F(\mu_s, \bar{Y}_s)) \bar{b}(F(\mu_s, \bar{Y}_s)) ds. \quad (3.13)$$

Note that functions F , H are the same in (3.10) and (3.13).

The main result of the paper is the following.

Theorem 3.1. 1) Assume that A1, A4, and A5 hold, X_t^ε and \bar{X}_t are the solutions of (3.9) and (3.12) respectively. Then for each $\omega \in \Omega$

$$\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t| \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (3.14)$$

2) Let, in addition, A2 holds. Then the set of the random variables

$$\frac{\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t|}{\varepsilon^{1/3}}, \quad \varepsilon > 0$$

is bounded in probability.

Proof. Functions F , $\frac{\partial H}{\partial x}$ are locally Lipschitz. Using (3.11) we obtain

$$\begin{aligned}
& |X_t^\varepsilon - \bar{X}_t| \leq C(\omega) |Y_t^\varepsilon - \bar{Y}_t| \\
(3.10), (3.13) \quad C(\omega) & \left| \int_0^t \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) b(X_s^\varepsilon, s/\varepsilon) ds - \int_0^t \frac{\partial H}{\partial x}(\mu_s, \bar{X}_s) \bar{b}(\bar{X}_s) ds \right| \\
& \leq C(\omega) \left| \int_0^t \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) (b(X_s^\varepsilon, s/\varepsilon) - b(\bar{X}_s, s/\varepsilon)) ds \right| \\
& \quad + C(\omega) \left| \int_0^t \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) (b(\bar{X}_s, s/\varepsilon) - \bar{b}(\bar{X}_s)) ds \right| \\
& \quad + C(\omega) \left| \int_0^t \left(\frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) - \frac{\partial H}{\partial x}(\mu_s, \bar{X}_s) \right) \bar{b}(\bar{X}_s) ds \right| \quad (3.15) \\
(2.8) \quad & \leq C(\omega) \int_0^t |b(X_s^\varepsilon, s/\varepsilon) - b(\bar{X}_s, s/\varepsilon)| ds + C(\omega) \left| \int_0^t \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) (b(\bar{X}_s, s/\varepsilon) - \bar{b}(\bar{X}_s)) ds \right| \\
& \quad + C(\omega) \sup |\bar{b}| \int_0^t \left| \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) - \frac{\partial H}{\partial x}(\mu_s, \bar{X}_s) \right| ds \\
A4.3) \quad & \leq C(\omega) \int_0^t |X_s^\varepsilon - \bar{X}_s| ds + C(\omega) \left| \int_0^t \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) (b(\bar{X}_s, s/\varepsilon) - \bar{b}(\bar{X}_s)) ds \right|.
\end{aligned}$$

Consider the second term of the last sum. Divide $[0, T]$ into n segments of length $\Delta = \frac{T}{n}$. We have

$$\begin{aligned}
I & := \left| \int_0^t \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) (b(\bar{X}_s, s/\varepsilon) - \bar{b}(\bar{X}_s)) ds \right| \\
& \leq \sum_{k=0}^{n-1} \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) (b(\bar{X}_s, s/\varepsilon) - \bar{b}(\bar{X}_s)) ds \right| \\
& \leq \sum_{k=0}^{n-1} \left(\left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} \left(\frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) - \frac{\partial H}{\partial x}(\mu_{k\Delta}, X_{k\Delta}^\varepsilon) \right) (b(\bar{X}_s, s/\varepsilon) - \bar{b}(\bar{X}_s)) ds \right| \right. \\
& \quad + \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} \frac{\partial H}{\partial x}(\mu_{k\Delta}, X_{k\Delta}^\varepsilon) (b(\bar{X}_s, s/\varepsilon) - b(\bar{X}_{k\Delta}, s/\varepsilon)) ds \right| \\
& \quad + \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} \frac{\partial H}{\partial x}(\mu_{k\Delta}, X_{k\Delta}^\varepsilon) (b(\bar{X}_{k\Delta}, s/\varepsilon) - \bar{b}(\bar{X}_{k\Delta})) ds \right| \\
& \quad \left. + \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} \frac{\partial H}{\partial x}(\mu_{k\Delta}, X_{k\Delta}^\varepsilon) (\bar{b}(\bar{X}_{k\Delta}) - \bar{b}(\bar{X}_s)) ds \right| \right) \\
& \quad := \sum_{k=0}^{n-1} (I_{1k} + I_{2k} + I_{3k} + I_{4k}), \\
I_{1k} & \stackrel{A4.4)}{\leq} C \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} \left| \frac{\partial H}{\partial x}(\mu_s, X_s^\varepsilon) - \frac{\partial H}{\partial x}(\mu_{k\Delta}, X_{k\Delta}^\varepsilon) \right| ds \\
& \leq C(\omega) \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} (|\mu_s - \mu_{k\Delta}| + |X_s^\varepsilon - X_{k\Delta}^\varepsilon|) ds, \\
I_{2k} & \stackrel{A4.3), (2.8)}{\leq} C(\omega) \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} |X_s^\varepsilon - X_{k\Delta}^\varepsilon| ds, \\
I_{3k} & = \left| \frac{\partial H}{\partial x}(\mu_{k\Delta}, X_{k\Delta}^\varepsilon) \right| \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} (b(\bar{X}_{k\Delta}, s/\varepsilon) - \bar{b}(\bar{X}_{k\Delta})) ds \right| \\
(2.8), s=r\varepsilon & \leq C(\omega) \varepsilon \left| \int_{((k\Delta \wedge t)/\varepsilon, ((k+1)\Delta \wedge t)/\varepsilon]} (b(\bar{X}_{k\Delta}, r) - \bar{b}(\bar{X}_{k\Delta})) dr \right| \stackrel{A5}{\leq} C(\omega) \varepsilon, \\
I_{4k} & \stackrel{(2.8)}{\leq} C(\omega) \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} |\bar{X}_s - \bar{X}_{k\Delta}| ds.
\end{aligned}$$

We have that $X_s^\varepsilon = F(\mu_s, Y_s^\varepsilon)$, where F is locally Lipschitz. From (2.7), (2.8) and A4.4) it follows that $|Y_t^\varepsilon - \bar{Y}_s^\varepsilon| \leq C(\omega)|t - s|$. Therefore,

$$|X_s^\varepsilon - X_{k\Delta}^\varepsilon| \leq C(\omega)(|\mu_s - \mu_{k\Delta}| + \Delta), \quad s \in (k\Delta \wedge t, (k+1)\Delta \wedge t].$$

The same estimate holds for \bar{X} in I_{4k} . We arrive at

$$I \leq C(\omega)n\varepsilon + C(\omega)n\Delta^2 + C(\omega) \sum_{k=0}^{n-1} \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} |\mu_s - \mu_{k\Delta}| ds.$$

Recall that $\Delta = \frac{T}{n}$. Using that $|\mu_s| \leq C(\omega)$, consider separately interval $(k\Delta, (k+1)\Delta] \ni t$, and we obtain

$$\begin{aligned} I &\leq J(n, \varepsilon) := C(\omega)n\varepsilon + C(\omega)\Delta + C(\omega)n\Delta^2 + C(\omega) \sum_{k=0}^{n-1} \int_{(k\Delta, (k+1)\Delta]} |\mu_s - \mu_{k\Delta}| ds \\ &= C(\omega)n\varepsilon + \frac{C(\omega)}{n} + C(\omega) \int_{(0, T]} |\mu_s - \mu_{[s/\Delta]\Delta}| ds. \end{aligned} \quad (3.16)$$

We have used the notation $[x]$ for the greatest integer not exceeding x .

From (3.15) and our considerations it follows that

$$|X_t^\varepsilon - \bar{X}_t| \leq C(\omega) \int_0^t |X_s^\varepsilon - \bar{X}_s| ds + J(n, \varepsilon).$$

Gronwall inequality implies that for all t, ε, n

$$|X_t^\varepsilon - \bar{X}_t| \leq C(\omega)J(n, \varepsilon). \quad (3.17)$$

For each $0 < \varepsilon \leq 1$ we can take $n = \lceil \varepsilon^{-1/2} \rceil$. Process μ_s has continuous paths, therefore

$$\int_{(0, T]} |\mu_s - \mu_{[s/\Delta]\Delta}| ds \rightarrow 0, \quad \Delta \rightarrow 0.$$

From (3.16) and (3.17) it follows statement 1) of our theorem.

Now let us prove part 2). We claim that if A1 and A2 hold then the set of values

$$\frac{1}{\sqrt{\Delta}} \int_{(0, T]} |\mu_s - \mu_{[s/\Delta]\Delta}| ds, \quad \Delta = \frac{T}{n}, \quad n \geq 1$$

is bounded in probability. The Cauchy inequality implies that

$$\int_{(0, T]} \frac{|\mu_s - \mu_{[s/\Delta]\Delta}|}{\sqrt{\Delta}} ds \leq C \left(\int_{(0, T]} \frac{|\mu_s - \mu_{[s/\Delta]\Delta}|^2}{\Delta} ds \right)^{1/2}.$$

Dividing each $(k\Delta, (k+1)\Delta]$ into segments of length $\alpha = \frac{\Delta}{m}$ for m large enough, we can approximate

$$\int_{(0, T]} \frac{|\mu_s - \mu_{[s/\Delta]\Delta}|^2}{\Delta} ds = \sum_{k=0}^{n-1} \int_{(k\Delta, (k+1)\Delta]} \frac{|\mu_s - \mu_{k\Delta}|^2}{\Delta} ds$$

by integral sums

$$\sum_{k=0}^{n-1} \sum_{j=1}^m \frac{|\mu_{k\Delta + j\alpha} - \mu_{k\Delta}|^2}{\Delta} \alpha = \sum_{k,j} \left(\int f_{k,j} d\mu \right)^2.$$

Here

$$f_{k,j} = \sqrt{\frac{\alpha}{\Delta}} \mathbf{1}_{(k\Delta, k\Delta + j\alpha]}, \quad \sum_{k,j} f_{k,j}^2 \leq \frac{\alpha}{\Delta} m = 1.$$

Lemma 2.2 implies our claim.

For each $0 < \varepsilon \leq 1$ we will choose $n = \lceil \varepsilon^{-2/3} \rceil$. From (3.16) it follows that then values of $J(n, \varepsilon)\varepsilon^{-1/3}$ are bounded in probability, applying of (3.17) finishes the proof. \square

Corollary 3.2. *If conditions of Theorem 3.1 2) hold then for each $\alpha < 1/3$*

$$\frac{\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t|}{\varepsilon^\alpha} \xrightarrow{\mathbb{P}} 0, \quad \varepsilon \rightarrow 0.$$

Remark 1. Note that for non-stochastic case $\mu_t = t$ convergence rate in (3.14) is $O(\varepsilon)$, see, for example, [20, Theorem 2.8.1].

Let $\mu_t = W_t$ be the Wiener process. Then A2 holds, equation (2.4) with Stratonovich integral is equivalent to the following equation with the Itô integral

$$dX_t = \sigma(X_t) dW_t + \hat{b}(X_t, t) dt, \quad \hat{b}(x, t) = b(x, t) + \frac{1}{2} \sigma(x) \sigma'(x). \quad (3.18)$$

Assumption A5 holds for b iff it is fulfilled for \hat{b} . Theorem 3.1 2) directly gives the convergence rate $O(\varepsilon^{1/3})$ for this Itô-type equation.

We can refine convergence in (3.14) if μ_t is Hölder continuous with exponent γ . In this case

$$\int_{(0, T]} |\mu_s - \mu_{[s/\Delta]\Delta}| ds \leq C(\omega) \Delta^\gamma = C(\omega) n^{-\gamma}.$$

From (3.16) and (3.17) for $n = \lceil \varepsilon^{-1/(1+\gamma)} \rceil$ we obtain

$$|X_t^\varepsilon - \bar{X}_t| \leq C(\omega) \varepsilon^{\gamma/(1+\gamma)}.$$

We can compare our results with rate of strong convergence obtained for systems of stochastic equations driven by the Wiener processes. In the case of slow-fast system of SDEs in [7] was obtained the rate $O(\varepsilon^{1/2})$. For SPDEs in [25] it was proved that $\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t| \varepsilon^{-1/2}$ are bounded in probability. For similar system in [2] rate of convergence of (3.14) is $O(\varepsilon^{1/2-})$. Note that in these papers non-generated case was studied, and we can not consider (3.18) as partial case of these systems.

For system of usual stochastic equations driven by Wiener and Poisson processes in [8] was obtained the rate $O(\varepsilon^{1/2})$, for SPDEs in [11] – $O(\varepsilon^{1/4})$.

Acknowledgements

The author is grateful to Prof. M. Zähle for fruitful discussions during the preparation of this paper and thanks the Friedrich-Schiller-University of Jena for its hospitality.

Funding

This work was supported by Alexander von Humboldt Foundation, grant 1074615.

References

- [1] J. Bao, G. Yin, and C. Yuan, *Two-time-scale stochastic partial differential equations driven by alpha-stable noises: Averaging principles*, Bernoulli 23 (2017), pp. 645–669.
- [2] C.E. Bréhier, *Strong and weak orders in averaging for SPDEs*, Stoch. Proc. Appl. 122 (2012), pp. 2553–2593.
- [3] S. Cerrai, *A Khasminskii type averaging principle for stochastic reaction-diffusion equations*, Ann. Appl. Probab. 19 (2009), pp. 899–948.
- [4] M.I. Freidlin and A.D. Wentzell, *Random Perturbations of Dynamical Systems*, A Series of Comprehensive Studies in Mathematics Vol. 260, Springer, 2012.

- [5] H. Fu, L. Wan, J. Liu, and X. Liu, *Weak order in averaging principle for stochastic wave equation with a fast oscillation*, Stoch. Proc. Appl. 128 (2018), pp. 2557–2580.
- [6] S. Kwapień and W.A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple*, Birkhäuser, Boston, 1992.
- [7] D. Liu, *Strong convergence of principle of averaging for multiscale stochastic dynamical systems*, Commun. Math. Sci. 8 (2010), pp. 999–1020.
- [8] D. Liu, *Strong convergence rate of principle of averaging for jump-diffusion processes*, Front. Math. China 7 (2012), pp. 305–320.
- [9] M. Maejima and C.A. Tudor, *Wiener integrals with respect to the Hermite process and a non-central limit theorem*, Stoch. Anal. Appl. 25 (2007), pp. 1043–1056.
- [10] T. Memin, Y. Mishura, and E. Valkeila, *Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion*, Statist. Probab. Lett. 51 (2001), pp. 197–206.
- [11] B. Pei, Y. Xu, and J.L. Wu, *Two-time-scales hyperbolic-parabolic equations driven by Poisson random measures: Existence, uniqueness and averaging principles*, J. Math. Anal. Appl. 447 (2017), pp. 243–268.
- [12] B. Pei, Y. Xu, and G. Yin, *Stochastic averaging for a class of two-time-scale systems of stochastic partial differential equations*, Nonlinear Anal. 160 (2017), pp. 159–176.
- [13] V. Radchenko, *Integrals with respect to General Stochastic Measures*, Institute of Mathematics, Kyiv, 1999, in Russian.
- [14] V. Radchenko, *Stochastic partial differential equations driven by general stochastic measures*, in *Modern Stochastics and Appl.*, V. Korolyuk, N. Limnios, Y. Mishura, L. Sakhno, and G. Shevchenko, eds., Springer, Cham Heidelberg, 2014, pp. 143–156.
- [15] V.M. Radchenko, *Parameter-dependent integrals with respect to general random measures*, Theory Probab. Math. Statist. 75 (2007), pp. 161–165.
- [16] V. Radchenko, *Stratonovich-type integral with respect to a general stochastic measure*, Stochastics 88 (2016), pp. 1060–1072.
- [17] F. Russo and P. Vallois, *Stochastic calculus with respect to continuous finite quadratic variation processes*, Stochastics 70 (2000), pp. 1–40.
- [18] A.M. Samoilenko, N.I. Makhmudov, and A.N. Stanzhitskii, *Averaging method and two-sided bounded solutions of itô stochastic systems*, Diff. Equat. 43 (2007), pp. 56–68.
- [19] G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman and Hall, London, 1994.
- [20] J.A. Sanders, F. Verhulst, and J. Murdock, *Averaging methods in nonlinear dynamical systems*, Appl. Math. Sci. Vol. 59, Springer, 2007.
- [21] A.V. Skorokhod, *Asymptotic methods in the theory of stochastic differential equations*, Translations of Mathematical Monographs Vol. 78, American Mathematical Soc., 2009.
- [22] C. Tudor, *Analysis of variations for self-similar processes: A stochastic calculus approach*, Springer, 2013.
- [23] C.A. Tudor, *Analysis of the Rosenblatt process*, ESAIM-Probab. Stat. 12 (2008), pp. 230–257.

- [24] C. Tudor, *On the Wiener integral with respect to a sub-fractional Brownian motion on an interval*, J. Math. Anal. Appl. 351 (2009), pp. 456–468.
- [25] W. Wang and A. Roberts, *Average and deviation for slow-fast stochastic partial differential equations*, J. Differ. Equations 253 (2012), pp. 1265–1286.
- [26] M. Zähle, *Integration with respect to fractal functions and stochastic calculus. I*, Probab. Theory Rel. 111 (1998), pp. 333–374.

DEPARTMENT OF MATHEMATICAL ANALYSIS, TARAS SHEVCHENKO NATIONAL UNIVERSITY OF
KYIV, KYIV 01601, UKRAINE
E-mail address: vradchenko@univ.kiev.ua