

OBSERVING A LÉVY PROCESS UP TO A STOPPING TIME

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ABSTRACT. It is proved that the law of a possibly killed Lévy process X , seen up to and including (resp. up to strictly before) a stopping time, determines already the law of X (resp. up to a compound Poisson component and killing).

1. INTRODUCTION

We fix a $d \in \mathbb{N}$ — the dimension of the Euclidean space \mathbb{R}^d in which our Lévy processes will live — and a $\partial \notin \mathbb{R}^d$ — it will play the role of a cemetery state. We agree that for $x \in \mathbb{R}^d \cup \{\partial\}$, $\partial \pm x$ and $x \pm \partial$ are all equal to ∂ .

Recall then that a stochastic process $X = (X_t)_{t \in [0, \infty)}$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is a possibly killed \mathbb{R}^d -valued Lévy process in the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, \infty)}$ satisfying $\mathcal{F}_\infty \subset \mathcal{G}$, if the following holds: (1) X takes values in $(\mathbb{R}^d \cup \{\partial\}, \mathcal{B}_{\mathbb{R}^d} \vee \sigma(\{\partial\}))$ and is \mathcal{F} -adapted; (2) $X = \partial$ on $[\zeta, \infty)$, where $\zeta := \inf\{t \in (0, \infty) : X_t = \partial\}$ is the lifetime of X ; (3) X has paths that are right-continuous and have left limits on $[0, \zeta)$; (4) \mathbb{P} -a.s. $X_0 = 0$ and $\zeta > 0$; and (5) $\mathbb{P}[g(X_t - X_s)\mathbb{1}_{\{s < \zeta\}} | \mathcal{F}_s] = \mathbb{1}_{\{s < \zeta\}} \mathbb{P}[g(X_{t-s})]$ a.s.- \mathbb{P} for all real $0 \leq s \leq t$ and all $g \in \mathcal{B}_{\mathbb{R}^d} / \mathcal{B}_{[0, \infty]}$ extended by 0 on $\{\partial\}$.¹ When these conditions prevail, then in fact for any \mathcal{F} -stopping time T with $\mathbb{P}(T < \zeta) > 0$, under the measure $\mathbb{P}(\cdot | T < \zeta)$, the process $\Delta_T X := (X_{T+t} - X_T)_{t \in [0, \infty)}$ is independent of $\mathcal{F}_T |_{\{T < \zeta\}}$ and has the same distribution as does X under \mathbb{P} . This is known as the strong Markov property of X . In particular there exists a necessarily unique $q \in [0, \infty)$ with $\zeta \sim_{\mathbb{P}} \text{Exp}(q)$.

Furthermore, X is called simply an \mathbb{R}^d -valued Lévy process in \mathcal{F} if $\zeta = \infty$ (corresponding to $q = 0$). In the latter case, if $\mathbf{e} \in \mathcal{G} / \mathcal{B}_{[0, \infty]}$ is independent of \mathcal{F}_∞ and exponentially distributed (with strictly positive mean) under \mathbb{P} , then $k_{\mathbf{e}}(X)$, the process X killed at the time \mathbf{e} (i.e. the process equal to X on $[0, \mathbf{e})$ and equal to ∂ on $[\mathbf{e}, \infty)$) is, in turn, a possibly killed Lévy process in the progressive enlargement of \mathcal{F} by \mathbf{e} , i.e. in the smallest enlargement of \mathcal{F} that makes \mathbf{e} a stopping time. Conversely, if we revert to X being just a possibly killed Lévy process, then there exists a unique

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¹Throughout we will write $\mathbb{Q}[W]$ for $\mathbb{E}_{\mathbb{Q}}[W]$, $\mathbb{Q}[W; A]$ for $\mathbb{E}_{\mathbb{Q}}[W\mathbb{1}_A]$, $\mathbb{Q}[W | \mathcal{H}]$ for $\mathbb{E}_{\mathbb{Q}}[W | \mathcal{H}]$, and $Z_*\mathbb{Q}$ for the law of Z under \mathbb{Q} w.r.t. a σ -field on the codomain that will be clear from context or made explicit. For σ -fields \mathcal{A} and \mathcal{B} , \mathcal{A}/\mathcal{B} will denote the set of \mathcal{A}/\mathcal{B} -measurable maps; \mathcal{B}_A is the Borel (under the standard topology) σ -field on A .

law \mathcal{L} of an \mathbb{R}^d -valued Lévy process such that for all $t \in [0, \infty)$, $(X|_{[0,t]})_\star \mathbf{P}(\cdot | t < \zeta) = (\xi|_{[0,t]})_\star \mathcal{L}$, where ξ is the canonical process.

We refer the reader to [8] for further general theory and terminology concerning Lévy processes (albeit without killing and in their natural filtrations). In particular, the reader will recall that, thanks to the stationary independent increments property, the one-dimensional distributions of a possibly killed Lévy process determine already its law.

Put differently, observing the laws of two possibly killed Lévy processes X^1 and X^2 up to a (and even just at a given) strictly positive deterministic time, we are able to say whether or not X^1 and X^2 have the same law. The result of Theorem 1 below — whose content was already described in informal terms in the abstract — provides a non-obvious (cf. Examples 3 and 4), though intuitively appealing complement to this observation, namely one in which a stopping time takes the role of a deterministic time. Remark 9 on p. 6 will comment on the related case of continuous-time Markov chains. Finally, another motivation for the investigations — and at the same time an application — of Theorem 1 is provided in Example 10 on p. 7.

2. RESULTS AND PROOFS

Notation-wise, in the statement of the theorem to follow, for a process $Z = (Z_t)_{t \in T}$ on a probability space $(\Theta, \mathcal{H}, \mathbf{Q})$, taking its values in $\mathbb{R}^d \cup \{\partial\}$, and defined temporally possibly only on some random subset T of the time axis $[0, \infty)$, by $Z_\star \mathbf{Q}$ we mean the \mathbf{Q} -law of the process Z' that is equal to Z on T and equal to some adjoined extra state \uparrow on $[0, \infty) \setminus T$, and we mean it on the space $((\mathbb{R}^d \cup \{\partial, \uparrow\})^{[0, \infty)}, (\mathcal{B}_{\mathbb{R}^d} \vee \sigma(\{\partial\}, \{\uparrow\}))^{\otimes [0, \infty)})$ [assuming of course Z' is \mathcal{H} -measurable w.r.t. the latter measurable structure]. Further, for laws $\mathcal{M}_1, \mathcal{M}_2, \mathcal{L}$ of \mathbb{R}^d -valued Lévy processes, and for $q \in [0, \infty)$: (I) \star denotes convolution of laws, viz. if $Y^1 \sim_{\mathbf{Q}} \mathcal{M}_1$ and $Y^2 \sim_{\mathbf{Q}} \mathcal{M}_2$, with Y_1 independent of Y^2 under \mathbf{Q} , then $\mathcal{M}_1 \star \mathcal{M}_2 = (Y^1 + Y^2)_\star \mathbf{Q}$, and (II) k_q is the operator of adding a killing at rate q , viz. if $Y \sim_{\mathbf{Q}} \mathcal{L}$ and $\mathbf{e} \sim_{\mathbf{Q}} \text{Exp}(q)$, Y independent of \mathbf{e} under \mathbf{Q} , then $k_q(\mathcal{L}) = (k_{\mathbf{e}}(Y))_\star \mathbf{Q}$.

Here is now the result of this note:

Theorem 1. *For $i \in \{1, 2\}$ let $X^i = (X_t^i)_{t \in [0, \infty)}$ be a possibly killed \mathbb{R}^d -valued Lévy process, defined on a probability space $(\Omega^i, \mathcal{G}^i, \mathbf{P}^i)$ in the filtration $\mathcal{F}^i = (\mathcal{F}_t^i)_{t \in [0, \infty)}$, and let T^i be an \mathcal{F}^i -stopping time with $\mathbf{P}^i(T^i > 0) > 0$.*

- (i) *If $(X^1|_{[0, T^1] \cap [0, \infty)})_\star \mathbf{P}^1 = (X^2|_{[0, T^2] \cap [0, \infty)})_\star \mathbf{P}^2$, then $X^1_\star \mathbf{P}^1 = X^2_\star \mathbf{P}^2$.*
- (ii) *If $(X^1|_{[0, T^1)})_\star \mathbf{P}^1 = (X^2|_{[0, T^2)})_\star \mathbf{P}^2$, then there exist a law \mathcal{L} of a Lévy process, laws \mathcal{L}_1 and \mathcal{L}_2 of compound Poisson processes (allowing the zero process), and $\{q_1, q_2\} \subset [0, \infty)$, such that $(X^i)_\star \mathbf{P}^i = k_{q_i}(\mathcal{L} \star \mathcal{L}_i)$ for $i \in \{1, 2\}$, i.e. “the laws of X^1 and X^2 differ only modulo compound Poisson processes and killing”.*

Before giving the proof of this theorem, some (counter)examples and comments.

Example 2. Even if, for $i \in \{1, 2\}$, T^i is finite \mathbf{P}^i -a.s., there can be no hope of having just $(X_{T^1}^1 \mathbb{1}_{\{T^1 < \infty\}}, T^1)_\star \mathbf{P}^1 = (X_{T^2}^2 \mathbb{1}_{\{T^2 < \infty\}}, T^2)_\star \mathbf{P}^2$ imply $X^1_\star \mathbf{P}^1 = X^2_\star \mathbf{P}^2$. Indeed, if, on a common

probability space, B is a linear Brownian motion, $\mathbf{0}$ is the the zero process, and S is the first hitting time of 0 by B after time 1, then a.s. $S < \infty$, S is a stopping time of the completed natural filtration of B in which both B and $\mathbf{0}$ are Lévy processes, $B_S \mathbb{1}_{\{S < \infty\}} = \mathbf{0} = \mathbf{0}_S \mathbb{1}_{\{S < \infty\}}$ a.s., yet of course B and $\mathbf{0}$ do not have the same law.

Example 3. For (i) the stopping time property is essential. If $\mathbf{0}$ is the zero process and N is a homogeneous Poisson process, both defined on a common probability space, then letting S be the first jump time of N , one has $\mathbf{0} = 0 = N$ on $[0, S/2] \cap [0, \infty)$ and $S/2 > 0$ a.s., yet $\mathbf{0}$ and N do not have the same law.

Example 4. Also for (ii) the stopping time property is essential. Indeed, by a result of Williams [7, Theorem 55.9], for any given $c \in (0, \infty)$, on a common probability space, one may construct a Brownian motion with drift $-c$, B^1 , a Brownian motion with drift c , B^2 , and an exponentially distributed random time γ of rate $2c$, such that γ is independent of B^1 , is equal to the time of the overall infimum of B^2 , and with $B^1|_{[0, \gamma)} = B^2|_{[0, \gamma)}$. (Of course this is also another counterexample for (i).)

Example 5. In (i), even if $(\Omega^1, \mathcal{G}^1, \mathbf{P}^1) = (\Omega^2, \mathcal{G}^2, \mathbf{P}^2)$, $\mathcal{F}^1 = \mathcal{F}^2$, and a.s. $T^1 = T^2$ & $X^1 = X^2$ on $[0, T^1] \cap [0, \infty)$, still the conclusion cannot be strengthened to a.s. equality. To exemplify this, take, on a common probability space, a standard one-dimensional Brownian motion B^1 , a random time T independent of B^1 , positive and finite with a positive probability, and let B^2 be got from B^1 by changing B^1 into an independent standard linear Brownian motion after time T . Then B^1 and B^2 are both standard univariate Brownian motions in their completed joint natural filtration of which T is a stopping time, they a.s. agree on $[0, T] \cap [0, \infty)$, but they are not a.s. equal.

Example 6. The conclusion of (ii) cannot be improved. For instance, if, on a common probability space, N is a homogeneous Poisson process, while M is zero up to and then killed at the first jump time S of N , then N and M are Lévy processes in the completed natural filtration of N of which S is a stopping time, a.s. $N = 0 = M$ on $[0, S)$ and $S > 0$, yet N and M “differ by killing and by a compound Poisson process”.

Remark 7. The content of Theorem 1 makes sense also for a possibly killed random walk (in the obvious interpretation of that qualification), but in that case it is trivial. Indeed, if $Z = (Z_n)_{n \in \mathbb{N}_0}$ is a possibly killed \mathbb{R}^d -valued random walk, in a filtration $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}_0}$, under a probability \mathbf{Q} , and if S is a stopping time of \mathcal{H} that is positive with a positive \mathbf{Q} -probability, then Z is independent under \mathbf{Q} of $\mathcal{H}_0 \ni \{S > 0\}$. Therefore, if the \mathbf{Q} -law of $Z|_{[0, S] \cap \mathbb{N}_0}$ is known, then for any $f \in \mathcal{B}_{\mathbb{R}^d} / \mathcal{B}_{[0, \infty]}$ extended by 0 on $\{\partial\}$, the quantity $\mathbf{Q}[f(Z_1)]\mathbf{Q}(S > 0) = \mathbf{Q}[f(Z_1); S > 0] = \mathbf{Q}[f(Z_1^S); S > 0]$, and hence the \mathbf{Q} -law of Z is known. On the other hand the knowledge of the \mathbf{Q} -law of $Z|_{[0, S) \cap \mathbb{N}_0}$ clearly need not determine the \mathbf{Q} -law of Z at all, since one can take $S = 1$.

Remark 8. (i) implies that, for any killed (lifetime $\zeta < \infty$ a.s.) Lévy process X in a filtration \mathcal{F} under a measure \mathbf{P} , and any \mathcal{F} -stopping time S with $\mathbf{P}(S > 0) > 0$, one has $\mathbf{P}(S \geq \zeta) > 0$.

We turn now to the

Proof of Theorem 1. Let $i \in \{1, 2\}$. Replacing both T^i with $T^i \wedge 1$ if necessary we may assume each T^i is finite. Then take the product space $(\Omega^{i,\infty}, \mathcal{G}^{i,\infty}, \mathbf{P}^{i,\infty}) := ((\Omega^i)^\mathbb{N}, (\mathcal{G}^i)^{\otimes \mathbb{N}}, \times_{\mathbb{N}} \mathbf{P}^i)$. Set $X^{i,n} := X^i \circ \text{pr}_n$ and $T_n^i := T^i \circ \text{pr}_n$ for $n \in \mathbb{N}$. By the law of large numbers, since $\mathbf{P}^i(T^i > 0) > 0$, and discarding a negligible set if necessary, we may assume that $S_n^i := \sum_{k=1}^n T_k^i \uparrow \infty$ as $n \rightarrow \infty$ over \mathbb{N}_0 .

Next we define the process $Y^{i,n} = (Y_t^{i,n})_{t \in [0, \infty)}$ on $\Omega^{i,\infty}$, with $n \in \mathbb{N}$, as follows: $Y^{i,1} := X^{i,1}$ and then inductively, for $n \in \mathbb{N}$, $Y^{i,n+1} = Y^{i,n}$ on $[0, S_n^i)$, while $Y^{i,n+1} = Y_{S_n^i}^{i,n} + X_{-S_n^i}^{i,n+1}$ on $[S_n^i, \infty)$. In words, still for $n \in \mathbb{N}$, $Y^{i,n}$ starts at $X_0^{i,1}$ and then, up to hitting ∂ , for $k \in \{1, \dots, n-1\}$, the increments of $Y^{i,n}$ on $[S_{k-1}^i, S_k^i]$ are those of $X^{i,k}$ on $[0, T_k^i]$, while on $[S_n^i, \infty)$, they are those of $X^{i,n}$ on $[0, \infty)$.

It is then clear that, as $n \rightarrow \infty$, the $Y^{i,n}$ are converging pointwise to a process, that we denote by Y^i ; we claim furthermore, that for each $n \in \mathbb{N}$, $Y^{i,n}$ (and therefore, in the limit, Y^i) has the same law under $\mathbf{P}^{i,\infty}$ as does X^i under \mathbf{P}^i .

We need only prove the latter for “ $i = 1$ ” (it is the same for “ $i = 2$ ”); and then we drop, in the next paragraph only, the superscript “ $i = 1$ ” to ease the notation.

Take then $\{n, k\} \subset \mathbb{N}$, $\{g_1, \dots, g_k\} \subset \mathcal{B}_{\mathbb{R}^d} / \mathcal{B}_{[0, \infty]}$ extended by zero on $\{\partial\}$, and real numbers $0 = t_0 < \dots < t_k$; we are to show that $\mathbf{P}[g_1(X_{t_1}) \cdots g_k(X_{t_k})] = \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_k(Y_{t_k}^n)]$. We compute:

$$\begin{aligned}
& \mathbf{P}^\infty[g_1(Y_{t_1}^{n+1}) \cdots g_k(Y_{t_k}^{n+1})] \\
&= \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_k(Y_{t_k}^n); t_k \leq S_n] + \sum_{l=1}^k \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_{l-1}(Y_{t_{l-1}}^n) g_l(Y_{t_l}^{n+1}) \cdots g_k(Y_{t_k}^{n+1}); t_{l-1} \leq S_n < t_l] \\
&= \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_k(Y_{t_k}^n); t_k \leq S_n] + \\
& \sum_{l=1}^k \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_{l-1}(Y_{t_{l-1}}^n) g_l(Y_{S_n}^n + X_{t_l - S_n}^{n+1}) \cdots g_k(Y_{S_n}^n + X_{t_k - S_n}^{n+1}); t_{l-1} \leq S_n < t_l, T_n < \zeta_n] \\
&= \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_k(Y_{t_k}^n); t_k \leq S_n] + \\
& \sum_{l=1}^k \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_{l-1}(Y_{t_{l-1}}^n) g_l(Y_{S_n}^n + (\Delta_{T_n} X^n)_{t_l - S_n}) \cdots g_k(Y_{S_n}^n + (\Delta_{T_n} X^n)_{t_k - S_n}); t_{l-1} \leq S_n < t_l, T_n < \zeta_n] \\
&= \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_k(Y_{t_k}^n); t_k \leq S_n] + \sum_{l=1}^k \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_{l-1}(Y_{t_{l-1}}^n) g_l(Y_{t_l}^n) \cdots g_k(Y_{t_k}^n); t_{l-1} \leq S_n < t_l] \\
&= \mathbf{P}^\infty[g_1(Y_{t_1}^n) \cdots g_k(Y_{t_k}^n)],
\end{aligned}$$

where crucially in the third equality: we used (I) the strong Markov property for X at time T , plus the various independences coming from the construction of \mathbf{P}^∞ , to establish that under \mathbf{P}^∞ , conditionally on $\{T_n < \zeta_n\}$, the process $\Delta_{T_n} X^n$ has the same law as X^{n+1} and is, like X^{n+1} , independent of $(X^1, \dots, X^{n-1}, (X^n)^{T_n}, T_1, \dots, T_n)$; noting that (II) $Y_{S_n}^n = X_{T_1}^1 + \dots + X_{T_n}^n$, $S_n = T_1 + \dots + T_n$, and for each $l \in \{1, \dots, k\}$ and $m \in \{1, \dots, l-1\}$, $g_m(Y_{t_m}^n) \mathbb{1}_{\{t_{l-1} \leq S_n\}}$ are all measurable w.r.t. $\sigma(X^1, \dots, X^{n-1}, (X^n)^{T_n}, T_1, \dots, T_n)$: it is only not obvious for the latter –

to check it, write $Y_{t_m}^n \mathbb{1}_{\{t_{l-1} \leq S_n\}} = \sum_{w=1}^n Y_{t_m}^n \mathbb{1}_{\{S_{w-1} < t_m \leq S_w, t_{l-1} \leq S_n\}} = \sum_{w=1}^n (X_{T_1}^1 + \cdots + X_{T_{w-1}}^{w-1} + (X^w)_{t_m - S_{w-1}}^{T_w}) \mathbb{1}_{\{S_{w-1} < t_m \leq S_w, t_{l-1} \leq S_n\}}$.² An inductive argument allows to conclude.

As we have noted, this now establishes that, for $i \in \{1, 2\}$, $Y^i \star \mathbf{P}^{i, \infty} = X^i \star \mathbf{P}^i$; it will also be helpful to keep in mind that, up to hitting ∂ , for $k \in \mathbb{N}$, the increments of Y^i on $[S_{k-1}^i, S_k^i]$ are those of $X^{i, k}$ on $[0, T_k^i]$.

(i). Y^1 and Y^2 are seen to be the same measurable transformation of the sequences $((X_t^{i, n})_{t \in [0, T_n^i]})_{n \in \mathbb{N}}$ with $i = 1$ and $i = 2$, respectively. Furthermore, for $i \in \{1, 2\}$, under $\mathbf{P}^{i, \infty}$, by construction, the sequence $((X_t^{i, n})_{t \in [0, T_n^i]})_{n \in \mathbb{N}}$ consists of i.i.d. random elements. Besides, $((X_t^{1, 1})_{t \in [0, T_1^1]}) \star \mathbf{P}^{1, \infty} = ((X_t^1)_{t \in [0, T^1]}) \star \mathbf{P}^1 = ((X_t^2)_{t \in [0, T^2]}) \star \mathbf{P}^2 = ((X_t^{2, 1})_{t \in [0, T_1^2]}) \star \mathbf{P}^{2, \infty}$ by the assumption of (i). Therefore $X^1 \star \mathbf{P}^1 = Y^1 \star \mathbf{P}^{1, \infty} = Y^2 \star \mathbf{P}^{2, \infty} = X^2 \star \mathbf{P}^2$.

(ii). Let $i \in \{1, 2\}$ and denote by ζ^i the lifetime of X^i . Replacing T^i with $T^i \wedge \zeta^i = \inf\{t \in (0, T^i) : X_t^i = \partial\} \wedge T^i$ if necessary we may assume that $T^i \leq \zeta^i$; then (possibly by enlarging the underlying space and filtration) we may assume that $\zeta^i = \infty$. Let next ν^i be the Lévy measure of X^i under \mathbf{P}^i (and hence of Y^i under $\mathbf{P}^{i, \infty}$). Set $\gamma := \nu^1 + \nu^2$, $f^i := \frac{d\nu^i}{d\gamma}$ for $i \in \{1, 2\}$, and $\nu := (f^1 \wedge f^2) \cdot \gamma$. We check that (A) $\nu^i - \nu$, $i \in \{1, 2\}$, are finite measures.

Suppose per absurdum, and then without loss of generality, that $\nu^{1'} := \nu^1 - \nu = (f^1 - f^1 \wedge f^2) \cdot \gamma$ is infinite. The measure $\nu^{1'}$ is locally finite in $\mathbb{R}^d \setminus \{0\}$, hence there exists a sequence $(A_k)_{k \in \mathbb{N}}$ in $\mathcal{B}_{\mathbb{R}^d \setminus \{0\}}$ such that $A_k \subset \{f^1 > f^1 \wedge f^2\} = \{f^1 > f^2\}$ for each $k \in \mathbb{N}$ and such that $[0, \infty) \ni \nu^{1'}(A_k) = \nu^1(A_k) - \nu^2(A_k) \uparrow \infty$ as $k \rightarrow \infty$. For $k \in \mathbb{N}$ and $i \in \{1, 2\}$, set ξ_k^i equal to the number of jumps of Y^i during the time interval $(0, 1]$ that fall into the Borel set A_k ; then $\xi_k^i \sim_{\mathbf{P}^{i, \infty}} \text{Pois}_{\mathbb{N}_0}(\nu^i(A_k))$ and in particular $\mathbf{P}^{i, \infty}[\xi_k^i] = \nu^i(A_k)$. Consequently, for each $k \in \mathbb{N}$, $\mathbf{P}^{1, \infty}[\xi_k^1] - \mathbf{P}^{2, \infty}[\xi_k^2] = \nu^1(A_k) - \nu^2(A_k)$, which is $\uparrow \infty$ as $k \rightarrow \infty$. On the other hand, setting $N := \sum_{k \in \mathbb{N}} \mathbb{1}_{\{S_k^1 \leq 1\}}$, it is clear by construction of the processes Y^1 and Y^2 and from $(X^1|_{[0, T^1]}) \star \mathbf{P}^1 = (X^2|_{[0, T^2]}) \star \mathbf{P}^2$, that $\mathbf{P}^{1, \infty}[\xi_k^1] - \mathbf{P}^{2, \infty}[\xi_k^2] \leq \mathbf{P}^{1, \infty}[N]$ for all $k \in \mathbb{N}$. At the same time, the T_k^1 , $k \in \mathbb{N}$, are i.i.d. under $\mathbf{P}^{1, \infty}$, hence by renewal theory [6], since $\mathbf{P}^{1, \infty}(T_1^1 > 0) = \mathbf{P}^1(T^1 > 0) > 0$, it follows that $\mathbf{P}^{1, \infty}[N] < \infty$, a contradiction.

Next, by the Lévy-Itô decomposition one can write, for each $i \in \{1, 2\}$, \mathbf{P}^i -a.s.:

$$X_t^i = B_t^i + \Gamma^i t + \lim_{\epsilon \downarrow 0} \int_{(0, t] \times B(\epsilon, 1)} x [J^i(ds, dx) - ds \nu^i(dx)] + \int_{(0, t] \times B(1, \infty)} x J^i(ds, dx), \quad t \in [0, \infty),$$

for a d -dimensional (possibly non-standard, of course) \mathbf{P}^i -Brownian motion B^i , a $\Gamma^i \in \mathbb{R}^d$, and with J^i being the Poisson random measure of the jumps of X^i . Furthermore, by (A), the limit $L := \lim_{\epsilon \downarrow 0} \left[\int_{B(\epsilon, 1)} x \nu^1(dx) - \int_{B(\epsilon, 1)} x \nu^2(dx) \right]$ is well-defined in \mathbb{R}^d . Set $X_t^{1, c} := B_t^1 + \Gamma^1 t$ and $X_t^{2, c} := B_t^2 + \Gamma^2 t + Lt$ for $t \in [0, \infty)$. Then, with $i \in \{1, 2\}$, on the time interval $[0, T^i)$, the processes $X^{i, c}$ can be extracted from the processes X^i by the same measurable transformation (because this is true of the jumps). Therefore $(X^{1, c}|_{[0, T^1]}) \star \mathbf{P}^1 = (X^{2, c}|_{[0, T^2]}) \star \mathbf{P}^2$ and hence by

²It is tempting to think that one could somehow bypass the strong Markov property and still prove that $Y \star \mathbf{P}^\infty = X \star \mathbf{P}$ without assuming that T is an \mathcal{F} -stopping time. But it is false. For instance, if, under \mathbf{P} , X is a linear Brownian motion with strictly negative drift, and if T is the last time that X is at 0, then under \mathbf{P}^∞ , 0 is recurrent for the process Y , while it is transient for the process X under \mathbf{P} .

sample-path continuity $(X^{1,c}|_{[0,T^1] \cap [0,\infty)})_* \mathbf{P}^1 = (X^{2,c}|_{[0,T^2] \cap [0,\infty)})_* \mathbf{P}^2$. Now, $X^{1,c}$ and $X^{2,c}$ are still Lévy processes in the filtrations \mathcal{F}^1 and \mathcal{F}^2 , respectively. Thus, by part (i), (B) $X^{1,c} \star \mathbf{P}^1 = X^{2,c} \star \mathbf{P}^2$.

Combining (A)-(B), by the independence between the “jump” and “continuous” part present in the Lévy-Itô decomposition, the desired conclusion follows. \square

Remark 9. The proof of Theorem 1(i) can be tweaked to handle the case of continuous-time Markov chains, though the result is less definitive in this context. Let us look at this in more detail.

Fix a countable set E – it will be the state space; fix also ∂ – it will be the cemetery state – a $\partial \notin E$. Recall then that a process $X = (X_t)_{t \in [0,\infty)}$, defined on a measurable space (Ω, \mathcal{G}) , is a (minimal) continuous-time E -valued Markov chain, in a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ with $\mathcal{F}_\infty \subset \mathcal{G}$, under the probabilities $\mathbf{P} = (\mathbf{P}_x)_{x \in E}$ on (Ω, \mathcal{G}) , provided: (1) X takes values in $E \cup \{\partial\}$, endowed with the discrete topology and measurable structure, and it is \mathcal{F} -adapted; (2) X has paths that are right-continuous, X is E -valued on $[0, \zeta)$, and $X = \partial$ on $[\zeta, \infty)$, where $\zeta := \lim_{n \rightarrow \infty} J_n$, $(J_n)_{n \in \mathbb{N}}$ being the sequence of the consecutive jump times of X ; (3) $\mathbf{P}_x(X_0 = x) = 1$ for all $x \in E$; (4) for $\{t, s\} \subset [0, \infty)$, $x \in E$, and $g \in 2^E / \mathcal{B}_{[0,\infty]}$ extended by 0 on $\{\partial\}$, one has $\mathbf{P}_x[g(X_{t+s}) | \mathcal{F}_t] = \mathbf{P}^{X_t}[g(X_s)]$ a.s.- \mathbf{P}_x on $\{t < \zeta\}$. As is well-known, such an X then has the strong Markov property: for any \mathcal{F} -stopping time T , $x \in E$ and $g \in (2^{E \cup \{\partial\}})^{\otimes [0,\infty)} / \mathcal{B}_{[0,\infty]}$, $\mathbf{P}_x[g(X_{T+\cdot}) | \mathcal{F}_T] = \mathbf{P}^{X_T}[g(X)]$ a.s.- \mathbf{P}_x on $\{T < \zeta\}$.

Suppose then that the system $(\Omega, \mathcal{G}, \mathcal{F}, \mathbf{P}, X)$ constitutes such a continuous-time Markov chain and let T be an \mathcal{F} -stopping time. Take the measure $\mathbf{P}^\infty := \times_{n \in \mathbb{N}} \times_{x \in E} \mathbf{P}_x$ on $(\prod_{n \in \mathbb{N}} \prod_{x \in E} \Omega, (\mathcal{G}^{\otimes E})^{\otimes \mathbb{N}})$ and let $X^{x,n} := X \circ \text{pr}_x \circ \text{pr}_n$, $T^{x,n} := T \circ \text{pr}_x \circ \text{pr}_n$ for $x \in E$, $n \in \mathbb{N}$ (in words, we take denumerably many independent copies of X for each starting position). Additionally set $X^{\partial,n} \equiv \partial$ and $T^\partial \equiv \infty$ for $n \in \mathbb{N}$. Fix next an $x \in E$. Define $Y^{x,1} := X^{x,1}$, $S_1^x := T^{x,1}$, and then recursively, for $n \in \mathbb{N}$, $Y^{x,n+1} := Y^{x,n}$ on $[0, S_n^x)$, $Y^{x,n+1} := X_{-S_n^x}^{Y^{x,n}, n+1}$ on $[S_n^x, \infty)$, and $S_{n+1}^x := S_n^x + T_{S_n^x}^{Y^{x,n}, n+1}$. (Note the denumerable state space ensures suitable measurability of these objects and it ensures that \mathbf{P}^∞ -a.s. $Y_{S_n^x}^{x,n+1} = Y_{S_n^x}^{x,n}$ for all $n \in \mathbb{N}$.) Set furthermore $S^x := \lim_{n \rightarrow \infty} S_n^x$ and assume that $\mathbf{P}^\infty(S^x = \infty) = 1$. (It would be an interesting question in its own right to investigate under which conditions does $\mathbf{P}^\infty(S^x = \infty) = 1$ in fact obtain, however this will not be pursued here.) It is then clear that the $Y^{x,n}$ converge \mathbf{P}^∞ -a.s. to a process Y^x as $n \rightarrow \infty$. Furthermore, using the strong Markov property, similarly to how we did in the proof of the Lévy case, we may show that, for each $n \in \mathbb{N}$, $Y^{x,n}$, and hence in the limit Y^x , has the same law under \mathbf{P}^∞ as does X under \mathbf{P}_x . We leave the (grantedly more tedious when compared to the Lévy case) details of these computations to the interested reader.

As a consequence of the preceding we obtain then, just as in the proof of Theorem 1(i), the statement:

For $i \in \{1, 2\}$, let $(\Omega^i, \mathcal{G}^i, \mathcal{F}^i, \mathbf{P}^i, X^i)$ be a continuous-time E -valued Markov chain in the sense made precise above, and let T^i be an \mathcal{F}^i -stopping time; associate to it $\mathbf{P}^{i,\infty}$ and the times $(S^{i,x})_{x \in E}$ in the obvious way, as above. Assume $(X^1|_{[0,T^1] \cap [0,\infty)})_* \mathbf{P}_x^1 =$

$(X^2|_{[0,T^2] \cap [0,\infty)})_* \mathbf{P}_x^2$ for all $x \in E$. Let $x \in E$. If $\mathbf{P}^{1,\infty}(S^{1,x} = \infty) = 1$ (and hence $\mathbf{P}^{2,\infty}(S^{2,x} = \infty) = 1$), then $X^1_* \mathbf{P}_x^1 = X^2_* \mathbf{P}_x^2$.

The case of discrete-time Markov chains is again trivial in this context (cf. Remark 7). The analogue of Theorem 1(ii) is of no interest in the context of Markov chains.

Example 10. We close this paper with an example in the context of self-similar Markov processes in which Theorem 1 produces non-trivial information.

To this end, let $X = (X_t)_{t \in [0,\infty)}$ be a one-dimensional stable Lévy process under the probabilities $(\mathbf{P}_x)_{x \in \mathbb{R}}$ in a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ satisfying the usual hypotheses. We make the standing assumption that X has jumps of both signs in order to avoid triviality, and refer the reader to [2, Chapter 13] [4] [3, Section 2] for any unexplained terminology and facts that we shall state without proof below: introducing everything properly here would not be consistent with the scope of this paper.

We consider the following two processes: Y , which is X sent to 0 on hitting $(-\infty, 0]$ (and then stopped); and Z , which is X sent to 0 on hitting 0 (and then stopped). It is then well-known that Y is a positive self-similar Markov process under the probabilities $(\mathbf{P}_x)_{x \in (0,\infty)}$ and that Z is a real self-similar Markov process under the probabilities $(\mathbf{P}_x)_{x \in \mathbb{R} \setminus \{0\}}$, both in the filtration \mathcal{F} . Moreover, defining $T_0 := \inf\{s \in (0, \infty) : Y_s = 0\}$, $R_0 := \inf\{s \in (0, \infty) : Z_s = 0\}$,

$$\tau_t := \inf \left\{ s \in (0, R_0) : \int_0^s |Z_v|^{-\alpha} dv > t \right\} \wedge R_0$$

and $\mathcal{G}_t := \mathcal{F}_{\tau_t}$ for $t \in [0, \infty)$, we have as follows:

- (1) Put $\gamma := \int_0^{R_0} |Z_v|^{-\alpha} dv$ and define the processes $\mu = (\mu_t)_{t \in [0,\infty)}$ and $J = (J_t)_{t \in [0,\infty)}$, by setting $\mu_t := \log(|Z_{\tau_t}|)$ and $J_t := \text{sgn}(Z_{\tau_t})$ for $t \in (0, \gamma)$, μ and J being killed on the time-interval $[\gamma, \infty)$. Then [1] (μ, J) is a possibly killed Markov additive process (MAP) under the probabilities $(\mathbf{P}_{je^x})_{(x,j) \in \mathbb{R} \times \{-1,1\}}$ in the filtration $(\mathcal{G}_t)_{t \in [0,\infty)}$. This is known as the Lamperti-Kiu transform. Further, let (G, M) be the ascending ladder MAP of (μ, J) with associated local time at the maximum L ; then, under \mathbf{P}_1 , G killed at the first time M changes its sign is a killed subordinator in the filtration $(\mathcal{G}_{L_s^{-1}})_{s \in [0,\infty)}$. It is the killed Lévy process H^1 which describes the movement of G as long as the modulating chain M is in state 1.
- (2) Similarly put $\zeta := \int_0^{T_0} Y_v^{-\alpha} dv$ and define the process $\xi = (\xi_t)_{t \in [0,\infty)}$ by setting $\xi_t := \log(Y_{\tau_t})$ for $t \in (0, \zeta)$, ξ being killed on the time-interval $[\zeta, \infty)$. Then [5] ξ is a possibly killed Lévy process in the filtration $(\mathcal{G}_t)_{t \in [0,\infty)}$ under the probabilities $(\mathbf{P}_{e^x})_{x \in \mathbb{R}}$. This is known as the Lamperti transform. Further, let H be the ascending ladder height process of ξ under a normalization of the local time at the maximum that is consistent with that of L ; then, under \mathbf{P}_1 , H is a killed subordinator in the filtration $(\mathcal{G}_{L_s^{-1}})_{s \in [0,\infty)}$.

Besides, it is clear from the pathwise construction of Y and Z , that H and H^1 agree on $[0, L_\zeta)$ with L_ζ being a stopping time of $(\mathcal{G}_{L_s^{-1}})_{s \in [0, \infty)}$. Indeed L_ζ is the lifetime of H , and albeit it is not the lifetime of H^1 , it is certainly majorized by the latter.

It follows then from Theorem 1 that the laws of H and H^1 differ only by killing and compound Poisson components, which yields an a priori insight into the non-trivial probabilistic structure of the MAP (G, M) in terms of the much simpler object H . (A fully explicit description of the law of (G, M) , viz. of the MAP exponent of (G, M) , is non-trivial, see [3].)

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