INVARIANT METRICS ON THE COMPLEX ELLIPSOID

GUNHEE CHO

Abstract. We provide a class of geometric convex domains on which the Carathéodory-Reiffen metric, the Bergman metric, the complete Kähler-Einstein metric of negative scalar curvature are uniformly equivalent, but not proportional to each other. In a two-dimensional case, we provide a full description of curvature tensors of the Bergman metric on the weakly pseudoconvex boundary point and show that invariant metrics are proportional to each other if and only if the geometric convex domain is the Poincaré-disk.

Contents

1. Introduction and results

In this paper, we study the invariant metrics on complex ellipsoid $E = E(m, n, p) = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : |z|^2 + |w|^{2p} < 1\}$ with $p > 0$.

On the unit disk B_n in $\mathbb{C}^n$, the Poincaré-metric is the primary example for invariant metrics since invariant metrics on unit disk are just the Poincaré-metric up to some constant. Precisely, let's denote by γ_{B_n}, χ_{B_n}, $g^B_{B_n}$, $g^{KE}_{B_n}$ the Carathéodory-Reiffen metric, the Kobayashi-Royden metric, the Bergman metric, and the complete Kähler-Einstein metric respectively. These metrics are all invariant under biholomorphisms and we have

$$\gamma_{B_n}(a; v) = \chi_{B_n}(a; v) < \sqrt{g^B_{B_n}((a; v), (a; v))} = \sqrt{g^{KE}_{B_n}((a; v), (a; v))} = c\gamma_{B_n}(a; v) \tag{1.1}$$

for any non-zero tangent vector $(a;v)$ where $c = c(n) > 0$. Hence if instead of the unit disk in $\mathbb{C}^n$ one considers more general complex manifolds, the four metrics provide characterizations into several classes. For a class of pseudoconvex domains in $\mathbb{C}^n$, one should expect that the relation among intrinsic metrics is different from (1.1) but some common phenomenon can be captured.

With invariant metrics on complex manifolds, there is one long-standing open problem in complex geometry. That is, prove that the Carathéodory-Reiffen metric on a simply-connected complete Kähler manifold (M,ω) with negative Riemannian sectional curvature range is equivalent to other invariant metrics and related progress on this problem have been made. Recently, D. Wu and S.T. Yau showed that for this class of Kähler manifolds (M,ω), the base Kähler metric ω is uniformly equivalent to the Bergman metric, the complete Kähler-Einstein metric, and the Kobayashi metric (see [20]). Based on this result, it is reasonable to ask whether such (M,ω) must be biholomorphic to a C^k-bounded strictly pseudoconvex-domain in $\mathbb{C}^n$ with reasonable $k \geq 0$, since it is known that the metrics are uniformly equivalent to each other for these domains (see, for example, [[5], [6], [9], [11], [16], [21]] and references therein). In this paper, we show that the complex ellipsoid serves as a bounded weakly pseudoconvex domain (with a further restriction $p > 1$ from our results) in $\mathbb{C}^n$ which is not the strictly pseudoconvex domain, but those invariant metrics are uniformly equivalent and they are not proportional to each other.

Let's denote the complete Kähler-Einstein metric of the Ricci eigenvalue -1, the Bergman metric and the Kobayashi-Royden metric on $E = E(m,n,p)$ by g_E^{KE}, g_E^B and χ_E respectively. Here are our results:

Theorem 1. *Let $E = E(m,n,p)$ for any $m,n \in \mathbb{N}$ with $p > 1/2$. Then there exists $C > 0$ such that g_E^{KE}, g_E^B and χ_E are uniformly equivalent to each other by $C > 0$.*

Moreover, the proof of Theorem 1 yields the equivalence of two invariant metrics on closed submanifolds of E:

Corollary 2. *Let $E = E(m,n,p)$ for any $m,n \in \mathbb{N}$ with $p > 1/2$. Then for any closed complex submanifold S in E, there exists $C > 0$ such that g_S^{KE} and χ_S are uniformly equivalent by $C > 0$.*

For the comparison of the Kähler-Einstein metric and the Bergman metric, S. Fu and B. Wong showed that for a simply-connected strictly pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary, if the Bergman metric is Kähler-Einstein, then the domain must be biholomorphic to the disk (see [10]). This is also claimed to be true for $\mathbb{C}^n$ (see, for example, [12]). Since the complex ellipsoid $E = E(m,n,p)$ with $p > 1$ is a weakly pseudoconvex domain because of the special boundary points $|z| = 1$, further investigation should be made to compare the Kähler-Einstein metric with the Bergman metric on $E = E(1,1,p)$. Here is our result:

Theorem 3. *Let $E = E(1,1,p) = \{(z,w) \in \mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1\}$ with $p > 0$. Then $g_E^B \neq \lambda g_E^{KE}$ for any $\lambda > 0$ if and only if $p \neq 1$.*

For the proof of Theorem 1 and Theorem 3, the explicit formula of Bergman kernel on E is used which was obtained by J.P. D'Angelo (See [8]). We also provide the explicit formula of the Bergman metric and curvature tensors for Theorem 3 in Section 4.

For the comparison of Carathéodory-Reiffen metric and the Kähler-Einstein metric on E, we have the following:

Theorem 4. *Let $E = E(m, n, p)$ for any $m, n \in \mathbb{N}$ with $p > 1/2$. Denote γ_E be the Carathéodory-Reiffen metric. Then*

$$\gamma_E \lneq \sqrt{g_E^{KE}}. \tag{1.2}$$

Furthermore, if we exclude $p = 1$, then for any $\lambda > 0$,

$$\gamma_E \neq \lambda\sqrt{g_E^{KE}}.$$

By combining the Theorems 1, 3, 4, we obtain:

Corollary 5. *On the complex ellipsoid $E = E(1, 1, p) = \{(z, w) \in \mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1\}$, with $1 \neq p > 1/2$. there exists $C > 0$ such that the followings hold: for any $\lambda > 0$,*

$$\begin{aligned}
\frac{1}{C}\sqrt{g_E^B} < \chi_E < C\sqrt{g_E^B},\\
\frac{1}{C}\sqrt{g_E^{KE}} < \chi_E < C\sqrt{g_E^{KE}},\\
\frac{1}{C}g_E^{KE} < g_E^B < Cg_E^{KE},\\
\gamma_E = \chi_E,\\
g_E^B \neq \lambda g_E^{KE},\\
\gamma_E \neq \lambda\sqrt{g_E^{KE}},\\
\gamma_E \neq \lambda\sqrt{g_E^B},\\
\gamma_E \lneq \sqrt{g_E^{KE}},\\
\gamma_E \lneq \sqrt{g_E^B}.
\end{aligned} \tag{1.3}$$

The geometric convexity of E when $p > 1/2$ implies that the Carathéodory-Reiffen metric and the Kobayashi-Royden metric are the same (see [16]). Also It is known that (1.3) holds for any bounded domains in $\mathbb{C}^n$ (e.g, see [13]).

Since it is known that the Riemannian sectional curvature of the Kähler-Einstein metric on $E = E(1, 1, p)$ is negatively pinched (see [4]), we have the following corollary which is related to the long-standing problem:

Corollary 6. *There exists a simply connected, weakly (but not strictly) pseudoconvex domain in $\mathbb{C}^2$ with negative Riemannian sectional curvature range with respect to the*

Kähler-Einstein metric such that the Bergman metric, the Kähler-Einstein metric, the Kobayashi-Royden metric are uniformly equivalent but those are not proportional to each other.

2. Prelinimaries

In this section, we collect the necessary definitions that we use to prove our results.

Let G be a domain in $\mathbb{C}^n$. A pseudometric $F(z,u) : G \times \mathbb{C}^n \to [0,\infty]$ on a domain G in $\mathbb{C}^n$ is called (biholomorphically) invariant if $F(z,\lambda u) = |\lambda| F(z,u)$ for all $\lambda \in \mathbb{C}^n$, and $F(z,u) = F(f(z), f(z)u)$ for any biholomorphism $f : G \to G$. The Caratheodory-Reiffen metric, Kobayashi-Royden metric, Bergman, Kähler-Einstein metric of negative scalar curvature are examples of invariant metrics on bounded weakly pseudoconvex domains in $\mathbb{C}^n$.

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Let $z \in G$ and $v \in T_zG$ a tangent vector at z. Define the Carathéodory-Reiffen metric by

$$\gamma_G(z;v) = \sup\{|df(z)v| : f \in \mathrm{Hol}(G,\mathbb{D})\}.$$

The Kobayashi-Royden metric is defined by

$$\chi_G(z;v) = \inf\{\frac{1}{\alpha} : \alpha > 0, f \in \mathrm{Hol}(\mathbb{D},G), f(0) = z, f'(0) = \alpha v\}. \tag{2.1}$$

Let $\rho_{\mathbb{D}}(a,b)$ denotes the distance between two points $a,b \in \mathbb{D}$ with respect to the Poincáre metric of constant holomorphic sectional curvature -4.

The Carathéodory pseudo-distance c_G on G is defined by

$$c_G(x,y) := \sup_{f \in \mathrm{Hol}(G,\mathbb{D})} \rho_{\mathbb{D}}(f(x), f(y)).$$

Here, $\rho_{\mathbb{D}}(a,b)$ denotes the integrated Poincaré-distance on the unit disk $\mathbb{D}$.

The Kobayashi pseudo-distance k_G on G is defined by

$$k_G(x,y) := \inf_{f_i \in \mathrm{Hol}(\mathbb{D},G)} \left\{ \sum_{i=1}^{n} \rho_{\mathbb{D}}(a_i,b_i) \right\}$$

where $x = p_0, ..., p_n = y$, $f_i(a_i) = p_{i-1}$, $f_i(b_i) = p_i$.

The inner-Carathéodory length and the Kobayashi length of a piecewise C^1 curve $\sigma : [0,1] \to G$ are given by

$$l^c(\sigma) := \int_0^1 \gamma_G(\sigma,\sigma')dt,$$

and

$$l^k(\sigma) := \int_0^1 \chi_G(\sigma,\sigma')dt$$

respectively. The inner-Carathéodory pseudo-distance and the inner-Kobayashi pseudo-distance on G are defined by

$$c^i_G(x,y) := \inf\{l^c(\sigma)(x,y)\} \text{ and } k^i_G(x,y) := \inf\{l^k(\sigma)(x,y)\},$$

where the infimums are taken over all piece-wise C^1 curves in G joining x and y.

The following relation is true in general:

$$0 \leq c_G \leq c^i_G \leq k^i_G = k_G. \tag{2.2}$$

Note that if G is a bounded domain, then k_G, c^i_G, c_G are non-degenerate and the topology induced by theses distances is the Euclidean topology.

For a bounded domain G in $\mathbb{C}^n$, denote $A^2(G)$ the holomorphic functions in $L^2(G)$. Let $\{\varphi_j : j \in \mathbb{N}\}$ be an orthonormal basis for $A^2(G)$ with respect to the L^2-inner product. The Bergman kernel K_G associated to G is given by

$$K_G(z, \overline{z}) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(z)}.$$

Note that K_G does not depend on the choice of orthonormal basis, gives rise to an invariant metric, the Bergman metric on G as follows:

$$g^B_G(\xi, \xi) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log K_G(z, \overline{z})}{\partial z_\alpha \partial \overline{z_\beta}} \xi_\alpha \overline{\xi_\beta}. \tag{2.3}$$

We say a domain $G \in \mathbb{C}^n$ is weakly pseudoconvex if G has a continuous plurisubharmonic exhaustion function. In particular, every geometric convex set is a weakly pseudoconvex domain.

The existence of the complete Kähler-Einstein metric on a bounded pseudoconvex domain was given in the main theorem in [17]. Based on this result, we can always find the unique complete Kähler-Einstein metric of the Ricci curvature -1 on a bounded weakly pseudoconvex domain G. i.e., g^{KE}_G satisfies $g^{KE}_G = -Ric_{g^{KE}_G}$ as a two tensor.

3. Equivalence of invariant metrics on ellipsoids

To show the equivalence of Kobayashi-Royden metric, the Kähler-Einstein metric and the Bergman metric on $E = E(m, n, p)$, W. Yin's complete invariant Kähler metric Y will be used. Precisely, Y is the complete invariant Kähler metric Y on $E = E(m, n, p)$ generated by the potential function

$$K((z, w), \overline{(z, w)}) = (1 - X)^{-\lambda}(1 - |z|^2)^{-N}$$

on $E = E(m, n, p)$, where $X = X(z, w) = |w|^2(1 - |z|^2)^{-1/p}$, $N_1 = (n + 1)p + m$, $N = N_1/p$. Here, we will take λ as $\lambda \geq \max\{m_1, m_2\}$, where

$$m_1 = \max_{0 \leq X < 1}\{\frac{F'(X)(1 - X)}{F(X)}\},$$

$$m_2 = \max_{0 \leq X < 1}\{\frac{[F(X)F''(X) - F(X)^2](1 - X)^2}{F(X)^2}\}.$$

Then Y satisfies

$$Y \geq g^B_E$$

(see [24] for details). From [24], it was shown that there exists $C > 0$ such that the holomorphic sectional curvature of Y on E is bounded above by $-C$. Then by the generalized Schwarz lemma, there exists $C' > 0$ satisfying

$$\sqrt{Y(v,v)} \le C'\chi_E(v)$$

for any vector $v \in TE$. Here, C' can be taken by $\sqrt{\frac{2}{C}}$ (see the Lemma 19 in [20] or [23]). Consequently, with Lempert's classical theorem on convex domains (see [16]),

$$\sqrt{g_E^B(v)} \le \sqrt{Y(v,v)} \le C'\chi_E(v) = C'\gamma_E(v) \le C'\sqrt{g_E^B(v,v)} \tag{3.1}$$

for any vector $v \in TE$. We have observed that Y is uniformly equivalent to the Bergman metric and the Kobayashi-Royden metric on E, and It remains to establish the equivalence between the Kähler-Einstein metric and Y. Now, once we have the following Proposition 7, Theorem 3 in [20] implies Theorem 1 for $S = E$.

Proposition 7. *There exists $D > 0$ such that the holomorphic sectional curvature of Y on E is bounded below by $-D$.*

Proof. Note that the holomorphic sectional curvature is invariant under the biholomorphic maps and for any $(z,w) \in E$, there exists an automorphism f on E such that $f(z,w) = (0,w')$. Thus it suffices to compute the holomorphic sectional curvature when $z = 0$. The formula of the holomorphic sectional curvature $\omega[(z,w),d(z,w)]_{z=0}$ of Y is explcitly given in [24]: for any $D > 0$,

$$\omega[(z,w),d(z,w)]_{z=0} = -D - \frac{\omega_1[(z,w),d(z,w)]}{[p^{-1}(XW' + N_1)|dz|^2 + W'|dw|^2 + W''|w\overline{dw}|^2]^2},$$

where $W' = \lambda(1-X)^{-1}$, $W'' = \lambda(1-X)^{-2}$,

$$\begin{aligned}\omega_1[(z,w),d(z,w)] =& P_1^*|w\overline{dw}|^4 + P_{12}^*|w\overline{dw}|^2|dw|^2 + P_2^*|dw|^4\\ &+ Q_1^*|dz|^2|dw|^2 + Q_2^*|wdw|^2|dz|^2 + R^*|dz|^4,\end{aligned}$$

and P_1^*, P_{12}^*, P_2^*, Q_1^*, Q_2^*, R^* are explicitly given as follows:

$$\begin{aligned}P_1^* =& aN_1(1-X)^{-4}(2 - DaN_1),\\ P_{12}^* =& 2aN_1(1-X)^{-3}(2 - DaN_1),\\ P_2^* =& aN_1(1-X)^{-2}(2 - DaN_1),\\ Q_1^* =& 2p^{-1}aN_1(1-X)^{-1}[(2 - DaN_1)(1-X)^{-1} - DN_1(1-a)],\\ Q_2^* =& 2p^{-1}(XW' + N_1)^{-1}(1-X)^{-2}aN_1^2[(1-X)^{-2}(2a - Da^2N_1)\\ &+(1-X)^{-1}[4(1-a) - 2DaN_1(1-a)] - D(1-a)^2N_1],\\ R^* =& p^{-2}N_1a[(1-X)^{-2}(2 - DaN_1) + (1-X)^{-1}2(p-1\\ &+DN_1(a-1)) + (1-1/a)[-2p - D(a-1)N_1]\end{aligned} \tag{3.2}$$

with $\lambda = aN_1$. We will claim that P_1^*, P_{12}^*, P_2^*, Q_1^*, Q_2^* and R^* are all non-positive for some $D > 0$. For simplicity, let $y := (1-X)^{-1}$.

For P_1^*, P_{12}^* and P_2^*, take $D > 2(aN_1)^{-1}$. Then $X \in [0,1)$ implies $y \in [1,\infty]$, thus P_1^*, P_{12}^* and P_2^* are non-positive.

For Q_1^*, let $f_{Q_1^*}(y) = (2 - DaN_1)y - DN_1(1-a)$. Since $D \geq 2(aN_1)^{-1}$, $f_{Q_1^*}(y)$ is decreasing which has the maximum $f_{Q_1^*}(1)$ on $[1,\infty]$. Then $f_{Q_1^*}(1) \leq 0$ is guaranted by taking $D \geq 2(N_1)^{-1}$.

For Q_2^*, let $f_{Q_2^*}(y) = (2a - Da^2N_1)y^2 + 2(1-a)(2-DaN_1)y - D(1-a^2)N_1$ from (3.2). If $D > 2(aN_1)^{-1}$, then $f_{Q_2^*}(y)$ has a maximum value $f_{Q_2^*}(y_0) = -2a^{-1}(a-1)^2 < 0$, where $y_0 = 1 - 1/a$.

Finally, for R^*, consider $f_{R^*}(y) = (2 - DaN_1)y^2 + 2(p - 1 + DN_1(a-1))y + (1 - 1/a)(-2p - D(a-1)N_1)$. Then $f'_{R^*}(y) = 2(2 - DaN_1)y + 2(p - 1 + DaN_1 - DN_1)$, thus $f'_{R^*}(y)$ is decreasing on $[1,\infty]$ if $D \geq 2(aN_1)^{-1}$. $f'_{R^*}(1) = 2(p + 1 - DN_1) \leq 0$ if $D \geq (p+1)/N_1$, and $f_{R^*}(1) = (1/a)(2p - DN_1) \leq 0$ if $D \geq (2p/N_1)$. Hence if $D \geq \max\{2(aN_1)^{-1}, (p+1)/N_1, (2p/N_1)\}$ then $f_{R^*}(y) \leq 0$.

In all, if $D \geq \max\{2(aN_1)^{-1}, 2(N_1)^{-1}, (p+1)/N_1, (2p/N_1)\}$, then P_1^*, P_{12}^*, P_2^*, Q_1^*, Q_2^* and R^* are all non-positive. Hence we obtain

$$\omega[(z,w), d(z,w)]_{z=0} \geq -D.$$

□

Remark 8. *From the proof of Theorem 1 with Theorem 3 in* [20], $C > 0$ *only depends on the negative holomorphic sectional curvature range of the W. Yin's complete invariant Kähler metric in* [24]*. In the special case of* $E = E(m,1,p)$ *with* $p \geq 1$*, we can also use the negative Riemannian sectional curvature range of the Kähler-Einstein metric on* E *to determine* $C > 0$ *(see Theorem 4 in* [4]*).*

Remark 9. *Theorem 1 can be proved by an alternative approach, which combines Theorem 1 in* [14] *and Theorem 7.2 in* [15] *or Theorem 2 in* [22]*. However, our approach has the further consequence which holds to any closed complex submanifold* S *in* $E = E(m,n,p)$*: Since we can restrict W. Yin's complete invariant Kähler metric* Y *to* S*, which still has the negative pinched holomorphic sectional curvature range on* S*. Then by Theorem 2 and Theorem 3 in* [20]*, we have Corollary 2.*

4. Bergman metric and its curvatures for two-dimensional ellipsoids

In this section, we will investigate the two-dimensional complex ellipsoid $E = E(1,1,p) = \{(z,w) \in \mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1\}, p > 0$. In order to prove Theorem 3, we will provide a detailed description of curvature tensors of the Bergman metric near to the special boundary points $|z| = 1$ on E. Recall that with the global coordinate $(z,w) \in E$ in $\mathbb{C}^2$, let $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\}$ be the basis on $T_0^{1,0}E$. In [8], the formula of the Bergman kernel K_E on $E = E(1,1,p) = \{(z,w) \in \mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1\}$ is explicitly known:

$$K_E((z,w), \overline{(z,w)}) = c_1 \frac{(1-|z|^2)^{-2+\frac{1}{p}}}{((1-|z|^2)^{1/p} - |w|^2)^2} + c_2 \frac{(1-|z|^2)^{-2+\frac{2}{p}}}{((1-|z|^2)^{1/p} - |w|^2)^3}, \tag{4.1}$$

where c_2, c_1 are given by

$$c_1 = \frac{1}{p\pi^2}(p-1), c_2 = \frac{2}{p\pi^2}.$$

For computations, define $\phi(z,\overline{z},w,\overline{w}) = 1 - z\overline{z}$ and $\psi(z,\overline{z},w,\overline{w}) = (1-z\overline{z})^{\frac{1}{p}} - w\overline{w}$. Then $K_E = c_1\phi^{-2+\frac{1}{p}}\psi^{-2} + c_2\phi^{-2+\frac{2}{p}}\psi^{-3}$. For convenience, let's establish the notation. We will denote $\frac{\partial}{\partial z}$ by ∂_1, and $\frac{\partial}{\partial \overline{z}} =: \partial_{\overline{1}}$, $\frac{\partial}{\partial \overline{w}} =: \partial_{\overline{2}}$, $\frac{\partial}{\partial \overline{w}} =: \partial_{\overline{2}}$. From (2.3), the metric component $g^B_{i\overline{j}}$ of g^B_E is given by

$$g^B_{i\overline{j}} = \frac{\partial^2 \log K_E(z,\overline{z})}{\partial z_i \partial \overline{z_j}} = K_E^{-2}(K_E\partial^2_{i\overline{j}}K_E - \partial_i K_E \partial_{\overline{j}} K_E), i = 1,2. \tag{4.2}$$

Hence with (4.1), the elementary computation gives the following proposition.

Proposition 10. *The components of g^B_E for any $(z,w) \in E$ are given as follows:*

$$\begin{aligned} g_{1\overline{1}} &= a_1a_2a_3, \\ g_{1\overline{2}} &= a_2a_4a_5, \\ g_{2\overline{1}} &= a_2a_4a_6, \\ g_{2\overline{2}} &= a_2a_4a_7, \end{aligned}$$

where each $a_{i\overline{j},k}$ is a function of $(z,\overline{z},w,\overline{w})$ as below:

$$\begin{aligned} a_1 =& 2p^4\psi^4 + 5p^3\psi^3(\phi^{\frac{1}{p}} + w\overline{w}) \\ &+ 2p^2\psi^2(w\overline{w}(z\overline{z}+5)\phi^{\frac{1}{p}} + 2\phi^{\frac{2}{p}} + 2(w\overline{w})^2) \\ &+ 4z\overline{z}w\overline{w}\phi^{\frac{1}{p}}(w\overline{w}\phi^{\frac{1}{p}} + \phi^{\frac{2}{p}} + (w\overline{w})^2) \\ &+ p(\phi^{\frac{2}{p}} - (w\overline{w})^2)(2w\overline{w}(3z\overline{z}+2)\phi^{\frac{1}{p}} + \phi^{\frac{2}{p}} + (w\overline{w})^2), \\ a_2 =& \psi^{-2}(p\psi + \phi^{\frac{1}{p}} + w\overline{w})^{-2}, \\ a_3 =& p^{-2}\phi^{-2}, \\ a_4 =& p^2\psi^2 + 3p(\phi^{\frac{2}{p}} - (w\overline{w})^2) + 2(w\overline{w}\phi^{\frac{1}{p}} + \phi^{\frac{2}{p}} + (w\overline{w})^2), \\ a_5 =& 2p^{-1}\overline{z}w\phi^{-1+\frac{1}{p}}, \\ a_6 =& 2p^{-1}\overline{w}z\phi^{-1+\frac{1}{p}}, \\ a_7 =& 2\phi^{\frac{1}{p}}. \end{aligned}$$

Proof. Notice that all of the first-order derivatives and second-order derivatives of K_E are written as linear combinations of $h\phi^\alpha\psi^\beta$ with some functions $h = h(z,\overline{z},w,\overline{w})$ and $\alpha, \beta \in \mathbb{R}$. We will provide each term of those in order to proceed the computation: the first-order derivatives of ϕ and ψ are given as follows:

$$\begin{aligned} &\partial_1\phi = -\overline{z}, \partial_{\overline{1}}\phi = -z, \partial_2\phi = \partial_{\overline{2}}\phi = 0, \\ &\partial_1\psi = -\frac{1}{p}\overline{z}\phi^{\frac{1}{p}-1}, \partial_{\overline{1}}\psi = -\frac{1}{p}z\phi^{\frac{1}{p}-1}, \partial_2\psi = -\overline{w}, \partial_{\overline{2}}\psi = -w. \end{aligned}$$

Then by direct computations, the first-order derivatives of K_E are written as:

$$\begin{aligned}
\partial_1 K_E =&\frac{(p-1)(2p-1)}{\pi^2p^2}\overline{z}\phi^{-3+\frac{1}{p}}\psi^{-2}+\frac{2(3p-5)}{\pi^2p^2}\overline{z}\phi^{-3+\frac{2}{p}}\psi^{-3}+\frac{6}{\pi^2p^2}\overline{z}\phi^{-3+\frac{3}{p}}\psi^{-4},\\
\partial_{\overline{1}} K_E =&\frac{(p-1)(2p-1)}{\pi^2p^2}z\phi^{-3+\frac{1}{p}}\psi^{-2}+\frac{2(3p-5)}{\pi^2p^2}z\phi^{-3+\frac{2}{p}}\psi^{-3}+\frac{6}{\pi^2p^2}z\phi^{-3+\frac{3}{p}}\psi^{-4},\\
\partial_2 K_E =&\frac{2(p-1)}{p\pi^2}\overline{w}\phi^{-2+\frac{1}{p}}\psi^{-3}+\frac{6}{p\pi^2}\overline{w}\phi^{-2+\frac{2}{p}}\psi^{-4},\\
\partial_{\overline{2}} K_E =&\frac{2(p-1)}{p\pi^2}w\phi^{-2+\frac{1}{p}}\psi^{-3}+\frac{6}{p\pi^2}w\phi^{-2+\frac{2}{p}}\psi^{-4}.
\end{aligned}$$

The second-order derivatives of K_E are written as:

$$\begin{aligned}
\partial^2_{1\overline{1}} K_E =&\frac{(p-1)(2p-1)}{\pi^2p^2}\phi^{-3+\frac{1}{p}}\psi^{-2}+\frac{6(p-1)}{\pi^2p^2}\phi^{-3+\frac{2}{p}}\psi^{-3}+\frac{6}{\pi^2p^2}\phi^{-3+\frac{3}{p}}\psi^{-4}\\
&+\frac{(p-1)(2p-1)(3p-1)}{\pi^2p^3}\phi^{-4+\frac{1}{p}}\psi^{-2}+\frac{2(p-1)(11p-7)}{\pi^2p^3}z\overline{z}\phi^{-4+\frac{2}{p}}\psi^{-3}\\
&+\frac{6(p-1)(3p+1)}{\pi^2p^3}z\overline{z}\phi^{-4+\frac{3}{p}}\psi^{-4}+\frac{24}{\pi^2p^3}z\overline{z}\phi^{-4+\frac{4}{p}}\psi^{-5},\\
\partial^2_{1\overline{2}} K_E =&\frac{4p^2-6p+2}{\pi^2p^2}w\overline{z}\phi^{-3+\frac{1}{p}}\psi^{-3}+\frac{18(p-1)}{\pi^2p^2}w\overline{z}\phi^{-3+\frac{2}{p}}\psi^{-4}+\frac{24}{\pi^2p^2}pw\overline{z}\phi^{-3+\frac{3}{p}}\psi^{-5},\\
\partial^2_{2\overline{1}} K_E =&\frac{4p^2-6p+2}{\pi^2p^2}z\overline{w}\phi^{-3+\frac{1}{p}}\psi^{-3}+\frac{18(p-1)}{\pi^2p^2}z\overline{w}\phi^{-3+\frac{2}{p}}\psi^{-4}+\frac{24}{\pi^2p^2}z\overline{w}\phi^{-3+\frac{3}{p}}\psi^{-5},\\
\partial^2_{2\overline{2}} K_E =&\frac{2(p-1)}{p\pi^2}\phi^{-2+\frac{1}{p}}\psi^{-3}+\frac{6(p-1)}{p\pi^2}w\overline{w}\phi^{-2+\frac{1}{p}}\psi^{-4}+\frac{6}{p\pi^2}\phi^{-2+\frac{2}{p}}\psi^{-4}+\frac{24}{p\pi^2}w\overline{w}\phi^{-2+\frac{2}{p}}\psi^{-5}.
\end{aligned}$$

Notice that $\phi^{-4+\frac{2}{p}}\psi^{-6}$ becomes the common factor to the numerator and the denominator of $\frac{K_E\partial^2_{i\overline{j}}K_E-\partial_iK_E\partial_{\overline{j}}K_E}{{K_E}^2}$ for indices $(i,\overline{j})=(1,\overline{1}),(1,\overline{2}),(2,\overline{1}),(2,\overline{2})$. Thus by multiplying ψ^2 on both sides after canceling the common factor $\phi^{-4+\frac{2}{p}}\psi^{-6}$, we get the common denominator-term a_2 for any $g_{i\overline{j}}$. Other two terms a_k, a_l of $g_{i\overline{j}}=a_2a_ka_l$ can be obtained from direct computation. For the rest of the proof, we proceed the concrete computation for $g_{1\overline{1}}$ and other metric components follow from similar computations. From (4.2),

$$\begin{aligned}
g_{1\overline{1}}=\frac{K_E\partial^2_{1\overline{1}}K_E-\partial_1K_E\partial_{\overline{1}}K_E}{{K_E}^2}&=\frac{\phi^{-4+\frac{2}{p}}\psi^{-6}(\widehat{K_E\partial^2_{1\overline{1}}}K_E-\partial_1\widehat{K_E\partial_{\overline{1}}}K_E)}{\phi^{-4+\frac{2}{p}}\psi^{-6}(\widehat{{K_E}^2})}\\
&=\frac{\psi^2(\widehat{K_E\partial^2_{1\overline{1}}}K_E-\partial_1\widehat{K_E\partial_{\overline{1}}}K_E)}{\psi^2(\widehat{{K_E}^2})},
\end{aligned}$$

where

$$\psi^2(\widehat{{K_E}^2})=\psi^2(p\psi+\phi^{\frac{1}{p}}+w\overline{w})^2=\frac{1}{a_2},$$

and

$$\psi^2(K_E\widehat{\partial^2_{1\overline{1}}K_E} - \partial_1\widehat{K_E\partial_{\overline{1}}}K_E) \tag{4.3}$$
$$= \frac{(p-1)^2(2p-1)(3p-1)-(2p^2-3p+1)^2z\overline{z}}{p^2}\phi^{-2}\psi^4 + \frac{(2p^2-p)(p-1)^2}{p^2}\phi^{-1}\psi^4$$
$$\frac{-2(p-1)(p^2-8p+3)z\overline{z}+2(p-1)(2p-1)(3p-1)}{p^2}\phi^{-2+\frac{1}{p}}\psi^3 + \frac{10p^3-18p^2+8p}{p^2}\phi^{-1+\frac{1}{p}}\psi^3$$
$$+ \frac{18(p^2-p)}{p^2}\phi^{-1+\frac{2}{p}}\psi^2 + \frac{18p^3-46p^2+90p+78}{p^2}z\overline{z}\phi^{-2+\frac{2}{p}}\psi^2$$
$$+ \frac{12(3p^2-6p+7)}{p^2}z\overline{z}\phi^{-2+\frac{3}{p}}\psi + \frac{12}{p}\phi^{-1+\frac{3}{p}}\psi + \frac{12}{p^2}z\overline{z}\phi^{-2+\frac{4}{p}}.$$

Take $a_3 = \frac{1}{p^2\phi^2}$, then one can see that with $z\overline{z} = 1-\phi$, the term of the highest degree of ψ of the rest terms of (4.3) becomes $2p^4\psi^4$ and further simplication with $\psi = \phi^{1/p} - w\overline{w}$ yields a_1. □

From this proposition, we can observe that two vector fields $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{w}}$ are orthogonal to each other with respect to the Bergman metric g^B_E if $a_5 = a_6 = 0$. Hence it is reasonable to evaluate the curvature tensors with the choice of points $(z,w) \in E$ satisfying

$$z = \overline{z} \text{ and } w = 0.$$

or

$$w = \overline{w} \text{ and } z = 0.$$

Since it's interesting to know those tensor components on C^2 weakly pseudoconvex boundary points $|z| = 1$ in the case $p > 1$, we proceed the computation with the former case.

First, let us compute the components of Ricci curvature tensor of the Bergman metric g^B_E. From the well-known formula of the Ricci curvature tensor with the Kähler metric g^B_E,

$$\begin{aligned}\mathrm{Ric}_{i\overline{j}}(g^B_E) &= -\partial_{i\overline{j}}\log\det g^B_E \\ &= (\det g^B_E)^{-2}(\partial_i \det g^B_E \partial_{\overline{j}} \det g^B_E - (\det g^B_E)\partial_{i\overline{j}}\det g^B_E).\end{aligned} \tag{4.4}$$

By previous propositions, many terms in the components of Ricci curvatures of g^B_E are vanished, so that we can compute to obtain the following propositions:

Proposition 11. *On the point $(z,w) \in E$ satisfying $z = \overline{z}, w = 0$, the components of the Ricci tensor of g^B_E are given as follows:*

$$\begin{aligned}&\mathrm{Ric}_{1\overline{1}} = -\frac{2p+1}{p\,(1-z\overline{z})^2},\\ &\mathrm{Ric}_{1\overline{2}} = \mathrm{Ric}_{2\overline{1}} = 0,\\ &\mathrm{Ric}_{2\overline{2}} = -\frac{2\left(2p^3+10p^2+10p+5\right)}{(p+1)(p+2)(2p+1)\,(1-z\overline{z})^{1/p}}.\end{aligned}$$

Proof. From (4.4), we should establish the formulas of the zero, first, and second-order derivatives of $\det g_E^B$. To do so, from the formula

$$\det g_E^B = g_{1\overline{1}}g_{2\overline{2}} - g_{1\overline{2}}g_{2\overline{1}} = a_1(a_2)^2a_3a_4a_7 - {a_2}^2{a_4}^2a_5a_6,$$

it is necessary to determine all formulas of zero, first, and second-order derivatives of $a_i, i = 1, ..., 7$ which are given in Proposition 10. For the zero-order derivatives of a_i's, by putting $z = \overline{z}, w = 0$ in formulas in Proposition 10, we have:

$$\begin{aligned}
&a_1 = p(p+1)^2(2p+1)\left(1-z\overline{z}\right)^{4/p},\\
&a_2 = \frac{1}{(p+1)^2\left(1-z\overline{z}\right)^{4/p}}, a_3 = \frac{1}{p^2\left(1-z\overline{z}\right)^2},\\
&a_4 = \left(p^2+3p+2\right)\left(1-z\overline{z}\right)^{2/p}, a_7 = 2\left(1-z\overline{z}\right)^{\frac{1}{p}},\\
&a_i = 0, i = 5, 6
\end{aligned}$$

Then by using $a_5 = a_6 = 0$,

$$\det g_E^B = a_1(a_2)^2a_3a_4a_7 = \frac{2(2p+1)\left(p^2+3p+2\right)\left(1-z\overline{z}\right)^{-1/p}}{p(p+1)^2\left(z\overline{z}-1\right)^2}.$$

Next, we compute the first-order derivatives of a_i's, then substituting $z = \overline{z}, w = 0$ yields the following:

$$\begin{aligned}
&\partial_1 a_1 = \partial_{\overline{1}} a_1 = -4(p+1)^2(2p+1)z\left(1-z\overline{z}\right)^{\frac{4}{p}-1},\\
&\partial_1 a_2 = \partial_{\overline{1}} a_2 = \frac{4z\left(1-z\overline{z}\right)^{-\frac{4}{p}-1}}{p(p+1)^2},\\
&\partial_1 a_3 = \partial_{\overline{1}} a_3 = \frac{2z}{p^2\left(1-z\overline{z}\right)^3},\\
&\partial_1 a_4 = \partial_{\overline{1}} a_4 = -\frac{2\left(p^2+3p+2\right)z\left(1-z\overline{z}\right)^{\frac{2}{p}-1}}{p},\\
&\partial_2 a_5 = \partial_{\overline{2}} a_6 = \frac{2z\left(1-z\overline{z}\right)^{\frac{1}{p}-1}}{p},\\
&\partial_1 a_7 = \partial_{\overline{1}} a_7 = -\frac{2z\left(1-z\overline{z}\right)^{\frac{1}{p}-1}}{p},\\
&\partial_2 a_1 = \partial_{\overline{2}} a_1 = \partial_2 a_2 = \partial_{\overline{2}} a_2 = \partial_2 a_3 = \partial_{\overline{2}} a_3 = \partial_2 a_4 = \partial_{\overline{2}} a_4\\
&\qquad = \partial_1 a_5 = \partial_{\overline{1}} a_5 = \partial_{\overline{2}} a_5 = \partial_1 a_6 = \partial_{\overline{1}} a_6 = \partial_2 a_6 = \partial_2 a_7 = \partial_{\overline{2}} a_7 = 0.
\end{aligned}$$

In particular, from those vanishing terms

$$\partial_i a_1 = \partial_i a_2 = \partial_i a_3 = \partial_i a_4 = \partial_i a_7 = 0, i = 2, \overline{2},$$

and $a_5 = a_6 = 0$, we have

$$\partial_2 \det g_E^B = \partial_{\overline{2}} \det g_E^B = 0.$$

On the other hand, from $\partial_1 a_i = \partial_{\overline{1}} a_i, i = 1,2,3,4$ with zero, and first-order derivatives of a_i's, computation yields

$$\partial_1 \det g_E^B = \partial_{\overline{1}} \det g_E^B = \frac{2(p+2)(2p+1)^2 z\,(1-z\overline{z})^{-\frac{1}{p}-3}}{p^2(p+1)}.$$

Now, we compute the second-order derivatives of a_i's, then substituting $z = \overline{z}, w = 0$ yields the following:

$$\begin{aligned}
\partial^2_{1\overline{1}} a_1 &= -\frac{4(p+1)^2(2p+1)\,(p-4z\overline{z})\,(1-z\overline{z})^{\frac{4}{p}-2}}{p},\\
\partial^2_{2\overline{2}} a_1 &= -2(p+1)\,(1-z\overline{z})^{3/p}\left(4p^3+p^2-p\,(z\overline{z}+2)-2z\overline{z}\right),\\
\partial^2_{1\overline{1}} a_2 &= \frac{4\,(1-z\overline{z})^{-\frac{2(p+2)}{p}}\,(p+4z\overline{z})}{p^2(p+1)^2},\\
\partial^2_{2\overline{2}} a_2 &= \frac{4p\,(1-z\overline{z})^{-5/p}}{(p+1)^3},\\
\partial^2_{1\overline{1}} a_3 &= \frac{4z\overline{z}+2}{p^2\,(z\overline{z}-1)^4},\\
\partial^2_{1\overline{1}} a_4 &= -\frac{2\left(p^2+3p+2\right)(p-2z\overline{z})\,(1-z\overline{z})^{\frac{2}{p}-2}}{p^2},\\
\partial^2_{2\overline{2}} a_4 &= -2\left(p^2-1\right)(1-z\overline{z})^{\frac{1}{p}},\\
\partial^2_{2\overline{1}} a_5 = \partial^2_{1\overline{2}} a_6 &= \frac{2\,(1-z\overline{z})^{\frac{1}{p}-2}\,(p-z\overline{z})}{p^2},\\
\partial^2_{1\overline{1}} a_7 &= \frac{2\,(1-z\overline{z})^{\frac{1}{p}-2}\,(z\overline{z}-p)}{p^2},\\
\partial^2_{1\overline{2}} a_1 &= \partial^2_{2\overline{1}} a_1 = \partial^2_{1\overline{2}} a_2 = \partial^2_{2\overline{1}} a_2 = \partial^2_{1\overline{2}} a_3 = \partial^2_{2\overline{1}} a_3 = \partial^2_{2\overline{2}} a_3 = \partial^2_{1\overline{2}} a_4 = \partial^2_{2\overline{1}} a_4\\
&= \partial^2_{1\overline{1}} a_5 = \partial^2_{1\overline{2}} a_5 = \partial^2_{2\overline{2}} a_5 = \partial^2_{1\overline{1}} a_6 = \partial^2_{2\overline{1}} a_6 = \partial^2_{2\overline{2}} a_6 = \partial^2_{1\overline{2}} a_7 = \partial^2_{2\overline{1}} a_7 = \partial^2_{2\overline{2}} a_7 = 0.
\end{aligned}$$

For two terms $\partial^2_{1\overline{2}} \det g_E^B$ and $\partial^2_{2\overline{1}} \det g_E^B$, one can check the following: expand $\partial^2_{1\overline{2}} \det g_E^B$ and $\partial^2_{2\overline{1}} \det g_E^B$ as linear combinations of a_i's and derivatives of those. Then each term in expressions contains at least one zero term. Consequently, we have

$$\partial^2_{1\overline{2}} \det g_E^B = \partial^2_{2\overline{1}} \det g_E^B = 0.$$

For the other two terms, direct computation with second-order derivatives of a_i's yields:

$$\begin{aligned}
\partial^2_{1\overline{1}} \det g_E^B &= \frac{2(p+2)(2p+1)^2\,(1-z\overline{z})^{-\frac{1}{p}-4}\,(2pz\overline{z}+p+z\overline{z})}{p^3(p+1)},\\
\partial^2_{2\overline{2}} \det g_E^B &= \frac{\left(8p^3+40p^2+40p+20\right)(1-z\overline{z})^{-\frac{2(p+1)}{p}}}{p(p+1)^2}.
\end{aligned}$$

By combining with (4.4), we have all desired formulas from direct computations. □

From the next Proposition, we have Theorem 3 as a Corollary. Moreover, not as in the smoothly bounded strictly pseudoconvex domain case, the Bergman metric even fails to be asymptotically Kähler-Einstein on E (also, see [6]).

Proposition 12. *On the point $(z,w)\in E$ satisfying $z=\overline{z}, w=0$, we have*

$$\frac{\mathrm{Ric}_{1\overline{1}}}{g_{1\overline{1}}}=-1,$$

$$\frac{\mathrm{Ric}_{2\overline{2}}}{g_{2\overline{2}}}=-\frac{2p^3+10p^2+10p+5}{(p+2)^2(2p+1)},$$

In particular, $\frac{\mathrm{Ric}_{1\overline{1}}}{g_{1\overline{1}}}=\frac{\mathrm{Ric}_{2\overline{2}}}{g_{1\overline{1}}}$ if and only if $p=1$.

Proof. From the formulas of a_i's which are obtained in the proof in Proposition 11 at $(z,\overline{z},0,0)\in E$ with $z=\overline{z}$, we have metric components

$$g_{1\overline{1}}=a_1a_2a_3=\frac{2p+1}{p\left(1-z\overline{z}\right)^2},$$

$$g_{2\overline{2}}=a_2a_4a_7=\frac{2(p+2)}{(p+1)\left(1-z\overline{z}\right)^{1/p}}.$$

Then by combining the formulas of Ricci curvature in Proposition 11, the result follows from the direct computation. □

The holomorphic sectional curvature h on l-dimensional complex hermitian manifold (M,g) in the holomorphic tangent vector $\xi=\sum_{i=1}^{l}\xi_i\frac{\partial}{\partial z_i}$ is given by

$$h(\xi)=\frac{2R_{\xi\overline{\xi}\xi\overline{\xi}}}{g(\xi,\overline{\xi})^2}=\frac{2\sum_{a,b,c,d=1}^{l}R_{a\overline{b}c\overline{d}}\xi_a\overline{\xi}_b\xi_c\overline{\xi}_d}{\sum_{a,b,c,d=1}^{l}g_{a\overline{b}}g_{c\overline{d}}\xi_a\overline{\xi}_b\xi_c\overline{\xi}_d},$$

where the components of curvature tensor R associated with g is given by

$$R_{a\overline{b}c\overline{d}}=-\frac{\partial^2 g_{a\overline{b}}}{\partial z_c\partial\overline{z}_d}+\sum_{p,q=1}^{l}g^{q\overline{p}}\frac{\partial g_{a\overline{p}}}{\partial z_c}\frac{\partial g_{q\overline{b}}}{\partial\overline{z}_d}. \tag{4.5}$$

Proposition 13. *On the point $(z,w)\in E$ satisfying $z=\overline{z}, w=0$, the components of the holomorphic sectional curvatures h of g_E^B are given as follows:*

$$h(\frac{\partial}{\partial z})=-\frac{2p}{1+2p},$$

$$h(\frac{\partial}{\partial w})=-\frac{1+4p+p^2}{(2+p)^2}.$$

In particular, $h(\frac{\partial}{\partial z})=h(\frac{\partial}{\partial w})$ if and only if $p=1$.

Proof. From the formulas of $a_i,\partial_j a_i$'s which are obtained in the proof in Proposition 11, $g_{1\overline{2}}=g_{2\overline{1}}=0$ because of $a_4=a_5=0$. Also, from $\partial_1 a_5=\partial_{\overline{1}}a_6=\partial_2 a_5=\partial_{\overline{2}}a_6=0$, we have

$$\partial_1 g_{1\overline{2}}=\partial_{\overline{1}}g_{2\overline{1}}=\partial_2 g_{1\overline{2}}=\partial_{\overline{2}}g_{2\overline{1}}=0.$$

Hence the components of the curvature tensor in (4.5) become

$$R_{1\bar{1}1\bar{1}} = -\partial^2_{1\bar{1}} g_{1\bar{1}} + g^{1\bar{1}} \partial_1 g_{1\bar{1}} \partial_{\bar{1}} g_{1\bar{1}},$$
$$R_{2\bar{2}2\bar{2}} = -\partial^2_{2\bar{2}} g_{2\bar{2}} + g^{2\bar{2}} \partial_2 g_{2\bar{2}} \partial_{\bar{2}} g_{2\bar{2}}.$$

From the formulas of a_i's which are obtained in the proof in Proposition 11 and Proposition 12,

$$g^{1\bar{1}} = \frac{g_{2\bar{2}}}{\det g_E^B} = \frac{p\,(1 - z\bar{z})^2}{2p+1},$$
$$g^{2\bar{2}} = \frac{g_{1\bar{1}}}{\det g_E^B} = \frac{(p+1)\,(1 - z\bar{z})^{\frac{1}{p}}}{2(p+2)}.$$

Combining with necessary formulas of $\partial_j a_i, \partial^2_{i\bar{j}} a_k$'s in the proof in Proposition 11, with the formulas of the inverse metrics given above, the result follows from the direct computation. □

Proposition 13 yields the following consequence:

Corollary 14. *the Bergman metric is not proportional to the Kobayashi-Royden metric on $E = E(1, 1, p)$ with $1 \neq p > 1/2$.*

Proof. Suppose the Bergman metric is proportional to the Kobayashi-Royden metric. i.e., $\chi_E = \lambda\sqrt{g_E^B}$ for some $\lambda > 0$. From the geometric-convexity of E, the Carathéodory-Reiffen metric is the same as the Kobayashi-Royden metric. Then the holomorphic sectional curvature of the Bergman metric must be the constant by applying Theorem 1 in [18], which is only possible when $p = 1$ from Proposition 13. □

5. The Carathéodory-Reiffen metric on geometric convex domains

In the following proposition, we consider bounded geometric convex domains Ω in $\mathbb{C}^n$ and distinguish the Carathéodory-Reiffen metric from Kähler-Einstein metric. Then Proposition 15 directly implies Theorem 4.

Proposition 15. *For any bounded geometric convex domain Ω in $\mathbb{C}^n$, let g_Ω^{KE} be the complete Kähler-Einstein metric of Ricci curvature -1. Then we have*

$$\gamma_\Omega(a; v) \leq \sqrt{g_\Omega^{KE}((a; v), (a; v))} \text{ for all nonzero tangent vectors } (a; v),$$

and

$$\gamma_\Omega(a; v) < \sqrt{g_\Omega^{KE}((a; v), (a; v))} \text{ for some nonzero tangent vector } (a; v).$$

Furthermore, if Ω is not biholomorphic to the unit disk in $\mathbb{C}^n$, then for any $\lambda > 0$,

$$\gamma_\Omega(a; v) \neq \lambda\sqrt{g_\Omega^{KE}((a; v), (a; v))} \text{ for some nonzero tangent vector } (a; v).$$

Proof. Notice that the first inequality is the consequence of the generalized Schwarz lemma (see [23]). To show the second inequality, suppose

$$\gamma_\Omega(a;v) = \sqrt{g_\Omega^{KE}((a;v),(a;v))} \text{ for all nonzero tangent vector } (a;v).$$

Then we have

$$c_\Omega^i = d_\Omega^{KE}.$$

Since Ω is geometric convex, the Carathéodory-Reifen metric is same as the Kobayashi-Royden metric. In particular, this forces Carathéodory pseudo-distance must be inner. Then from (2.2), $c_\Omega = c_\Omega^i = d_\Omega^{KE}$, which contradicts the main theorem in [7]. In the case that a bounded geometric convex domain Ω is not biholomorphic to the disk, if we assume further that $\gamma_\Omega(a;.) = \lambda\sqrt{g_\Omega^{KE}(.,.)}$, for some $\lambda > 0$, then Ω must be biholomorphic to the unit disk by Theorem 2 in [18], which is impossible. □

Acknowledgements. This work was partially supported by NSF grant DMS-1611745. I would like to thank my advisor Professor Damin Wu for many insights and deep encouragements. Also, I appreciate Prof. Yunhui Wu who pointed out the alternative proof of the Theorem 1 after submitting the paper on ArXiv. I appreciate to Hyun Chul Jang who is the Ph.D. candidate in Uconn for helpful discussions.

References

[1] B. Blank, D. Fan, D. Klein, S. Krantz, D. Ma, and M.-Y. Pang *The Kobayashi metric of a complex ellipsoid in C2*, Experimental Math. 1 4755, 1992.
[2] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^2$ with non-compact automorphism group*, Indiana Univ. Math. J. 47 149-222, 1999.
[3] J.S. Bland, *On the existence of bounded holomorphic functions on complete Kähler manifolds*, Invent. math. 81, 555-566 (1985).
[4] J.S. Bland, *The Einstein-Kähler Metric on* $\{|z|^2 + |w|^{2p} < 1\}$, Michigan math, J. 33, 1986.
[5] M. Beals, C. Fefferman, and R. Grossman. *Strictly pseudoconvex domains in $\mathbb{C}^n$*. Bull. Amer. Math. Soc. (N.S.), 8(2):125322, 1983.
[6] S.Y. Cheng and S.T. Yau *On the existence of a complete kähler metric on non-compact complex manifolds and the regularity of Feffermanss equation*, Comm, Pure Appl. Math., 33(4):507-544, 507-544, 1980.
[7] G. Cho *Comparison of the Carathéodory-distance, the Kähler-Einstein distance and the Kobayashi-distance on complete Reinhardt domains* arXiv:1806.06311.
[8] J.P. D'Angelo *An Explicit Computation of the Bergman Kernel Function* J.of Geometric Analysis, 4(1), 23-34, 1994.
[9] K. Diederich. *Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudokonvexen Gebieten* Math. Ann., 187:936, 1970.
[10] S. Fu, B. Wong *On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics* Mathematical Research Letters 4, 697703, 1997.
[11] I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary*, Trans. Amer. Math. Soc., 207:219 240, 1975.
[12] X. Huang, M. Xiao, *A remark on Bergman-Einstein metrics*, arXiv preprint arXiv:1604.07065.
[13] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*, volume 9 of de Gruyter Expositions in Mathematics. Walter de Gruyter GmbH and Co.KG, Berlin, extended edition, 2013.
[14] K. Kim AND L. Zhang, *On the Uniform Squeezing Property of Bounded Convex Domains in $\mathbb{C}^n$*. Pacific Journal of Mathematics Vol. 282, No. 2, 2016.

[15] K. Liu, X. Sun, and S.T. Yau *Canonical metrics on the moduli space of Riemann Surfaces I*, J. Differential Geom. Volume 68, Number 3, 571-637, 2004.
[16] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la doule*, Bull. Soc. Math. France, 109(4):427-474, 1981.
[17] N. Mok, S.T. Yau, *Completeness of the Kähler-Einstein Metric on Bounded Domains and the Characterization of Domains of Holomorphy by Curvature Conditions*, Proceedings of Symposia in Pure Mathematics, Vol **39**, Part 1, 41-59, 1983.
[18] B. Wong, *On the holomorphic curvature of some intrinsic metrics*, Proc. Amer. Math. Soc. 65, 57-61, 1977.
[19] B. Wong, *Characterization of the Unit Ball in $\mathbb{C}^n$ by its Automorphism Group*, Inventiones mathematicae 41: 253-258, 1977.
[20] D. Wu and S.T. Yau *Invariant metrics on negatively pinched complete Kähler manifolds*, appear in Journal of the American Mathematical Society.
[21] H. Wu, *Old and new invariant metrics on complex manifolds*, In Several complex variables (Stockholm, 1987/1988), volume 38 of Math. Notes, pages 640682. Princeton Univ. Press, Princeton, NJ, 1993.
[22] S.K. Yeung. *Geometry of domains with the uniform squeezing property*, Adv. Math., 221(2):547-569, 2009.
[23] S.T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math.,**100**(1):197-203, 1978.
[24] W. Yin, *The comparison theorem for the bergman and kobayashi metrics on certain pseudoconvex domains*, Complex Variables and Elliptic Equations, 34:4, 351-373, 1997.

Department of Mathematics, University of Connecticut, 196 Auditorium Road, Storrs, CT 06269-3009, USA

E-mail address: gunhee.cho@uconn.edu