A geometric criterion on the equality between BKK bound and intersection index

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Abstract

The Bernshtein-Kushnirenko-Khovanskii theorem provides a generic root count for system of Laurent polynomials in terms of the mixed volume of their Newton polytopes (i.e., the BKK bound). A recent and far-reaching generalization of this theorem is the study of birationally invariant intersection index by Kaveh and Khovanskii. This short note establishes a simple geometric condition on the equality between the BKK bound and the intersection index for a system of vector spaces of Laurent polynomials. Applying this, we show that the intersection index for the algebraic Kuramoto equations equals their BKK bound.

Keywords: BKK bound, Bernshtein-Kushnirenko-Khovanskii theorem, Newton polytope, mixed volume, Kuramoto equations

1. Introduction

The Bernshtein-Kushinirenko-Khovanskii theorem [2, 8, 9, 11, 12] relates the root counting problem for systems of polynomial equations and the theory of convex bodies. In particular, it states that the generic (and hence maximum) number of isolated solutions a system of Laurent polynomial equations has in the algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ equals the mixed volume of their Newton polytopes. This is the Bernshtein-Kushnirenko-Khovanskii (BKK) bound.

Recently, a far-reaching generalization of this theorem is developed in a series of works [5, 6, 7] by K. Kaveh and A. Khovanskii where the root counting question is considered for much more general spaces of rational functions. Given an irreducible n-dimensional complex algebraic variety X and n-tuple of finite dimensional vector spaces $(L_1, \ldots, L_n)$ of rational functions on X, for generic elements $f_i \in L_i$ for $i = 1, \ldots, n$, the number of common solutions a system $f_1 = \cdots = f_n = 0$ has in X is closely related to the geometry of the Newton-Okunkov bodies associated with $L_1, \ldots, L_n$. This generic root count is given the name *birationally invariant intersection index* (or simply, intersection index).

In this short note we establish a simple geometric criterion on the equality between the BKK bound and the more refined intersection index, even when the space of functions are not generated by monomials.

2. Preliminaries

For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{a} = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}^\top \in \mathbb{Z}^n$, we use the notation $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for the *Laurent monomial.* For a *Laurent polynomial* $f(\mathbf{x}) = \sum_{\mathbf{a} \in S} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, $\operatorname{Newt}(f) := \operatorname{conv}(S)$ is its *Newton polytope.* With respect to a vector $\mathbf{v} \in \mathbb{R}^n$, its *initial form* $\operatorname{init}_{\mathbf{v}}(f)$ is the expression $\sum_{\mathbf{a} \in (S)_{\mathbf{v}}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ where $(S)_{\mathbf{v}} \subset S$ is the subset on which the linear functional $\langle \mathbf{v}\,,\,\cdot \rangle$ is minimized. $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ denotes the set of all Laurent polynomials in $x_1, \ldots, x_n$.

The natural space to study this root counting question is the *algebraic torus* $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. While the root count of a Laurent polynomial system can vary greatly depending on the coefficients, for "generic" coefficients the $(\mathbb{C}^*)^n$-root count remains a constant and only depends on its monomial structure. D. Bernshtein showed this constant is precisely the mixed volume[1] of their Newton polytopes [2]. This is also an upper bound on the number of isolated $\mathbb{C}^*$-zeros such a Laurent polynomial system could have, and it is known as the *Bernshtein-Kushnirenko-Khovanskii* (BKK) bound, after a circle of closely related works by Bernshtein [2], Kushnirenko [9, 11, 12], and Khovanskii [8]. The arguably more important part of Bernshtein's paper [2] is his second theorem:

Theorem 1 (Bernshtein 1975 [2])**.** *Given a Laurent polynomial system* $\mathbf{f} = (f_1, \ldots, f_n)$ *in* $\mathbf{x} = (x_1, \ldots, x_n)$*, if for all* $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^n$*, the initial system* $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ *has no zero in* $(\mathbb{C}^*)^n$*, then all zeros of* $\mathbf{f}$ *in* $(\mathbb{C}^*)^n$ *are isolated, and the total number, counting multiplicity, is the mixed volume* $\operatorname{MVol}(\operatorname{Newt}(f_1), \ldots, \operatorname{Newt}(f_n))$.

Systems satisfying this condition are said to be *Bernshtein-general.* Bernshtein showed Bernshtein-generalness hold for generic choices of coefficients:

Lemma 2 (Bernshtein 1975 [2])**.** *Given a Laurent polynomial system* $\mathbf{f} = (f_1, \ldots, f_n)$ *in* $\mathbf{x} = (x_1, \ldots, x_n)$*, there is a nonempty Zariski-open set of the coefficients for which the initial system* $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ *has no solution in* $(\mathbb{C}^*)^n$ *for any nonzero vector* $\mathbf{v} \in \mathbb{R}^n$.

These results have been generalized into the theory of *birationally invariant intersection index* [6, 7]. Instead of considering generic linear combinations of Laurent monomials, one could start with $\mathbb{C}$-vector spaces $L_1, \ldots, L_n$ each spanned by finitely many rational functions on an irreducible toric variety X. Then for generic choices of functions $f_1 \in L_1, \ldots, f_n \in L_n$, the number of common zeros $(f_1, \ldots, f_n)$ has (away from base locus of the system) in X is a constant that is independent of the choices. This is the *intersection index* of $L_1, \ldots, L_n$ and is denoted by $[L_1, \ldots, L_n]$. This grand theory relates the root counting problem to the geometric properties of *Newton-Okunkov bodies*, and the BKK bound is thus a special case of this intersection index in the situations where each L_i is spanned by Laurent monomials. In the following, we extend the BKK bound to certain cases where each L_i is spanned by Laurent polynomials.

[1] Here, we follow the convention that the mixed volume $\operatorname{MVol}(P_1, \ldots, P_n)$ of n convex polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$ is the coefficient of the mixed term $\lambda_1 \cdots \lambda_n$ in the homogenous polynomial $\operatorname{Vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ [13], where Vol is the Euclidean volume form.

3. The main theorem

The goal here is to show the equality of the intersection index and the BKK bound under a simple geometric condition and thereby generalize the theory of BKK bound. We focus on the cases where $X = (\mathbb{C}^*)^n$ and $L_1, \ldots, L_n$ are spanned by finitely many Laurent polynomials. That is,

$$L_i = \operatorname{span}_{\mathbb{C}}\{P_{ij}\}_{j=1}^{m_i} \tag{1}$$

where each $P_{ij} \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ is nonzero and $m_i \in \mathbb{Z}^+$. This setup is a generalization of the situation in Theorem 1 where each L_i is only spanned by a set of Laurent monomials. That is, if each P_{ij} is a Laurent monomial, then $[\,L_1, \ldots, L_n\,]$ is exactly the BKK bound. The main result of this note is a geometric condition under which the BKK bound remains sharp even when each P_{ij} is a Laurent polynomial.

A generic element $f_i \in L_i$, is a Laurent polynomial $f_i = \sum_{j=1}^{m_i} c_{ij} P_{ij}$ with generic choice of the coefficients $c_{i1}, \ldots, c_{im_i}$. It is easy to see that among the terms within such a generic element, there are no cancellations and consequently $\operatorname{Newt}(f_i) = \operatorname{conv}\left(\cup_{j=1}^{m_i} \operatorname{Newt}(P_{ij})\right)$. It is therefore reasonable to use the notation

$$\operatorname{Newt}(L_i) := \operatorname{conv}\left(\cup_{j=1}^{m_i} \operatorname{Newt}(P_{ij})\right).$$

By Bernshtein's Theorem [2],

$$[\,L_1, \ldots, L_n\,] \leq \operatorname{MVol}(\operatorname{Newt}(L_1), \ldots, \operatorname{Newt}(L_n)).$$

The equality does not hold in general. The main result of this note is a sufficient condition on the equality between the two root counts stated in terms of the "exposure" of each Newton polytope $\operatorname{Newt}(P_{ij})$ on the boundary of $\operatorname{Newt}(L_i)$. This condition is, intentionally, chosen to only rely on the geometric information that can be obtained from the individual Newton polytopes $\operatorname{Newt}(L_1), \ldots, \operatorname{Newt}(L_n)$ and does not require information from any of their mixed subdivisions. In other words, the condition to be established here is purely polytopal, not tropical.

Theorem 3. *Let $L_1, \ldots, L_n$ be vector spaces of rational functions with $L_i = \operatorname{span}_{\mathbb{C}}\{P_{ij}\}_{j=1}^{m_i}$ where each $P_{ij} \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ and $m_i \in \mathbb{Z}^+$ as described above. If for each $i = 1, \ldots, n$, we have*

1. $\dim(\operatorname{Newt}(L_i)) = n$,
2. *functions in L_i have no common zeros in $(\mathbb{C}^*)^n$,*
3. *every positive-dimensional proper faces of $\operatorname{Newt}(L_i)$ intersect $\operatorname{Newt}(P_{ij})$ at no more than one point for each $j = 1, \ldots, m_i$,*

then

$$[\,L_1, \ldots, L_n\,] = \operatorname{MVol}(\,\operatorname{Newt}(L_1), \ldots, \operatorname{Newt}(L_n)\,). \tag{2}$$

Proof. Let $f_1, \dots, f_n$ be generic elements in $L_1, \dots, L_n$ respectively, i.e., $f_i = \sum_{j=1}^{m_i} c_{ij} P_{ij}$ for generic coefficients $\{c_{ij}\}$. Since it is also assumed that Laurent polynomials in each L_i have no common roots in $(\mathbb{C}^*)^n$, the common root count of the system $\mathbf{f} = (f_1, \dots, f_n)$ in $(\mathbb{C}^*)^n$ equals the intersection index $[\, L_1, \dots, L_n \,]$. It is therefore sufficient to show the root count of $\mathbf{f}$ in $(\mathbb{C}^*)^n$ matches the BKK bound, i.e., $\mathbf{f}$ satisfies the conditions in Theorem 1.

Let $\mathbf{v} \in \mathbb{R}^n$ be a nonzero vector such that $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ does not contain a unit (i.e., no component of $\mathbf{f}$ is a single Laurent monomial term). Since $\operatorname{Newt}(f_i) = \operatorname{Newt}(L_i)$ is assumed to be full-dimensional for $i = 1, \dots, n$, $\mathbf{v}$ must be a common inner normal vector for n proper positive dimensional faces $F_1, \dots, F_n$ of $\operatorname{Newt}(f_1), \dots, \operatorname{Newt}(f_n)$ respectively.

For each $i = 1, \dots, n$, let $A_{ij} = F_i \cap \operatorname{Newt}(P_{ij})$, then, by assumption, each A_{ij} contains at most one point. Without loss of generality, after re-indexing P_{ij}'s, we can assume that for a fixed i, $A_{ij} = \{\mathbf{a}_{ij}\}$ for $j = 1, \dots, m'_i$ and $A_{ij} = \varnothing$ for $j = m'_i + 1, \dots, m_i$ where $m'_i \in \mathbb{Z}^+$ and $m'_i \leq m_i$ (since $F_i \cap \operatorname{Newt}(P_{ij})$ may be empty for some j). With this definition, $\{\mathbf{a}_{i,1}, \dots, \mathbf{a}_{i,m'_i}\} = \bigcup_{j=1}^{m_i} A_{ij}$, and consequently,

$$\operatorname{init}_{\mathbf{v}}(f_i) \in \operatorname{span}_{\mathbb{C}}\{\mathbf{x}^{\mathbf{a}_{i1}}, \dots, \mathbf{x}^{\mathbf{a}_{im'_i}}\}.$$

Moreover, the set of coefficients is a subset of the coefficients in f_i. Indeed,

$$\operatorname{init}_{\mathbf{v}}(f_i) = \sum_{j=1}^{m'_i} c_{ij} \mathbf{x}^{\mathbf{a}_{i,j}}$$

with the exponent vectors $\mathbf{a}_{i1}, \dots, \mathbf{a}_{im'_i}$ all lie in a proper face of $\operatorname{Newt}(L_i)$ and the coefficients c_{ij}'s being independent from one another. By lemma 2, there exists a nonempty Zariski open set in the coefficient space $\{c_{ij}\}$ for which the initial system $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ has no solution in $(\mathbb{C}^*)^n$.

Note that there are only finitely many distinct initial systems for $\mathbf{f}$. By taking the intersection of a finite number of nonempty Zariski open set, we can see that there remains a nonempty Zariski open set in the coefficient space $\{c_{ij}\}$ such that for all choices in this set, $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ either contains a unit or has no solution in $(\mathbb{C}^*)^n$ for any nonzero vector $\mathbf{v} \in \mathbb{R}^n$.

By Theorem 1, for generic choices of the coefficients $\{c_{ij}\}$, the BKK bound for $\mathbf{f}$ is exact, i.e., the common root count in $(\mathbb{C}^*)^n$ for this system is exactly $\operatorname{MVol}(\operatorname{Newt}(f_1), \dots, \operatorname{Newt}(f_n))$. Recall that each f_i is a generic member of L_i. This shows

$$[\, L_1, \dots, L_n \,] = \operatorname{MVol}(\, \operatorname{Newt}(L_1), \dots, \operatorname{Newt}(L_n) \,). \qquad \square$$

Remark 4. *Condition 2 of Theorem 3 is needed because the root count of a system in* $(L_1, \dots, L_n)$ *may include points at which all functions in* L_i *vanish for some* i*, while these points are excluded from the intersection index* $[\, L_1, \dots, L_n]$.

Neither condition 1 nor 3 are necessary. In this note, we focus on the problem of detecting the equality between the BKK bound and the intersection index using geometric information of the individual Newton polytopes in $\operatorname{Newt}(L_1), \dots, \operatorname{Newt}(L_n)$*, rather than the more refined tropical information. In Remark 6, we highlight some more general results.*

4. Generic root count of Kuramoto equations

The Kuramoto model [10] is a ubiquitous model for studying the phenomenon of spontaneous synchronization of networks of coupled oscillators. It has long been known that when simple harmonic oscillators are coupled with one another, complicated collective behaviors emerge. Biological examples include pacemaker cells in the heart and the formation of circadian rhythm in the brain. Such models have since found important applications in many other seemingly independent research fields.

An oscillator can be modeled as a moving point on the complex plane circling 0. A swarm of such points interacting with one another thus form a network of coupled oscillators. For weakly coupled and nearly identical oscillators, Winfree intuited that there is natural separation of timescales: On the short timescale, oscillators are approximated by their limit cycles and thus can be represented by their phases [14]. This is derived rigorously by Kuramoto [10] using perturbation methods. Kuramoto singled out the simplest case with the governing equations

$$\dot{\theta}_i = \omega_i - \sum_{j=0}^{n} k_{ij} \sin(\theta_i - \theta_j) \quad \text{for } i = 0, \ldots, n, \tag{3}$$

in which $\theta_0, \ldots, \theta_n \in [0, 2\pi)$ are the phases of the oscillators, ω_i's are their natural frequencies (i.e., their limit cycle frequencies), where $k_{ij} = k_{ij}$ are constant coupling coefficients. This model has since been called the Kuramoto model.

In the study of this model, one important problem is the classification of "frequency synchronization configurations", which correspond to the equilibria of (3). Although the equilibrium equations are not algebraic, through a change of variables and a relaxation to include complex configurations, we can consider the algebraic synchronization equation [1], given by

$$0 = f_i(\mathbf{x}) = \omega_i - \sum_{j=0}^{n} a_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \quad \text{for } i = 1, \ldots, n \tag{4}$$

where $a_{ij} = \frac{k_{ij}}{2}$ are complex constants, $x_0 = 1$, and $x_i = e^{\mathrm{i}\theta_i}$. The complex zero set of $(f_1, \ldots, f_n)$ captures the equilibria of (3) which correspond to the synchronization configurations. The problem of counting such configurations thus relaxes to a root counting problem. Since each f_i in (4) is a linear combination of 1 and Laurent polynomials $x_i x_j^{-1} - x_j x_i^{-1}$, it is natural to view the generic root count for this system as the intersection index $[\,L_1, \ldots, L_n\,]$ where

$$L_i = \operatorname{span}_{\mathbb{C}}\{1\} \cup \{x_i x_j^{-1} - x_j x_i^{-1}\}_{j=0}^{n} \quad \text{for } i = 1, \ldots, n. \tag{5}$$

Through a "modified Bézout technique", Baillieul and Byrnes first computed this intersection index ([1, Theorem 4.1]). Their proof employed some rather deep results in modern algebraic geometry. In the following, we demonstrate the potential usefulness of Theorem 3 by providing a simple alternative proof.

Proposition 5. *Let L_i be the vector spaces of Laurent polynomials as defined in* (5)*. Then $[\,L_1, \ldots, L_n\,]$ equal to the BKK bound of* (4)*.*

Proof. For each $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, n\}$ we define

$$P_{ij} = \frac{x_i}{x_j} - \frac{x_j}{x_i}.$$

Then each P_{ij} is a Laurent polynomial in the variables $\mathbf{x} = (x_1, \ldots, x_n)$, and $\text{Newt}(P_{ij}) = \text{conv}\{\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_j - \mathbf{e}_i\}$ where $\mathbf{e}_0 = \mathbf{0}$. So for each pair (i, j), $\text{Newt}(P_{ij})$ is a line segment through the origin.

For each $i = 1, \ldots, n$, we consider the vector space of rational functions

$$L_i = \text{span}_{\mathbb{C}} \{1\} \cup \{P_{ij}\}_{j=0}^{n}.$$

Then functions in L_i have no common zeros in $(\mathbb{C}^*)^n$ It is easy to verify that $f_i \in L_i$ for each $i = 1, \ldots, n$. Therefore, the statement to be proved is equivalent to the claim that $[L_1, \ldots, L_n]$ equals the BKK bound of the system $(f_1, \ldots, f_n)$.

By definition,

$$\text{Newt}(L_i) = \text{conv}(\mathbf{0} \ \cup \ \{\mathbf{e}_i - \mathbf{e}_j \, , \, \mathbf{e}_j - \mathbf{e}_i\}_{j=0}^{n}),$$

and, in it, $\mathbf{0}$ is an interior point. For $n > 1$, $\text{Newt}(L_i)$ is the convex hull of n affinely independent line segments through the origin, and thus $\dim(\text{Newt}(L_i)) = n$ for every i. Moreover, fixing i, for each $j = 0, \ldots, n$ and $j \neq i$, $\text{Newt}(P_{ij})$ is a line segment passing though an interior point, the origin, of $\text{Newt}(L_i)$. Therefore, for each proper positive dimensional face F of $\text{Newt}(L_i)$, $F \cap \text{Newt}(P_{ij})$ is either empty or a single point. By theorem 3, the generic root count in $(\mathbb{C}^*)^n$, i.e., $[L_1, \ldots, L_n]$ is exactly the BKK bound. □

Remark 6. *Though Kuramoto originally only considered complete networks, in which every oscillator is influenced by every other oscillator, recent research activities have shifted toward sparse networks. Sparsity corresponds to the requirement that certain coefficients in* (4) *are zero. The generalizations of this result to sparse networks are developed in Refs. [3, 4] using other tools.*

5. Conclusions

In this short note, we established a sufficient geometric condition under which the intersection index of a system of vector spaces of Laurent polynomials equals its BKK bound. This condition is stated purely in terms of the geometric information in the Newton polytopes of the polynomials involved and can be checked easily using simple algorithms from convex geometry without the information from corresponding tropical intersection. It shows that certain algebraic relations among the coefficients have no effect on the exactness of the BKK bound. The usefulness of this result is demonstrated through an application to the algebraic Kuramoto equations — a well studied family of equations used to model spontaneous synchronization phenomenon in many fields. With this theorem, we easily established the equality between the BKK bound and the more refined intersection index of the algebraic Kuramoto equations, even though this system has inherent algebraic relations among the coefficients.

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References

[1] J. Baillieul and C. Byrnes. Geometric critical point analysis of lossless power system models. *IEEE Transactions on Circuits and Systems*, 29(11):724–737, nov 1982.

[2] D. N. Bernshtein. The number of roots of a system of equations. *Functional Analysis and its Applications*, 9(3):183–185, 1975.

[3] T. Chen, R. Davis, and D. Mehta. Counting Equilibria of the Kuramoto Model Using Birationally Invariant Intersection Index. *SIAM Journal on Applied Algebra and Geometry*, 2(4):489–507, jan 2018.

[4] T. Chen, E. Korchevskaia, and J. Lindberg. On the typical and atypical solutions to the Kuramoto equations. *arXiv*, 2022.

[5] K. Kaveh and A. Khovanskii. Convex bodies and algebraic equations on affine varieties. *arXiv preprint arXiv:0804.4095*, page 44, apr 2008.

[6] K. Kaveh and A. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Annals of Mathematics*, 176(2):925–978, sep 2012.

[7] K. Kaveh and A. G. Khovanskii. Mixed volume and an extension of theory of divisors. *Moscow Mathematical Journal*, 10(2):343–375, 2010.

[8] A. G. Khovanskii. Newton polyhedra and the genus of complete intersections. *Functional Analysis and Its Applications*, 12(1):38–46, 1978.

[9] A. G. Kouchnirenko. Polyèdres de Newton et nombres de Milnor. *Inventiones Mathematicae*, 32(1):1–31, 1976.

[10] Y. Kuramoto. Self-entrainment of a population of coupled non-linear oscillators. Lecture Notes in Physics, pages 420–422. Springer Berlin Heidelberg, 1975.

[11] A. G. Kushnirenko. A Newton polyhedron and the number of solutions of a system of k equations in k unknowns. *Usp. Math. Nauk*, 30:266–267, 1975.

[12] A. G. Kushnirenko. Newton polytopes and the Bezout theorem. *Functional Analysis and Its Applications*, 10(3):233–235, jul 1976.

[13] H. Minkowski. Theorie der konvexen Korper, insbesondere Begrundung ihres Oberflachenbegriffs. *Gesammelte Abhandlungen von Hermann Minkowski*, 2:131–229, 1911.

[14] A. T. Winfree. Biological rhythms and the behavior of populations of coupled oscillators. *Journal of Theoretical Biology*, 16(1):15–42, 1967.