

DERIVED CATEGORY OF PROJECTIVIZATION AND GENERALIZED LINEAR DUALITY

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ABSTRACT. In this note, we generalize the linear duality between vector subbundles (or equivalently quotient bundles) of dual vector bundles to coherent quotients $V \rightarrow \mathcal{L}$ considered in [JL18c], in the framework of Kuznetsov's homological projective duality (HPD). As an application, we obtain a generalized version of the fundamental theorem of HPD for the $\mathbb{P}(\mathcal{L})$ -sections and the respective dual sections of a given HPD pair.

1. INTRODUCTION

Let S be a fixed scheme, which for simplicity we assume to be smooth over an algebraically closed field \mathbb{k} of characteristic zero, and V be a vector bundle of rank $N \geq 2$ over S . Denote $V^\vee := \mathcal{H}om(V, \mathcal{O}_S)$ the dual vector bundle. For a short exact sequence of vector bundles

$$0 \rightarrow K \rightarrow V \rightarrow L \rightarrow 0,$$

over S , there is a dual short exact sequence of vector bundles

$$(1.1) \quad 0 \rightarrow L^\vee \rightarrow V^\vee \rightarrow K^\vee \rightarrow 0.$$

The *linear duality* refers to the duality of subbundles $\{K \subset V\} \leftrightarrow \{L^\vee \subset V^\vee\}$, or equivalently the quotient bundles $\{V \twoheadrightarrow L\} \leftrightarrow \{V^\vee \twoheadrightarrow K^\vee\}$. If we use Grothendieck convention $\mathbb{P}(\mathcal{E}) := \text{Proj}_S \text{Sym}_{\mathcal{O}_S}^\bullet \mathcal{E}$ for a coherent sheaf \mathcal{E} on S , then linear duality equivalently refers to the reflexive relationship between all projective linear subbundles of $\mathbb{P}(V)$ and $\mathbb{P}(V^\vee)$:

$$\{\mathbb{P}(L) \subset \mathbb{P}(V)\} \longleftrightarrow \{\mathbb{P}(L)^\perp := \mathbb{P}(K^\vee) \subset \mathbb{P}(V^\vee)\}.$$

Question. *What should be the dual of coherent quotient sheaves $\{V \twoheadrightarrow \mathcal{L}\}$, or equivalently the subschemes $\{\mathbb{P}(\mathcal{L}) \subset \mathbb{P}(V)\}$, where $\mathcal{L} := \text{coker}(K \rightarrow V)$ is not necessarily locally free?*

In this case we still have a short exact sequence of \mathcal{O}_S -modules $0 \rightarrow K \rightarrow V \rightarrow \mathcal{L} \rightarrow 0$; however the sequence (1.1) is now replaced by a four-term exact sequence:

$$(1.2) \quad 0 \rightarrow \mathcal{L}^\vee \rightarrow V^\vee \rightarrow K^\vee \rightarrow \mathcal{E}xt_S^1(\mathcal{L}, \mathcal{O}_S) \rightarrow 0,$$

where $\mathcal{L}^\vee := \mathcal{H}om_S(\mathcal{L}, \mathcal{O}_S)$, and $\mathcal{E}xt_S^1(\mathcal{L}, \mathcal{O}_S)$ is supported on the singular locus $\text{Sing}(\mathcal{L}) := \{s \in S \mid \text{rank } \mathcal{L}(s) > \ell\} \subset S$ of \mathcal{L} , where ℓ is the generic rank of \mathcal{L} .

In this note, we answer the above question in the framework homological projective duality:

Theorem 1.1 (See Thm. 3.2). *The homological projective dual (HPD) of $\mathbb{P}(\mathcal{L}) \subset \mathbb{P}(V)$ is given by $\tilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$, the blowing up of $\mathbb{P}(K^\vee)$ along the $\mathbb{P}(\mathcal{E}xt_S^1(\mathcal{L}, \mathcal{O}_S)) \subset \mathbb{P}(K^\vee)$.*

The homological projective dual (HPD) of a Lefschetz variety $X \rightarrow \mathbb{P}(V)$, introduced by Kuznetsov [K07], denoted by $Y = X^\natural \rightarrow \mathbb{P}(V^\vee)$, is a homological modification of the classical projective dual variety $X^\vee \subset \mathbb{P}(V^\vee)$ of $X \rightarrow \mathbb{P}(V)$, see §2.5 for precise definitions.

The HPD relation is *reflexive*: $(X^\natural)^\natural \simeq X$, see [K07, JLX17]; And HPD *extends* the previously discussed linear duality $\{\mathbb{P}(L) \subset \mathbb{P}(V)\} \leftrightarrow \{\mathbb{P}(L)^\perp := \mathbb{P}(K^\vee) \subset \mathbb{P}(V^\vee)\}$ between projective subbundles: $\mathbb{P}(L)^\natural = \mathbb{P}(L)^\perp$, see [K07, Cor. 8.3], [JLX17, Cor. 5.16], but notice that our theorem (in the case when \mathcal{L} is locally free) also provides a different proof of this fact. Therefore, thanks to above theorem, it makes sense to denote:

$$\mathbb{P}(\mathcal{L})^\perp := \widetilde{\mathbb{P}}(K^\vee) = \mathbb{P}(\mathcal{L})^\natural \rightarrow \mathbb{P}(V^\vee)$$

and regard it as the *dual* of $\mathbb{P}(\mathcal{L}) \subset \mathbb{P}(V)$. The relation $\mathbb{P}(\mathcal{L}) \leftrightarrow \mathbb{P}(\mathcal{L})^\perp$ hence generalizes the usual linear duality.

An immediate consequence of our theorem is the following generalization of the fundamental theorem of HPD from linear sections to the above generalized linear system $V \rightarrow \mathcal{L}$.

Theorem 1.2 (Fundamental theorem of HPD for $V \rightarrow \mathcal{L}$). *Let \mathcal{A} be a $\mathbb{P}(V^\vee)$ -linear Lefschetz category of length m with Lefschetz components \mathcal{A}_i 's, and \mathcal{A}^\natural be its HPD category, which is a $\mathbb{P}(V)$ -linear Lefschetz category of length n with Lefschetz components \mathcal{A}_j^\natural 's. Then for $1 \leq \ell \leq N$, there are semiorthogonal decompositions*

$$\begin{aligned} \mathcal{A}_{\mathbb{P}(\mathcal{L})^\perp} &= \langle \text{prim}(\mathcal{A}_{\mathbb{P}(\mathcal{L})^\perp}), \mathcal{A}_1^\epsilon(H), \dots, \mathcal{A}_{\ell-1}^\epsilon((\ell-1)H), \\ &\quad \langle \mathcal{A}_\ell, \mathcal{A}_\ell^\epsilon(\ell H), \dots, \langle \mathcal{A}_{m-1}, \mathcal{A}_{m-1}^\epsilon \rangle((m-1)H) \rangle, \\ \mathcal{A}_{\mathbb{P}(\mathcal{L})}^\natural &= \langle \mathcal{A}_{1-n}^\natural((\ell-n)H'), \dots, \mathcal{A}_{-\ell}^\natural(-H'), (\mathcal{A}_{\mathbb{P}(\mathcal{L})}^\natural)^{\text{prim}} \rangle. \end{aligned}$$

Furthermore, there is an equivalence of categories of the primitive components:

$$\text{prim}(\mathcal{A}_{\mathbb{P}(\mathcal{L})^\perp}) \simeq (\mathcal{A}_{\mathbb{P}(\mathcal{L})}^\natural)^{\text{prim}}.$$

If \mathcal{L} is locally free, then the “correction terms” $\mathcal{A}_i^\epsilon = \emptyset$, and the theorem reduces to the usual fundamental theorem of HPD (see [K07, JLX17, R17, P18]).

If \mathcal{L} is not locally free, then there are nontrivial “correction terms”:

$$\mathcal{A}_i^\epsilon := (\mathcal{A}_i)|_{\mathbb{P}(\mathcal{E}xt^1(\mathcal{L}, \mathcal{O}))} = \mathcal{A}_i \boxtimes_S D(\mathbb{P}(\mathcal{E}xt^1(\mathcal{L}, \mathcal{O}))), \quad \text{for } i = 1, \dots, m-1,$$

supported on $\text{Sing}(\mathcal{L}) \subset S$. Our theorem shows that, after taking these corrections into consideration, the fundamental theorem of HPD still holds.

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2. PRELIMINARIES

2.1. Conventions. Let S be a fixed scheme, for simplicity we assume to be smooth over an algebraically closed field \mathbb{k} of characteristic zero. All schemes considered in this paper will be S -schemes. Let V be a fixed vector bundle of rank $N \geq 2$ over S , and V^\vee be the dual vector bundle. We use Grothendieck convention $\mathbb{P}(\mathcal{E}) := \text{Proj}_S \text{Sym}_{\mathcal{O}_S}^\bullet \mathcal{E}$ for a coherent sheaf \mathcal{E} on S . We use $D(X) := D_{\text{coh}}^b(X)$ to denote the bounded derived categories of coherent sheaves on a scheme X .

Let X, Y be S -schemes, and $f : X \rightarrow Y$ a proper S -morphism, then (whenever well defined) denote $\mathbb{R}f_*$ and $\mathbb{L}f^*$ the right and respectively left derived functors of usual push-forward $f_* : \text{coh } X \rightarrow \text{coh } Y$ and pullback $f^* : \text{coh } Y \rightarrow \text{coh } X$. Denote by \otimes , $\mathcal{H}om(-, -)$ the tensor and sheaf (internal) Hom on $\text{coh } X$, and $\otimes^{\mathbb{L}}$ and $\mathbb{R}\mathcal{H}om(-, -)$ the derived functors. A *Fourier-Mukai functor* is an exact functor between $D(X)$ and $D(Y)$ of the form

$$\Phi_{\mathcal{P}}^{X \rightarrow Y}(-) = \Phi_{\mathcal{P}}(-) := \mathbb{R}\pi_{Y*}(\mathbb{L}\pi_X^*(-) \otimes^{\mathbb{L}} \mathcal{P}) : D(X) \rightarrow D(Y),$$

where $\mathcal{P} \in D(X \times Y)$ is called the *Fourier-Mukai kernel*, and $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are natural projections.

2.2. Generalities. The readers are referred to [Huy, Căl, K14] for basic notations and properties of derived categories of coherent sheaves, and semiorthogonal decompositions.

A full triangulated subcategory \mathcal{A} of a triangulated category \mathcal{T} is called *admissible* if the inclusion functor $i = i_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ has both a right adjoint $i^! : \mathcal{T} \rightarrow \mathcal{A}$ and a left adjoint $i^* : \mathcal{T} \rightarrow \mathcal{A}$. If $\mathcal{A} \subset \mathcal{T}$ is admissible, then $\mathcal{A}^\perp = \{T \in \mathcal{T} \mid \text{Hom}(\mathcal{A}, T) = 0\}$ and ${}^\perp\mathcal{A} = \{T \in \mathcal{T} \mid \text{Hom}(T, \mathcal{A}) = 0\}$ are both admissible, and $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$.

A *semiorthogonal decompositions* (SOD) for a triangulated category \mathcal{T} , written as

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle.$$

a sequence of admissible full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$, such that (i) $\text{Hom}(a_j, a_i) = 0$ for all $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$, $j > i$, and, (ii) they generate the whole $D(X)$. Starting with a semiorthogonal decomposition of \mathcal{T} , one can obtain a whole collection of new decompositions by functors called *mutations*. The functor $\mathbf{L}_{\mathcal{A}} := i_{\mathcal{A}^\perp} i_{\mathcal{A}}^*$ (resp. $\mathbf{R}_{\mathcal{A}} := i_{{}^\perp\mathcal{A}} i_{\mathcal{A}}^!$) is called the *left* (resp. *right*) *mutation through* \mathcal{A} . For any $b \in \mathcal{T}$, by there are exact triangles

$$i_{\mathcal{A}} i_{\mathcal{A}}^!(b) \rightarrow b \rightarrow \mathbf{L}_{\mathcal{A}} b \xrightarrow{[1]}, \quad \mathbf{R}_{\mathcal{A}} b \rightarrow b \rightarrow i_{\mathcal{A}} i_{\mathcal{A}}^*(b) \xrightarrow{[1]}.$$

$(\mathbf{L}_{\mathcal{A}})|_{\mathcal{A}} = 0$ and $(\mathbf{R}_{\mathcal{A}})|_{\mathcal{A}} = 0$ are the zero functors; $(\mathbf{L}_{\mathcal{A}})|_{{}^\perp\mathcal{A}} : {}^\perp\mathcal{A} \rightarrow \mathcal{A}^\perp$ and $(\mathbf{R}_{\mathcal{A}})|_{\mathcal{A}^\perp} : \mathcal{A}^\perp \rightarrow {}^\perp\mathcal{A}$ are mutually inverse equivalences of categories. Starting with a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_k, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle$ of admissible subcategories, one can

obtain other sods through mutations, for $k \in [1, n]$:

$$\begin{aligned} \mathcal{T} &= \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-2}, \mathbf{L}_{\mathcal{A}_{k-1}}(\mathcal{A}_k), \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle \\ &= \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \mathbf{R}_{\mathcal{A}_{k+1}}(\mathcal{A}_k), \mathcal{A}_{k+2}, \dots, \mathcal{A}_n \rangle \end{aligned}$$

We refer the reader to [BK, K07] for more about mutations.

For a S -scheme $a: X \rightarrow S$ be a, $D(X)$ is naturally equipped with S -linear structure, given by $A \otimes a^*F$, for any $F \in D(S)$ and $A \in D(X)$. An admissible subcategory $\mathcal{A} \subset D(X)$ is called S -linear if $A \otimes a^*F \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $F \in D(S)$. Such an admissible subcategory \mathcal{A} will be referred as an S -linear category. An S -linear functor between S -linear categories is an exact functor functorially preserving S -linear structures. An S -linear SOD $D(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ for a S -scheme X is a SOD such that all \mathcal{A}_i 's are S -linear subcategories. See [K11] for more about linear categories. Many geometric operations (projective bundles, blowing up, etc) can be performed on linear categories, see [JL18a]. See also [P18] for discussions in the Lurie's framework of stable ∞ -categories.

2.3. Generalized universal hyperplane section and Orlov's results. The references are [T15, O05], see also [JL18c, §2.3], and [JL18a, §3.4] for noncommutative cases.

Let \mathcal{E} to be a locally free sheaf of rank r on a regular scheme X , and $s \in H^0(X, \mathcal{E})$ be a regular section. Denote $Z := Z(s)$ the zero locus of the section s . Then through $H^0(X, \mathcal{E}) = H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$, the section s corresponds to a section f_s of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$. The zero loci $\mathcal{H}_s := Z(f_s) \subset \mathbb{P}(\mathcal{E})$ is called the *generalized universal hyperplane*, which comes with projection $\pi: \mathcal{H}_s \rightarrow X$. The general fiber of this projection is a projective space \mathbb{P}^{r-2} , and the fiber dimensions of π jumps exactly over Z . If we denote $i: Z \hookrightarrow X$ the inclusion, then its normal sheaf is $\mathcal{N}_i \simeq \mathcal{E}|_Z$, and it is direct to see $\pi^{-1}(Z) = \mathbb{P}(\mathcal{N}_i)$.

The above situation is called *Cayley's trick*. The situation is categorified by Orlov to obtain relationships between $D(Z)$ and $D(\mathcal{H}_s)$ (see also [JL18c, JL18a]).

Theorem 2.1 (Orlov, [O05, Prop. 2.10]). *In the above situation, then the functors $\mathbb{R}j_*p^*: D(Z) \rightarrow D(\mathcal{H}_s)$ and $\mathbb{L}\pi^*(-) \otimes \mathcal{O}_{\mathcal{H}_s}(k): D(X) \rightarrow D(\mathcal{H}_s)$ are fully faithful, where $k = 1, \dots, r-1$, $\mathcal{O}_{\mathcal{H}_s}(k) := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)|_{\mathcal{H}_s}$, and there is a semiorthogonal decomposition:*

$$\begin{aligned} D(\mathcal{H}_s) &= \langle \mathbb{R}j_*p^*D(Z), \mathbb{L}\pi^*D(X) \otimes \mathcal{O}_{\mathcal{H}_s}(1), \dots, \pi^*D(X) \otimes \mathcal{O}_{\mathcal{H}_s}(r-1) \rangle, \\ &= \langle \mathbb{L}\pi^*D(X) \otimes \mathcal{O}_{\mathcal{H}_s}(2-r), \dots, \mathbb{L}\pi^*D(X), \mathbb{R}j_*p^*D(Z) \rangle. \end{aligned}$$

2.4. Blowing up, and relation with Cayley's trick. Suppose Z is a codimension $r \geq 2$ locally complete intersection of a smooth variety X , the *blowing up of Z along X* is $\pi: \text{Bl}_Z X := \mathbb{P}(\mathcal{I}_Z) \rightarrow X$, where \mathcal{I}_Z is the ideal sheaf of Z inside X . The *exceptional divisor* is $i_E: E := \text{Bl}_Z X \times_X Z \hookrightarrow \text{Bl}_Z X$. Since $\mathcal{I}_Z|_Z = \mathcal{N}_{Z/X}^\vee$, therefore $E = \mathbb{P}(\mathcal{N}_{Z/X}^\vee)$. Denote $p: E \rightarrow Z$ be the projection. The following is due to Orlov [O92] (see also [JL18a] for the case without smoothness condition on Z and for the noncommutative case).

Theorem 2.2 (Blowing up formula, Orlov [O92]). *In the above situation, then the functors $\mathbb{L}\pi^* : D(X) \rightarrow D(\mathrm{Bl}_Z X)$ and $\mathbb{R}i_{E*} \mathbb{L}p^*(-) \otimes \mathcal{O}(-kE) : D(Z) \rightarrow D(\mathrm{Bl}_Z X)$ are fully faithful, $k \in \mathbb{Z}$. Denote the image of the latter to be $D(Z)_k$, then*

$$\begin{aligned} D(\mathrm{Bl}_Z X) &= \langle \mathbb{L}\pi^* D(X), D(Z)_0, D(Z)_1, \dots, D(Z)_{r-2} \rangle; \\ &= \langle D(Z)_{1-r}, \dots, D(Z)_{-2}, D(Z)_{-1}, \mathbb{L}\pi^* D(X) \rangle. \end{aligned}$$

2.4.1. *Relationship with Cayley's trick.* There is a wonderful geometry relating blowing ups with Cayley's trick [AW]. In the situation of Cayley's trick (§2.3), if we pull back $\pi : \mathcal{H}_s \rightarrow X$ along the blow-up $\beta : \mathrm{Bl}_Z X \rightarrow X$ of X along Z , then the fiber product $\mathrm{Bl}_Z X \times_X \mathcal{H}_s$ will have two irreducible components: one is $\mathbb{P}(\mathcal{N}_i) \times_Z \mathbb{P}(\mathcal{N}_i^\vee)$, the other is the *strict transform* of \mathcal{H}_s along the blow-up β , $\mathcal{U} := \overline{(\mathcal{H}_s \setminus \mathbb{P}(\mathcal{N}_i)) \times_X \mathrm{Bl}_Z X} \subset \mathrm{Bl}_Z X \times_X \mathcal{H}_s$. Then the projection $\pi_{\mathcal{U}} : \mathcal{U} \rightarrow \mathrm{Bl}_Z X$ will be a projective bundle of fiber \mathbb{P}^{r-2} , and its restriction to $\mathbb{P}(\mathcal{N}_i^\vee)$ is nothing but the fiberwise incidence quadric $Q_Z \subset \mathbb{P}(\mathcal{N}_i) \times_Z \mathbb{P}(\mathcal{N}_i^\vee)$, which is defined fiberwisely over $z \in Z$ by incidence relation $\{(n, n^\vee) \in \mathbb{P}(\mathcal{N}_i|_z) \times \mathbb{P}(\mathcal{N}_i^\vee|_z) \mid \langle n, n^\vee \rangle = 0\}$. From blowing up closure lemma, \mathcal{U} is the blowing up of \mathcal{H}_s along $\mathbb{P}(\mathcal{N}_i)$:

$$\mathcal{U} = \mathrm{Bl}_{\mathbb{P}(\mathcal{N}_i)} \mathcal{H}_s, \quad j_Q : Q_Z \hookrightarrow \mathcal{U} \text{ is the exceptional divisor.}$$

Therefore we have a commutative diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\gamma} & \mathcal{H}_s \\ \pi_{\mathcal{U}} \downarrow & & \downarrow \pi \\ \mathrm{Bl}_Z X & \xrightarrow{\beta} & X \end{array}$$

relating the projection $\pi : \mathcal{H}_s \rightarrow X$ from the universal hyperplane with the projection $\pi_{\mathcal{U}}$ of a projective bundle, via the two blow-ups β and γ . Notice the pullback $\tilde{q} : \mathrm{Bl}_Z X \times_X \mathbb{P}(\mathcal{E}) \rightarrow \mathrm{Bl}_Z X$ of projective bundle q along β is also projective bundle over $\mathrm{Bl}_Z X$, and also the divisor inclusion $\iota_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathrm{Bl}_Z X \times_X \mathbb{P}(\mathcal{E})$ is defined by fiberwise quadric incidence relation (between $\mathrm{Bl}_Z X \subset \mathbb{P}(\mathcal{E}^\vee)$ and $\mathbb{P}(\mathcal{E})$), i.e. \mathcal{U} is the *universal hyperplane* for $\mathrm{Bl}_Z X \subset \mathbb{P}(\mathcal{E}^\vee)$ over X in the language of HPD §2.5.

Remark 2.3. Another way of understanding this picture ([AW]) is: $\mathrm{Bl}_Z X$ is the connected component of *Hilbert scheme* parametrizing the deformations of a general fiber \mathbb{P}^{r-2} inside \mathcal{H}_s ; \mathcal{U} is the *universal family*, therefore a projective bundle with fiber \mathbb{P}^{r-2} over $\mathrm{Bl}_Z X$.

Lemma 2.4 ([CT15, Prop. 3.4], [JL18c, Lem. 2.9]). *In the situation of blowing up formula Thm. 2.2, for any $\mathcal{E}^\bullet \in D(X)$, $k \in \mathbb{Z}$, we have the following equalities in $D(\mathrm{Bl}_Z X)$:*

$$\begin{aligned} \mathbf{L}_{D(Z)_k}(\mathbb{L}\pi^* \mathcal{E}^\bullet \otimes \mathcal{O}(-(k+1)E)) &= \mathbb{L}\pi^* \mathcal{E}^\bullet \otimes \mathcal{O}(-kE), \\ \mathbf{R}_{D(Z)_k}(\mathbb{L}\pi^* \mathcal{E}^\bullet \otimes \mathcal{O}(-kE)) &= \mathbb{L}\pi^* \mathcal{E}^\bullet \otimes \mathcal{O}(-(k+1)E). \end{aligned}$$

2.5. Lefschetz varieties and HPD. Lefschetz categories are the key ingredients for HPD theory. A variety $X \rightarrow \mathbb{P}(V)$ is said to admit a (*right*) *Lefschetz decomposition* with respect to $\mathcal{O}_{\mathbb{P}(V)}(1)$ if there is a semiorthogonal decomposition of the form:

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{m-1}(m-1) \rangle,$$

with $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \supset \mathcal{A}_{m-1}$ a descending sequence of admissible subcategories, where $\mathcal{A}_*(k) = \mathcal{A}_* \otimes \mathcal{O}_{\mathbb{P}(V)}(k)$ denotes the image of \mathcal{A}_* under the autoequivalence $\otimes \mathcal{O}_{\mathbb{P}(V)}(k)$ for $k \in \mathbb{Z}$. Dually, a *left Lefschetz decomposition* of $D(X)$ is a SOD of the form:

$$D(X) = \langle \mathcal{A}_{1-m}(1-m), \dots, \mathcal{A}_{-1}(-1), \mathcal{A}_0 \rangle,$$

with $\mathcal{A}_{1-m} \subset \dots \subset \mathcal{A}_{-1} \subset \mathcal{A}_0$ an ascending sequence of admissible subcategories.

The variety $X \rightarrow \mathbb{P}(V)$ is said to be a *Lefschetz variety*, or to admit a *Lefschetz structure* if $D(X)$ admits both right and left Lefschetz decompositions (with same \mathcal{A}_0 and m) as above. If X is a smooth S -scheme, then X is a Lefschetz variety if it admits either a right or a left Lefschetz decomposition. The number m is called the *length* of the Lefschetz structure. See [K07, K08, JLX17, P18, JL18a] for more about Lefschetz decompositions.

Let $Q = \{(x, [H]) \mid x \in H\} \subset \mathbb{P}(V) \times_S \mathbb{P}(V^\vee)$ be the universal quadric for $\mathbb{P}(V)$ (or equivalently for $\mathbb{P}(V^\vee)$). Then the *universal hyperplane* \mathcal{H}_X for $X \rightarrow \mathbb{P}(V)$ is defined to be

$$\mathcal{H}_X := X \times_{\mathbb{P}(V)} Q \subset X \times_S \mathbb{P}(V^\vee).$$

Denote $\iota_{\mathcal{H}}: \mathcal{H}_X \rightarrow X \times_S \mathbb{P}(V^\vee)$ the inclusion, then it is easy to show there is a $\mathbb{P}(V^\vee)$ -linear semiorthogonal decomposition (see [K07, T15, JLX17]):

$$D(\mathcal{H}_X) = \langle \mathcal{C}, \iota_{\mathcal{H}}^*(\mathcal{A}_1(1) \boxtimes_S D(\mathbb{P}(V^\vee))), \dots, \iota_{\mathcal{H}}^*(\mathcal{A}_{m-1}((m-1)) \boxtimes_S D(\mathbb{P}(V^\vee))) \rangle.$$

Definition 2.5. The category \mathcal{C} is called the *HPD category* of $D(X)$, denoted by $D(X)^\natural$. If there exists a variety Y with $Y \rightarrow \mathbb{P}(V^\vee)$, and a Fourier-Mukai kernel $\mathcal{P} \in D(Y \times_{\mathbb{P}(V^\vee)} \mathcal{H}_X)$ such that the $\mathbb{P}(V^\vee)$ -linear Fourier Mukai functor $\Phi_{\mathcal{P}}^{Y \rightarrow \mathcal{H}_X}: D(Y) \rightarrow D(\mathcal{H})$ induces an equivalence of categories $D(Y) \simeq D(X)^\natural$, then $Y \rightarrow \mathbb{P}(V^\vee)$ is called the *homological projective dual variety* or *HPD variety* of $X \rightarrow \mathbb{P}(V)$.

The HPD is a *reflexive* relation: $(X^\natural)^\natural \simeq X$, see [K07, JLX17]. The primary output of the HPD theory is the Kuznetsov's fundamental theorem of HPD for linear sections [K07]; we refer the readers to the references [K07, K14, T15, JLX17, JL18a] for the precise statement of the theorem and its various applications.

Remark 2.6. The HPD theory can be set up in the noncommutative setting for a $\mathbb{P}(V)$ -linear Lefschetz category \mathcal{A} , which is a $\mathbb{P}(V)$ -linear category (with proper enhancement) together with a right and left Lefschetz decomposition as above, see [P18, JL18a]. Then one can similarly define the HPD category \mathcal{A}^\natural of \mathcal{A} , and the fundamental theorem of HPD still holds for dual linear sections of \mathcal{A} and \mathcal{A}^\natural , see [JLX17, R17, P18].

3. GENERALIZED LINEAR DUALITY

As in the introduction, let V and K be vector bundles over S of rank $N \geq 2$ and $k \leq N$ respectively, $\sigma \in \text{Hom}_S(K, V)$ be an injective \mathcal{O}_S -module morphism and $\mathcal{L} = \text{coker}(\sigma)$ be the cokernel. Denote $\ell = \text{rank } \mathcal{L}$, therefore $k = N - \ell$. There is a short exact sequence:

$$0 \rightarrow K \rightarrow V \rightarrow \mathcal{L} \rightarrow 0,$$

and the dual sheaves fit into a four-term exact sequence given by (1.2). Further denote

$$Z := \mathbb{P}(\mathcal{E}xt^1(\mathcal{L}, \mathcal{O}_S)) \subset \mathbb{P}(K^\vee),$$

which is a desingularization of the degeneracy locus $S_\sigma = \text{Sing}(\mathcal{L}) \subset S$, and denote by

$$\tilde{\mathbb{P}}(K^\vee) := \text{Bl}_Z \mathbb{P}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$$

the blowing up of $\mathbb{P}(K^\vee)$ along $Z \subset \mathbb{P}(K^\vee)$.

Lemma 3.1 (Lefschetz decomposition). *The blowing up $\tilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$ admits a S -linear Lefschetz decomposition with respect to the action of $\mathcal{O}_{\mathbb{P}(V^\vee)}(1)$:*

$$(3.1) \quad D(\tilde{\mathbb{P}}(K^\vee)) = \langle \mathcal{A}_0, \mathcal{A}_1(1) \dots, \mathcal{A}_{r-2}(r-2) \rangle, \quad r = \max\{N, k+1\},$$

where (i) denotes the twist by $\mathcal{O}_{\mathbb{P}(V^\vee)}(i)$, $i \in \mathbb{Z}$, $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \supset \mathcal{A}_{r-2}$ are given by:

$$\begin{aligned} \mathcal{A}_0 = \dots = \mathcal{A}_{k-1} &= \langle \mathbb{L}\pi^* D(S), D(Z)_0 \rangle, & \mathcal{A}_k = \dots = \mathcal{A}_{N-2} &= D(Z)_0, & \text{if } k \leq N-1, \\ \mathcal{A}_0 = \dots = \mathcal{A}_{k-2} &= \langle \mathbb{L}\pi^* D(S), D(Z)_0 \rangle, & \mathcal{A}_{N-1} &= \mathbb{L}\pi^* D(S), & \text{if } k = N. \end{aligned}$$

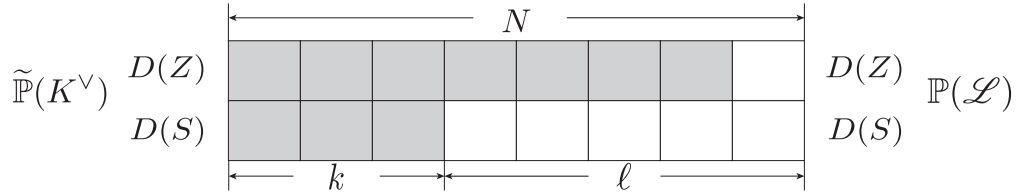
where $D(Z)_0$ is the image of $D(Z)$ under fully faithful embedding $\mathbb{R}j_* \mathbb{L}p^*$.

Proof. This follows directly from performing right mutations Lem. 2.4 to Orlov's formula Thm. 2.2 for the blowing up $\tilde{\mathbb{P}}(K^\vee)$. (Cf. [CT15, Prop. 3.1], [JL18a, Prop. 4.4]). \square

Our main result is the following generalization of linear duality:

Theorem 3.2 (Generalized linear duality). *The S -linear scheme $\mathbb{P}(\mathcal{L}) \hookrightarrow \mathbb{P}(V)$ is homological projective dual to $\tilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$ with respect to the Lefschetz decomposition (3.1).*

The HPD relation between $\tilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$ and $\mathbb{P}(\mathcal{L}) \hookrightarrow \mathbb{P}(V)$ of the theorem can be visualized in the following diagram using Kuznetsov's convention [K07]:



The SOD for $D(\mathbb{P}(\mathcal{L}))$ obtained from HPD theory applied to Lem. 3.1 agrees with the projectivization formula of [JL18c, Thm. 3.1] (up to mutations). Therefore the above

theorem shows the duality between the projectivization formula of [JL18c] and the blowing up formula of [O92], and one can deduce one formula from the other based on results of “chess game” [JLX17].

If \mathcal{L} is locally free, then the above theorem reduces to the usual linear duality” $\mathbb{P}(L)^\natural = \mathbb{P}(L)^\perp \equiv \mathbb{P}(K^\vee)$. If $K = \mathcal{O}_S$, and $\sigma = s : \mathcal{O} \rightarrow V$ a regular section, then this is the HP duality between generalized universal hyperplane $\mathcal{H}_s \subset \mathbb{P}(V)$ and blowing up $\text{Bl}_Z S \subset \mathbb{P}(V^\vee)$ (cf. [CT15, Prop. 3.2]), which can be visualised using Kuznetsov’s convention as

$$\text{Bl}_Z S = \mathbb{P}(\mathcal{I}_Z) \quad \begin{array}{c} D(Z) \\ D(S) \end{array} \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \quad \begin{array}{c} D(Z) \\ D(S) \end{array} \quad \mathcal{H}_s = \mathbb{P}(\mathcal{L}).$$

If $N = k$, then theorem implies $S_\sigma^+ = \mathbb{P}(\mathcal{L}) \subset \mathbb{P}(V)$ is homological projective dual to $\text{Bl}_{S_\sigma^-} \mathbb{P}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$, the blowing up along a (in general) different resolution $S_\sigma^- = Z$ of singularities of the degeneracy locus S_σ . If $k = N - 1$, then $S_\sigma \subset S$ is a Cohen-Macaulay subscheme of codimension 2, and $\mathbb{P}(\mathcal{L}) = \text{Bl}_{S_\sigma} S$ is the blowing up along S_σ . The theorem states the HPD between the two blowing-ups $\text{Bl}_{S_\sigma} S \subset \mathbb{P}(V)$ and $\text{Bl}_Z \mathbb{P}(K^\vee) \rightarrow \mathbb{P}(V^\vee)$.

Proof of Thm. 3.2. The situation can be regarded as a relative situation of [CT15, JL18a], and a similar strategy can be applied. Apply the construction of §2.3 to the scheme $X = \mathbb{P}(K^\vee)$ and the zero locus $i : Z \hookrightarrow \mathbb{P}(K)$ of the canonical regular section of vector bundle $\mathcal{E} = V \boxtimes \mathcal{O}_{\mathbb{P}(K^\vee)}(1)$, then the generalized universal hyperplane $\iota : \mathcal{H} := \mathcal{H}_s \subset \mathbb{P}(V) \times_S \mathbb{P}(K^\vee)$ is a divisor of the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(K^\vee)}(1)$. Consider the blowing up $\beta : \tilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(K^\vee)$. Its exceptional divisor is given by $\mathbb{P}(\mathcal{N}_i^\vee) = \mathbb{P}(V^\vee) \times_S Z$. Apply the geometry of §2.4.1, we get that the blowing up $\gamma : \mathcal{U} \rightarrow \mathcal{H}$ of \mathcal{H} along $j : \mathbb{P}(V) \times_S Z \hookrightarrow \mathcal{H}$ is the universal hyperplane for $\tilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(V)$, i.e. $\iota_{\mathcal{U}} : \mathcal{U} = \mathcal{H}_{\tilde{\mathbb{P}}(K^\vee)} \hookrightarrow \tilde{\mathbb{P}}(K^\vee) \times_S \mathbb{P}(V)$, and the exceptional locus of γ is $j_Q : Q_Z \hookrightarrow \mathcal{U}$, where $Q_Z \subset \mathbb{P}(V) \times_S \mathbb{P}(V^\vee) \times_S Z$ is the base-change of the universal quadric $Q \subset \mathbb{P}(V) \times_S \mathbb{P}(V^\vee)$ along map $Z \rightarrow S$. The situation is summarized in the following diagram, with notation of maps as indicated:

(3.2)

$$\begin{array}{ccccc} & & \mathbb{P}(V) \times_S Z & \xleftarrow{j} & \mathcal{H} & \xleftarrow{\iota} & \mathbb{P}(V) \times_S \mathbb{P}(K^\vee) \\ & \nearrow \pi_Q & \downarrow p & \nearrow \gamma & \downarrow \pi_{\mathcal{H}} & \nearrow \tilde{\beta} & \\ Q_Z & \xleftarrow{j_Q} & \mathcal{U} & \xleftarrow{\iota_{\mathcal{U}}} & \mathbb{P}(V) \times_S \tilde{\mathbb{P}}(K^\vee) & & \\ \downarrow \tilde{\pi}_Q & & \downarrow \pi_{\mathcal{U}} & & \downarrow & & \\ & \nearrow \tilde{p} & Z & \xleftarrow{i} & \mathbb{P}(K^\vee) & & \\ & & \downarrow \tilde{j} & & \downarrow \beta & & \\ \mathbb{P}(V^\vee) \times_S Z & \xleftarrow{\tilde{j}} & \tilde{\mathbb{P}}(K^\vee) & & & & \end{array}$$

(Note: The diagram also includes a curved arrow labeled q from $\mathbb{P}(V) \times_S \mathbb{P}(K^\vee)$ to $\mathbb{P}(K^\vee)$ and a curved arrow labeled \tilde{q} from $\tilde{\mathbb{P}}(K^\vee)$ to $\mathbb{P}(K^\vee)$.)

In the rest of the proof we will write derived functors as *underived*, for simplicity of notations. From blowing up formula for $\gamma: \mathcal{U} \rightarrow \mathcal{H}$, we have

$$(3.3) \quad D(\mathcal{U}) = \langle \gamma^* D(\mathcal{H}), D(\mathbb{P}(V) \times_S Z)_0, \dots, D(\mathbb{P}(V) \times_S Z)_{N-3} \rangle.$$

where $D(\mathbb{P}(V) \times_S Z)_k = j_{Q*} \pi_Q^* D(\mathbb{P}(V) \times_S Z) \otimes \mathcal{O}(-kE_Q)$ (where Q_Z is the exceptional divisor and E_Q denotes the divisor class of Q_Z). It follows directly from $\mathcal{O}(-E_Q) = \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(-1)$ and the diagram that

$$D(\mathbb{P}(V) \times_S Z)_k = D(Z)_k \boxtimes_S \mathbb{P}(V)|_{\mathcal{U}}.$$

On the other hand, as observed in [JL18c], \mathcal{H} is also the generalized universal hyperplane for the scheme $X_1 = \mathbb{P}(V)$ and the zero locus $i_1: \mathbb{P}(\mathcal{L}) \hookrightarrow \mathbb{P}(V)$ of a canonical section of the vector bundle $\mathcal{E}_1 = K^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$. Denote $\pi_1: \mathcal{H} \rightarrow \mathbb{P}(V)$ the projection, $j_1: \mathbb{P}_{\mathbb{P}(\mathcal{L})}(\mathcal{N}_{j_1}) \hookrightarrow \mathcal{H}$ the inclusion and $p_1: \mathbb{P}_{\mathbb{P}(\mathcal{L})}(\mathcal{N}_{j_1}) \rightarrow \mathbb{P}(\mathcal{L})$ the projection. Then by Thm. 2.1,

$$(3.4) \quad D(\mathcal{H}) = \langle \Phi_1(D(\mathbb{P}(\mathcal{C}_\sigma)), \pi_1^* D(\mathbb{P}(V)) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}, \dots, \pi_1^* D(\mathbb{P}(V)) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(k-2) \rangle,$$

where $\Phi_1 = j_{1*} p_1^*(-) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(-1)$. From diagram (3.2) we have

$$(3.5) \quad \gamma^*(\pi_1^* D(\mathbb{P}(V)) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(k)) = (\pi^* D(S) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(k)) \boxtimes_S \mathbb{P}(V)|_{\mathcal{U}}.$$

By Lem. 2.4, each time one right mutates (3.5) passing through some $D(\mathbb{P}(V) \times_S Z)_{k'}$ inside (3.3) will result in tensoring (3.5) with $\mathcal{O}(-E_Q)$ and thus gets

$$(\pi^* D(S) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(k-1) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1)) \boxtimes_S D(\mathbb{P}(V))|_{\mathcal{U}}.$$

Repeating this process of mutations inside (3.3) and substitute the category $D(\mathcal{H})$ by (3.4), we end up with the following SOD:

$$D(\mathcal{U}) = \langle \Psi(D(\mathbb{P}(\mathcal{C}_\sigma)), (\mathcal{A}_1(1) \boxtimes_S D(\mathbb{P}(V)))|_{\mathcal{U}}, \dots, (\mathcal{A}_{r-2}(r-2) \boxtimes_S D(\mathbb{P}(V)))|_{\mathcal{U}} \rangle,$$

where \mathcal{A}_i 's are given by Lem. 3.1, and $\Psi = \mathbb{L}\gamma^* \mathbb{R}j_{1*} \mathbb{L}p_1^*(-) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \otimes \mathcal{O}_{\mathbb{P}(K^\vee)}(-1) : D(\mathbb{P}(\mathcal{L})) \hookrightarrow D(\mathcal{U})$. By definition of HPD (Def. 2.5), we are done. \square

Proof of Thm. 1.2. Apply the categorical Plücker formula of [JLX17] to the two HPD pairs $(\mathcal{A}/\mathbb{P}(V^\vee), \mathcal{A}^\natural/\mathbb{P}(V))$ and $(\mathbb{P}(\mathcal{L}) \subset \mathbb{P}(V^\vee), \mathbb{P}(\mathcal{L})^\perp := \widetilde{\mathbb{P}}(K^\vee) \rightarrow \mathbb{P}(V))$, then the theorem 1.2 immediately follows. ¹ \square

¹Notice that one could also apply the nonlinear HPD theorem of [KP18, JL18b] to our theorem 3.2 and obtain similar results in a slightly different formulation.

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