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SPACES OF ABELIAN DIFFERENTIALS AND HITCHIN'S SPECTRAL COVERS

M. Bertola^{†‡[1](#page-0-0)}, D. Korotkin^{† [2](#page-0-1)},

† *Department of Mathematics and Statistics, Concordia University 1455 de Maisonneuve W., Montr´eal, Qu´ebec, Canada H3G 1M8*

> ‡ *SISSA/ISAS, Area of Mathematics via Bonomea 265, 34136 Trieste, Italy*

Abstract

Using the embedding of the moduli space of generalized $GL(n)$ Hitchin's spectral covers to the moduli space of meromorphic Abelian differentials we study the variational formulæ of the period matrix, the canonical bidifferential, the prime form and the Bergman tau function. This leads to residue formulæ which generalize the Donagi-Markman formula for variations of the period matrix. The computation of second derivatives of the period matrix reproduces the formula derived in [\[2\]](#page-20-0) using the framework of topological recursion.

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¹Marco.Bertola@{concordia.ca, sissa.it}

²Dmitry.Korotkin@concordia.ca

1. INTRODUCTION

The geometry of spaces of Abelian differentials on Riemann surfaces has attracted interest in relationship with the theory of Teichmüller flow $[15, 16, 7]$ $[15, 16, 7]$ $[15, 16, 7]$. Methods inspired by the theory of integrable systems were applied to the study of these spaces in [\[14,](#page-20-4) [19,](#page-21-1) [12\]](#page-20-5) where an appropriate version of deformation theory of Riemann surfaces and the formalism of tau functions was developed. In particular, variations of moduli and of various canonical objects associated to a Riemann surface were computed in [\[14\]](#page-20-4) (holomorphic case) and in [\[12\]](#page-20-5) (meromorphic case). The *Bergman tau function* introduced in [\[14\]](#page-20-4) is a natural generalization of Dedekind's eta-function to higher genus.

The origin of Hitchin's spectral covers and their moduli spaces is the dimensional reduction of self-dual Yang-Mills equations on a four-dimensional space represented as the product of a Riemann surface and \mathbb{R}^2 [\[10\]](#page-20-6). Such a dimensional reduction gives a family of completely integrable systems associated to families of Riemann surfaces of arbitrary genus [\[11\]](#page-20-7). Hamiltonians of such integrable systems (we consider here only the $GL(n)$) gauge group) are encoded in the n-sheeted *spectral cover* of a Riemann surface. The moduli space of spectral covers for a base Riemann surface of given genus was also intensively studied (see [\[1,](#page-20-8) [6\]](#page-20-9)). In particular, the *Donagi-Markman cubic* describes variations of the period matrix of the spectral cover for fixed base, answering the question posed in [\[1\]](#page-20-8). Variations of the canonical meromorphic bi-differential on these spaces were derived in [\[2\]](#page-20-0) using the formalism developed in [\[9\]](#page-20-10).

The space of Hitchin's spectral covers admits a natural embedding in a space of Abelian differentials; this embedding was used in [\[18\]](#page-21-2) to define a natural version of Bergman tau functions on spaces of spectral covers (with variable or fixed base) and find the class of the locus of degenerate covers (the universal Hitchin's discriminant) in the Picard group of the universal moduli space of spectral covers.

In this paper we further exploit this embedding to show how variational formulæ for the period matrix, the canonical bidifferential and the prime form on the moduli spaces of generalized Hitchin's systems (when the coefficients of the equation defining the spectral cover are allowed to be meromorphic differentials) can be deduced from variational formulæ on moduli spaces of meromorphic Abelian differentials derived in [\[14,](#page-20-4) [12\]](#page-20-5). In the special case of regular Hitchin's systems we reproduce residue formulæ for the canonical bidifferential obtained in [\[2\]](#page-20-0) and for the period matrix (given by the Donagi-Markman cubic [\[6\]](#page-20-9)). We also derive residue formulæ for variations of Bergman tau function of spaces of spectral covers for the holomorphic case.

The formulas for the second derivatives of the period matrix (in holomorphic case) found in our formalism coincide with expressions derived in [\[2\]](#page-20-0) using the formalism of topological recursion of [\[8\]](#page-20-11). These formulæ are rather cumbersome in contrast to analogous formulæ on spaces of Abelian differentials. This suggest a possibility of existence of a natural simple structure on spaces of Abelian differentials which underlie the topological recursion framework on spaces of spectral covers.

2. SPACES OF GENERALIZED SPECTRAL COVERS

Denote by C a Riemann surface of genus g, with m marked points y_1, \ldots, y_m on C and associated corresponding multiplicities $k_1, \ldots, k_m, k_j \geq 1$. The Higgs bundle on C is a pair (E, Φ) where E is a holomorphic vector bundle and Φ (the *Higgs field*) is a holomorphic (or meromorphic, depending on the specific setting) Ad_E -valued 1-form on C [\[11,](#page-20-7) [6\]](#page-20-9). For a given base curve C and a degree of the bundle E the space of pairs (E, Φ) is called the moduli space of Higgs bundles.

Consider a meromorphic $GL(n)$ Higgs field Φ with poles at y_j 's of the corresponding order k_j , $j = 1, \ldots, m$. We also assume a generic form of the singular parts of Φ near these poles. The *spectral curve* \widehat{C} is defined as a locus in T^*C by the equation $\det(\Phi - v \, \text{Id}) = 0$, which can be written as

(2.1)
$$
v^{n} + Q_{1}v^{n-1} + \cdots + Q_{n} = 0
$$

where Q_ℓ is a meromorphic ℓ -differential on C with pole of order ℓk_j at the point y_j thanks to the genericity assumption.

For fixed C and $\{y_j\}_{j=1}^{\tilde{m}}$ we denote by $\mathcal{M}^n_H[\mathbf{k}]$ the moduli space of curves [\(2.1\)](#page-2-1) which can be identified with the moduli space of sets of the differentials Q_{ℓ} with poles of appropriate order at the points y_j . Namely, denoting by Ω_ℓ the vector space of ℓ -differentials on C with poles of order ℓk_j at y_j , we have

$$
\mathcal{M}^n_{H}[\mathbf{k}]=\bigoplus_{\ell=1}^n\Omega_\ell
$$

Denote by π the projection $\hat{C} \to C$. Assuming that the branch points of \hat{C} do not coincide with y_j we have $\pi^{-1}(y_j) = \{y_j^{(s)}\}$ ${j \choose j}_{s=1}^n$.

The meromorphic Abelian differential v has, on \widehat{C} , poles of order k_j at all $y_j^{(s)}$ $j^{\langle s \rangle}$. Denote by χ_j a local coordinate on C near y_j ; since we have assumed that y_j is a not a branch point of $\widehat C$ we can use χ_j also as local coordinate near each $y_j^{(s)}$ $j_j^{(s)}$ for $s = 1, \ldots, n$. Consider the singular parts of v at $y_j^{(s)}$ $j^{(s)}$:

(2.2)
$$
v(\zeta_j) = \left(\frac{C_j^{(s),k_j}}{\chi_j^{k_j}} + \frac{C_j^{(s),k_j-1}}{\chi_j^{k_j-1}} + \dots + \frac{C_j^{(s),1}}{\chi_j} + O(1)\right) d\chi_j.
$$

The discriminant W of the equation [\(2.1\)](#page-2-1) is a meromorphic $n(n - 1)$ differential on C which has pole of order $n(n - 1)k_j$ at y_j . Therefore, the total degree of poles of W is $n(n-1)\sum_{j=1}^mk_j$ and the number of its zeros (i.e. the number of branch points of $\widehat C)$ is

(2.3)
$$
p = n(n-1) \left(2g - 2 + \sum_{j=1}^{m} k_j \right).
$$

It follows from the Riemann–Hurwitz formula that the genus of \widehat{C} equals

(2.4)
$$
\hat{g} = n^2(g-1) + 1 + \frac{n(n-1)}{2} \sum_{j=1}^{m} k_j
$$

The degree of the divisor of zeroes of the Abelian differential v on \widehat{C} is

(2.5)
$$
r = 2\hat{g} - 2 + n \sum_{j=1}^{m} k_j
$$

The dimension of $\mathcal{M}_{H}^{n}[\mathbf{k}]$ equals to the sum of dimensions of spaces of coefficients of [\(2.1\)](#page-2-1), which is computed as

$$
\left(\sum k_j - 1 + g\right) + \left(2\sum k_j + 3(g-1)\right) + \cdots + \left(n\sum k_j + (2n-1)(g-1)\right).
$$

Assuming that at least one $k_j > 0$, the above gives

(2.6)
$$
\dim \mathcal{M}_{H}^{n}[\mathbf{k}] = \frac{n(n+1)}{2} \sum k_{j} + n^{2}(g-1) = \hat{g} + n \sum_{j=1}^{m} k_{j} - 1.
$$

On the moduli space $\mathcal{M}_{H}^{n}[\mathbf{k}]$ we introduce the following local coordinates:

$$
(2.7) \quad \left\{ \{A_{\alpha}\}_{\alpha=1}^{\hat{g}}, \{C_{j}^{(s),\ell}\}, j=1,\ldots,m, \ s=1,\ldots,n, \ \ell=1,\ldots,k_j, \ (j,s,\ell)\neq(1,1,1) \right\}
$$

where $C_i^{(s),\ell}$ $j^{(s),\ell}_j$ are coefficients in singular parts of v near $y_j^{(k)}$ $j^{(k)}$ [\(2.2\)](#page-2-2) (these coefficients of course depend on the choice of local coordinates χ_j near y_j on C), and A_α are a -periods of v under an arbitrary choice of Torelli marking:

$$
(2.8) \t\t A_{\alpha} = \int_{a_{\alpha}} v \ .
$$

The coefficient $C_1^{(1),1}$ $I_1^{(1),1}$ is not an independent coordinate since the sum of residues of v on C vanishes:

(2.9)
$$
\sum_{j=1}^{m} \sum_{s=1}^{k_j} C_j^{(s),1} = 0.
$$

We observe that the number of coordinates [\(2.7\)](#page-3-0) coincides with the dimension [\(2.6\)](#page-3-1) of $\mathcal{M}_{H}^{n}[\mathbf{k}].$

Subordinate to the choice of Torelli marking we also define the normalized first-kind Abelian differentials (holomorphic) v_α with the property

$$
\oint_{a_{\beta}} v_{\alpha} = \delta_{\alpha\beta}.
$$

We similarly define the normalized second-kind differentials $w_i^{(s),l}$ $j^{(s),i}$ on C with prescribed singular part:

$$
(2.11) \quad \oint_{a_{\alpha}} w_j^{(s), \ell} = 0, \qquad w_j^{(s), \ell}(x) = \left(\frac{1}{\chi_j^l} + O(1)\right) d\chi_j \ , \qquad x \sim y_j^{(s)} \ , \qquad \ell = 2, \dots, k_j
$$

and the normalized differentials of the third kind $u_i^{(s)}$ $j^{(s)}(x)$ on C which have simple poles at $y_1^{(1)}$ $j_1^{(1)}$ and $y_j^{(s)}$ with residues $-1, +1$, respectively.

Since the moduli of the base curve \overline{C} are kept constant, we can define unambiguously the derivative with respect to the moduli of our space for any Abelian differential w on \widehat{C} . To wit, we fix a local chart D on C with a local coordinate ξ and lift D to all sheets of \widehat{C} . Then in any connected component of $\pi^{-1}(D)$ we can use ξ as a local coordinate away from ramification points. We express the differential w in such coordinate $w = f(\xi)d\xi$ and define

$$
\frac{dw}{dz_k} = \frac{df(\xi)}{dz_k} d\xi
$$

where the coordinate ξ remains fixed under differentiation. Clearly, the definition [\(2.12\)](#page-4-0) is independent of the choice of the local coordinate ξ because the moduli of the base curve are kept constant. Keeping this in mind we formulate the following proposition.

Proposition 2.1. *The following variational formulæ of* v *with respect to coordinates [\(2.7\)](#page-3-0) on* $\mathcal{M}_{H}^{n}[\mathbf{k}]$ hold:

$$
\frac{\partial v}{\partial A_{\alpha}} = v_{\alpha} \; ,
$$

(2.14)
$$
\frac{\partial v}{\partial C_j^{(s),l}} = w_j^{(s),l}, \qquad \ell = 2,\ldots,k_j
$$

where $w_i^{(s),\ell}$ $\int_j^{(s),\ell}$ are normalized (i.e. with $\int_{a_\alpha} w_j^{(s),\ell} = 0$) differentials of second kind defined by [\(2.11\)](#page-4-1) *and*

$$
\frac{\partial v}{\partial C_j^{(s),1}} = u_j^{(s)}
$$

where $j = 1, \ldots, m$ and $s = 1, \ldots, n$; $u_j^{(s)}$ $j_{j}^{\left(s\right) }\left(x\right)$ are the normalized differentials of the third kind on \widehat{C} *defined after* [\(2.11\)](#page-4-1)*.*

Proof. First notice that the differential v vanishes at all branch points x_j of \widehat{C} ; generically these zeros are of first order. This is due the fact that a coefficient Q_k of equation [\(2.1\)](#page-2-1) is a k-differential on C. Being lifted from C to \widehat{C} , it gains a zero of order k at each branch point since near the ramification point x_j the local coordinate on \widehat{C} is given by $(\xi - \xi_j)^{1/2}$ where ξ is the local coordinate on C near $\pi(x_i)$ (ξ is assumed to be independent of coordinates [\(2.7\)](#page-3-0)) and $\xi_i = \xi(\pi(x_i))$. In particular, the *n*-differential Q_n , being lifted to \hat{C} , has zeros of order *n* at all branch points (as well as zeros lifted to \widehat{C} from its zeros on C).

Therefore, locally near x_j , t we have

$$
v(\xi) = (\xi - \xi_j)^{1/2} (a_0 + a_1(\xi - \xi_j)^{1/2} + \dots) d(\xi - \xi_j)^{1/2} = \frac{1}{2} (a_0 + a_1(\xi - \xi_j)^{1/2} + \dots) d\xi.
$$

Although ξ is independent of the moduli coordinates [\(2.7\)](#page-3-0), the coordinate ξ_i of the branch point $\pi(x_i)$ does depend on them, and differentiation with respect to any coordinate z from the list [\(2.7\)](#page-3-0) gives

$$
\frac{\partial v}{\partial z} = \frac{1}{4} \left(-\frac{a_1(\xi_j)_z}{(\xi - \xi_j)^{1/2}} + O(1) \right) d\xi = -\frac{1}{2} (a_1(\xi_j)_z + o(1)) d\sqrt{\xi - \xi_j}
$$

which is holomorphic (although generically non-vanishing) at x_j . It then follows that all the differentials $\partial v/\partial z$ are holomorphic at the branch points, and can have poles only at the $y_j^{(s)}$ $j^{(s)}{}'s.$

The differentials $\partial v/\partial A_i$ are holomorphic since the coefficients of the singular parts of v near all $y_i^{(s)}$ $i^{(s)}_i$ are independent of A_j . Moreover, all *a*-periods of $\partial v/\partial A_j$ vanish except for the period over a_j , which equals 1. Therefore, we deduce [\(2.13\)](#page-4-2).

Consider $\partial v/\partial C_j^{(s),\ell}$ for $l\geq 2.$ The only singularity of this differential is at $y_j^{(s)}$ $j^{(s)}$ and its singular part there coincides with the one of $w_i^{(s)}$ $j^{(s)}$. Moreover, since the A_α and the $C_j^{(s),\ell}$ j coordinates are independent of each other, all a-periods of $\partial v/\partial C_j^{(s),\ell}$ vanish; thus this differential coincides with $w_j^{(s)}$ $j^{(s)}$.

Similarly, one verifies that the differential $\partial v/\partial C_j^{(s),1}$ coincides with the third kind differential $u_i^{(s)}$ j . In the second control of the second control of the second control of the second control of the second control

We are going to combine this proposition with the variational formulæ on moduli spaces of meromorphic Abelian differentials obtained in [\[14,](#page-20-4) [12\]](#page-20-5) which we discuss next.

3. VARIATIONAL FORMULÆ AND BERGMAN TAU FUNCTION ON MODULI SPACES OF MEROMORPHIC ABELIAN DIFFERENTIALS

Denote by $\mathcal{H}_{\hat{g}}[d_1,\ldots,d_q]$ the moduli space of pairs (\widehat{C}, v) where \widehat{C} is a Riemann surface of genus \hat{g} and v is a meromorphic differential on \hat{C} with q poles y_1, \ldots, y_q of orders d_1, \ldots, d_q , respectively, and simple zeros x_1, \ldots, x_r where $r = 2\hat{g} - 2 + \sum_{i=1}^q d_i$. The notations \widehat{C} and \widehat{g} are now used in agreement with the previous discussion. The dimension of $\mathcal{H}_{\hat{g}}[d_1,\ldots,d_q]$ is the sum of: 3 $\hat{g}-3$ moduli parameters of C , q positions of the singularities, $\sum_{j=1}^{q} d_j - 1$ coefficients of the singular parts and \hat{g} additional moduli corresponding to the addition of an arbitrary holomorphic differential to v . Altogether, we get

(3.1)
$$
\dim \mathcal{H}_{\hat{g}}[d_1, ..., d_q] = 4\hat{g} - 4 + q + \sum_{j=1}^q d_j
$$

The dimension of $\mathcal{H}_{\hat{q}}[d_1,\ldots,d_q]$ coincides with the dimension of the relative homology group

(3.2)
$$
H_1(\widehat{C} \setminus \{y_j\}_{j=1}^q, \{x_i\}_{i=1}^r)
$$

A set of generators of this group can be chosen as follows:

(3.3)
$$
\{s_i\}_{i=1}^{\dim \mathcal{H}_g[d_1,\dots,d_q]} = \left\{ \{a_\alpha, b_\alpha\}_{\alpha=1}^{\hat{g}}, \{c_i\}_{i=2}^q, \{l_i\}_{i=1}^{r-1} \right\}
$$

where $\{a_{\alpha}, b_{\alpha}\}\$ form a Torelli marking on \hat{C} , c_i are small counter-clockwise contours around y_i and each contour l_i connects x_r with x_i .

The homology group dual to [\(3.2\)](#page-6-0) is

(3.4)
$$
H_1(\widehat{C} \setminus \{x_i\}_{j=1}^r, \{y_j\}_{j=1}^q)
$$

and the set of generators dual to the set [\(3.3\)](#page-6-1) with the intersection index

$$
s_i^* \cdot s_j = \delta_{ij}
$$

is given by

(3.5)
$$
s_i^* = \left\{ \{-b_\alpha, a_\alpha\}_{\alpha=1}^g , \ \{-\tilde{l}_i\}_{i=2}^q , \ \{\tilde{c}_i\}_{i=1}^{r-1} \right\}
$$

where \tilde{l}_i is the contour connecting the pole y_1 with y_i ; \tilde{c}_i is a small counter-clockwise contour around x_i .

The set of *homological*, or *period* coordinates on $\mathcal{H}_{\hat{q}}[d_1, \ldots, d_q]$ is given by integrals of v over the basis $\{s_i\}$ [\(3.3\)](#page-6-1):

(3.6)
$$
\mathcal{P}_i = \int_{s_i} v , \qquad i = 1, \ldots, \dim \mathcal{H}_{\hat{g}}[d_1, \ldots, d_q] .
$$

Introduce the following objects on \hat{C} : the prime-form $E(x, y)$, canonical bidifferential $B(x, y)$ (see for example [\[9\]](#page-20-10), Ch. II, for the definition and properties of E and B), holomorphic Abelian differentials v_α normalized via $\int_{a_\alpha} v_\beta = \delta_{\alpha\beta}$ and the period matrix $\Omega_{\alpha\beta} = \int_{b_{\alpha}} v_{\beta}.$

Choose a fundamental polygon of \widehat{C} with vertex at x_r and dissected along paths connecting x_r with poles y_j (having only x_r as common point); denote the resulting simply connected domain by \tilde{C} ; on it we define the "flat" coordinate

$$
(3.7) \t\t\t z(x) = \int_{x_r}^x v
$$

which can be used as local coordinate on \widehat{C} outside of zeros and poles of v.

Proposition 3.1. [\[14,](#page-20-4) [12\]](#page-20-5) *The following variational formulæ for the period matrix* Ω *on the space* $\mathcal{H}_{\hat{q}}[d_1,\ldots,d_q]$ hold:

.

(3.8)
$$
\frac{\partial \Omega_{\alpha\beta}}{\partial \mathcal{P}_j} = \int_{s_j^*} \frac{v_\alpha v_\beta}{v}
$$

To present variational formulæ for v_{α} , B and E we need to define their variations: for v_{α} we define

(3.9)
$$
\frac{\partial v_{\alpha}}{\partial \mathcal{P}_j}(x) = \frac{\partial}{\partial \mathcal{P}_j} \left(\frac{v_{\alpha}(x)}{v(x)} \right) \Big|_{z(x) = const} v(x) .
$$

The result is a differential in \tilde{C} with discontinuities across all the dissecting cuts of \hat{C} where the discontinuity is the addition of a constant depending which boundary component of the dissection we are crossing. Analogously we define variations of $B(x, y)$ and $E(x, y)$ in \mathcal{P}_i .

Proposition 3.2. [\[14,](#page-20-4) [12\]](#page-20-5) *The following variational formulæ on the space* $\mathcal{H}_{\hat{q}}[d_1,\ldots,d_q]$ *hold*

(3.10)
$$
\frac{\partial v_{\alpha}(x)}{\partial P_i} = \frac{1}{2\pi i} \int_{t \in s_i^*} \frac{v_{\alpha}(t)B(x,t)}{v(t)},
$$

(3.11)
$$
\frac{\partial B(x,y)}{\partial \mathcal{P}_i} = \frac{1}{2\pi i} \int_{t \in s_i^*} \frac{B(x,t)B(t,y)}{v(t)},
$$

(3.12)
$$
\frac{\partial}{\partial \mathcal{P}_i} \ln \left(E(x, y) \sqrt{v(x)} \sqrt{v(y)} \right) = -\frac{1}{4\pi i} \int_{t \in s_i^*} \frac{1}{v(t)} \left[d_t \ln \frac{E(x, t)}{E(y, t)} \right]^2.
$$

In the next section we show to deduce variational formulæ on spaces of spectral covers by restriction of the above ones.

On the subspace of $\mathcal{H}_{\hat{g}}^0[d_1,\ldots,d_q]$ of $\mathcal{H}_{\hat{g}}[d_1,\ldots,d_q]$ defined by the vanishing of the residues of v we define the Bergman tau-function via the system of differential equations [\[14,](#page-20-4) [12\]](#page-20-5):

(3.13)
$$
\frac{\partial}{\partial \mathcal{P}_j} \ln \tau_B(\widehat{C}, v) = \int_{s_j^*} \frac{B_{reg}^v(x, x)}{v(x)}
$$

where

(3.14)
$$
B_{reg}^v(x,x) = \left(B(x,y) - \frac{v(x)v(y)}{(\int_x^y v)^2} \right) \Big|_{x=y}.
$$

We refer to [\[14,](#page-20-4) [12\]](#page-20-5) for explicit formula for τ_B and to [\[19,](#page-21-1) 12] for its properties and applications.

4. VARIATIONAL FORMULÆ ON SPACES OF GENERALIZED HITCHIN'S COVERS

We first discuss the variations of the period matrix $\hat{\Omega}$ of \widehat{C} on the moduli space $\mathcal{M}_{H}^{n}[{\bf k}]$ of spectral covers: these formulæ are obtained by pullback of the variational formulæ on the space $\mathcal{H}_{\hat{q}}(d)$ of Abelian differentials on Riemann surfaces of genus \hat{g} where the vector d is given by

(4.1)
$$
\mathbf{d} = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_n, \ldots, k_n)
$$

where each k_i is repeated n times. Thus in the context of previous section we have $k = nm$, and the set of poles $\{y_j\}$ coincides with the set $\{y_j^{(s)}\}$ ${j \choose j}, j = 1, \ldots, m, s = 1, \ldots, n.$

Assume that the branch points of \widehat{C} , i.e., zeros of the discriminant W of [\(2.1\)](#page-2-1), are also simple. We have $(v) = D_{br} + D_0$ where D_{br} is the divisor of ramification points of \hat{C} . The projection of D_{br} on C coincides with the divisor of the discriminant $W: \pi(D_{br}) = (W)$. The projection of D_0 on C coincides with the divisor of the *n*-differential Q_n : $\pi(D_0)$ = (Q_0) . Then $\text{deg} D_0 = n(2g - 2) + n \sum_{j=1}^n k_j$ i.e. $\text{deg} D_{br} + \text{deg} D_0 = m$ as expected. Let us enumerate their points as follows:

$$
D_{br} = \{x_i\}_{i=1}^{\deg D_{br}}, \qquad D_0 = \{x_i\}_{\deg D_{br}+1}^m.
$$

We now consider first the case of variations of the period matrix.

The map of $\mathcal{M}_{H}^{n}[{\bf k}]$ to $\mathcal{H}_{\hat{g}}(d_1,\ldots,d_{mn})$ is defined by assigning to a point of $\mathcal{M}_{H}^{n}[{\bf k}]$ the pair (\widehat{C}, v) ; for a generic point of $\mathcal{M}^n_H[\mathbf{k}]$ all zeros of v are simple.

Theorem 4.1. *The variations of the period matrix* Ω *with respect to the coordinates [\(2.7\)](#page-3-0) on* $\mathcal{M}_{H}^{n}[\mathbf{k}]$ are given by:

(4.2)
$$
\frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} = -2\pi i \sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)} (x_i) \operatorname{res}_{x_i} \frac{v_{\alpha}v_{\beta}}{v},
$$

(4.3)
$$
\frac{\partial \Omega_{\alpha\beta}}{\partial C_j^{(s),l}} = -2\pi i \sum_{x_i \in D_{br}} \frac{w_j^{(s),l}}{d\ln(v/d\xi)}(x_i) \operatorname{res}_{x_i} \frac{v_\alpha v_\beta}{v},
$$

(4.4)
$$
\frac{\partial \Omega_{\alpha\beta}}{\partial C_j^{(s),1}} = -2\pi i \sum_{x_i \in D_{br}} \frac{u_j^{(s)}}{d \ln(v/d\xi)} (x_i) \operatorname{res}_{x_i} \frac{v_\alpha v_\beta}{v}
$$

 $\,$ where in these formulæ $\,$ ξ denotes a local coordinate on C near $x_i^{\,3}$ $x_i^{\,3}$ $x_i^{\,3}$; the right-hand side of [\(4.5\)](#page-8-1) is independent of the choice of these coordinates near $x_i.$

The formula [\(4.2\)](#page-8-2) can be written alternatively in the following more familiar form:

(4.5)
$$
\frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} = -2\pi i \sum_{x_i \in D_{br}} \text{res}_{x_i} \frac{v_{\alpha} v_{\beta} v_{\gamma}}{d\xi d(v/d\xi)}
$$

and analogous versions of [\(4.3\)](#page-8-3) and [\(4.4\)](#page-8-4) where v_{γ} is replaced by $w^{(s), \ell}_j$ $_j^{(s),\ell}$ and $u_j^{(s)}$ $j^{(s)}$, respectively.

On the submanifold $\mathcal{M}_{H}^{n}[\mathbf{k}]$ of $\mathcal{H}_{\hat{g}}(\mathbf{d})$ we use the set of independent coordinates given by [\(2.7\)](#page-3-0) so that the period coordinates [\(3.3\)](#page-6-1) on $\mathcal{M}_{H}^{n}[{\bf k}]$ become functions of [\(2.7\)](#page-3-0) defined implicitly by the condition that the moduli of the base curve C are constants.

For the proof of Theorem [4.1](#page-8-5) we need the following Lemma.

³We did not carry in the notation the dependence on i for brevity of notation.

Lemma 4.2. *Denote by* s^k *a contour from the list [\(3.3\)](#page-6-1) which does not coincide with with a contour connecting* x_r *with* x_i *with* $x_i \in D_{br}$ (a branchpoint). The derivatives of the integrals of v over *the basis [\(3.3\)](#page-6-1) with respect to the coordinates [\(2.7\)](#page-3-0) are then given by*

(4.6)
$$
\frac{\partial(\int_{s_k} v)}{\partial z_j} = \int_{s_k} \frac{\partial v}{\partial z_j}
$$

where z^j *is any coordinate from the list [\(2.7\)](#page-3-0) and the periods of the right-hand side are given by standard formulæ taking into account [\(2.13\)](#page-4-2), [\(2.14\)](#page-4-3), [\(2.15\)](#page-4-4).*

If x_i is a branch point then the derivatives have the following additional contributions: $\,$

(4.7)
$$
\frac{\partial(\int_{x_r}^{x_i} v)}{\partial A_\alpha} = \int_{x_r}^{x_i} v_\alpha - \frac{v_\alpha}{d \ln(v/d\xi)}(x_i) ,
$$

(4.8)
$$
\frac{\partial(\int_{x_r}^{x_i} v)}{\partial C_j^{(s), \ell}} = \int_{x_r}^{x_i} w_j^{(s), \ell} - \frac{w_j^{(s), \ell}}{d \ln(v/d\xi)}(x_i) ,
$$

(4.9)
$$
\frac{\partial(\int_{x_r}^{x_i} v)}{\partial C_j^{(s),1}} = \int_{x_r}^{x_i} u_j^{(s)} - \frac{u_j^{(s)}}{d \ln(v/d\xi)}(x_i)
$$

where the coordinate ξ *is assumed to be invariant under the deformation. The expressions [\(4.7\)](#page-9-0)- [\(4.9\)](#page-9-1) are independent of the choice of local coordinate* ξ *on* C*.*

Proof. We start from [\(4.6\)](#page-9-2): if the contour s is closed (i.e. coincides with one of a- or b-cycles or a small contour around one of $y_k^{(s)}$ $\binom{S}{k}$ then the differentiation commutes with integration. If s connects x_r with another zero x_i which is not a branch point of \widehat{C} then s can be projected on C , and in a local coordinate on C the integrand vanishes at both endpoints. Therefore, the differentiation commutes with integration in this case, too.

The only case when the dependence of the endpoint on the differentiation variable gives a non-trivial contribution is the case when s connects x_r with one of the branch points x_i of \widehat{C} . Below we prove [\(4.7\)](#page-9-0); the proof of [\(4.8\)](#page-9-3) and [\(4.9\)](#page-9-1) is almost identical.

Let $x_i \in \widehat{C}$ be a ramification point of \widehat{C} and $\xi_i = \xi(\pi(x_i)) \in C$ be the corresponding critical value in some local coordinate ξ on C which remains fixed under deformation of \hat{C} ; let $\zeta = \xi - \xi_i$ be a coordinate on C vanishing at $\pi(x_i)$ (the coordinate ζ deforms when \widehat{C} varies). A suitable local coordinate on \widehat{C} near x_i can then be chosen to be $\widehat{\zeta}(x) = \zeta^{1/2}$.

Then the differentiation with respect to A_{α} of the endpoint also gives a contribution to $\partial z_k/\partial A_\alpha$ and we get

(4.10)
$$
\frac{\partial(\int_{x_r}^{x_i} v)}{\partial A_\alpha} = \int_{x_r}^{x_i} v_\alpha + \frac{\partial \xi_i}{\partial A_\alpha} \frac{v}{d\xi}(\xi_i)
$$

for $k = 1, \ldots, \text{deg} D_{br}$.

To compute the derivative $\partial \xi_i/\partial A_\alpha$ we follow [\[3\]](#page-20-12) and we write $v(\xi)$ near ξ_i in the form

$$
(4.11) \t v = (a + b\sqrt{\xi - \xi_i} + \dots) d\xi
$$

(recall that v has simple zero in the local parameter $\sqrt{\xi - \xi_i}$ and $d\xi$ has already a simple zero). Thus

(4.12)
$$
b = \frac{d(v/d\xi)}{d\hat{\zeta}_i}\Big|_{\xi=\xi_i} = \frac{d(v/d\xi)}{d\sqrt{\xi-\xi_i}}\Big|_{\xi=\xi_i}
$$

and

(4.13)
$$
v_{\alpha} = \frac{\partial v}{\partial A_{\alpha}} = \left(a_{A_{\alpha}} - \frac{\xi_{iA_{\alpha}}}{2\sqrt{\xi - \xi_{i}}}b + \dots \right) d\xi.
$$

Therefore,

(4.14)
$$
-\left.\frac{v_{\alpha}}{d\sqrt{\xi-\xi_i}}\right|_{\xi=\xi_i}=b\frac{\partial\xi_i}{\partial A_{\alpha}}
$$

and,

(4.15)
$$
\frac{\partial \xi_i}{\partial A_\alpha} = -\frac{v_\alpha/d\hat{\zeta}_i}{(v/d\xi)_{\hat{\zeta}_i}}(x_i) .
$$

Now [\(4.10\)](#page-9-4) takes the form

(4.16)
$$
\frac{\partial \int_{x_r}^{x_i} v}{\partial A_\alpha} = \int_{x_r}^{x_i} v_\alpha - \frac{v_\alpha / d\hat{\zeta}_i}{[\ln(v/d\xi)]_{\hat{\zeta}_i}}(x_i)
$$

for $i = 1, \ldots, \deg D_{br}$. This proves the lemma.

Proof of Theorem [4.1](#page-8-5). Let us prove [\(4.2\)](#page-8-2); the proofs of [\(4.3\)](#page-8-3) and [\(4.4\)](#page-8-4) are parallel.

On the space $\mathcal{M}^n_H[{\bf k}]$ the periods B_γ and $\int_{x_r}^{\bar{x_i}} v$ become functions of $\{A_\gamma\}_{\gamma=1}^{\hat{g}}.$ Therefore one can compute derivatives of the period matrix using the chain rule:

$$
(4.17) \qquad \frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} = \frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} \Big|_{B_{\gamma},(\int_{x_r}^{x_i} v) = const} + \sum_{\delta=1}^{\hat{g}} \frac{\partial \Omega_{\alpha\beta}}{\partial B_{\delta}} \frac{\partial B_{\delta}}{\partial A_{\gamma}} + \sum_{i=1}^{r-1} \frac{\partial \Omega_{\alpha\beta}}{\partial (\int_{x_r}^{x_i} v)} \frac{\partial (\int_{x_r}^{x_i} v)}{\partial A_{\gamma}}
$$

(since A_{α} and the residues of v are independent coordinates we omit the term involving these derivatives). Using [\(2.13\)](#page-4-2), [\(4.7\)](#page-9-0) together with variational formulæ [\(3.8\)](#page-6-2)

$$
(4.18) \qquad \frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} = -\int_{b_{\gamma}} \frac{v_{\alpha}v_{\beta}}{v} , \qquad \frac{\partial \Omega_{\alpha\beta}}{\partial B_{\gamma}} = \int_{a_{\gamma}} \frac{v_{\alpha}v_{\beta}}{v} , \qquad \frac{\partial \Omega_{\alpha\beta}}{\partial(\int_{x_r}^{x_i} v)} = 2\pi i \mathop{\rm res}_{x_i} \frac{v_{\alpha}v_{\beta}}{v}
$$

(where x_i runs through the set of all zeros of v) we rewrite [\(4.17\)](#page-10-0) as follows:

$$
\frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} = \sum_{\delta=1}^{\hat{g}} \left[-\left(\int_{a_{\delta}} v_{\gamma} \right) \left(\int_{b_{\delta}} \frac{v_{\alpha} v_{\beta}}{v} \right) + \left(\int_{b_{\delta}} v_{\gamma} \right) \left(\int_{a_{\delta}} \frac{v_{\alpha} v_{\beta}}{v} \right) \right] +
$$
\n
$$
(4.19) \qquad \qquad + 2\pi i \sum_{i=1}^{r-1} \left(\int_{x_r}^{x_i} v_{\gamma} \right) \left(\operatorname{res}_{x_i} \frac{v_{\alpha} v_{\beta}}{v} \right) - 2\pi i \sum_{i=1}^{\text{deg} D_{br}} \frac{v_{\gamma}}{d \ln(v/d\zeta)} (x_i) \operatorname{res}_{x_i} \frac{v_{\alpha} v_{\beta}}{v}
$$

Due to the Riemann bilinear identity the sum of the first three terms in [\(4.19\)](#page-10-1) vanishes. The remaining terms give [\(4.2\)](#page-8-2).

The formulas [\(4.3\)](#page-8-3) and [\(4.4\)](#page-8-4) are obtained in a similar way by applying Riemann bilinear identities to the pairs $(w^{(s),\ell}_i)$ $_j^{(s),\ell}, \frac{v_\alpha v_\beta}{v}$ $\frac{v^{v_\beta}}{v}$) and $(u^{(s)}_j)$ $\frac{v_{\alpha}v_{\beta}}{v}$ (v^{α}_{ν}) , respectively.

We give below the computation leading to [\(4.4\)](#page-8-4); the proof of [\(4.3\)](#page-8-3) requires only minimal modifications. Taking into account [\(4.9\)](#page-9-1) we get (recall that all $\it{a}\textrm{-periods}$ of $\it{u}_{i}^{(s)}$ $j^{\langle s \rangle}$ vanish)

.

$$
(4.20)\quad \frac{d\Omega_{\alpha\beta}}{dC_j^{(s),1}} = \frac{\partial \Omega_{\alpha\beta}}{\partial C_j^{(s),1}} + \sum_{\delta=1}^{\hat{g}} \left[\frac{\partial \Omega_{\alpha\beta}}{\partial A_{\delta}} \frac{\partial A_{\delta}}{\partial C_j^{(s),1}} + \frac{\partial \Omega_{\alpha\beta}}{\partial B_{\delta}} \frac{\partial B_{\delta}}{\partial C_j^{(s),1}} \right] + \sum_{k=1}^{r-1} \frac{\partial \Omega_{\alpha\beta}}{\partial (\int_{x_r}^{x_i} v)} \frac{\partial (\int_{x_r}^{x_i} v)}{\partial C_j^{(s),1}}
$$

We have $\partial A_\delta/\partial C_j^{(s),1}=0$ since all a -periods of $u_j^{(s)}$ $j_j^{(s)}$ vanish; according to [\(3.8\)](#page-6-2),

$$
\frac{\partial \Omega_{\alpha\beta}}{\partial C_j^{(s),1}} = -2\pi i \int_{y_1^{(1)}}^{y_j^{(s)}} \frac{v_\alpha v_\beta}{v} ,
$$

which gives

(4.21)
$$
\frac{d\Omega_{\alpha\beta}}{dC_j^{(s),1}} = -2\pi i \int_{y_1^{(1)}}^{y_j^{(s)}} \frac{v_\alpha v_\beta}{v} + \sum_{\delta=1}^{\hat{g}} \left[\left(\int_{b_\delta} u_j^{(s)} \right) \left(\int_{a_\delta} \frac{v_\alpha v_\beta}{v} \right) \right]
$$

$$
(4.22) \t+ 2\pi i \sum_{i=1}^{r-1} \left(\underset{x_i}{\text{res}} \frac{v_\alpha v_\beta}{v} \right) \left(\int_{x_r}^{x_i} u_j^{(s)} \right) - 2\pi i \sum_{i=1}^{\text{deg} D_{br}} \frac{u_j^{(s)}}{d \ln(v/d\zeta)} (x_i) \underset{x_i}{\text{res}} \frac{v_\alpha v_\beta}{v}.
$$

Again, the Riemann bilinear identities applied to the pair of differentials of third kind $u_i^{(s)}$ $j^{(s)}_j$ and $\frac{v_\alpha v_\beta}{v}$ prove the vanishing of the sum of all terms except the last one, leading to (4.4) (we notice that these two differentials have different positions of poles).

4.1. **Variations of Abelian differentials.** Here we are going to use variational formulæ [\(3.10\)](#page-7-1)-[\(3.12\)](#page-7-2) on moduli spaces of Abelian differentials to derive the following analogs of Theorem [4.1.](#page-8-5)

Theorem 4.3. *The variations of canonical differentials* v_α *with respect to coordinates* [\(2.7\)](#page-3-0) *on* $\mathcal{M}_{H}^{n}[\mathbf{k}]$ are expressed by the following formulæ:

(4.23)
$$
\frac{\partial v_{\alpha}(x)}{\partial A_{\gamma}} = -\sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)} (x_i) \operatorname{res}_{t=x_i} \frac{v_{\alpha}(t)B(t,x)}{v(t)}
$$

(4.24)
$$
\frac{\partial v_{\alpha}(x)}{\partial C_j^{(s),\ell}} = - \sum_{x_i \in D_{br}} \frac{w_j^{(s),\ell}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \frac{v_{\alpha}(t)B(t,x)}{v(t)}
$$

(4.25)
$$
\frac{\partial v_{\alpha}(x)}{\partial C_j^{(s),1}} = -\sum_{x_i \in D_{br}} \frac{u_j^{(s)}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \frac{v_{\alpha}(t)B(t,x)}{v(t)}
$$

where ξ *is a local coordinate on* C *near* xⁱ *as in Theorem [4.1.](#page-8-5) The right-hand side of [\(4.5\)](#page-8-1) is independent of the choice of these coordinates near* x_r .

Proof. Let us show how to derive [\(4.23\)](#page-11-1) from the variational formulæ [\(3.10\)](#page-7-1). In comparison with the variational formulæ for Ω proven above it is essential to carefully consider the dependence of v_α on the point of \hat{C} , since the latter is deforming. Moreover, the variation of v_α with respect to \mathcal{P}_j used in [\(3.10\)](#page-7-1) is defined by [\(3.9\)](#page-7-3) where the "flat" coordinate is kept fixed, while in [\(4.23\)](#page-11-1) the differentiation is performed according to the rule [\(2.12\)](#page-4-0) where ξ is a local parameter lifted to \widehat{C} from C which is assumed to be independent of moduli coordinates on $\mathcal{M}_{H}^{n}[\mathbf{k}]$.

Taking into account these differences, one can compute the left-hand side of [\(4.23\)](#page-11-1) as follows. Let $f_{\alpha}(x) = v_{\alpha}/v$: then the left-hand side of [\(4.23\)](#page-11-1) is rewritten as

(4.26)
\n
$$
\frac{\partial v_{\alpha}(x)}{\partial A_{\gamma}} = \frac{\partial (v f_{\alpha}(x))}{\partial A_{\gamma}} = \frac{\partial f_{\alpha}(x)}{\partial A_{\gamma}} \Big|_{\xi(x)} v(x) + f_{\alpha}(x) \frac{\partial v(x)}{\partial A_{\gamma}} \Big|_{\xi(x)}
$$
\n
$$
= \frac{\partial f_{\alpha}(x)}{\partial A_{\gamma}} \Big|_{z(x)} v(x) + \frac{\partial f_{\alpha}(x)}{\partial z(x)} \frac{\partial z(x)}{\partial A_{\gamma}} \Big|_{\xi(x)} v(x) + f_{\alpha}(x) v_{\gamma}(x)
$$
\n
$$
= \frac{\partial f_{\alpha}(x)}{\partial A_{\gamma}} \Big|_{z(x)} + \mathcal{A}_{\gamma}(x) d\left(\frac{v_{\alpha}}{v}\right) + \frac{v_{\alpha} v_{\gamma}}{v}(x)
$$

where $\mathcal{A}_{\gamma}(x) = \int_{x_r}^x v_{\gamma}$ is the component $\gamma \in \{1, \ldots, \hat{g}\}$ of the Abel map.

The computation of the first term in [\(4.26\)](#page-12-1) can then be performed in complete analogy to [\(4.19\)](#page-10-1) with the differential $\frac{v_{\alpha}v_{\beta}}{v}(t)$ replaced by the differential $\frac{1}{2\pi i}$ $v_{\alpha}(t)B(x,t)$ $\frac{\partial D(x,t)}{\partial(t)}$. Applying the Riemann bilinear relations to the differentials v_{γ} and $\frac{1}{2\pi i}$ $\frac{v_{\alpha}(t)B(x,t)}{v(t)}$ we obtain the sum of terms entering the right-hand side of [\(4.23\)](#page-11-1) minus the residue of $\frac{1}{2\pi i}$ $v_\alpha(t)B(x,t)$ $\frac{\partial v(B(x,t))}{\partial v(t)} \int_{x_r}^t v_{\gamma}$ at $t = x$. This residue is equal to the sum of the last two terms in [\(4.26\)](#page-12-1) with opposite sign. This gives [\(4.23\)](#page-11-1). The proofs of the formulæ [\(4.24\)](#page-11-2) and [\(4.25\)](#page-11-3) are parallel.

4.2. **Variations of prime-form and canonical bidifferential.** Variational formulæ for $E(x, y)$ and $B(x, y)$ can be proven in parallel to Th[.4.3.](#page-11-4)

As in the case of normalized canonical differential, we define the derivative of $B(x, y)$ and $E(x, y)$ with respect to any coordinate z_i on $\mathcal{M}_{H}^{n}[{\bf k}]$ as

(4.27)
$$
\frac{\partial B(x,y)}{\partial z_i} = \frac{\partial}{\partial z_i} \left(\frac{B(x,y)}{d\xi(x)d\xi(y)} \right) d\xi(x)d\xi(y)
$$

(4.28)
$$
\frac{\partial E(x,y)}{\partial z_i} = \frac{\partial}{\partial z_i} \left(E(x,y) [d\xi(x) d\zeta(y)]^{1/2} \right) [d\xi(x) d\zeta(y)]^{-1/2}
$$

where $\xi(x)$ and $\xi(y)$ are local coordinates lifted to \widehat{C} from moduli-independent local coordinates on C, and these coordinates remain fixed under differentiation.

Theorem 4.4. *The variations of the canonical bidifferential* $B(x, y)$ *with respect to the coordinates* (2.7) on $\mathcal{M}_{H}^{n}[\mathbf{k}]$ are given by:

(4.29)
$$
\frac{\partial B(x,y)}{\partial A_{\gamma}} = -\sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)} (x_i) \operatorname{res}_{t=x_i} \frac{B(x,t)B(t,y)}{v(t)}
$$

(4.30)
$$
\frac{\partial B(x,y)}{\partial C_j^{(s),\ell}} = - \sum_{x_i \in D_{br}} \frac{w_j^{(s),\ell}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \frac{B(x,t)B(t,y)}{v(t)}
$$

(4.31)
$$
\frac{\partial B(x,y)}{\partial C_j^{(s),1}} = - \sum_{x_i \in D_{br}} \frac{u_j^{(s)}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \frac{B(x,t)B(t,y)}{v(t)}
$$

where ξ *is a local coordinate on* C *near* xⁱ *; the right-hand side of [\(4.5\)](#page-8-1) is independent on the choice of these coordinates near* x_r .

Theorem 4.5. *The variations of the prime-form with respect to coordinates* [\(2.7\)](#page-3-0) on $\mathcal{M}_{H}^{n}[\mathbf{k}]$ are *given by:*

(4.32)
$$
\frac{\partial \ln E(x, y)}{\partial A_{\gamma}} = -\frac{1}{2} \sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)} (x_i) \operatorname{res}_{t=x_i} \frac{1}{v(t)} \left[d_t \ln \frac{E(x, t)}{E(y, t)} \right]^2
$$

(4.33)
$$
\frac{\partial \ln E(x, y)}{\partial C_j^{(s), \ell}} = -\frac{1}{2} \sum_{x_i \in D_{br}} \frac{w_j^{(s), \ell}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \frac{1}{v(t)} \left[d_t \ln \frac{E(x, t)}{E(y, t)} \right]^2
$$

(4.34)
$$
\frac{\partial \ln E(x, y)}{\partial C_j^{(s), 1}} = -\frac{1}{2} \sum_{x_i \in D_{br}} \frac{u_j^{(s)}}{d \ln(v/d\xi)}(x_i) \underset{t=x_i}{\text{res}} \left\{ \frac{1}{v(t)} \left[d_t \ln \frac{E(x, t)}{E(y, t)} \right]^2 \right\}
$$

where ξ *is a local coordinate on* C *near* xⁱ *; the right-hand side of [\(4.5\)](#page-8-1) is independent of the choice of these coordinates near* x_i *.*

4.3. **The Bergman tau-function on spaces of spectral covers.** The Bergman tau function on the moduli spaces of Abelian differentials is a natural higher genus analog of Dedekind's eta-function [\[17,](#page-21-3) [14,](#page-20-4) [19\]](#page-21-1). One can define two natural tau functions associated to the moduli space of spectral covers; in the case of holomorphic v these tau functions were introduced in [\[18\]](#page-21-2) and used to study the Picard group of the moduli spaces (in [\[18\]](#page-21-2) we considered the tau functions on universal spaces of spectral covers i.e. we allowed the base curve C to vary).

Here we restrict ourselves to the case of holomorphic v, namely, to moduli space \mathcal{M}_{H}^{n} of spectral covers of the ordinary Hitchin systems. In this case the equations for the Bergman tau functions take a similar form to the variational formulæ for the canonical objects considered above.

Denote the moduli space of ordinary Hitchin's spectral covers by \mathcal{M}_{H}^{n} ; in this case all coefficients Q_k of the equation [\(2.1\)](#page-2-1) are holomorphic k-differentials, the genus of the spectral cover is $\hat{g} = n^2(g-1) + 1$, the number of branch points is $p = n(n-1)(2g-2)$ and the total number of zeros of v is $r = 2\hat{g} - 2 = p + 2n(g - 1)$. The differential v is holomorphic, and the local coordinates on \mathcal{M}_{H}^{n} are given by the *a*-periods $A_{\gamma} = \int_{a_{\gamma}} v$.

Considering \mathcal{M}_{H}^{n} as a subspace of the space of holomorphic Abelian differentials with simple zeros $\mathcal{H}_{\hat{g}}$ we define the Bergman tau function on \mathcal{M}^n_H by restriction of the Bergman tau function [\(3.13\)](#page-7-4) from $\mathcal{H}_{\hat{a}}$.

The resulting equations for τ_B (this tau function is defined by the formula (4.3) of [\[18\]](#page-21-2)) as function of periods A_{γ} can be derived from [\(3.13\)](#page-7-4) in analogy to [\(4.2\)](#page-8-2):

Proposition 4.6. *The Bergman tau-function* $\tau_B(\widehat{C}, v)$ *on the space* \mathcal{M}_{H}^{n} *satisfies the following system of equations*

$$
(4.35) \qquad \frac{\partial \ln \tau_B}{\partial A_{\gamma}} = -2\pi i \sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)} (x_i) \operatorname{res}_{x_i} \left(\frac{B_{reg}^v}{v} \right) - \frac{\pi i}{8} \sum_{i=1}^r \operatorname{res}_{x=x_i} \left(\frac{v_{\gamma}(x)}{\int_{x_i}^x v} \right) .
$$

Proof. In parallel to [\(4.19\)](#page-10-1) we have, applying the chain rule to the equations [\(3.13\)](#page-7-4) and using [\(2.13\)](#page-4-2), [\(4.7\)](#page-9-0):

(4.36)
$$
\frac{\partial \ln \tau}{\partial A_{\gamma}} = \sum_{\delta=1}^{\hat{g}} \left[-\left(\int_{a_{\delta}} v_{\gamma} \right) \left(\int_{b_{\delta}} \frac{B_{reg}}{v} \right) + \left(\int_{b_{\delta}} v_{\gamma} \right) \left(\int_{a_{\delta}} \frac{B_{reg}}{v} \right) \right]
$$

$$
(4.36) \qquad + 2\pi i \sum_{i=1}^{r-1} \left(\int_{x_r}^{x_i} v_{\gamma} \right) \left(\underset{x_i}{\text{res}} \frac{B_{reg}}{v} \right) - 2\pi i \sum_{i=1}^{\text{deg}} \frac{v_{\gamma}}{d \ln(v/d\zeta)} (x_i) \underset{x_i}{\text{res}} \frac{B_{reg}}{v}
$$

Using Riemann bilinear relations the first sum in [\(4.36\)](#page-14-0) equals to the sum of the residues as follows

(4.37)
$$
-2\pi i \sum_{i=1}^r \underset{x_i}{\text{res}} \left(\frac{B_{reg}}{v} \int_{x_r}^x v_\gamma \right).
$$

The main difference with the proof of the variational formula [\(4.2\)](#page-8-2) for the period matrix is that the poles of $\frac{B_{reg}}{v}$ are of order 3 (as we see below) which leads to extra terms while computing the residues. Let us now represent B_{reg} via difference of two projective connections [\[14\]](#page-20-4):

(4.38)
$$
B_{reg}(x,x) = \frac{1}{6}(S_B - S_v)
$$

where S_B is the Bergman projective connection (this projective connection is holomorphic; it equals to the constant term in the asymptotics of $B(x, y)$ on the diagonal equals (1/6) S_B) and S_v is the meromorphic projective connection given by the Schwarzian derivative

$$
S_v(\xi) = \left\{ \int^x v, \xi \right\} = \left(\frac{v'}{v}\right)' - \frac{1}{2} \left(\frac{v'}{v}\right)^2
$$

in any local coordinate ξ on \widehat{C} . In a neighbourhood of a zero x_i of v we choose ξ such that $v = \xi d\xi$; then near x_i we have

$$
\frac{1}{6}\frac{S_B-S_v}{v}=\left(\frac{1}{4\xi^3}+\frac{1}{6\xi}S_B(\xi)\right)d\xi
$$

and

$$
\operatorname{res}_{x_i} \left(\frac{B_{reg}}{v} \int_{x_r}^x v_\gamma \right) = \left(\int_{x_r}^{x_i} v_\gamma \right) \operatorname{res}_{x_i} \frac{B_{reg}}{v} + \frac{1}{8} \left(\frac{v_\gamma}{d\xi} \right)_{\xi} (x_i)
$$

$$
= \left(\int_{x_r}^{x_i} v_\gamma \right) \operatorname{res}_{x_i} \frac{B_{reg}}{v} + \frac{1}{16} \operatorname{res}_{x=x_i} \left(\frac{v_\gamma(x)}{\int_{x_i}^x v} \right)
$$

and [\(4.36\)](#page-14-0) equals to

$$
-2\pi i \sum_{i=1}^{\deg D_{br}} \frac{v_{\gamma}}{d\ln(v/d\zeta)}(x_i) \operatorname{res}_{x_i} \frac{B_{reg}}{v} - \frac{\pi i}{8} \sum_{i=1}^r \operatorname{res}_{x=x_i} \left(\frac{v_{\gamma}(x)}{\int_{x_i}^x v}\right)
$$

which gives (4.35) .

5. Higher order derivatives on $\mathcal{H}_{\hat{g}}$ and \mathcal{M}_{H}^{n}

5.1. **Space** $\mathcal{H}_{\hat{g}}$. The higher order derivatives with respect to moduli on the space $\mathcal{H}_{\hat{q}}$ can be obtained by a simple iteration of first derivatives.

Let us consider first the multiple derivatives of the Bergman tau function. Using the coordinates $\mathcal{P}_i = \int_{s_i} v$, where $s_i \in H_1(\widehat{C}, \{x_i\}_{i=1}^r)$, and referring to [\(3.13\)](#page-7-4) and [\(3.11\)](#page-7-5) we find:

(5.1)
$$
\frac{\partial^2}{\partial \mathcal{P}_i \partial \mathcal{P}_j} \ln \tau_B = \frac{1}{2\pi i} \text{symm}_{i,j} \int_{s_i^*} \int_{s_j^*} \frac{B^2(x, y)}{v(x)v(y)}
$$

where the symmetrization is $1/2$ of the sum of the (ij) and (ji) terms. The symmetrization is necessary if the contours s_i^* and s_j^* have non-zero intersection index (see formulas (3.5) and (3.6) of [\[14\]](#page-20-4) for details about the extra term associated to the intersection point if the symmetrization is not assumed).

Further differentiation using [\(3.11\)](#page-7-5) gives

(5.2)
$$
\frac{\partial^3}{\partial \mathcal{P}_i \partial \mathcal{P}_j \partial \mathcal{P}_k} \ln \tau_B = \frac{2}{(2\pi i)^2} \text{symm}_{(i,j,k)} \int_{x \in s_i^*} \int_{y \in s_j^*} \int_{t \in s_k^*} \frac{B(x,y)B(x,t)B(t,y)}{v(x)v(y)v(t)}
$$

where the symmetrization is again understood as averaging over the 6 permutations of (i, j, k) . The *n*th derivatives of τ_B are given by

(5.3)
$$
\frac{\partial^{(n)}}{\partial \mathcal{P}_{i_1} \dots \partial \mathcal{P}_{i_n}} \ln \tau_B = \frac{1}{(2\pi i)^{n-1}} \text{symm}_{(i_1,\dots,i_n)} \int_{s_{i_1}^*} \dots \int_{s_{i_n}^*} \mathcal{Q}_n(z_1,\dots,z_n)
$$

where the completely symmetric multi-differential \mathcal{Q}_n is given by

(5.4)
$$
Q_n(z_1,\ldots,z_n) = 2 \sum_{all \Gamma} \frac{\prod_{j=1}^n B(z_{k_j}, z_{k_{j+1}})}{\prod_{j=1}^n v(z_j)}
$$

The sum runs over all $(n-1)!/2$ permutations $\Gamma = (k_1, \ldots, k_n)$ of z_1, \ldots, z_n which form a cycle of length n (two such permutations are considered equivalent if they are related by cyclic permutation i.e. we do not distinguish between (1234) and (2341)); k_{n+1} is identified with k_1 .

The multi-differentials $Q_n(z_1, \ldots, z_n)$ satisfy the relations

(5.5)
$$
\frac{\partial}{\partial \mathcal{P}_i} \mathcal{Q}_n(z_1,\ldots,z_n) = \frac{1}{2\pi i} \int_{t \in s_i^*} \mathcal{Q}_{n+1}(z_1,\ldots,z_n,t) .
$$

Another natural hierarchy of multi-differentials (although no longer completely symmetric) which are given by combinations of $B(x, y)$ can be obtained by differentiation of $B(x, y)$ itself. Namely, using the variational formula [\(3.11\)](#page-7-5) on the space $\mathcal{H}_{\hat{q}}$ we get

$$
(5.6) \qquad \frac{\partial^n}{\partial \mathcal{P}_{i_1} \dots \partial \mathcal{P}_{i_n}} B(x,y) = \frac{1}{(2\pi i)^n} \text{symm}_{(i_1,\dots,i_n)} \int_{s_{i_1}^*} \dots \int_{s_{i_n}^*} \mathcal{R}_{n+2}(x,z_1,\dots,z_n,y)
$$

where the multi-differentials \mathcal{R}_n with n arguments are given by

(5.7)
$$
\mathcal{R}_n(z_1,\ldots,z_n) = \sum_{all \ \tilde{\Gamma}} \frac{\prod_{j=1}^{n-1} B(z_{k_j}, z_{k_{j+1}})}{\prod_{j=2}^{n-1} v(z_j)}
$$

where in all products entering this sum, the indices k_1, k_n are given by $k_1 = 1$ and $k_n = n$; the sum goes over all $(n-2)!$ paths $\overline{\Gamma}$ connecting z_1 with z_n which go only once through every vertex representing the other arguments z_2, \ldots, z_{n-1} .

The multi-differentials \mathcal{R}_n are symmetric is under permutations of the intermediate arguments x_2, \ldots, x_{n-1} , but not fully symmetric, in contrast to \mathcal{Q}_n .

The families of multi-differentials \mathcal{Q}_n and \mathcal{R}_n as well as their variational formulæ resemble the structures arising in the framework of topological recursion of [\[8\]](#page-20-11) (the genus of the base curve C equals zero in the constructions of [\[8\]](#page-20-11)). Moreover, both Q_n 's and \mathcal{R}_n 's have second order poles when any two arguments coincide (in addition to generically simple poles at the branch points), while the multi-differentials W_n of [\[8\]](#page-20-11) have poles only at the ramification points of the cover.

The formula [\(5.6\)](#page-16-0) implies the following expression for the multiple derivatives of the period matrix Ω of \widehat{C} on $\mathcal{H}_{\widehat{g}}$:

(5.8)
$$
\frac{\partial^{(n)}}{\partial P_{i_1} \dots \partial P_{i_n}} \Omega_{\alpha\beta} = \frac{1}{(2\pi i)^{n-1}} \text{symm}_{(i_1,\dots,i_n)} \int_{s_{i_1}^*} \dots \int_{s_{i_n}^*} \mathcal{R}_n^{\alpha\beta}(z_1,\dots,z_n)
$$

where

$$
\mathcal{R}_n^{\alpha\beta}(z_1,\ldots,z_n)=\int_{x\in b_\alpha}\int_{y\in b_\beta}\mathcal{R}_{n+2}(x,z_1,\ldots,z_n,y)
$$

or

(5.9)
$$
\mathcal{R}_n^{\alpha\beta}(z_1,\ldots,z_n) = v_\alpha(z_1)v_\beta(z_n) \sum_{all \ \tilde{\Gamma}} \frac{\prod_{j=1}^{n-1} B(z_{k_j}, z_{k_{j+1}})}{\prod_{j=1}^n v(z_j)}
$$

where, as before, in all products entering this sum $k_1 = 1$ and $k_n = n$; the sum goes over all $(n-2)!$ paths Γ connecting x_1 with x_n which go only once through every vertex representing other arguments x_2, \ldots, x_{n-1} .

5.2. **The space** \mathcal{M}_{H}^{n} . On the spaces of spectral covers \mathcal{M}_{H}^{n} the multi-differentials \mathcal{Q}_{n} are related to \mathcal{Q}_{n+1} by formulæ which can be derived from [\(5.5\)](#page-16-1) in parallel to the proof of [\(4.23\)](#page-11-1):

$$
(5.10) \qquad \frac{\partial}{\partial A_{\gamma}} \mathcal{Q}_n(z_1,\ldots,z_n) = -\sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \left\{ \frac{1}{v(t)} \mathcal{Q}_{n+1}(z_1,\ldots,z_n,t) \right\}
$$

Similarly, the multi-differentials \mathcal{R}_n and \mathcal{R}_{n+1} are related by

$$
(5.11) \quad \frac{\partial}{\partial A_{\gamma}} \mathcal{R}_n(z_1,\ldots,z_n) = -\sum_{x_i \in D_{br}} \frac{v_{\gamma}}{d \ln(v/d\xi)}(x_i) \operatorname{res}_{t=x_i} \left\{ \frac{1}{v(t)} \mathcal{R}_{n+1}(z_1,\ldots,z_{n-1},t,z_n) \right\}
$$

Integrating [\(5.10\)](#page-17-1) over two *b*-cycles with respect to z_1 and z_2 we get similar formulæ for $\mathcal{R}_n^{\alpha\beta}$ [\(5.9\)](#page-16-2).

While higher derivatives of the period matrix, tau-function and canonical bidifferential on the space $\mathcal{H}_{\hat{q}}$ are given by a simple formulæ [\(5.3\)](#page-15-2), [\(5.6\)](#page-16-0) and [\(5.8\)](#page-16-3) their restriction to the space \mathcal{M}_{H}^{n} is much less trivial. As an example of such computation we find below the second derivatives of the period matrix.

5.2.1. *Second derivatives of* $\Omega_{\alpha\beta}$. The period matrix on the space $\Omega_{\alpha\beta}$ is known to be given by second derivatives of a single function (the "prepotential")

(5.12)
$$
F_0 = \frac{1}{2} \sum_{\gamma=1}^{\hat{g}} A_{\gamma} B_{\gamma} :
$$

$$
\Omega_{\alpha\beta} = \frac{\partial^2 F_0}{\partial A_{\alpha} \partial A_{\beta}}.
$$

We recall the proof of [\(5.12\)](#page-17-2): using the relation $\partial B_{\gamma}/\partial A_{\alpha} = \Omega_{\alpha\beta}$ we get

$$
\frac{\partial F_0}{\partial A_\alpha} = \frac{1}{2} \left(B_\alpha + \sum_{\gamma=1}^{\hat{g}} A_\gamma \Omega_{\alpha \gamma} \right)
$$

and

$$
\frac{\partial^2 F_0}{\partial A_\alpha \partial A_\beta} = \Omega_{\alpha\beta} + \frac{1}{2} \sum_{\gamma=1}^{\hat{g}} A_\gamma \frac{\partial \Omega_{\alpha\gamma}}{\partial A_\beta}.
$$

The last sum in this formula equals zero: indeed the formula [\(4.5\)](#page-8-1) for $\partial\Omega_{\alpha\beta}/\partial A_{\gamma}$ implies that this tensor is invariant under permutations of the indices α, β, γ and thus we have

(5.13)
$$
\sum_{\gamma=1}^{\hat{g}} A_{\gamma} \frac{\partial \Omega_{\alpha \gamma}}{\partial A_{\beta}} = \left(\sum_{\gamma=1}^{\hat{g}} A_{\gamma} \frac{\partial}{\partial A_{\gamma}} \right) \Omega_{\alpha \beta}.
$$

The last expression vanishes because it is the action of the scaling operator $\mathbb{E}=\sum_{\gamma=1}^{\hat{g}}A_{\gamma}\frac{\partial}{\partial A}$ ∂A_γ generating the map $v \mapsto \lambda v$ ($\lambda \in \mathbb{C}^{\times}$) and the period matrix is clearly invariant under such rescaling.

Due to [\(5.12\)](#page-17-2) all higher derivatives of $\Omega_{\alpha\beta}$ in A_{γ} 's are also completely symmetric with respect to all indices. It is convenient to use the following notation:

$$
y=\frac{v}{d\xi}
$$

which is a function defined on the union of small disks on \widehat{C} around ramification points x_i depending on the choice of local parameter ξ on C near each branch point x_i . Since v has a simple zero at x_i , in a neighbourhood of x_i y is a holomorphic function of the corresponding local parameter $\hat{\xi}_i = \sqrt{\xi - \xi(x_i)}$.

To compute second derivatives of $\Omega_{\alpha\beta}$ on \mathcal{M}^n_H one can differentiate the formula [\(4.5\)](#page-8-1)

(5.14)
$$
\frac{\partial \Omega_{\alpha\beta}}{\partial A_{\gamma}} = -2\pi i \sum_{x_i \in D_{br}} \operatorname{res}_{x_i} \frac{v_{\alpha} v_{\beta} v_{\gamma}}{d \xi dy}
$$

with respect to the coordinate A_{δ} using [\(2.13\)](#page-4-2) and [\(4.23\)](#page-11-1). Then due to (4.23) we have

$$
\frac{\partial (dy)}{\partial A_{\delta}} = d \left(\frac{v_{\delta}}{d \xi} \right)
$$

which has second order pole at x_i ($d\xi$ has a simple zero at the ramification point, and, therefore, $v_{\delta}/d\xi$ has a simple pole at x_i). We have then

$$
\frac{\partial^2 \Omega_{\alpha\beta}}{\partial A_{\delta} \partial A_{\gamma}} = 2\pi i \sum_{x_i, x_j \in Br} \underset{t=x_i}{\text{res}} \underset{\tilde{t}=x_j}{\text{res}} \left\{ B(t, \tilde{t}) \frac{v_{\delta}(\tilde{t}) v_{\gamma}(\tilde{t}) v_{\alpha}(t) v_{\beta}(t) + (perm \ of \ (\alpha, \beta, \gamma))}{(dy \, d\xi)(t) \, (dy \, d\xi)(\tilde{t})} \right\} + (5.15) \qquad + 2\pi i \sum_{x_i \in D_{Br}} \underset{x_i}{\text{res}} \left\{ \frac{v_{\alpha} v_{\beta} v_{\gamma}}{(dy)^2 d\xi} d\left(\frac{v_{\delta}}{d\xi}\right) \right\} .
$$

It is natural to treat the terms corresponding to $i = j$ in this double sum separately. First, we compute the residue at the first order pole arising from zero of $d\xi$ at $x_i.$ Namely, we have $d\xi = 2\hat{\xi}d\hat{\xi}$ near each x_i and using the notation $v_\alpha(x_i) = v_\alpha/d\hat{\xi}(x_i)$ we have

$$
\mathop{\mathrm{res}}\limits_{\tilde{t}=x_i} B(t,\tilde{t}) \frac{v_\delta(\tilde{t}) v_\gamma(\tilde{t})}{(dy\,d\xi)(\tilde{t})} = \frac{1}{2} \frac{B(t,x_i) v_\delta(x_i) v_\gamma(x_i)}{y'(x_i)}
$$

where, for any differential u on \widehat{C} , the notation $u(x_i)$ is used to denote $(u/d\hat{\xi})(x_i)$ and the prime denotes the derivative with respect to $\hat{\xi_i}.$

The resulting expression has a third order pole at x_i (arising from the double pole of $B(t, x_i)$ and the simple zero of $d\xi$):

$$
\mathop{\mathrm{res}}\limits_{t=x_i}\left\{\frac{B(t,x_i)}{(dyd\xi)(t)}v_\alpha(t)v_\beta(t)\right\} = \frac{1}{2}B_{reg}^{d\xi}(x_i)\frac{v_\alpha v_\beta}{y'}(x_i) + \frac{1}{4}\left(\frac{v_\alpha v_\beta}{y'}\right)''
$$

where

$$
B_{reg}^{d\hat{\xi}}(x_i) = \left(B(x, y) - \frac{d\hat{\xi}(x)d\hat{\xi}(y)}{(\hat{\xi}(x) - \hat{\xi}(y))^2} \right)\Big|_{x=y=x_i}
$$

is equal to $1/6$ of the Bergman projective connection computed at x_i in the coordinate $\hat{\xi}.$

To compute the last residue in [\(5.15\)](#page-18-0) we notice that the corresponding expression has a pole of third order at x_i . Starting from

$$
v_{\delta} = \left(v_{\delta}(x_i) + v'_{\delta}\hat{\xi}_i + \frac{v''_{\delta}}{2}\hat{\xi}^2 + \ldots\right)d\hat{\xi}_i, \qquad d\xi = 2\hat{\xi}_i d\hat{\xi}_i
$$

and

$$
\frac{1}{d\xi}d\left(\frac{v_\delta}{d\xi}\right)=-\frac{v_\delta}{4\hat{\xi}^3_i}+\frac{v''_\delta}{8\hat{\xi}}+\ldots
$$

the last term in [\(5.15\)](#page-18-0) can be computed as follows:

$$
\operatorname{res}_{x_i} \left\{ \frac{v_\alpha v_\beta v_\gamma}{(dy)^2} \left(-\frac{v_\delta}{4 \hat{\xi}_i^3} + \frac{v''_\delta}{8 \hat{\xi}} + \dots \right) \right\} = -\frac{1}{8} \left(\frac{v_\alpha v_\beta v_\gamma}{(y')^2} \right)'' v_\delta + \frac{1}{8} \frac{v_\alpha v_\beta v_\gamma v''_\delta}{(y')^2}.
$$

Now the formula [\(5.15\)](#page-18-0) can be written as

$$
\frac{1}{2\pi i} \frac{\partial^2 \Omega_{\alpha\beta}}{\partial A_{\delta} \partial A_{\gamma}} = \frac{1}{4} \sum_{x_i \neq x_j \in Br} \left\{ B(x_i, x_j) \frac{v_{\delta}(x_i) v_{\gamma}(x_i) v_{\alpha}(x_j) v_{\beta}(x_j) + (cycl \space of \space (\alpha, \beta, \gamma))}{y'(x_i) y'(x_j)} \right\}
$$

$$
+ \frac{1}{4} \sum_{x_i \in Br} \frac{B_{reg}^{d_{\xi}^{\xi}}(v_{\alpha} v_{\beta} v_{\gamma} v_{\delta} + (cycl(\alpha, \beta, \gamma))}{y'^2} (x_i) + \frac{1}{8} \sum_{x_i \in Br} \left(\frac{v_{\gamma} v_{\delta}}{y'} \left(\frac{v_{\alpha} v_{\beta}}{y'} \right)'' (x_i) + (cycl(\alpha, \beta, \gamma)) \right)
$$

$$
+ \frac{1}{8} \sum_{x_i \in Br} \left[-\left(\frac{v_{\alpha} v_{\beta} v_{\gamma}}{(y')^2} \right)'' v_{\delta} + \frac{v_{\alpha} v_{\beta} v_{\gamma} v_{\delta}''}{(y')^2} \right] .
$$

A straightforward computation by expanding the derivatives above, shows that the sum of the last two terms is equal to

$$
-\frac{1}{8} \sum_{x_i \in Br} \frac{y'''}{y'^3} v_\alpha v_\beta v_\gamma v_\delta + \frac{1}{8} \sum_{x_i \in Br} \frac{1}{y'^2} (v''_\alpha v_\beta v_\gamma v_\delta + (cycl \ (\alpha, \beta, \gamma, \delta)) \ .
$$

Therefore we get the following proposition:

Proposition 5.1.

$$
\frac{1}{2\pi i} \frac{\partial^2 \Omega_{\alpha\beta}}{\partial A_{\delta} \partial A_{\gamma}} = \frac{1}{4} \sum_{x_i \neq x_j \in Br} \left[B(x_i, x_j) \frac{v_{\delta}(x_i) v_{\gamma}(x_i) v_{\alpha}(x_j) v_{\beta}(x_j) + (cycl \ (\alpha, \beta, \gamma))}{y'(x_i) y'(x_j)} \right]
$$
\n
$$
(5.16) \qquad + \frac{1}{8} \sum_{x_i \in Br} \left[\left(\frac{6B_{reg}^{d\hat{\xi}}}{y'^2} - \frac{y'''}{y'^3} \right) v_{\alpha} v_{\beta} v_{\gamma} v_{\delta}(x_i) + \frac{1}{y'^2} (v''_{\alpha} v_{\beta} v_{\gamma} v_{\delta} + cycl \ (\alpha, \beta, \gamma, \delta)) \right].
$$

The formula [\(5.16\)](#page-19-0) coincides with the expression obtained in Theorem 7.5 of [\[2\]](#page-20-0) using the framework of topological recursion of [\[8\]](#page-20-11) (notice that $6B_{reg}^{d\hat{\xi}}$ is nothing but the Bergman projective connection S_B which enters the formula (7.4) of [\[2\]](#page-20-0).

To conclude, we have shown that the deformation calculus on spaces of Hitchin's spectral covers can be naturally induced from a much more transparent deformation theory on moduli spaces of holomorphic or meromorphic Abelian differentials on Riemann surfaces. In consideration of the close relationship between deformations of spectral covers and the theory of topological recursion of [\[8\]](#page-20-11), it is natural to expect that the topological recursion itself could be a manifestation of a much less involved structure associated to moduli spaces of Abelian differentials.

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