DEFORMATION THEORY OF HOLOMORPHIC CARTAN GEOMETRIES

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Abstract. We introduce the deformation theory of holomorphic Cartan geometries. The infinitesimal automorphisms, as well as the infinitesimal deformations, of holomorphic Cartan geometries are computed. We also prove the existence of a semi-universal deformation of a holomorphic Cartan geometry.

1. INTRODUCTION

Let G be a complex Lie group and $H < G$ a complex Lie subgroup with Lie algebras \mathfrak{g} , and \mathfrak{h} respectively. The quotient map $G \longrightarrow G/H$ defines a holomorphic principal H-bundle. Moreover, on the total space of this principal bundle, namely G , we have a tautological g–valued holomorphic 1-form (the Maurer-Cartan form), which is constructed by identifying g with the right-invariant vector fields. This 1-form is an isomorphism of the holomorphic tangent bundle of G with the trivial vector bundle $G \times \mathfrak{g} \longrightarrow G$. The restriction of this form to the fibers of the projection $G \longrightarrow G/H$ coincide with the $\n *h*-valued Maurer-Cartan form for the right-action of H on the fiber.$

A holomorphic Cartan geometry of type (G, H) on a compact complex manifold X is infinitesimally modeled on the above set-up. More precisely, a holomorphic Cartan geometry of type (G, H) on X consists of

- a holomorphic principal *H*-bundle E_H over X ,
- a holomorphic 1-form A on E_H with values in the Lie algebra g of G that induces a holomorphic isomorphism from the holomorphic tangent bundle of E_H to the trivial vector bundle on E_H with fiber g. This isomorphism is required to be H invariant and on each fiber of E_H it should be the Maurer-Cartan form for the action of H.

(see Definition [2.1](#page-3-0) in Section [2](#page-2-0) and for more details [\[Sh\]](#page-12-0)). The notion of holomorphic Cartan geometry extends to Sasakian manifolds [\[BDS\]](#page-11-0).

A fundamental result of E. Cartan shows that the obstruction for A to satisfy the Maurer-Cartan equation of G is a curvature tensor which vanishes if and only if (E_H, A) is locally isomorphic (not just infinitesimally) to the H-principal bundle $G \longrightarrow G/H$ endowed with the Maurer-Cartan form $[Sh]$. In this case X admits local coordinates with values in G/H which are well defined up to the canonical action of G on G/H . Indeed, Ehresmann proved that a flat Cartan geometry is equivalent with the following data: a holomorphic principal G-bundle over X endowed with a flat holomorphic connection and a H-subbundle transverse to the flat connection $|Eh|$ (see also the survey $|BD2|$). This

²⁰¹⁰ Mathematics Subject Classification. 32G13, 53C55.

Key words and phrases. Deformation theory, holomorphic Cartan geometry, Atiyah bundle.

implies that the pull-back of this G-principal bundle to the universal cover \tilde{X} of X is isomorphic to $\tilde{X} \times G$ and the H-subbundle is given by a holomorphic map $\tilde{X} \longrightarrow G/H$. Moreover, the transversality condition is equivalent to the fact that the previous map is a local biholomorphism, traditionally called the developing map of the (flat) Cartan geometry. The developing map is equivariant with respect to the action of the fundamental group $\pi_1(X)$ of X by deck transformations on X and through the monodromy morphism $\rho : \pi_1(X) \longrightarrow G$ (of the G-flat bundle) on the model space G/H [\[Eh\]](#page-11-1) (see also [\[BD2\]](#page-11-2)). Ehresmann's geometrical description leads to the following nice and useful description of the deformation space of flat Cartan geometries with given model type (G, H) on the (real) manifold X. This so-called Ehresmann-Thurston principle ensures that the map associating to each flat Cartan geometry its monodromy morphism $\rho : \pi_1(X) \longrightarrow G$ (uniquely defined up to inner conjugacy in G) is a local homeomorphism between the deformation space of flat Cartan geometries with model (G, H) on X and the space of group homomorphisms from $\pi_1(X)$ to G $|Go|$.

Since G is a complex Lie group and H a closed complex subgroup in G , the model manifold G/H inherits a G-invariant complex structure. Any flat Cartan geometry with model (G, H) induces on X an underlying complex structure. Hence there is a natural forgetful map from the deformation space of flat Cartan geometries with model complex Lie groups (G, H) into the Kuranishi space of X. In the particular case of complex projective structures on Riemann surfaces, this map played a major role in the understanding of the uniformization theorem for Riemann surfaces (see [\[Gu\]](#page-11-4) or [\[StG\]](#page-12-1)). More precisely, the uniformization theorem asserts the existence on any Riemann surface of a compatible complex projective structure with injective developing map. In the case where the Riemann surface is compact of genus $g \geq 2$, the developing map is an isomorphism between the universal cover of the surface and the unitary disk in C and the image of the monodromy morphism is a uniform lattice in $PSL(2,\mathbb{R})$ (the isometry group of the hyperbolic disk) [\[StG\]](#page-12-1).

More recently the deformation space of flat Cartan geometries with model $G = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and $H = SL(2,\mathbb{C})$, diagonally embedded, was used by Ghys in [\[Gh\]](#page-11-5) in order to compute the Kuranishi space of the parallelizable manifolds $SL(2,\mathbb{C})/\Gamma$, with Γ a uniform lattice in $SL(2,\mathbb{C})$. It was proved in [\[Gh\]](#page-11-5) that the deformation space of those flat Cartan geometries is locally isomorphic to the Kuranishi space and modeled on the germ at the trivial morphism in the algebraic variety of group homomorphisms (representations) from Γ into $SL(2,\mathbb{C})$. In particular, for any uniform lattice Γ with positive first Betti number this germ has positive dimension. Hence the corresponding parallelizable manifolds $SL(2,\mathbb{C})/\Gamma$ admit nontrivial deformations of the underlying complex structure. Those examples of flexible parallelizable manifolds associated to semi-simple complex Lie groups are exotic (the unique case not covered by Raghunathan's rigidity results [\[Ra\]](#page-11-6) is precisely that of a factor locally isomorphic to $SL(2,\mathbb{C})$.

Let us mention that the deformation theory of representations of fundamental groups of compact Kähler manifolds into real algebraic groups G was worked out in [\[GM\]](#page-11-7). Under various conditions on a given representation, it was proved in [\[GM\]](#page-11-7) that there exists a neighborhood of it in the algebraic variety of representations which is analytically equivalent to a cone defined by homogeneous quadratic equations. This is the case in the neighborhood of the monodromy morphism of a variation of Hodge structure. It is also the case for the monodromy of a flat principal bundle G which admits a reduction to a compact subgroup K such that the quotient G/K is a Hermitian symmetric space.

Our aim here is to introduce the deformation theory of holomorphic (not necessarily flat) Cartan geometries on a compact complex manifold. We compute the tangent cohomology of a holomorphic Cartan geometry (E_H, A) in degree zero and one. Infinitesimal automorphisms of a holomorphic Cartan geometry (E_H, A) consist of all the $ad(E_H)$ valued holomorphic vector fields ϕ satisfying the condition that the Lie derivative $L_{\phi}(A)$ vanishes. The space of infinitesimal deformations of (E_H, A) fits into a short exact sequence that we construct in the fourth section. The construction of the semi-universal deformation is worked out in the last section. Using the same methods a semi-universal deformation can also be constructed when the principal H -bundle and the underlying compact complex manifold are both moving.

While we have restricted ourselves to the holomorphic category, it should be mentioned that Cartan geometries are also defined in C^{∞} category. However in the C^{∞} category, the infinitesimal deformations of a Cartan geometry are not parameterized by a finite dimensional space.

Our project in the future is to use the deformation theory of holomorphic Cartan geometries on a compact complex manifold developed here in order to investigate the Kuranishi space of a compact complex manifold bearing a holomorphic Cartan geometry. Hopefully this will led to a better understanding of the classification of compact complex manifolds admitting holomorphic Cartan geometries and to uniformization theorems for those in higher dimension. It should be mentioned that Inoue, Kobayashi and Ochiai proved that compact complex surfaces bearing holomorphic affine connections (respectively, holomorphic projective connections) also admit flat holomorphic affine connections (respectively, flat holomorphic projective connections) with corresponding injective developing map. In particular, those complex surfaces are uniformized as quotients of open subsets in the complex affine plane (respectively, complex projective plane) by a discrete subgroup of affine transformations (respectively, projective transformations) acting properly and discontinuously [\[IKO,](#page-11-8) [KO1,](#page-11-9) [KO2\]](#page-11-10). We aim to prove, as a generalization of those results, that compact complex surfaces bearing holomorphic Cartan geometries with model (G, H) also admit flat holomorphic Cartan geometries with model (G, H) and with corresponding injective developing map into G/H . This would uniformize compact complex surfaces bearing holomorphic Cartan geometries with model (G, H) as compact quotients of open subsets U in G/H by discrete subgroups in G preserving $U \subset G/H$ and acting properly and discontinuously on U.

2. Deformations of holomorphic Cartan geometries – Definition

For any complex manifold M its holomorphic tangent bundle will be denoted by T_M . We shall denote by G a connected complex Lie group, and by $H < G$ a closed connected complex Lie subgroup. Furthermore fix a compact complex manifold X. Let E_H denote a holomorphic principal H-bundle over X. Let $E_G = E_H \times_H G$ be the holomorphic principal G-bundle over X obtained by extending the structure group of E_H using the inclusion of H in G .

The action of the group H on E_H produces an action of H on the tangent bundle T_{E_H} . In other words, for $p \in E_H$, a tangent vector $v \in T_p(E_H)$ and $h \in H$, we have $v \cdot h = R_{h*}v$, where R_h is the right multiplication by h. For $\gamma \in \mathfrak{g} = \text{Lie}(G)$ we have $h \cdot \gamma = ad(h^{-1}(\gamma)).$

Let $\pi : E_H \longrightarrow X$ be the projection from the total space of the principal H-bundle E_H , and let π_* : $T_{E_H} \longrightarrow T_X$ the corresponding induced map of tangent bundles. The adjoint bundle $ad(E_H)$ is defined to be the associated vector bundle $E_H \times_H \mathfrak{h}$, where $\mathfrak{h} = \text{Lie}(H)$. The space of vertical tangent vectors ker π_* is invariant under the action of H, and we have

$$
ad(E_H) = \ker(\pi_*)/H. \tag{2.1}
$$

Given a section of $ad(E_H)$ we shall use the same notation for its pull-back to a section of $\ker(\pi_*) \subset T_{E_H}.$

Definition 2.1. A holomorphic Cartan geometry of type (G, H) on X is a pair (E_H, A) , where E_H is a holomorphic principal H-bundle on X, and A is a C-linear holomorphic isomorphism of vector bundles

$$
A\,:\,T_{E_H}\,\stackrel{\sim}{\longrightarrow}\,E_H\times\mathfrak{g}
$$

over E_H such that

- (i) A is H -equivariant, and
- (ii) the restriction of A to the fibers of $E_H \longrightarrow X$ coincides with the Maurer-Cartan form on the fiber for the action of H.

We note that the above isomorphism A induces an isomorphism

$$
A_H: T_{E_H}/H \xrightarrow{\sim} (E_H \times \mathfrak{g})/H, \qquad (2.2)
$$

where T_{E_H}/H is, by definition, the Atiyah bundle $At(E_H)$, which fits into the Atiyah exact sequence

$$
0 \longrightarrow ad(E_H) \longrightarrow At(E_H) \longrightarrow T_X \longrightarrow 0
$$

(see [\[At\]](#page-11-11)). We also have $(E_H \times \mathfrak{g})/H \simeq ad(E_G)$, where the quotient is for the conjugation action mentioned earlier.

The isomorphism A_H in [\(2.2\)](#page-3-1) induces a holomorphic connection on E_G [\[Sh\]](#page-12-0), [\[BD1,](#page-11-12) (2.8) , which in turn produces a holomorphic connection on $ad(E_G)$. Therefore, we have a holomorphic differential operator

$$
D: ad(E_G) \longrightarrow \Omega^1_X \otimes ad(E_G) \tag{2.3}
$$

of order one.

An *isomorphism* $\Phi : (E_H, A) \longrightarrow (F_H, B)$ between holomorphic Cartan geometries (E_H, A) and (F_H, B) of same type (G, H) on X is given by a holomorphic isomorphism $\phi: E_H \longrightarrow F_H$ of principal H-bundles that takes A to B so that

$$
T_{E_H} \xrightarrow{A} E_H \times \mathfrak{g}
$$
\n
$$
X \xrightarrow{\phi_*}
$$
\n
$$
T_{F_H} \xrightarrow{B} F_H \times \mathfrak{g}
$$
\n
$$
(2.4)
$$

is a commutative diagram (where ϕ_* denotes the differential of ϕ).

Our aim is to define deformations of holomorphic Cartan geometries.

Let us first define families of Cartan geometries of type (G, H) on a given complex manifold X and isomorphisms between families.

Definition 2.2. Let $H < G$ be a closed connected complex subgroup of a complex Lie group, and S a complex space.

(i) A holomorphic family of principal H-bundles with holomorphic Cartan geometries over S (also called *holomorphic family of Cartan geometries*) is a pair $(\mathcal{E}_H, \mathcal{A}),$ where \mathcal{E}_H is a principal H-bundle over $X \times S$ together with a linear isomorphism $\mathcal A$ over S

such that all restrictions \mathcal{A}_s of $\mathcal A$ to fibers $T_{\mathcal{E}_{H,s}}$ over $s \in S$ define holomorphic Cartan geometries $(T_{\mathcal{E}_{H,s}}, \mathcal{A}_s)$ of type (G, H) on X.

(ii) An *isomorphism* Ξ : $(\mathcal{E}_{H,1}, \mathcal{A}_1) \longrightarrow (\mathcal{E}_{H,2}, \mathcal{A}_2)$ between holomorphic families of Cartan geometries $(\mathcal{E}_{H,1}, \mathcal{A}_1)$ and $(\mathcal{E}_{H,2}, \mathcal{A}_2)$ over S is given by an isomorphism $\xi : \mathcal{E}_{H,1} \stackrel{\sim}{\longrightarrow} \mathcal{E}_{H,2}$ of principal H-bundles such that $(\xi \times id_{\mathfrak{g}}) \circ \mathcal{A}_1 = \mathcal{A}_2 \circ T_{\xi}$, with T_{ξ} : $T \mathcal{E}_{H,1} \stackrel{\sim}{\longrightarrow} T \mathcal{E}_{H,2}$ being the differential of ξ .

Let us now to define deformations (over complex spaces) of a holomorphic Cartan geometry and the corresponding notion of isomorphism between deformations. We also define below (germs of) deformations (over germs of complex spaces) of a Cartan geometry.

Definition 2.3. Let (E_H, A) be a holomorphic Cartan geometry of type (G, H) on X.

- (i) Let (S, s_0) be a complex space with a distinguished point $s_0 \in S$. A deformation of (E_H, A) over (S, s_0) is a pair $((\mathcal{E}_H, \mathcal{A}), \Phi)$, where
	- (a) $(\mathcal{E}_H, \mathcal{A})$ is a holomorphic family of holomorphic Cartan geometries over S, and
	- (b) Φ : $(E_H, A) \longrightarrow (\mathcal{E}_{H,s_0}, \mathcal{A}_{s_0})$ is an isomorphism of holomorphic Cartan geometries.

An *isomorphism* $((\mathcal{E}_{H1}, \mathcal{A}_1), \Phi_1) \longrightarrow ((\mathcal{E}_{H1}, \mathcal{A}_2), \Phi_2)$ between deformations $((\mathcal{E}_{H1}, \mathcal{A}_1), \Phi_1)$ and $((\mathcal{E}_{H1}, \mathcal{A}_2), \Phi_2)$ of the Cartan geometry (E_H, A) is given by an isomorphism of families $\Xi : (\mathcal{E}_{H,1}, \mathcal{A}_1) \longrightarrow (\mathcal{E}_{H,2}, \mathcal{A}_2)$ such that $\Xi \circ \Phi_1 = \Phi_2$.

(ii) Let \underline{S} be a germ of a complex space represented by a complex space (S, s_0) with a distinguished point $s_0 \in S$. A *(germ of) deformation* $((\underline{\mathcal{E}}_H, \underline{\mathcal{A}}), \underline{\Phi})$ of (E_H, A) over S is an equivalence class of deformations of (E_H, A) over complex spaces with distinguished point s_0 . More precisely, a (germ) of deformation over S is represented by a deformation of (E_H, A) over a neighborhood $S_1 \subset S$ of s_0 . A further deformation over an open neighborhood $s_0 \in S_2 \subset S$ is equivalent, if there exists a neighborhood $s_0 \in S_3 \subset S_1 \cap S_2$ and an isomorphism of the restrictions to S_3 in the sense of (i).

3. Deformations of holomorphic Cartan geometries

The tangent cohomology of a H-principal bundle E_H is equal to $H^{\bullet}(X, ad(E_H))$, where $ad(E_H)$ is the adjoint bundle of E_H with fiber $\mathfrak h$ [\[Don\]](#page-11-13) (see also [\[BHH\]](#page-11-14)). In particular in degrees 0, 1, and 2 we have infinitesimal automorphisms, infinitesimal deformations, and the space containing obstructions respectively.

We deal now with the tangent cohomology for holomorphic Cartan geometries.

3.1. Infinitesimal automorphisms of holomorphic Cartan geometries. Let (E_H, A) be a holomorphic Cartan geometry of type (G, H) on X.

We deal first with the tangent cohomology $T^0(E_H, A)$ of degree zero for the holomorphic Cartan geometry (E_H, A) . Recall that A takes values in \mathfrak{g} . From [\(2.1\)](#page-3-2) it follows that any holomorphic section ψ of $ad(E_H)$ over $U \subset X$ gives rise to a H–invariant holomorphic vector field $\tilde{\psi}$ over $E_H|_U$ which is vertical for the projection π . We shall identify ψ with ψ . By L_{ψ} we denote the Lie derivative with respect to this vector field $\widetilde{\psi}$.

Proposition 3.1. The space of infinitesimal automorphisms is equal to

$$
T^{0}(E_{H}, A) = H^{0}(X, ad(E_{H}))_{A} = \{ \psi \in H^{0}(X, ad(E_{H})) \mid L_{\psi}(A) = 0 \}.
$$
 (3.1)

Proof. We consider [\(2.4\)](#page-3-3) for $F_H = E_H$ and the infinitesimal action of an element $\psi \in$ $H^0(X, ad(E_H))$ of the group of vertical infinitesimal automorphisms of E_H . This gives rise to the following digram of homomorphisms on E_H :

$$
T_{E_H} \xrightarrow{A} E_H \times \mathfrak{g}
$$
\n
$$
L_{\psi} \downarrow \qquad \qquad \downarrow \psi
$$
\n
$$
T_{E_H} \xrightarrow{A} E_H \times \mathfrak{g}
$$
\n(3.2)

Diagram [\(3.2\)](#page-5-0) is interpreted as follows. As mentioned before, holomorphic sections of $ad(E_H)$ are H-invariant vertical sections of the holomorphic tangent bundle T_{E_H} . We apply A to a holomorphic section v of T_{E_H} ; then ψ is applied to $A(v)$, which is simply the Lie bracket on g because ψ is a function on E_H with values in h. On the other hand, ψ acts on holomorphic sections of $ad(E_H)$ by applying the Lie derivative L_{ψ} , which is the infinitesimal version of the adjoint action.

The infinitesimal automorphism ψ is compatible with the holomorphic Cartan geometry, if [\(3.1\)](#page-5-1) commutes for all sections v of $ad(E_H)$, i.e.

$$
A(L_{\psi}(v)) = \psi(A(v)) = L_{\psi}(A)(v) + A(L_{\psi}(v))
$$

for all v. Therefore, [\(3.1\)](#page-5-1) commutes if and only if $L_{\psi}(A) = 0$.

3.2. Infinitesimal deformations of holomorphic Cartan geometries. Let $\mathbb{C}[\epsilon] =$ $\mathbb{C}[t]/(t^2)$, so that $\mathbb{C}[\epsilon] = \mathbb{C} \oplus \epsilon \cdot \mathbb{C}$ holds with $\epsilon^2 = 0$. The space $D = (\{0\}, \mathcal{O}_D)$ is also called double point with $\mathcal{O}_D = \mathbb{C}[\epsilon]$. The tangent space T_{S,s_0} of an arbitrary complex space S at a point $s_0 \in S$ can be identified with the space of all holomorphic mappings $D \longrightarrow S$ such that the underlying point 0 is mapped to s_0 .

An *infinitesimal deformation* is an isomorphism class of deformations over D considered as a complex space, or equivalently a deformation over the induced space germ. In

particular, an infinitesimal deformation of a Cartan geometry is a deformation over the germ of complex space represented by D in the sense of Definition [2.3](#page-4-0) (ii).

Let E_H be a principal H-bundle over X, and let \mathcal{E}_H be an infinitesimal deformation of E_H given by an H-principal bundle over $X \times D$, whose restriction to X (the fiber over the canonical base point) is equipped with an isomorphism to E_H over X. We describe such an object in more detail in the following:

Proposition 3.2. An infinitesimal deformation \mathcal{E}_H of E_H can be described in the following alternative ways.

- (i) The restriction map $\rho : \mathcal{E}_H \longrightarrow E_H$ of H-principal bundles defines an affine h-bundle aff (\mathcal{E}_H) over E_H with underlying vector bundle ad(E_H), whose affine structure is determined by a cocycle from $H¹(X, ad(E_H))$ up to isomorphism over E_H .
- (ii) Let $H_D = H \oplus \epsilon \mathfrak{h}$ be the Lie group defined by $\epsilon^2 = 0$ and the adjoint action. Then \mathcal{E}_H is an H_D-principal bundle on X with an underlying H-bundle E_H .

The adjoint bundle $ad(\mathcal{E}_H)$ taken from the H_D-bundle \mathcal{E}_H on X possesses the Lie algebra $\mathfrak{h}_D = \mathfrak{h} \oplus \epsilon \mathfrak{h}$ (determined by $\epsilon^2 = 0$) as fiber.

Proof. We need the proof for the fact that isomorphism classes of infinitesimal deformations of E_H correspond to elements of $H^1(X, ad(E_H))$ [\[Don\]](#page-11-13) in the following version.

Let $\mathfrak{U} = \{U_i\}$ be an open covering of X by contractible Stein subsets such that E_H is defined by a cocycle $\{g_{ij}(x)\}\$ of H-valued, holomorphic mappings on $U_{ij} = U_i \cap U_j$. Now an H-principal bundle on $X \times D$ with respect to the covering $U_i \times D$ is determined by a cocycle γ_{ij} such that for $x \in U_{ij}$

$$
\widetilde{\gamma}_{ij}(x)\,:\,\mathcal{O}_{H,g_{ij}(x)}\,\longrightarrow\,\mathcal{O}_{U_{ij},x}\oplus\,\epsilon\,\mathcal{O}_{U_{ij},x}
$$

with $\widetilde{\gamma}_{ij}(x) = \widetilde{g}_{ij}(x) + \epsilon \tau_{ij}(x)$. The condition $\epsilon^2 = 0$ implies

$$
\widetilde{\gamma}_{ij}(x)(\varphi \cdot \psi) = (\varphi \cdot \psi) \circ g_{ij} + \epsilon (\tau_{ij}(\varphi) \cdot (\psi \circ g_{ij}) + \tau_{ij}(\psi) \cdot (\varphi \circ g_{ij})) \tag{3.3}
$$

for all functions φ and ψ . Hence

$$
\tau_{ik} = L_{g_{ij}*}(\tau_{jk}) + R_{g_{jk}*}(\tau_{ij}),
$$

implies that τ_{ij} is a derivation, which can be readily written in terms of the Lie-algebra h of right-invariant vector fields

$$
\tau_{ik} = ad(g_{ij})(\tau_{jk}) + \tau_{ij}.
$$

The cocycle condition for the transition functions of the *H*-principle bundle \mathcal{E}_H yields that $\{\tau_{ij}\}\$ is an $ad(E_H)$ -valued cocycle. Now we can set

$$
\gamma_{ij} = g_{ij} + \epsilon \tau_{ij} : U_{ij} \times D \longrightarrow H,
$$

and interpret γ_{ij} as a truncated power series in powers of ϵ with coefficients that are holomorphic on U_{ij} . Here g_{ij} has values in H and τ_{ij} is \mathfrak{h} -valued. It can be reinterpreted as an H_D -valued cocycle, which proves (ii).

On the other hand, fixing E_H we apply the transition function to a section η_i with values in \mathfrak{h} . Then the transition equation for an \mathfrak{h} -valued holomorphic section η_i on U_i is

$$
\eta_j \, \longmapsto \, ad(g_{ij})(\eta_j) + \tau_{ij} \, .
$$

This defines an affine \mathfrak{h} -bundle with underlying vector bundle $ad(E_H)$.

Proposition 3.3. Let (E_H, A) be a holomorphic Cartan geometry of type (G, H) on X. Let \mathcal{E}_H be an isomorphism class of infinitesimal deformations of E_H . Then the obstructions to extend the holomorphic Cartan geometry A to \mathcal{E}_H are in

$$
H^1(E_H, \, Hom(T_{E_H}, \, E_H \times \mathfrak{g}))^H
$$

or equivalently in

$$
H^1(X, Hom(At(E_H), ad(E_G)) .
$$

Proof. We assume that an infinitesimal deformation \mathcal{E}_H of E_H is given like in the Proposition [3.2.](#page-6-0) We use again the notation in the proof of Proposition [3.2](#page-6-0) and we consider the affine h-bundle

$$
\mathcal{E}_H \xrightarrow[\mathfrak{h}]{\rho} E_H.
$$

Now $H \subset H_D$ is a subgroup so that over U_i there is a section σ_i of ρ . In particular there is a H-equivariant section of the above affine bundle over $\pi^{-1}(U_i)$, where $\pi : \mathcal{E}_H \longrightarrow X$ is the projection on X.

Furthermore we get an induced infinitesimal deformation of the tangent bundle T_{E_H} .

Now we assume that (E_H, A) defines a holomorphic Cartan geometry. Altogether this gives rise to exact sequences on E_H

$$
0 \longrightarrow \epsilon \cdot \mathbb{C} \longrightarrow \mathbb{C} \oplus \epsilon \cdot \mathbb{C} \longrightarrow \mathbb{C}
$$
\n
$$
0 \longrightarrow \epsilon \cdot T_{E_H} \xrightarrow{T_L} T_{\epsilon_H/D} \xrightarrow{T_{\rho}} T_{E_H} \longrightarrow 0
$$
\n
$$
\downarrow A \qquad \sim \downarrow A
$$
\n
$$
\epsilon_H \times \mathfrak{g} \xrightarrow[\rho \times id_{\mathfrak{g}}] E_H \times \mathfrak{g} \qquad ,
$$
\n(3.4)

where the existence of an isomorphism A has still to be discussed.

Note that the existence of a compatible morphism A is equivalent to the existence of a holomorphic Cartan geometry over D (there is only one fiber), which in turn defines an infinitesimal deformation of the given object.

Now we also use the induced splitting $T\sigma_i$ of $T\rho$ over U_i . Once the above splittings are fixed there exists an extension $\mathcal{A}_i : T_{\mathcal{E}_H/D} | \pi^{-1}(U_i) \longrightarrow E_H \times \mathfrak{g} | \pi^{-1}(U_i)$ of $A | \pi^{-1}(U_i)$. By \mathcal{O}_D -linearity any such map is unique when restricted to the kernel of $T\rho|\pi^{-1}(U_i)$. We consider $\mathcal{A}_i - \mathcal{A}_j$ over $\pi^{-1}(U_{ij})$. We apply Proposition [3.2\(](#page-6-0)i) and see that $\mathcal{A}_i - \mathcal{A}_j$ has values in $ad(E_H) \times \mathfrak{g}$. Hence it is induced by an H-invariant morphism \mathcal{A}_{ij} : $T_{E_H} |\pi^{-1}(U_{ij}) \longrightarrow$ $ad(E_H) \times \mathfrak{g}$ as

$$
\mathcal{A}_i\,=\,\mathcal{A}_{ij}\circ T\rho+\mathcal{A}_j
$$

over U_{ij} , where the addition is taken in the sense of the affine bundle structure $\mathcal{E}_H \longrightarrow E_H$. Suppose that \mathcal{A}_{ij} is a coboundary of the form $\mathcal{B}_j - \mathcal{B}_i|U_{ij}$, where \mathcal{B}_i : $T_{E_H}|U_i| \longrightarrow$ $E_H|U_i \times \mathfrak{g}$ are linear and H-invariant. Then, using the affine structure of $\rho : \mathcal{E}_H \longrightarrow^{\mathfrak{h}} E_H$ the morphisms A_i can be changed into $A_i + B_i \circ T \rho$, so that these fit together, and define the desired (global) map \mathcal{A} .

It can be verified immediately that the cohomology class of A_{ij} is uniquely determined by the infinitesimal deformation of the principal H -bundle E_H and the holomorphic Cartan geometry A. If we assume that a holomorphic deformation $(\mathcal{E}_H, \mathcal{A})$ of the holomorphic Cartan geometry (E_H, A) does exist, then A is unique up to an H-invariant morphism $T_{E_H} \longrightarrow E_H \times \mathfrak{g}.$

We denote by $H^1(X, ad(E_H))_A$ the space of isomorphism classes of infinitesimal deformations of E_H such that the Cartan geometry A of the central fiber can be extended, so $H^1(X, ad(E_H))_A \subset H^1(X, ad(E_H))$. Furthermore we denote by $T^1(E_H, A)$ the space of isomorphism classes of infinitesimal deformations of (E_H, A) .

Now Proposition [3.1](#page-5-2) and Proposition [3.3](#page-7-0) imply the following result.

Theorem 3.4 (Infinitesimal deformations). Let (E_H, A) be a holomorphic Cartan geometry. Then there is an exact sequence

$$
0 \longrightarrow H^0(E_H, Hom(T_{E_H}, E_H \times \mathfrak{g}))^H \longrightarrow T^1(E_H, A) \longrightarrow H^1(X, ad(E_H))_A \longrightarrow 0,
$$

where $H^0(E_H, Hom(T_{E_H}, E_H \times \mathfrak{g}))$ ^H can be identified with $H^0(X, Hom(At(E_H), ad(E_G)))$.

3.3. Semi-universal deformation of principal bundles. The aim of Section [4](#page-9-0) is to prove the existence of a semi-universal deformation of a holomorphic Cartan geometry. We begin here the construction with a semi-universal deformation of a principal H -bundle E_H .

Recall that a deformation over a complex space S with base point s_0 is given by a holomorphic family \mathcal{E}_H of principal H-bundles over $X \times S$, together with an isomorphism $\Xi : E_H \longrightarrow \mathcal{E}_H | X \times \{s_0\}.$

Semi-universality amounts to the following conditions (cf. also Definition [2.3](#page-4-0) for deformations over germs of complex spaces):

- (i) (Completeness) for any deformation ξ of E_H over a space (W, w_0) , given by a holomorphic family $\mathcal{E}_{H,W} \longrightarrow X \times W$ together with an isomorphism of the above type, there is a base change morphism $f : (W, w_0) \longrightarrow (S, s_0)$ such that (after replacing the base space with open neighborhoods of the base points, if necessary) the pull back $f^*\xi = (id_X \times f)^*(\mathcal{E}_H)$ and $\mathcal{E}_{H,W}$ are isomorphic with isomorphism inducing the identity map over $X \times \{w_0\}$.
- (ii) Let $(W, w_0) = (D, 0)$ be as in (i). Then any such base change f is uniquely determined by the given deformation. The set of isomorphism classes of deformations over $(D, 0)$ is called "tangent space of the deformation functor" or first tangent cohomology associated to the given deformation problem.

For the sake of completeness we mention the *Kodaira-Spencer map*:

Given a deformation ξ over (W, w_0) we identify a tangent vector v of W at w_0 with a holomorphic map $f : (D, 0) \longrightarrow (W, w_0)$. Then the Kodaira-Spencer map $\rho :$ $T_{W,w_0} \longrightarrow H^1(X, ad(E_H))$ maps ξ to the isomorphism class of $f^*\xi$. We already computed the tangent cohomologies of order zero and one.

4. Semi-universal deformation for holomorphic Cartan geometries

In this Section we prove the existence of a semi-universal deformation of a holomorphic Cartan geometry.

Let us first define a natural pull-back functor for holomorphic families and for morphisms between families.

4.1. A pull-back functor for holomorphic families. Let again X be a compact complex manifold, S a complex space. Let $\mathbf{A} \mathbf{n}_S$ be the category of complex analytic spaces over S. The objects of $\mathbf{A}_{\mathbf{R}_{\mathcal{S}}}$ are complex spaces W together endowed with holomorphic maps $W \longrightarrow S$ and the morphisms are compatible holomorphic mappings.

Denote by Sets the category of sets.

Now we define a *pull-back of families* functor F from the category $\mathbf{A}_{\mathbf{R}}$ to the category Sets.

Choose E, E' holomorphic vector bundles on $X \times S$. Consider the map

$F : \mathbf{An}_S \longrightarrow \mathbf{Sets}$

which assigns to each object $(f : W \to S) \in \mathbf{An}_{S}$ the set

$$
F(f) = F(f: W \to S) = Hom((id_X \times f)^*E, (id_X \times f)^*E').
$$

Proposition 4.1.

- (i) The map F is a morphism of categories;
- (ii) The functor F is representable.

Proof. (i) Let α be a morphism between the objects $(f_1 : W_1 \to S)$ and $(f_2 : W_2 \to S)$ in the category \mathbf{An}_{S} . Then α is given by a holomorphic mapping described in the following diagram:

$$
W_1 \xrightarrow{\alpha} W_2
$$

 $f_1 \searrow \searrow f_2$
 S

Let us set $E_{W_i} = (id_X \times f_i)^* E$ for $i = 1, 2$ and $E'_{W_i} = (id_X \times f_i)^* E'$ for $i = 1, 2$.

Let us denote by β the following diagram

and notice that the image of β through $F(\alpha)$ is given by:

This implies that F is a morphism of categories.

(ii) We shall use the representability of the morphism functor. In the algebraic case it was shown by Grothendieck in [\[Gr,](#page-11-15) EGAIII 7.7.8 and 7.7.9]. In the analytic case the corresponding theorem for complex spaces is due to Douady [\[Dou,](#page-11-16) 10.1 and 10.2]. For coherent (locally free) sheaves M, and N over a space $\mathcal{X} \longrightarrow S$, one considers morphisms of the simple extensions $\mathcal{X}[\mathcal{M}] \longrightarrow \mathcal{X}[\mathcal{N}].$

Fact. The functor F is representable by a complex space $g : R \longrightarrow S$. There is a universal object v in $F(q)$

such that any other object over $\widetilde{R} \longrightarrow S$ is isomorphic to \widetilde{g}^*v , where $\widetilde{g} : \widetilde{R} \longrightarrow R$ is a holomorphic map over S.

 \Box

4.2. Application to holomorphic Cartan geometries.

Theorem 4.2. Let X be a compact complex manifold, $H < G$ connected complex Lie groups, and (E_H, A) a a holomorphic Cartan geometry on X of type (G, H) . Then (E_H, A) possesses a semi-universal deformation.

Proof. We begin with a semi-universal deformation of E_H :

and look at the holomorphic vector bundles

to which we apply the representability of the homomorphism functor. We have the following universal homomorphism η . (It can be checked easily that At and ad commute with base change).

Now the holomorphic Cartan geometry A_H : $At(E_H) \longrightarrow ad(E_G)$ amounts to a point $r_0 \in R$, which is mapped to s_0 under $R \longrightarrow S$ (observing the deformation theoretic isomorphisms of the given objects and distinguished fibers of the semi-universal families). We take the connected component of R through r_0 and restrict it to an open neighborhood, where the universal morphism is an isomorphism. Going through the construction, we see that the restricted family yields a semi-universal deformation of the holomorphic Cartan geometry (E_H, A) .

ACKNOWLEDGEMENTS

We thank the referee for helpful comments to improve the exposition. The first-named author thanks Université Côte d'Azur and Philipps-Universität Marburg for their hospitality while this was carried out. He is partially supported by a J. C. Bose Fellowship.

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