

THE DIRICHLET PROBLEM FOR m -SUBHARMONIC FUNCTIONS ON COMPACT SETS

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ABSTRACT. We characterize those compact sets for which the Dirichlet problem has a solution within the class of continuous m -subharmonic functions defined on a compact set, and then within the class of m -harmonic functions.

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1. INTRODUCTION

A fundamental tool in the study of uniform algebras is the class of subharmonic functions defined on compact sets, and its dual, the Jensen measures. In [7], Gamelin presented a model that can be used both for subharmonic as well as pluri-subharmonic functions defined on compact sets. In this note we shall use this model to investigate the Dirichlet problem for m -(sub)harmonic functions. Our inspiration is the work of Poletsky [12] and especially Poletsky-Sigurdsson [14, 15].

Two natural types of boundaries in potential theory are the Choquet boundary (Definition 3.1) with respect to a given class of Jensen measures, and the Šilov boundary (Definition 3.6). In our study of the Dirichlet problem these boundaries have a prominent role, and therefore we shall in Section 3 put extra attention on them in terms of for example peak points and harmonic m -measures. We shall then, in Section 4, characterize those compact sets for which the Dirichlet problem has a solution within the class of continuous m -subharmonic functions (Theorem 4.2 and Theorem 4.3). We end this note in Section 5 with a Dirichlet problem for m -harmonic functions (Theorem 5.10). In the 1-subharmonic case, these results were obtained by Hansen among others (see e.g. [6, 8, 11] and the references therein), and in the n -subharmonic case these results were proved by Poletsky-Sigurdsson [14, 15]. In this note we prove the $1 < m < n$ cases. We start in Section 2 by stating some basic definitions and facts.

2. JENSEN MEASURES AND ENVELOPES

Let $\mathcal{SH}_m^o(X)$ denote the set of functions that are the restriction to X of functions that are m -subharmonic and continuous on some neighborhood of $X \subseteq \mathbb{C}^n$. Furthermore, let $\mathcal{USC}(X)$ be the set of upper semicontinuous functions defined on X . For a background on m -subharmonic functions defined on an open set see

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e.g. [1, 10]. Recall that

$$\mathcal{PSH}(\Omega) = \mathcal{SH}_n(\Omega) \subset \cdots \subset \mathcal{SH}_1(\Omega) = \mathcal{SH}(\Omega),$$

where Ω is an open domain in \mathbb{C}^n , \mathcal{PSH} denotes the plurisubharmonic functions, \mathcal{SH} denotes the subharmonic functions and \mathcal{SH}_m denotes the m -subharmonic functions defined on Ω .

Next, we define a class of Jensen measures.

Definition 2.1. Let X be a compact set in \mathbb{C}^n , $1 \leq m \leq n$, and let μ be a non-negative regular Borel measure defined on X with $\mu(X) = 1$. We say that μ is a *Jensen measure with barycenter* $z \in X$ w.r.t. $\mathcal{SH}_m^o(X)$ if

$$u(z) \leq \int_X u d\mu \quad \text{for all } u \in \mathcal{SH}_m^o(X).$$

The set of such measures will be denoted by $\mathcal{J}_z^m(X)$.

With the help of the Jensen measures defined in Definition 2.1 we can now define m -subharmonic functions defined on compact sets. For more results about these functions, see [3].

Definition 2.2. Let X be a compact set in \mathbb{C}^n . An upper semicontinuous function u defined on X is said to be *m -subharmonic on X* , $1 \leq m \leq n$, if

$$u(z) \leq \int_X u d\mu, \quad \text{for all } z \in X \text{ and all } \mu \in \mathcal{J}_z^m(X).$$

The set of m -subharmonic functions defined on X will be denoted by $\mathcal{SH}_m(X)$. A function $h : X \rightarrow \mathbb{R}$ is called *m -harmonic* if h , and $-h$, are m -subharmonic. The set of all m -harmonic functions defined on X will be denoted by $\mathcal{H}_m(X)$. We shall call n -harmonic functions *pluriharmonic*, and denote it by $\mathcal{PH}(X) = \mathcal{H}_n(X)$.

Remark. By definition, we see that $\mathcal{SH}_m^o(X) \subseteq \mathcal{SH}_m(X)$.

Remark. It follows from Definition 2.1 that for any $z \in X$ we have

$$\mathcal{J}_z^m(X) = \bigcap_{X \subset U} \mathcal{J}_z^m(U) = \bigcap_{X \subset U} \mathcal{J}_z^{m,c}(U),$$

where U denotes an *open* set in \mathbb{C}^n . Recall that $\mu \in \mathcal{J}_z^m(U)$ ($\mathcal{J}_z^{m,c}(U)$) if μ is a probability measure with compact support in U , and for all $u \in \mathcal{SH}_m(U)$ ($u \in \mathcal{SH}_m(U) \cap \mathcal{C}(U)$) it holds that

$$u(z) \leq \int u d\mu.$$

Thanks to Theorem 2.2 in [4] we have that $\mathcal{J}_z^m(U) = \mathcal{J}_z^{m,c}(U)$.

Remark. Sometimes we will leave out the parenthesis and only write \mathcal{J}_z^m instead of $\mathcal{J}_z^m(X)$, where X is a compact set.

Remark. From Definition 2.2 it follows that a continuous function $h : X \rightarrow \mathbb{R}$ is m -harmonic if, and only if, for every $z_0 \in \mathcal{J}_{z_0}^m$ we have that

$$h(z_0) = \int h d\mu.$$

Remark. For Borel probability measures let us define the following two classes

$$\mathcal{M}_1 = \left\{ \mu : u(z) \leq \int_X u d\mu \text{ for all } u \in \mathcal{SH}_m(X) \right\}$$

$$\mathcal{M}_2 = \left\{ \mu : u(z) \leq \int_X u d\mu \text{ for all } u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X) \right\}.$$

It follows from the proof of Theorem 2.8 in [3] that

$$\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{J}_z^m(X).$$

This means that the class of Jensen measures can be generated by the class of m -subharmonic functions on X or by the class of *continuous* m -subharmonic functions on X .

In Definition 2.3 we introduce two useful envelope constructions.

Definition 2.3. Assume that $X \subseteq \mathbb{C}^n$ is a compact set, and $1 \leq m \leq n$. For $f \in \mathcal{C}(X)$ we define

$$\mathbf{S}_f(z) = \sup \{v(z) : v \in \mathcal{SH}_m(X), v \leq f\},$$

and similarly

$$\mathbf{S}_f^c(z) = \sup \{v(z) : v \in \mathcal{SH}_m(X) \cap \mathcal{C}(X), v \leq f\}.$$

We shall need the following version of Edwards' celebrated duality theorem (see Theorem 2.8 in [3]).

Theorem 2.4. Let X be a compact subset in \mathbb{C}^n , $1 \leq m \leq n$, and let f be a real-valued lower semicontinuous function defined on X . Then we have that

(a)

$$\sup \{\psi(z) : \psi \in \mathcal{SH}_m^o(X), \psi \leq f\} = \inf \left\{ \int f d\mu : \mu \in \mathcal{J}_z^m(X) \right\}, \text{ and}$$

(b)

$$\mathbf{S}_f(z) = \mathbf{S}_f^c(z) = \inf \left\{ \int f d\mu : \mu \in \mathcal{J}_z^m(X) \right\}.$$

In Theorem 3.5 we shall use the following lemma.

Lemma 2.5. Let X be a compact subset in \mathbb{C}^n , $1 \leq m \leq n$, and let f be a real-valued lower semicontinuous function defined on X . Then there exists a sequence $u_j \in \mathcal{SH}_m^o(X)$ such that $u_j \nearrow \mathbf{S}_f$, as $j \rightarrow \infty$.

Proof. The proof follows from Theorem 2.4, and Choquet's lemma (see e.g. Lemma 2.3.4 in [9]). □

3. THE CHOQUET AND ŠILOV BOUNDARIES OF COMPACT SETS

The Choquet boundary (Definition 3.1) of a compact set X w.r.t. $\mathcal{J}_{z_0}^m$ and its topological closure the Šilov boundary (Definition 3.6) are central concepts in the Dirichlet problems studied in Section 4 and Section 5, so in this section we shall characterize these boundaries in terms of peak points and m -harmonic measures.

Definition 3.1. Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . The *Choquet boundary* of X w.r.t. $\mathcal{J}_{z_0}^m$ is defined as

$$O_X^m = \{z \in X : \mathcal{J}_{z_0}^m = \{\delta_z\}\}.$$

From Lemma 1.10 in [7] it follows that O_X^m is a G_δ -set. Let Ω be a bounded domain in \mathbb{C}^n , and set $X = \bar{\Omega}$. Then the Choquet boundary is contained in the topological boundary, i.e. $O_X^m \subseteq \partial X$.

Next we introduce the concept of m -subharmonic peak points.

Definition 3.2. Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . We say that a point $z \in X$ is a *m -subharmonic peak point* (or simply a *peak point*) if there exists a function $u \in \mathcal{SH}_m(X)$ such that $u(z) = 0$, and $u(w) < 0$ for $w \in X \setminus \{z\}$. The function u is then called a *peak function*.

Using Gamelin's more general setting we can, from Theorem 1.13 in [7], draw the conclusion that: A point $z \in X$ is a m -subharmonic peak point if, and only if, there exists a function $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ such that $u(z) = 0$ and $u(w) < 0$ for $w \in X \setminus \{z\}$.

We shall later use Lemma 3.3 in Theorem 4.2, and Lemma 3.4 is used in Theorem 3.5.

Lemma 3.3. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . Then $z \in O_X^m$ if, and only if, for every $f \in \mathcal{C}(X)$ we have that $\mathbf{S}_f(z) = f(z)$.*

Proof. Cf. page 10 in [7]. □

Lemma 3.4. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . A point $z \in X$ is a m -subharmonic peak point if, and only if, for any neighborhood V of z there exists $u \in \mathcal{SH}_m(X)$ such that $u < 0$, $u(z) = -1$ and $u \leq -2$ on $X \setminus V$.*

Proof. The implication \Rightarrow is immediate. To prove the converse implication take $z_0 \in X$, and let $\epsilon_j \searrow 0$, as $j \rightarrow \infty$, be a sequence such that

$$\epsilon_j < \frac{1}{2} \frac{\left(\frac{3}{4}\right)^j}{1 - \left(\frac{3}{4}\right)^j}.$$

Let $Y_n \subseteq X$ be closed subsets such that

$$X \setminus \{z_0\} = \bigcup_{n=1}^{\infty} Y_n.$$

Now we shall define a sequence of functions from $\mathcal{SH}_m(X)$. Let $u_0 = -1$. Suppose that we already have chosen $u_0, \dots, u_j \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$, such that for $1 \leq k \leq j$ we have that $u_k < 0$, $u_k(z_0) = -1$, and

$$u_k \leq -2 \quad \text{on } U_{k-1} \cup Y_{k-1},$$

where

$$U_{k-1} = \left\{ w \in X : \max_{1 \leq l \leq k-1} u_l(w) \geq -1 + \epsilon_{k-1} \right\}.$$

Note that $\{U_k\}$ is an increasing sequence of closed sets. Now take a function $u_{j+1} \in \mathcal{SH}_m(X)$ such that $u_{j+1} \leq -2$ on $Y_j \cup U_j$, and $u_{j+1}(z_0) = -1$. Let us then define

$$u(z) = \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j u_j.$$

This construction then implies that $u \in \mathcal{SH}_m(X)$ and $u(z_0) = -1$. Now suppose that $w \neq z_0$ and $w \notin \bigcup_{k=1}^{\infty} U_k$, that $u_j(w) \leq -1$ for all j , and $u_j(w) \leq -2$ for at least one j , therefore $u(w) < -1$. Assume next that $w \in \bigcup_{k=1}^{\infty} U_k$. If $w \in U_l \setminus U_{l-1}$, then

$$\begin{aligned} u_j(w) &\leq -1 + \epsilon_{l-1}, \text{ for } 1 \leq j \leq l-1, \\ u_l(w) &< 0, \\ u_j(w) &\leq -2, \text{ for } j > l. \end{aligned}$$

Now we have that

$$\begin{aligned} u(w) &\leq \frac{1}{4} \left((-1 + \epsilon_l) \sum_{j=0}^{l-1} \left(\frac{3}{4}\right)^j - 2 \sum_{j=l+1}^{\infty} \left(\frac{3}{4}\right)^j \right) \\ &= -1 + \epsilon_l \left(1 - \left(\frac{3}{4}\right)^l \right) - \frac{1}{2} \left(\frac{3}{4}\right)^l < -1. \end{aligned}$$

This means that $u + 1$ is a peak function. \square

In Theorem 3.5 we characterize the Choquet boundary of X w.r.t. $\mathcal{J}_{z_0}^m$ in terms of peak points.

Theorem 3.5. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . Then $z \in O_X^m$ if, and only if, z is a peak point.*

Proof. If z is a peak point, then there exists a function $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ such that $u(z) = 0$ and $u(w) < 0$, for all $w \neq z$. Let $\mu \in \mathcal{J}_z^m$. Then it holds that

$$0 = u(z) \leq \int_X u d\mu \leq 0,$$

which implies that $\text{supp } \mu \subseteq \{w : u(w) = 0\}$. Hence, $\mu = \delta_z$.

On the other hand, let $z_0 \in O_X^m$ and let V be any neighborhood of z_0 . Furthermore, let $f \in \mathcal{C}(X)$ be such that $f < 0$, $f(z_0) = -1$, and $f < -4$ on $X \setminus V$. Then, $\mathbf{S}_f(z_0) = -1$, and $\mathbf{S}_f < -4$ on $X \setminus V$. From Lemma 2.5 it follows that one can find a function $v \in \mathcal{SH}_m^o(X)$ such that $v \leq \mathbf{S}_f$, and $-1 > v(z_0) > -2$. Then the function u defined by

$$u(z) = -\frac{v(z)}{v(z_0)}$$

satisfies $u(z_0) = -1$, and $u < -2$ on $X \setminus V$. The proof is finished by Lemma 3.4. \square

Remark. It follows from Theorem 3.5 that O_X^m is non-empty.

Next, we introduce the Šilov boundary of a compact set X w.r.t. $\mathcal{J}_{z_0}^m$.

Definition 3.6. Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . The *Šilov boundary*, B_X^m , of X is defined to be the topological closure of O_X^m .

Remark. Obviously, it holds that $O_X^m \subseteq B_X^m$, and O_X^m is closed if, and only if, $O_X^m = B_X^m$.

It is not always true that a m -subharmonic function must attain its maximum on the potential boundary, B_X^m (see the example before Theorem 4.3 in [14]). But we have at least the following weak maximum principle that we shall use in our study of the Šilov boundary and the m -harmonic measure.

Theorem 3.7. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . If $u \in \mathcal{SH}_m(X)$, then*

$$u(z) \leq \sup_{w \in B_X^m} u(w) \quad \text{for all } z \in X.$$

Proof. See Theorem 1.12 in [7]. \square

Definition 3.8. Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . The m -harmonic measure of a subset $E \subseteq X$ is defined as the function

$$\omega(z, E, X) = \inf_{\substack{V \supseteq E \\ V \text{ open}}} \sup_{\mu \in \mathcal{J}_z^m} \mu(V).$$

We have the following estimate.

Theorem 3.9. *Let $1 \leq m \leq n$, K and k be constants and let X be a compact set in \mathbb{C}^n . If $u \in \mathcal{SH}_m(X)$ satisfies $u \leq K$ on X , and $u \leq k < K$ on some set $Y \subseteq X$, then*

$$u(z) \leq k\omega(z, Y, X) + K(1 - \omega(z, Y, X)), \quad z \in X.$$

Proof. Fix $\epsilon > 0$, and set

$$U = \{z \in X : u(z) < k + \epsilon\}.$$

Then U is an open set such that $Y \subseteq U$. For all $z \in X$, there exists a measure $\mu \in \mathcal{J}_z^m(X)$, and an open set V , such that $Y \subseteq V \subseteq U$ and

$$\omega(z, Y, X) \leq \mu(V) \leq \omega(z, Y, X) + \epsilon.$$

Then we have that

$$\begin{aligned} u(z) &\leq \int_X u d\mu = \int_V u d\mu + \int_{X \setminus V} u d\mu \\ &\leq (k + \epsilon)(\omega(z, Y, X) + \epsilon) + K(1 - \omega(z, Y, X)). \end{aligned}$$

If we let $\epsilon \rightarrow 0^+$, then we get the desired conclusion. \square

Theorem 3.10. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . The Šilov boundary, B_X^m , is the smallest closed set E such that $\omega(z, E, X)$ is identically 1.*

Proof. First we shall prove that $\omega(z, B_X^m, X) = 1$ for all $z \in X$. To prove this assume by contradiction that there exists a $z_0 \in X$ such that $\omega(z_0, B_X^m, X) < 1$. Then there exists an open neighborhood V of B_X^m , and $0 < c < 1$, such that for all $\mu \in \mathcal{J}_{z_0}^m$ it holds that

$$\mu(V) < c.$$

Let W be an open set such that $B_X^m \Subset W \Subset V$, and let $f \in \mathcal{C}(X)$ be such that $f = -1$ on W , and $f = 0$ on $X \setminus V$. Then we have that

$$\mathbf{S}_f \leq f = -1 \text{ on } W, \tag{3.1}$$

and thanks to Edwards' theorem (Theorem 2.4) we have that

$$\mathbf{S}_f(z_0) = \inf \left\{ \int \phi d\mu : \mu \in \mathcal{J}_{z_0}^m(X) \right\}.$$

For given $\mu \in \mathcal{J}_{z_0}^m$ it holds that

$$\int_X f d\mu = \int_{X \setminus V} f d\mu + \int_V f d\mu > 0 + (-1)(1 - c) = -1 + c,$$

so

$$\mathbf{S}_f(z_0) > -1 + c. \quad (3.2)$$

Therefore, by (3.1) and (3.2) we conclude that there exists a function $u \in \mathcal{SH}_m(X)$ such that $u < -1$ on W , and $u(z_0) > -1$. But this is impossible since by Theorem 3.7 each m -subharmonic function must attain its maximum on B_X^m . This ends the first part of the proof.

Next, assume that there exists a proper closed subset E of B_X^m such that for all $z \in X$ we have that $\omega(z, E, X) = 1$. Then there exist a point $z_0 \in O_X^m \setminus E$, and a neighborhood V of E such that $z_0 \notin V$. Then since $\mathcal{J}_{z_0}^m = \{\delta_{z_0}\}$, we get that $\omega(z_0, E, X) = 0$ and a contradiction is obtained. \square

Corollary 3.11. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . The Šilov boundary, B_X^m , is the smallest closed set such that for every $z \in X$ there exists a Jensen measure $\mu \in \mathcal{J}_z^m$ such that $\text{supp } \mu \subseteq B_X^m$.*

Proof. Assume that Y is a subset of X such that for every $z \in X$ there exists a Jensen measure $\mu \in \mathcal{J}_z^m$ such that $\text{supp } \mu \subseteq Y$. For z from the Choquet boundary we have that $\mathcal{J}_z^m = \{\delta_z\}$. Therefore it follows that $O_X^m \subseteq Y$, and hence $B_X^m \subseteq Y$. For $z \in X$, and for any neighborhood V of Y , we have that

$$\sup_{\mu \in \mathcal{J}_z^m} \mu(V) = 1,$$

and therefore $\omega(z, Y, X) = 1$. If Y is the smallest closed set with the assumed property it now follows by using Theorem 3.10 that $B_X^m = Y$. \square

In Definition 3.12 we introduce the subset, $\mathcal{J}_z^{b,m}$, of Jensen measures \mathcal{J}_z^m whose support is contained in the Šilov boundary, B_X^m . We shall need $\mathcal{J}_z^{b,m}$ in Lemma 5.7, Proposition 5.9, and in Theorem 5.10.

Definition 3.12. Let $1 \leq m \leq n$, X be a compact set in \mathbb{C}^n and $z \in X$. Then we define

$$\mathcal{J}_z^{b,m} = \{\mu \in \mathcal{J}_z^m : \text{supp } \mu \subseteq B_X^m\}.$$

A direct consequence of Corollary 3.11 is that $\mathcal{J}_z^{b,m}$ is non-empty.

Corollary 3.13. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . For every $z \in X$ we have that $\mathcal{J}_z^{b,m}$ is non-empty.*

Proof. This follows from Corollary 3.11. \square

In solving the Dirichlet problem in the case when the Choquet boundary is the whole compact set (Theorem 4.3) we shall need $\mathcal{I}_X^m(z)$, defined below, together with Proposition 3.15. The inspiration behind $\mathcal{I}_X^m(z)$ is from potential theory, and it is explained in the remark after Proposition 3.15.

Definition 3.14. Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . For $z \in X$ let us define the following set

$$\mathcal{I}_X^m(z) = \{w \in X : \omega(z, \bar{B}(w, r) \cap X, X) > 0, \text{ for all } r > 0\}.$$

Proposition 3.15. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . Then for $z \in X$ we have that*

- (1) $\mathcal{I}_X^m(z)$ is a closed set;
- (2) if $\mu \in \mathcal{J}_z^m(X)$ then $\text{supp } \mu \subseteq \mathcal{I}_X^m(z)$;

(3) $\mathcal{I}_X^m(z) = \{z\}$ if, and only if, $z \in O_X^m$.

Proof. (1) First note that

$$\mathcal{I}_X^m(z) = \bigcap_{r>0} Y_r, \quad \text{where } Y_r = \{w \in X : \omega(z, \bar{B}(w, r) \cap X, X) > 0\},$$

and therefore it is sufficient to prove that the sets Y_r are closed. Let $Y_r \ni x_j \rightarrow x \in X$. Then for every j there exists an open set $V_j \supset \bar{B}(x_j, r) \cap X$, and $\mu_j \in \mathcal{J}_z^m(X)$ such that $\mu_j(V_j) > 0$. From a compactness argument there exists a j_0 such that $V_{j_0} \supset \bar{B}(x, r) \cap X$, and therefore $\omega(z, \bar{B}(x, r) \cap X, X) > 0$, so $x \in Y_r$.

(2) Fix $r > 0$. Let $\mu \in \mathcal{J}_z^m(X)$, and let $w \in \text{supp } \mu$, then $\mu(\bar{B}(w, r) \cap X) > 0$. This means that $\omega(z, \bar{B}(w, r) \cap X, X) > 0$, hence $w \in \mathcal{I}_X^m(z)$.

(3) It follows from (2) that if $\mathcal{I}_X^m(z) = \{z\}$, then $\mathcal{J}_z^m(X) = \{\delta_z\}$. Thus, $z \in O_X^m$. On the other hand, if $z \in O_X^m$, then for $w \neq z$ we have that $\omega(z, \bar{B}(w, r), X) = 0$ if $r < \|z - w\|$. Therefore, $w \notin \mathcal{I}_X^m(z)$, which implies that $\mathcal{I}_X^m(z) = \{z\}$. \square

Remark. There is a very nice characterization of those points for which $\mathcal{I}_X^1(z) = \{z\}$ in the case of subharmonic functions. Namely, $\mathcal{I}_X^1(z) = \{z\}$ if, and only if, X^c is not thin at z (see Theorem 3.3 in [13]). A similar result for m -subharmonic functions, $m > 1$, is not possible. To see this look at Example 5.5: Then for all $z \in \partial \mathbb{D}^n$ the set $(\mathbb{D}^n)^c$ is not m -thin at z , but $\mathcal{I}_{\mathbb{D}^n}^m(z) \neq \{z\}$, $m > 1$, if e.g. $z \in \mathbb{D} \times \cdots \times \mathbb{D} \times \partial \mathbb{D}$.

4. THE DIRICHLET PROBLEM FOR CONTINUOUS m -SUBHARMONIC FUNCTIONS

In Theorem 4.2, we characterize those compact sets X for which the Dirichlet problem has a solution within the class of continuous m -subharmonic functions defined on a compact set. To obtain this we need the notion of O^m -regular compact sets (Definition 4.1). We end this section with Theorem 4.3 where we consider the case when X is equal to its Choquet boundary.

Definition 4.1. Let $1 \leq m \leq n$. We say that a compact set X in \mathbb{C}^n is O^m -regular if O_X^m is a closed subset of X .

The next theorem provides the characterization of O^m -regular sets.

Theorem 4.2. *Let $1 \leq m \leq n$. Let X be a compact set in \mathbb{C}^n . Then the following conditions are equivalent:*

- (1) X is a O^m -regular set;
- (2) for every $f \in \mathcal{C}(O_X^m)$ there exists a function $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ such that $u = f$ on O_X^m ;
- (3) for every $f \in \mathcal{C}(B_X^m)$ there exists a function $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ such that $u = f$ on B_X^m .

Proof. The implication (1) \Rightarrow (2) follows from Theorem 3.3 in [15]. To prove the implication (2) \Rightarrow (3) note that if $f \in \mathcal{C}(B_X^m)$, then $f \in \mathcal{C}(O_X^m)$, and therefore there exists $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ such that $u = f$ on O_X^m . Since both functions u and f are continuous we obtain that $u = f$ on B_X^m . To prove the last implication (3) \Rightarrow (1) suppose that there exists $z_0 \in B_X^m \setminus O_X^m$. By Lemma 3.3 there exists a function $f \in \mathcal{C}(X)$ such that $\mathbf{S}_f(z_0) < f(z_0)$. By assumption there exists a function $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ such that $u = f$ on B_X^m . Then we get that

$$\mathbf{S}_f(z_0) \geq u(z_0) = f(z_0),$$

and a contradiction is obtained. \square

Next, we consider the case when X is equal to its Choquet boundary.

Theorem 4.3. *Let X be a compact set in \mathbb{C}^n , and $1 \leq m \leq n$. The following conditions are then equivalent:*

- (1) $O_X^m = X$;
- (2) $B_X^m = X$;
- (3) $\mathcal{C}(X) = \mathcal{SH}_m(X)$;
- (4) $\mathcal{C}(X) = \mathcal{H}_m(X)$;
- (5) $f(z) = \|z\|^2 \in \mathcal{H}_m(X)$;
- (6) $g(z) = -\|z\|^2 \in \mathcal{SH}_m(X)$;
- (7) $\mathcal{I}_X^m(z) = \{z\}$ for all $z \in X$.

Proof. The following implications are obvious: (1) \Rightarrow (2), (3) \Rightarrow (4), (4) \Rightarrow (5), and (3) \Rightarrow (6). We have that (2) \Rightarrow (3) follows from Theorem 4.2. For implication (5) \Rightarrow (1) take $z_0 \in X$. Since $-\|z\|^2 \in \mathcal{H}_m(X)$, then also $-\|z - z_0\|^2$ is m -harmonic and it is also a peak function for z_0 . Thus, $z_0 \in O_X^m$. To note implication (6) \Rightarrow (5): Since $\|z\|^2$ is m -subharmonic, and by assumption $-\|z\|^2$ is also m -subharmonic we have that $\|z\|^2$ is m -harmonic. Finally, the equivalence between (1) and (7) follows from Proposition 3.15. \square

5. THE DIRICHLET PROBLEM FOR m -HARMONIC FUNCTIONS

In this section we shall characterize those compact sets for which the Dirichlet problem has a solution for m -harmonic functions (Theorem 5.10). First let us compare m -harmonic functions defined on a compact set with m -harmonic functions defined on an open set.

It was proved in [3] that every m -harmonic function defined on an *open* set is pluriharmonic. The situation is different for the function theory on compact sets. We give in Example 5.1 an example of a 2-harmonic function defined on a compact set that is not pluriharmonic (3-harmonic). On the other hand, in Proposition 5.2 we show that there are compact sets X for which $\mathcal{H}_m(X) = \mathcal{PH}(X)$.

Example 5.1. Let $X = \{(0, 0, z_3) \in \mathbb{C}^3 : |z_3| \leq 1\}$, and let u be a function defined on X by $u(z_1, z_2, z_3) = -|z_3|^2$. Then $-u$ is plurisubharmonic, and also 2-subharmonic. Furthermore, u is the restriction of a 2-subharmonic function defined in \mathbb{C}^3 ; namely

$$u(z_1, z_2, z_3) = 2 \left(|z_1|^2 + |z_2|^2 - \frac{1}{2}|z_3|^2 \right), \quad (z_1, z_2, z_3) \in X.$$

Finally, note that u is not plurisubharmonic (3-subharmonic) on X . To prove this assume by contradiction that $u \in \mathcal{SH}_3(X)$. By assumption there exists a decreasing sequence $u_j \in \mathcal{SH}_3^o(X)$ such that $u_j \rightarrow u$, as $j \rightarrow \infty$. But then u_j must be subharmonic on the set $Y = X \cap \{|z_3| < 1\}$, and therefore u must be also subharmonic on Y , and a contradiction is obtained. \square

Proposition 5.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded B -regular domain in the sense of Sibony [16]. Then we have that $\mathcal{H}_m(\bar{\Omega}) = \mathcal{PH}(\bar{\Omega})$.*

Proof. Recall that if Ω is a B -regular domain, then for all $z \in \partial\Omega$ we have that $\mathcal{J}_z^n(\Omega) = \{\delta_z\}$. Take any $h \in \mathcal{H}_m(\bar{\Omega})$, then $h \in \mathcal{H}_m(\Omega)$, so $h \in \mathcal{PH}(\Omega)$. By the assumption of B_n -regularity we have also that $h \in \mathcal{PH}(\partial\Omega)$, which implies that $h \in \mathcal{PH}(\bar{\Omega})$. \square

One of the main notions in Theorem 5.10 is so called m -Poisson sets defined as follows.

Definition 5.3. Let $1 \leq m \leq n$. A compact set X in \mathbb{C}^n is called a m -Poisson set if for every $f \in \mathcal{C}(B_X^m)$, there exists a function $u \in \mathcal{H}_m(X)$ such that $u = f$ on B_X^m .

In Example 5.4 we see that the topological closure of the unit ball in \mathbb{C}^n is a O^m -regular set, but not a m -Poisson set. Furthermore, in Example 5.5 we see that the topological closure of the unit polydisc in \mathbb{C}^n , $n \geq 3$, is a n -Poisson set.

Example 5.4. Let X be the topological closure of the unit ball \mathbb{B} in \mathbb{C}^n , and let $1 \leq m \leq n$. Then we have that

$$O_X^m = B_X^m = \partial\mathbb{B}.$$

Hence, X is a O^m -regular set. But X is not a m -Poisson set, since it is not always possible to extend a function $f \in \mathcal{C}(\partial\mathbb{B})$ to the inside so that it is m -pluriharmonic (see e.g. [5]). \square

Example 5.5. Let X be the closure of the unit polydisc \mathbb{D}^n in \mathbb{C}^n , and let $1 \leq m \leq n$. Then

$$\begin{aligned} O_X^1 &= B_X^1 = \partial\mathbb{D}^n, \\ O_X^n &= B_X^n = \partial\mathbb{D} \times \cdots \times \partial\mathbb{D}, \end{aligned}$$

and for $1 < m < n$ we get that

$$O_X^m = B_X^m = \bigcup_{1 \leq j_1 < \cdots < j_m \leq n} \partial\mathbb{D} \times \cdots \times \overbrace{\mathbb{D}}^{j_1} \times \cdots \times \overbrace{\mathbb{D}}^{j_m} \times \cdots \times \partial\mathbb{D}.$$

Thus, O_X^n , and B_X^n are equal to the distinguished boundary of \mathbb{D}^n . For $1 < m < n$, the above statement follows from the fact that any m -subharmonic function in $\Omega \subset \mathbb{C}^n$ is $(m-1)$ -subharmonic on any hyperplane passing by Ω (see [1]). In particular, X is a O^m -regular set. Furthermore, for $n \geq 3$ the compact set X is a n -Poisson set, since for every $f \in \mathcal{C}(\partial\mathbb{D} \times \cdots \times \partial\mathbb{D})$ we can always find a pluriharmonic function u defined on \mathbb{D}^n such that $u = f$ on $\partial\mathbb{D} \times \cdots \times \partial\mathbb{D}$ (see e.g. [2, 3]). \square

Let us next define an (partial) order in the cone of Jensen measures.

Definition 5.6. Let μ and ν be Jensen measures. We say that μ is *subordinated* to ν , and denote it with $\mu \preceq \nu$, if for all $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ it holds that

$$\int_X u d\mu \leq \int_X u d\nu.$$

Remark. Note that \preceq is indeed an (partial) order, see e.g. [7].

Lemma 5.7 shall later be used in Proposition 5.9 and in Theorem 5.10. The set $\mathcal{J}_z^{b,m}$ of Jensen measures used in Lemma 5.7 was defined in Definition 3.12.

Lemma 5.7. *Let $1 \leq m \leq n$, and let X be a compact set in \mathbb{C}^n . For every $\mu \in \mathcal{J}_z^m$ there exists a measure $\nu \in \mathcal{J}_z^{b,m}$ with $\mu \preceq \nu$.*

Proof. Cf. Theorem 1.17 in [7]. □

In contrast with Definition 2.3 we shall in Definition 5.8 introduce an envelope construction where the given function is only defined on the Šilov boundary, B_X^m . We name this envelope as customary after Oskar Perron and Hans-Joachim Bremermann. In Proposition 5.9, we prove some elementary, but useful facts about this envelope.

Definition 5.8. Let $1 \leq m \leq n$, and X be a compact set in \mathbb{C}^n . For given $f \in \mathcal{C}(B_X^m)$ we define the *Perron-Bremermann envelope*, PB_f , as

$$PB_f = \sup \{v(z) : v \in \mathcal{SH}_m(X) \cap \mathcal{C}(X), v \leq f \text{ on } B_X^m\}.$$

Proposition 5.9. *Let $1 \leq m \leq n$, and X be a compact set in \mathbb{C}^n . For every $f \in \mathcal{C}(B_X^m)$ we have that*

- (1) *PB_f is a lower semicontinuous function, and that for any $z \in X$ and for any $\mu \in \mathcal{J}_z^m$ it holds that*

$$PB_f(z) \leq \int PB_f d\mu ;$$

- (2) *if X is a O^m -regular set, then*

$$PB_f(z) = \inf \left\{ \int f d\mu; \mu \in \mathcal{J}_z^{b,m} \right\}.$$

Proof. Part (1) is an immediate consequence of the definition. Next, to part (2). For $f \in \mathcal{C}(B_X^m)$, let

$$I_f(z) = \inf \left\{ \int f d\mu; \mu \in \mathcal{J}_z^{b,m} \right\}.$$

By construction we have that if $v \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$, and $v \leq f$ on B_X^m , then for any $\mu \in \mathcal{J}_z^{b,m}$ it holds that

$$v(z) \leq \int v d\mu \leq \int f d\mu,$$

and therefore $PB_f(z) \leq I_f(z)$. By Theorem 4.2, there exists a function u such that $-u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ and $u = f$ on B_X^m . Thanks to Lemma 5.7 it holds that for every $\mu \in \mathcal{J}_z^m$ there is a Jensen measure $\nu \in \mathcal{J}_z^{b,m}$ such that

$$\int u d\mu \geq \int u d\nu.$$

Thus,

$$PB_f(z) \geq \mathbf{S}_u(z) = \inf \left\{ \int u d\mu; \mu \in \mathcal{J}_z^m \right\} = I_u(z) = I_f(z) \geq PB_f(z).$$

□

We end this note with Theorem 5.10, and the characterization of those compact sets X for which the Dirichlet problem has a solution within the class of m -harmonic functions defined on a compact set.

Theorem 5.10. *Let X be a compact set in \mathbb{C}^n , and let $1 \leq m \leq n$. Then the following conditions are equivalent:*

- (1) X is a m -Poisson set;
- (2) for every $z \in X$, the set $\mathcal{J}_z^{b,m}$ contains exactly one measure, P_z^m ;
- (3) for every $f \in \mathcal{C}(B_X^m)$ we have that

$$PB_{-f} = -PB_f \quad \text{on } X.$$

Proof. To prove the implication (1) \Rightarrow (2) assume that X is a m -Poisson set, $z \in X$, and $\mu, \nu \in \mathcal{J}_z^{b,m}$. By the assumptions we have that for any $f \in \mathcal{C}(B_X^m)$ there exists a function $h \in \mathcal{H}_m(X)$ with $h = f$ on B_X^m . Hence,

$$u(z) = \int f d\mu = \int f d\nu,$$

which implies that $\mu = \nu$. Next, we shall verify the implication (2) \Rightarrow (3). First we shall prove that X is an O^m -regular set. Let $O_X^m \ni z_j \rightarrow z \in X$, as $j \rightarrow \infty$. Then from the sequence of measures δ_{z_j} we can extract a subsequence (denoted also by δ_{z_j}) such that δ_{z_j} is weak*-convergent to some measure $\mu \in \mathcal{J}_z^{b,m}$. By assumption the measure $\mu = P_z^m$ is unique. Then for every $f \in \mathcal{C}(X)$ we have that

$$\int f dP_z^m = \lim_{j \rightarrow \infty} \int f d\delta_{z_j} = \lim_{j \rightarrow \infty} f(z_j) = f(z) = \int f d\delta_z,$$

which means that $P_z^m = \delta_z$, so $z \in O_X^m$. Thus, X is an O^m -regular set. Now for $f \in \mathcal{C}(B_X^m)$ let us define the following function

$$h(z) = \int f dP_z^m.$$

We are going to prove that h is a m -harmonic function. First we show that h is continuous. Let $X \ni z_j \rightarrow z \in X$. We can assume that $P_{z_j}^m$ is weak*-convergent to some measure $\mu \in \mathcal{J}_z^{b,m}$, if necessary we extract a subsequence. Therefore, by assumption $\mu = P_z^m$, and it follows that

$$\lim_{j \rightarrow \infty} h(z_j) = \lim_{j \rightarrow \infty} \int f dP_{z_j}^m = \int f dP_z^m = h(z).$$

Proposition 5.9 part (2), gives us that for every $z \in X$ it holds that

$$h(z) = \int f dP_z^m = \inf \left\{ \int f d\mu; \mu \in \mathcal{J}_z^{b,m} \right\} = PB_f(z), \quad (5.1)$$

and therefore again by Proposition 5.9 part (1) we can conclude that $h \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$. On the other hand, for any $z \in X$ and any $\mu \in \mathcal{J}_z^m$ we have by Lemma 5.7 that

$$h(z) \leq \int f d\mu \leq \int h dP_z^m = \int f dP_z^m = h(z).$$

Hence, $h \in \mathcal{H}_m(X)$. Note also that it follows from (5.1) that

$$PB_{-f}(z) = \int -f dP_z^m = - \int f dP_z^m = -PB_f(z).$$

Next we shall prove the implication (3) \Rightarrow (1). By Proposition 5.9 part (1), the envelopes PB_f and PB_{-f} are lower semicontinuous and therefore we can conclude that $PB_f \in \mathcal{C}(X)$. Furthermore, we have that for any $z \in X$ and any $\mu \in \mathcal{J}_z^m$ it holds that

$$-PB_f(z) = PB_{-f}(z) \leq \int PB_{-f} d\mu = \int -PB_f d\mu \leq -PB_f(z).$$

Hence, $PB_f \in \mathcal{H}_m(X)$. To conclude that (1) holds note that $PB_f \leq f$, and $PB_{-f} \leq -f$, on B_X^m and therefore we must have that $PB_f = f$ on B_X^m . \square

Corollary 5.11. *Let $1 \leq m \leq n$. Then every compact set X in \mathbb{C}^n that is a m -Poisson set is also an O^m -regular set.*

Proof. This follows immediately from Theorem 4.2 and Theorem 5.10. \square

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