SINGULARITIES OF DIVISORS OF LOW DEGREE ON SIMPLE ABELIAN VARIETIES

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ABSTRACT. It is known by results of Kollár, Ein, Lazarsfeld, Hacon and Debarre that effective divisors representing principal and other low degree polarizations on complex abelian varieties have mild singularities. In this note, we extend these results to all polarizations of degree $< g$ on simple g-dimensional abelian varieties, settling a conjecture of Debarre and Hacon.

It is known by results of Kollár, Ein, Lazarsfeld, Hacon and Debarre that effective divisors representing principal and other low degree polarizations on complex abelian varieties have mild singularities ([K, Theorem 17.13], [EL], [H1], [H2],[DH]). In this note, we prove another result in the same direction, conjectured by Debarre and Hacon in [DH, §6] and proved by them for $\chi(l) < 2\sqrt{g} - 1$ and also for low values of g.

Theorem A. *Let (A, l) be a complex g-dimensional simple polarized abelian variety with $\chi(l) < g$. Then*

(1) *every effective divisor E representing l is prime* (Debarre-Hacon, [DH, Proposition 2]) *and normal with rational singularities.*

(2) *Let $m \geq 2$ and let D be an effective divisor representing ml. Then, unless $D = mE$ with E representing l, one has $\lfloor \frac{1}{m}D \rfloor = 0$ ([DH, Corollary 2]) and the pair $(A, \frac{1}{m}D)$ is log terminal.*

We refer to the previously quoted works, especially the introductions of [EL] and [DH], and to Section 10.1.B of the book [L] for history, motivation, and applications.

The proof makes use of all the ingredients of the previously quoted papers, in particular (generic) vanishing theorems involving adjoint and multiplier ideals and the linearity theorem for their cohomological support loci. In this way, Theorem A is a standard consequence of Theorem B below, which is the main content of the present note.

Let us recall that, given a coherent sheaf $\mathcal{F}$ on an abelian variety A, its cohomological support loci are the following subvarieties

$$V^i(A, \mathcal{F}) = \{\alpha \in \widehat{A} \mid H^i(A, \mathcal{F} \otimes P_\alpha) \neq 0\}$$

of $\widehat{A} := \mathrm{Pic}^0 A$, where P_α denotes the line bundle parametrized by $\alpha \in \widehat{A}$ via the choice of a Poincaré line bundle. We also set

$$V_{>0}(A, \mathcal{F}) = \bigcup_{i>0} V^i(A, \mathcal{F}).$$

Finally, we recall that a subvariety X of an abelian variety A is said to be *geometrically non-degenerate* if, for all abelian subvarieties K of A, $\dim(X + K) = \min\{\dim A, \dim X + \dim K\}$ ([R, Lemma II.12] , [D1, (1.11)]).

Theorem B. *Let (A, l) be a polarized g-dimensional abelian variety and let L be a line bundle representing l. Let Z be a non-trivial subscheme of A with geometrically non-degenerate support. Assume also that Z is not a divisor representing l.*

   (1) If $V_{>0}(A, \mathcal{I}_Z \otimes L)$ is empty then $\chi(l) \geq g + 1$.
   (2) If $V_{>0}(A, \mathcal{I}_Z \otimes L)$ is 0-dimensional then $\chi(l) \geq g$.

The proof of Theorem B is based on the study of the Fourier-Mukai-Poincaré transform of the derived dual of the sheaf $\mathcal{I}_Z \otimes L$. This operation produces (see §1 below) a coherent sheaf in cohomological degree g on the dual abelian variety $\widehat{A}$, whose generic rank is $\chi(\mathcal{I}_Z \otimes L)$. Such a sheaf is usually denoted by $\widehat{(\mathcal{I}_Z \otimes L)^\vee}$. From a body of results on the Fourier-Mukai-Poincaré transform, it follows that in case (1), $\widehat{(\mathcal{I}_Z \otimes L)^\vee}$ is an ample vector bundle, while in case (2), it is a k-syzygy sheaf, with k sufficiently high. Applying to $\widehat{(\mathcal{I}_Z \otimes L)^\vee}$ the Le Potier vanishing theorem in the former case and the Evans-Griffith syzygy theorem in the latter case,[1] we obtain a lower bound for the generic rank of the sheaf $\widehat{(\mathcal{I}_Z \otimes L)^\vee}$, hence for $\chi(\mathcal{I}_Z \otimes L)$. This, combined with an upper bound for $\chi(\mathcal{I}_Z \otimes L)$ due to Debarre-Hacon (inequality (2.4) below), proves Theorem B.

Acknowledgments.   The author thanks Giulio Codogni, Zhi Jiang, Mihnea Popa, and Stefan Schreieder for their useful and interesting comments.

1. BACKGROUND ON THE FOURIER-MUKAI-POINCARÉ TRANSFORM

In this section we briefly recall the necessary background about some sheaf-theoretic properties of the Fourier-Mukai-Poincaré transform on abelian varieties. We refer to the papers quoted below or to the survey [P] for more details.

A Poincaré line bundle $\mathcal{P}$ on $A \times \widehat{A}$ defines a Fourier-Mukai functor

$$\Phi_{\mathcal{P}}^{A \to \widehat{A}} : D(A) \to D(\widehat{A})$$

which is an equivalence of categories (Mukai [M1]). Its inverse is $\Phi_{\mathcal{P}^\vee}^{\widehat{A} \to A}[g]$, which can be expressed as

$$(1.1) \qquad\qquad \Phi_{\mathcal{P}^\vee}^{\widehat{A} \to A}[g] = (-1)_A^* \circ \Phi_{\mathcal{P}}^{\widehat{A} \to A}[g] \,,$$

where $(-1)_A$ denotes the natural involution on A. This follows from the fact that $\mathcal{P}^\vee = (-1, 1)_{A \times \widehat{A}}^* \mathcal{P} = (1, -1)_{A \times \widehat{A}}^* \mathcal{P}$ ([BL, Lemma 14.1.2]).

Definition 1.1. ([PP1, Definition 3.1]) Let $\mathcal{F}$ be a coherent sheaf of A. The *gv-index* of $\mathcal{F}$ is the integer

$$gv(\mathcal{F}) = \min_{i>0} (\operatorname{codim}_{\widehat{A}} V^i(A, \mathcal{F}) - i)$$

Moreover:
- if $gv(\mathcal{F}) \geq 0$ then $\mathcal{F}$ is said to be a *Generic Vanishing sheaf*, or simply *GV*;
- if $gv(\mathcal{F}) \geq 1$ then $\mathcal{F}$ is said to be *Mukai-regular*, or simply *M-regular*;
- if $V_{>0} = \emptyset$ then $\mathcal{F}$ is said to *verify the Index Theorem with index 0*, or simply *IT(0)*.

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[1]Interestingly, the theorems of Le Potier and Evans-Griffith are related. In fact, via linear complexes as in [LP, Remark 4.2], one can show that the application of the Evans-Griffith theorem needed here is in turn implied by Le Potier vanishing via an argument of Ein ([L, Example 7.3.10]).

We set $\mathcal{F}^\vee := R\mathcal{H}om_A(\mathcal{F}, \mathcal{O}_A)$. We have the following duality result.

Theorem 1.2. (Hacon, Pareschi-Popa, see [PP1, Theorem 2.2], [PP2, Theorem A]) *Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A. Then $\mathcal{F}$ is GV if and only if $\Phi_{\mathcal{P}^\vee}^{A \to \widehat{A}}(\mathcal{F}^\vee)$ is a sheaf in degree g, i.e.*

$$\Phi_{\mathcal{P}^\vee}^{A \to \widehat{A}}(\mathcal{F}^\vee) = R^g \Phi_{\mathcal{P}^\vee}^{A \to \widehat{A}}(\mathcal{F}^\vee)[-g].$$

If this is the case, following Mukai, we use the following notation

$$\widehat{\mathcal{F}^\vee} := R^g \Phi_{\mathcal{P}^\vee}^{A \to \widehat{A}}(\mathcal{F}^\vee).\ ^2$$

Remark 1.3. (a) Assume that $\mathcal{F}$ is GV. By base change and Serre duality the support of the sheaf $\widehat{\mathcal{F}^\vee}$ is $V^0(A, \mathcal{F})$. Therefore the subvariety $V^0(A, \mathcal{F})$ is non-empty as soon as $\mathcal{F}$ is non-zero, since otherwise $\Phi_{\mathcal{P}^\vee}^{A \to \widehat{A}}(\mathcal{F}^\vee)$ would be zero, hence $\mathcal{F}$ itself would be zero.
(b) For a GV sheaf $\mathcal{F}$, the Euler characteristic $\chi(\mathcal{F})$ is equal to the generic value of $h^0(A, \mathcal{F} \otimes P_\alpha)$, for $\alpha \in \widehat{A}$. Therefore $\chi(\mathcal{F}) \geq 0$ and $\chi(\mathcal{F})$ coincides with the generic rank of $\widehat{\mathcal{F}^\vee}$.

For a GV sheaf $\mathcal{F}$, the dictionary between the gv-index and the sheaf-theoretic properties of the transform $\widehat{\mathcal{F}^\vee}$ is summarized in the following statement.

Theorem 1.4. *Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A.*
(1) (Pareschi-Popa, [PP1, Corollary 3.2]) For $k \geq 0$, $gv(\mathcal{F}) \geq k$ if and only if $\widehat{\mathcal{F}^\vee}$ is a k-syzygy sheaf.[3]
(2) $\mathcal{F}$ is M-regular if and only if $\widehat{\mathcal{F}^\vee}$ is torsion-free (hence, in particular, $\chi(\mathcal{F}) > 0$).
(3) $\mathcal{F}$ is IT(0) if and only if $\widehat{\mathcal{F}^\vee}$ is locally free; equivalently, if and only if $\Phi_{\mathcal{P}}^{A \to \widehat{A}}(\mathcal{F}) = R^0 \Phi_{\mathcal{P}}^{A \to \widehat{A}}(\mathcal{F})$.

Note that (2) is a particular case of (1) since a 1-syzygy sheaf is simply a torsion-free sheaf. Item (3) is elementary: it follows immediately from Grauert's theorem on cohomology and base-change.

A key ingredient of the proof of Theorems 1.2 and 1.4 is the identification

(1.2) $$R^i \Phi_{\mathcal{P}}^{A \to \widehat{A}}(\mathcal{F}) = \mathcal{E}xt_{\widehat{A}}^i(\widehat{\mathcal{F}^\vee}, \mathcal{O}_{\widehat{A}}),$$

consequence of Grothendieck duality, (see [P, Proposition 1.6(b)] or [PP1, Corollary 3.2]).

Remark 1.5. From (1.2) and base-change, it follows that the support of the sheaf $\mathcal{E}xt_{\widehat{A}}^i(\widehat{\mathcal{F}^\vee}, \mathcal{O}_{\widehat{A}})$ is contained in $V^i(A, \mathcal{F})$.

Remark 1.6. Clearly the roles of A and $\widehat{A}$ can be exchanged, and all of the above could have been said for a sheaf $\mathcal{G}$ on $\widehat{A}$ as well, starting from the Fourier-Mukai equivalence $\Phi_{\mathcal{P}}^{\widehat{A} \to A} : D(\widehat{A}) \to D(A)$. For example, the cohomological support loci of the sheaf $\mathcal{G}$ are

$$V^i(\widehat{A}, \widehat{\mathcal{G}}) = \{a \in A \mid H^i(\widehat{A}, \mathcal{G} \otimes P_a) \neq 0\}.$$

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[2]In some of the quoted references, $\widehat{\mathcal{F}^\vee}$ denotes instead the sheaf $R^g \Phi_{\mathcal{P}}^{A \to \widehat{A}}(\mathcal{F}^\vee)$. However this will not cause trouble since, as in (1.1), $\Phi_{\mathcal{P}^\vee}^{A \to \widehat{A}}[g] = (-1)_{\widehat{A}}^* \circ \Phi_{\mathcal{P}}^{A \to \widehat{A}}$, hence the two transforms produce essentially the same results.

[3]We refer to [OSS, Chapter 2, §1.1] or the Appendix in [PP1] for k-syzygy sheaves. See also [EG2]).

Finally we recall that a sheaf $\mathcal{F}$ on an abelian variety A is *homogeneous* if it has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a line bundle in $\widehat{A}$ for all $i \in \{1, \ldots, n\}$. The following proposition is certainly well known, but we could not find a reference.

Proposition 1.7. *A sheaf $\mathcal{F}$ on A is a homogeneous vector bundle if and only if $\chi(\mathcal{F}) = 0$ and $\dim V_{>0}(A, \mathcal{F}) \leq 0$.*

Proof. The direct implication is obvious. Conversely, if $V_{>0}(A, \mathcal{F})$ is 0-dimensional or empty then $\mathcal{F}$ is GV. Thus, by Remark 1.3 the condition $\chi(\mathcal{F}) = 0$ means that the locus $V^0(A, \mathcal{F})$ is a proper subvariety of $\widehat{A}$ (non-empty if $\mathcal{F}$ is non-zero). It is known that the GV condition has the following pleasant consequence: any component W of $V^0(A, \mathcal{F})$ of codimension j is also a component of $V^j(A, \mathcal{F})$ (see [PP2, Proposition 3.15] or [P, Lemma 1.8]). Therefore, since $V_{>0}(A, \mathcal{F})$ has dimension 0, W must be 0-dimensional (in fact an isolated point in $V^g(A, \mathcal{F})$). Therefore, $\dim V^0(A, \mathcal{F}) = 0$ and $\widehat{\mathcal{F}^\vee}$ is a $\mathcal{O}_{\widehat{A}}$-module of finite length (it is supported at $V^0(A, \mathcal{F})$). By a result of Mukai ([M2, Theorems 4.17, 4.19]), this means that $\mathcal{F}^\vee$ is a homogeneous vector bundle. Equivalently, $\mathcal{F}$ is a homogeneous vector bundle. $\square$

2. Proof of Theorems A and B

Proof of Theorem B. From the exact sequence

$$(2.1) \qquad 0 \to \mathcal{I}_Z \otimes L \to L \to L_{|Z} \to 0,$$

it follows that for $i \geq 1$

$$(2.2) \qquad V^i(A, L_{|Z}) = V^{i+1}(A, \mathcal{I}_Z \otimes L).$$

Therefore the hypotheses on $V_{>0}(\mathcal{I}_Z \otimes L)$ imply that in both cases (1) and (2), the sheaf $L_{|Z}$ is M-regular, hence $\chi(L_{|Z}) > 0$ (see Theorem 1.4(2)). Hence $\chi(\mathcal{I}_Z \otimes L) < \chi(L)$. On the other hand, $\chi(\mathcal{I}_Z \otimes L) > 0$ since otherwise, by Proposition 1.7, the sheaf $\mathcal{I}_Z \otimes L$ would be a homogeneous vector bundle, and it is easy to verify that this happens if and only if Z is a divisor representing l. In conclusion, we have

$$(2.3) \qquad 0 < \chi(\mathcal{I}_Z \otimes L) < \chi(L).$$

Note that the generic values of $h^0(\mathcal{I}_Z \otimes L \otimes P_\alpha)$ and $h^0(L_{|Z} \otimes P_\alpha)$, for $\alpha \in \mathrm{Pic}^0 A$, coincide with $\chi(\mathcal{I}_Z \otimes L)$ and $\chi(L_{|Z})$. By a result of Debarre-Hacon, [DH, Lemma 5(e)],[4] the inequalities (2.3) imply that

$$(2.4) \qquad \chi(\mathcal{I}_Z \otimes L) \leq \chi(L) - 1 - \dim Z.$$

Proof of (1). The hypothesis means that $\mathcal{I}_Z \otimes L$ is IT(0) (Definition 1.1). By Theorem 1.4(3), one has

$$\Phi_{\mathcal{P}}^{A \to \widehat{A}}(\mathcal{I}_Z \otimes L) = R^0 \Phi_{\mathcal{P}}^{A \to \widehat{A}}(\mathcal{I}_Z \otimes L) := \mathcal{G}$$

and $\mathcal{G}$ is a locally free sheaf on $\widehat{A}$ of rank equal to $\chi(\mathcal{I}_Z \otimes L)$. Therefore, taking the inverse functor,

$$\Phi_{\mathcal{P}^\vee}^{\widehat{A} \to A}(\mathcal{G}) = R^g \Phi_{\mathcal{P}^\vee}^{\widehat{A} \to A}(\mathcal{G})[-g] = \mathcal{I}_Z \otimes L[-g].$$

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[4]This Lemma is stated under the assumption that the abelian variety is simple, but in fact what is needed is that the support of Z is geometrically non-degenerate.

This means that the dual vector bundle $\mathcal{G}^\vee$ is a GV sheaf on $\widehat{A}$ (by Theorem 1.2 applied to the equivalence $\Phi_{\mathcal{P}}^{\widehat{A} \to A}$, see also Remark 1.6). More, since the sheaf $\widehat{\mathcal{G}} = \mathcal{I}_Z \otimes L$ is torsion-free, $\mathcal{G}^\vee$ is M-regular (Theorem 1.4(2)). But a result of Debarre ([D2, Corollary 3.2]) says that a M-regular sheaf is ample. Therefore the vector bundle $\mathcal{G}^\vee$ is ample. Thus, by Le Potier's vanishing, its cohomological support loci $V^i(\widehat{A}, \mathcal{G}^\vee)$ are empty for $i > \operatorname{rk} \mathcal{G}^\vee - 1 = \chi(\mathcal{I}_Z \otimes L) - 1$.

On the other hand, by (1.2) and Remark 1.5, the cohomological support locus $V^i(\widehat{A}, \mathcal{G}^\vee)$ contains the support of $\mathcal{E}xt_{\widehat{A}}^i(\mathcal{I}_Z, \mathcal{O}_{\widehat{A}}) = \mathcal{E}xt_{\widehat{A}}^{i+1}(\mathcal{O}_Z, \mathcal{O}_{\widehat{A}})$. As soon as Z has an $(i+1)$-codimensional component such a sheaf is non-zero, hence $V^i(\widehat{A}, \mathcal{G}^\vee)$ is non-empty. Therefore

$$g - (\dim Z + 1) \le \chi(\mathcal{I}_Z \otimes L) - 1.$$

Together with (2.4), this proves (1).

Proof of (2). For a coherent sheaf $\mathcal{F}$ on A, we set

$$i_{\max}(\mathcal{F}) = \max\{i \mid V^i(A, \mathcal{F}) \ne \emptyset\}.$$

Assume first that $i_{\max}(\mathcal{I}_Z \otimes L) = 1$. By hypothesis, this implies that its gv-index (Definition 1.1) is

$$gv(\mathcal{I}_Z \otimes L) = g - 1.$$

Therefore, by Theorem 1.4(1), $\widehat{(\mathcal{I}_Z \otimes L)^\vee}$ is a non-locally free $(g-1)$-syzygy sheaf. Therefore, by the Evans-Griffith syzygy theorem ([EG1, Corollary 1.7], see also [PP1, Appendix]) its generic rank is $\ge g - 1$, i.e.

$$\chi(\mathcal{I}_Z \otimes L) \ge g - 1.$$

As we know that $\chi(L_{|Z}) \ge 1$, (2) follows in this case.

Assume otherwise that $i_{\max}(\mathcal{I}_Z \otimes L) > 1$. As above, we have

$$gv(\mathcal{I}_Z \otimes L) = g - i_{\max}(\mathcal{I}_Z \otimes L).$$

Again by Theorem 1.4(1) and Evans-Griffith, we get

$$\chi(\mathcal{I}_Z \otimes L) \ge g - i_{\max}(\mathcal{I}_Z \otimes L).$$

On the other hand, by (2.2), we have $i_{\max}(\mathcal{I}_Z \otimes L) = i_{\max}(L_{|Z}) + 1 \le \dim Z + 1$. Therefore

$$\chi(\mathcal{I}_Z \otimes L) \ge g - \dim Z - 1.$$

Together with (2.4), this proves (2) and concludes the proof of Theorem B.

Remark 2.1. (a) In Theorem B, the hypothesis that Z is not a divisor representing the polarization l is clearly necessary.
(b) The inequalities of Theorem B are sharp in both cases (1) and (2), as it is shown by taking for Z a point $p \in A$. Indeed a general polarized abelian variety of type $(1, \ldots, 1, g+1)$ is base point free ([DHS, Proposition 2]). Since the line bundles $L \otimes P_\alpha$ are the translates of L, this is easily seen to be equivalent to the fact that $V_{>0}(A, \mathcal{I}_p \otimes L)$ is empty. Similarly, a general polarized abelian variety of type $(1, \ldots, 1, g)$ has a 0-dimensional base locus ([DHS, Remark 3(a)]). As above, this means that $V_{>0}(A, \mathcal{I}_p \otimes L)$ is 0-dimensional.

Proof of Theorem A. The fact that Lemma B implies Theorem A is known. We review this for the sake of self-containedness, referring to [L, §10.1.B] and [DH] for more details. Indeed the last assertion of (1) (respectively the last assertion of (2)) of Theorem A is equivalent to the triviality of the adjoint ideal of E (respectively of the multiplier ideal of the $\mathbb{Q}$-divisor $\frac{1}{m}D$). We claim that

both ideals satisfy all hypotheses of Lemma B and therefore Lemma B implies Theorem A. To prove the claim, let us denote $\mathcal{I}$ both ideals.

To begin with, $\mathcal{I}$ cannot be $\mathcal{O}_A(-E)$, where E is a divisor representing L. This is obvious for the adjoint ideal, and it holds for the multiplier ideal of $\frac{1}{m}D$ (if D is not equal to mE, with E representing l) because $\lfloor \frac{1}{m}D \rfloor = 0$ ([DH, Corollary 3]).

Moreover, by definition, any subvariety of a simple abelian variety is geometrically non-degenerate.

It remains only to prove that the locus $V_{>0}(A, \mathcal{I} \otimes L)$ is either empty or 0-dimensional. This follows from the hypothesis that the abelian variety A is simple and the fact that $\mathcal{I}$ satisfies the following property:

(*)   *the locus $V_{>0}(A, \mathcal{I} \otimes L)$ is either empty or a proper linear subvariety, i.e. a finite union of translates of proper abelian subvarieties of* $\mathrm{Pic}^0 A$.

To prove (*), we recall that the generic vanishing and linearity theorems of Green and Lazarsfeld ([GL],[EL, Remark 1.6]), combined with the Grauert-Riemenschneider vanishing theorem, say that for a smooth projective variety X, equipped with a generically finite morphism $f : X \to A$, the locus $V_{>0}(A, f_*\omega_X)$ is either empty or a proper linear subvariety, i.e. a finite union of translates of proper abelian subvarieties of $\mathrm{Pic}^0 A$. Hence for the adjoint ideal, the property (*) follows from via the exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{I}_Z \otimes L \to f_*\omega_{E'} \to 0$$

where $f : E' \to E$ is any resolution of singularities of E ([L, Proposition 9.3.48]). If instead $\mathcal{I}$ is the multiplier ideal of the $\mathbb{Q}$-divisor $\frac{1}{m}D$, property (*) follows from the same theorems of Green and Lazarsfeld via the fact that such a multiplier ideal (twisted by the canonical bundle of A, which is trivial) is a direct summand of the pushforward of the canonical bundle of a smooth variety via a generically finite morphism. This in turn goes back to the work of Esnault-Viehweg ([EV, (3.13)]). A more explicit reference is [DH, page 226].[5] This proves (*).

References

[BL]   C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften 302, Springer, 2004.

[B]   N. Budur, Unitary local systems, multiplier ideals, and polynomial periodicity of Hodge numbers, Adv. Math. **221** (2009) 217 - 250.

[D1]   O. Debarre, Théorèmes de connexité et variétés abéliennes, Amer. J. Math. **117** (1995) 787 - 805.

[D2]   O. Debarre, On coverings of simple abelian varieties, Bull. Soc. Math. France **134** (2006) 253 - 260.

[DH]   O. Debarre and Ch. Hacon, Singularities of divisors of low degree on abelian varieties, Manuscripta Math. **122** (2007) 217 - 228.

[DHS] O. Debarre, K. Hulek and J. Spandaw, Very ample linear systems on abelian varieties, Math. Ann. **300** (1994) 181 - 202.

[EG1] E.G. Evans and P. Griffith, The syzygy problem, Ann. of Math. **114** (1981) 323 - 333.

[EG2] E.G. Evans and P. Griffith, *Syzygies*, London Math. Soc. Lecture Note Ser. **106**, Cambridge Univ. Press, 1985.

[EL]   L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, J. Amer. Math. Soc. **10** (1997) 243 - 258.

[EV]   H. Esnault and E. Viehweg, Dyson's lemma for polynomials in several variables (and the theorem of Roth), Invent. Math. **78** (1984) 75 - 86.

[GL]   M. Green and R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, J. Amer. Math. Soc. **1** (1991) 87 - 103.

[L]   R. Lazarsfeld, *Positivity in Algebraic Geometry II*, Springer, 2004.

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[5]See also [B, Theorem 1.3] for a more general linearity theorem along these lines.

[LP] R. Lazarsfeld, M. Popa, Derivative complex, BGG correspondence, and numerical inequalities for compact Kähler manifolds, Invent. Math. **182** (2010) 605 - 633.
[H1] Ch. Hacon, Divisors on principally polarized abelian varieties, Compos. Math **119** (1999) 321 - 329.
[H2] Ch. Hacon, Fourier transforms, generic vanishing theorem and polarizations of abelian varieties, Math. Z. **235** (2000) 717 - 726.
[K]   J. Kollár, Shafarevich maps and automorphic forms, M.B. Porter Lectures, Princeton Univ. Press, 1995.
[M1] S.Mukai, Duality between $D(X)$ and $D(X^\vee)$ with its application to Picard sheaves, Nagoya Math. J. **81** (1981) 153 - 175.
[M2] S.Mukai, Semi-homogeneous vector bundles on abelian varieties, J. Math. Kyoto Univ. **18** (1978) 239 - 272.
[P]   G. Pareschi, Basic results on irregular varieties via Fourier-Mukai methods, in *Current developments in Algebraic Geometry*, MSRI Publications **59** (2011) 279 - 403.
[PP1] G. Pareschi and M. Popa, Strong generic vanishing and a higher dimensional Castelnuovo-de Franchis inequality, Duke Math. J. **150** (2009) 269 - 285.
[PP2] G. Pareschi and M. Popa, GV-sheaves, Fourier-Mukai transform, and generic vanishing, Amer. J. Math. **133** (2011) 235 - 271.
[OSS] C. Okonek, M. Schneider, and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Birkäuser, Boston, 1980.
[R]   Z. Ran, On subvarieties of abelian varieties, Invent. Math. **62** (1981) 459 - 479.

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