

PERIOD RELATIONS FOR RIEMANN SURFACES WITH MANY AUTOMORPHISMS

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ABSTRACT. By employing the theory of vector-valued automorphic forms for non-unitarizable representations, we provide a new bound for the number of linear relations with algebraic coefficients between the periods of an algebraic Riemann surface with many automorphisms. The previous best-known general bound for this number was the genus of the Riemann surface, a result due to Wolfart. Our new bound significantly improves on this estimate, and it can be computed explicitly from the canonical representation of the Riemann surface. As observed by Shiga and Wolfart, this bound may then be used to estimate the dimension of the endomorphism algebra of the Jacobian of the Riemann surface. We demonstrate with a few examples how this improved bound allows one, in some instances, to actually compute the dimension of this endomorphism algebra, and to determine whether the Jacobian has complex multiplication.

1. INTRODUCTION

The theory of vector-valued modular forms, though nascent in the work of various nineteenth century authors, is a relatively recent development in mathematics. Perhaps the leading motivation for working out a general theory in this area comes from two-dimensional conformal field theory, or more precisely from the theory of vertex operator algebras (VOAs). Indeed, in some sense the article [Zhu96] – in which Zhu proved that the graded dimensions of the simple modules for a rational VOA constitute a weakly holomorphic vector-valued modular function – created a demand for understanding how such objects work in general, and what may be learned about them by studying the representations according to which they transform. Knopp and Mason [KM03, KM04, Mas07] were the first to give a systematic treatment of vector-valued modular forms, with significant contemporary contributions by Bantay and Gannon [BG07, Ban09] as well. This initial motivation from theoretical physics leads one naturally to arithmetic considerations, and more recently a number of authors have utilized representations of the modular group to study the *bounded denominator conjecture* for noncongruence modular forms [Mas12, FM14, Mar15, FM16, FGM18], the vector-valued version of which originated with Mason as well.

In this article we utilize the vector-valued point of view for another arithmetic application, this time to the classical theory of Riemann surfaces and in particular to the study of the periods of an algebraic Riemann surface. More precisely, suppose X is a compact Riemann surface of genus g defined over $\overline{\mathbb{Q}}$. For any holomorphic differential ω defined over $\overline{\mathbb{Q}}$ and any homology class $[\gamma] \in H_1(X, \mathbb{Z})$, represented by a closed loop γ on X , the *period* associated to ω and γ is the complex number $\int_{\gamma} \omega$. Following [Wol02],

let

$$V_X := \text{span}_{\overline{\mathbb{Q}}} \left\{ \int_{\gamma} \omega : \gamma \in H_1(X, \mathbb{Z}), \omega \text{ defined over } \overline{\mathbb{Q}} \right\},$$

viewed as a $\overline{\mathbb{Q}}$ -subspace of \mathbb{C} . Since the dimension of the space of $\overline{\mathbb{Q}}$ -differentials is g and the rank of $H_1(X, \mathbb{Z})$ is $2g$, we deduce that $\dim_{\overline{\mathbb{Q}}} V_X \leq 2g^2$. This bound is saturated, and in fact it is realized by generic compact Riemann surfaces of genus g .

There are, however, special classes of algebraic curves where the above bound is a substantial overestimate. For example, in [Wol02] Wolfart studies Riemann surfaces with *many automorphisms*. These curves are characterized by the fact that if one fixes any desired genus and automorphism group, the resulting family of curves is zero-dimensional. For such a curve X , it is proven in [Wol02] that

$$(1) \quad \dim_{\overline{\mathbb{Q}}} V_X \leq g,$$

reflecting the fact that the larger automorphism group imposes additional linear relations among the periods.

In this article we improve the bound (1), by inserting terms depending on the *canonical representation* of X . More precisely, let X be a curve with many automorphisms, let G be the group of automorphisms of X , and denote by

$$\rho_X : G^{\text{op}} \longrightarrow \text{GL}(\Omega^1(X))$$

be the g -dimensional complex representation given by the action of $g \in G$ on differentials given by pull-back (note that pull-back is contravariant, thus the ‘op’ is needed to obtain a group representation). This is called the *canonical representation* of X , since $\Omega^1(X)$ is also known as the *canonical bundle* of X . Its decomposition into irreducible characters can readily be computed using the *Chevalley-Weil* formula [CW34] (see [Can18] for a modern account). Now any Riemann surface X with many automorphisms can actually be uniformized as a quotient [Wol02]

$$X(N) := \mathbb{H}/N, \quad N \triangleleft \Delta$$

where \mathbb{H} denotes the complex upper half-plane and $N \triangleleft \Delta$ is a finite-index normal subgroup of a *co-compact Fuchsian triangle group*

$$\Delta = \Delta(p, q, r) = \{\delta_0, \delta_1, \delta_\infty : \delta_0^p = \delta_1^q = \delta_\infty^r = \delta_0\delta_1\delta_\infty\}$$

acting as linear fractional transformations on \mathbb{H} , so that G appears as the finite quotient Δ/N . In this presentation there are now three generators $\delta_0, \delta_1, \delta_\infty$ of G , which, by abuse of notation, represent the images of the corresponding generators of Δ . In terms of these generators, we prove:

Theorem 1.1. *Let X be a curve with many automorphisms, uniformized as $X = X(N)$ and with canonical representation $\rho_N := \rho_{X(N)}$. Then*

$$\dim_{\overline{\mathbb{Q}}} V_X \leq g - d_0 - d_1 - d_\infty$$

where $d_x = \dim_{\mathbb{C}}(\ker(\rho_N(\delta_x) - I_g))$ is the geometric multiplicity of the eigenvalue 1 in $\rho_N(\delta_x)$, $x \in \{0, 1, \infty\}$.

In practice, the numbers d_0, d_1, d_∞ are easy to compute given the character of ρ_N , and they tend to be linear in g , providing a significant improvement over the bound (1). We demonstrate sample computations of the bound of Theorem 1.1 in Section 8 below.

The proof of Theorem 1.1 builds on the ideas of [CHMY18], and it relies crucially on the theory of vector-valued modular forms for non-unitarizable representations, as pioneered by Knopp and Mason [KM03], [KM04]. The proof of the Theorem, together with the connection with modular forms, can be found in Sections 5 and 7 below.

As demonstrated already by Shiga and Wolfart in [SW95], the knowledge of the integer $\dim_{\overline{\mathbb{Q}}} V_X$ is essentially equivalent to the knowledge of the dimension of the \mathbb{Q} -algebra of endomorphisms of $\text{Jac}(X)$ (see Theorem 3.2 below for a precise statement). There are many open questions about the endomorphism algebras of Jacobians, such as Coleman’s conjecture regarding Riemann surfaces with *complex multiplication* and the Ekedahl-Serre question regarding Riemann surfaces whose Jacobian is isogenous to a product of elliptic curves. These questions may be viewed as the main motivation for studying linear relations between the periods of X . We demonstrate in Section 8 how Theorem 1.1, in conjunction with the theorems of Shiga-Wolfart, allow one to compute the decomposition of the Jacobians of some Riemann surfaces with many automorphisms.

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2. RIEMANN SURFACES WITH MANY AUTOMORPHISMS

Suppose X is a compact, connected Riemann surface. By the uniformization theorem of Riemann surfaces, X is isomorphic to a quotient $\Gamma \backslash \mathcal{U}$, where \mathcal{U} is either the complex upper half-plane \mathbb{H} , the Riemann sphere \mathbb{P}^1 , or the complex plane \mathbb{C} and $\Gamma \subseteq \text{Aut}(\mathcal{U})$ is a discrete subgroup. By Riemann’s existence theorem, X also possesses the structure of an algebraic curve over \mathbb{C} : that is, it can be defined as the zero locus of a collection of polynomials with complex coefficients inside a suitable projective space. Suppose that these polynomials can be chosen to all have coefficients in the field of algebraic numbers $\overline{\mathbb{Q}}$. Then we say that X is *defined* over $\overline{\mathbb{Q}}$. In this case, we can be more specific about the group Γ uniformizing X :

Theorem 2.1 (Belyi [Bel79], Wolfart [Wol02]). *Let X be a compact, connected Riemann surface defined over $\overline{\mathbb{Q}}$. Then X is isomorphic to $\Gamma \backslash \mathbb{H}$, for some finite-index subgroup Γ of a co-compact Fuchsian triangle group $\Delta \subseteq \text{PSL}_2(\mathbb{R}) \simeq \text{Aut}(\mathbb{H})$.*

For any choice of positive integers $p, q, r > 0$, the *triangle group* $\Delta = \Delta(p, q, r)$ is defined abstractly as the infinite group generated by three elements $\delta_0, \delta_1, \delta_\infty$ with

presentation

$$\Delta = \Delta(p, q, r) = \{\delta_0, \delta_1, \delta_\infty : \delta_0^p = \delta_1^q = \delta_\infty^r = \delta_0\delta_1\delta_\infty\}.$$

The triangle group is *Fuchsian* if it can be embedded in $\mathrm{PSL}_2(\mathbb{R})$, and it is *co-compact* if the corresponding quotient $\Delta \backslash \mathbb{H}$ is compact. In this article Δ will always be Fuchsian, but not necessarily co-compact. For example, it is customary to allow any of p, q, r to be ∞ whenever the corresponding generator is of infinite order, so that, for example,

$$\Delta(\infty, \infty, \infty) = F_2 = \Gamma(2), \quad \Delta(2, 3, \infty) = \mathrm{PSL}_2(\mathbb{Z}).$$

When this happens the triangle groups are no longer co-compact, in which case we apply a suitable compactification to the quotient $\Delta \backslash \mathbb{H}$ by adding *cusps* (as is the case for *modular curves*, for example). In either case, we let $X(\Delta)$ denote the corresponding Riemann surface, and similarly we denote by $X(\Gamma)$ the compact Riemann surface corresponding to finite-index subgroups $\Gamma \leq \Delta$, each of which yields a finite ramified cover $X(\Gamma) \rightarrow X(\Delta)$.

Theorem 2.1 is remarkable in that it allows one to study algebraic Riemann surfaces from the point of view of group theory and the representation theory of the triangle groups $\Delta(p, q, r)$, which are very much amenable to computations. For example, given a fixed triangle group Δ , we get a 1-1 correspondence

$$\{\text{finite-index normal subgroups } N \triangleleft \Delta\} \longleftrightarrow \{\text{ramified Galois covers } X(N) \rightarrow X(\Delta)\}.$$

As is the case in algebraic number theory, where questions about general finite extensions L/K of fields can be tackled by first studying *Galois* extensions, here too the study of an algebraic Riemann surface $X(N) \rightarrow X(\Delta)$ can often be reduced to studying *Galois* covers, by taking a suitable ‘Galois closure’. By the above correspondence, that is the same as restricting to *normal* subgroups $N \subseteq \Delta$. For simplicity we will do so in this article, keeping in mind that all the results presented can be extended to the case of a general Riemann surface, not necessarily uniformized by a normal subgroup.

Now, when $N \triangleleft \Delta$ is normal, the finite quotient $G = \Delta/N$ acts as a group of automorphisms on $X(N)$, which partially justifies the following:

Definition 2.2 (Wolfart [Wol02]). A Riemann surface X defined over $\bar{\mathbb{Q}}$ has *many automorphisms* if there is an isomorphism $X \simeq X(N)$, where $N \triangleleft \Delta$ is a normal subgroup of a triangle group Δ .

In general, let $A(G)$ be the locus of genus g Riemann surfaces with automorphism group equal to G , viewed as a subvariety of the moduli space of all Riemann surfaces of genus g (see, e.g., [MSSV02]). Then X has many automorphisms if and only if $\dim A(G) = 0$.

3. ENDOMORPHISM ALGEBRAS OF ABELIAN VARIETIES AND TRANSCENDENCE

Suppose now that A is an abelian variety of dimension $g > 0$ defined over $\bar{\mathbb{Q}}$. Equivalently, A is a complex torus \mathbb{C}^g/Λ together with an embedding into projective space defined by polynomial equations with coefficients in $\bar{\mathbb{Q}}$. In this case, the g -dimensional \mathbb{C} -vector space of holomorphic 1-forms for A contains the $\bar{\mathbb{Q}}$ -vector space of 1-forms that

are *defined over* $\bar{\mathbb{Q}}$, that is, those differentials that come by base-change from the regular, Kähler differentials on the underlying algebraic variety. These $\bar{\mathbb{Q}}$ -differentials can be integrated over homology classes of cycles $\gamma \in H^1(A, \mathbb{Z})$, and the resulting complex number

$$\int_{\gamma} \omega \in \mathbb{C}$$

is a *period* of ω . This integration process is only possible by viewing A as a complex manifold, a highly transcendental operation, and therefore periods tend to be transcendental numbers even if ω is defined over $\bar{\mathbb{Q}}$. To keep track of ‘how many’ transcendental periods we get, we make the following important definition:

Definition 3.1 ([Wol02]). Let A be an abelian variety defined over $\bar{\mathbb{Q}}$. The *period $\bar{\mathbb{Q}}$ -span* of A is the $\bar{\mathbb{Q}}$ -vector space

$$V_A := \text{span}_{\bar{\mathbb{Q}}} \left\{ \int_{\gamma} \omega : \gamma \in H^1(A, \mathbb{Z}), \omega \text{ defined over } \bar{\mathbb{Q}} \right\} \subseteq \mathbb{C}$$

If we regard \mathbb{C} as an infinite-dimensional vector space over $\bar{\mathbb{Q}}$, then V_A is seen to be a $\bar{\mathbb{Q}}$ -subspace of \mathbb{C} . It is clearly finite-dimensional: indeed, the $\bar{\mathbb{Q}}$ -space of algebraic differential 1-forms for A is g -dimensional, by base-change, and the rank of $H^1(A, \mathbb{Z})$ as a free \mathbb{Z} -module is $2g$. Therefore

$$\dim_{\bar{\mathbb{Q}}} V_A \leq 2g^2$$

for any abelian variety A defined over $\bar{\mathbb{Q}}$. We call this the *trivial bound* on $\dim_{\bar{\mathbb{Q}}} V_A$.

How can we get ‘extra’ linear $\bar{\mathbb{Q}}$ -relations on V_A ? If ϕ is a non-scalar endomorphism of A (defined over $\bar{\mathbb{Q}}$) then it is easy to show that ϕ induces a $\bar{\mathbb{Q}}$ -relation via its action by pull-back on ω and its natural action on $H^1(A, \mathbb{Z})$. The amazing fact is that *all* such linear relations are given in this way [SW95], as follows from Wüstholz’ analytic subgroup theorem. To state the precise result, let

$$\text{End}_0(A) := \text{End}(A) \otimes \mathbb{Q}$$

be the endomorphism algebra of A .

Theorem 3.2 (Shiga-Wolfart, [SW95]). *Suppose A is a simple abelian variety of dimension g . Then*

$$\dim_{\bar{\mathbb{Q}}} V_A = \frac{2g^2}{\dim_{\mathbb{Q}} \text{End}_0(A)}.$$

More generally, if A is isogenous to a product $A_1^{k_1} \times \cdots \times A_m^{k_m}$ of simple abelian varieties, each of dimension g_i , then

$$\dim_{\bar{\mathbb{Q}}} V_A = \sum_{i=1}^m \frac{2g_i^2}{\dim_{\mathbb{Q}} \text{End}_0(A_i)}.$$

Therefore, the knowledge of $\dim_{\bar{\mathbb{Q}}} V_A$ gives information about the dimension of the endomorphism algebra of A , and it is often enough to actually determine $\dim_{\mathbb{Q}} \text{End}_0(A)$.

Example 3.3. Suppose $g = 1$, so that $A = E$ is an elliptic curve over $\bar{\mathbb{Q}}$. In this case the space of $\bar{\mathbb{Q}}$ -differentials is one-dimensional, say, generated by ω , and the integral homology $H^1(E, \mathbb{Z})$ is a rank 2 free \mathbb{Z} -module generated by, say, γ_1 and γ_2 . The periods of ω are the nonzero complex numbers

$$\Omega_1 := \int_{\gamma_1} \omega, \quad \Omega_2 := \int_{\gamma_2} \omega.$$

Consider the ratio $\tau := \Omega_2/\Omega_1$. Generically, this ratio will be transcendental, so that $\dim_{\bar{\mathbb{Q}}} V_E = 2$ and the elliptic curve attains the trivial bound of $2g^2$. By Theorem 3.2, this forces $\text{End}_0(E) = \mathbb{Q}$, that is, the only endomorphisms of E act as scalars. On the other hand, when $\tau \in \bar{\mathbb{Q}}$, then Theorem 3.2 implies that $\dim_{\bar{\mathbb{Q}}} \text{End}_0(E) = 2$. As is well-known [Sil86], in this case $\text{End}(E)$ is an order inside the ring of integers of an imaginary quadratic extension of \mathbb{Q} , and the elliptic curve is said to have *complex multiplication*. We thus recover the famous theorem of Schneider from 1936:

$$\tau \in \bar{\mathbb{Q}} \iff E \text{ has complex multiplication.}$$

Example 3.4. In general, when $\dim_{\bar{\mathbb{Q}}} V_A = 2g^2$ then Theorem 3.2 implies that $\text{End}(A) = \mathbb{Z}$.

4. JACOBIANS WITH MANY AUTOMORPHISMS

Suppose now that X is a Riemann surface of genus $g > 0$ and let $J(X)$ be its *Jacobian variety*, the abelian variety of dimension g canonically associated to X by taking the torus \mathbb{C}^g/Λ , where Λ is the rank $2g$ lattice spanned by the periods of X . If $X = X(N) \rightarrow X(\Delta)$ has many automorphisms (Def. 2.2), then X is defined over $\bar{\mathbb{Q}}$, and its Jacobian variety $J(X)$ is an abelian variety also defined over $\bar{\mathbb{Q}}$ [Mil86]. In this case, by using the fact that the group of automorphisms of the Galois cover $X(N) \rightarrow X(\Delta)$ acts transitively on $H_1(X(N), \mathbb{Z})$, it is possible to give a sharper general bound on the dimension of the period $\bar{\mathbb{Q}}$ -span of $\text{Jac}(X)$:

Theorem 4.1 (Wolfart [Wol02]). *Suppose X is a Riemann surface of genus $g > 0$ with many automorphisms, let $J(X)$ be its Jacobian variety and let $V_X := V_{\text{Jac}(X)}$ be the period $\bar{\mathbb{Q}}$ -span of $\text{Jac}(X)$. Then*

$$\dim_{\bar{\mathbb{Q}}} V_X \leq g.$$

This bound on $\dim_{\bar{\mathbb{Q}}} V_X$ is so far the only known for a general Riemann surface with many automorphisms. Though weak, it can already be used to deduce special structures within the endomorphism algebras of Jacobians with many automorphisms, as in the two examples below.

Example 4.2. Suppose X has genus $g = 1$, and that X has many automorphisms. Then $\dim_{\bar{\mathbb{Q}}} V_X = 1$ so that $\text{Jac}(X) \simeq X$ has complex multiplication (compare to Example 3.3 in Section 3).

Example 4.3. In general, if X has many automorphisms then $\text{End}(\text{Jac}(X)) \neq \mathbb{Z}$ always (compare to Example 3.4 in Section 3).

5. VECTOR-VALUED MODULAR FORMS AND PERIODS

An excellent reference for the Riemann surface and automorphic function theory described in this section is Shimura’s classic text [Shi94], and the vector-valued point of view we utilize here emerges naturally out of work of Knopp and Mason [KM03, KM04, Mas07, Mas08]. We also note that the construction of Theorem 5.1 below is a generalization of an idea used in Theorem 1.4 of the recent article [CHMY], where such a result was proven in a different manner for subgroups of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

Suppose N is a normal, finite index subgroup of a Fuchsian group of the first kind $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$, and let \mathbb{H}^* denote the union of \mathbb{H} with the cusps of N in $\mathbb{R} \cup \{\infty\}$ (note that if Γ is cocompact in $\mathrm{PSL}_2(\mathbb{R})$ then so is N , and this set of cusps is empty). Let $X(N) = N \backslash \mathbb{H}^*$ denote the compact Riemann surface $X(N)$ associated to N , and assume that the genus of $X(N)$ is $g \geq 1$. The finite group $G = \Gamma/N$ acts as automorphisms of the cover $X(N) \rightarrow X(\Gamma)$ and, via pullback (which is contravariant), acts on the vector space of differential forms for $X(N)$. This defines a g -dimensional linear representation $G^{\mathrm{op}} \rightarrow \mathrm{GL}_g(\mathbb{C})$, the *canonical representation* of the cover. We may lift this representation to a group representation

$$(2) \quad \rho_N : \Gamma^{\mathrm{op}} \rightarrow \mathrm{GL}_g(\mathbb{C})$$

via the quotient map $\Gamma \rightarrow \Gamma/N$. Let $\{\omega_1, \dots, \omega_g\}$ be a basis for the space $\Omega^1(X(N))$ of holomorphic 1-forms for $X(N)$. Under the uniformization isomorphism $X(N) \simeq N \backslash \mathbb{H}^*$, we can lift $\omega_1, \dots, \omega_g$ to a basis $\{f_1, \dots, f_g\}$ of weight two cusp forms on N . The vector

$$F := (f_1, \dots, f_g)^t$$

is then a weight two vector-valued cusp form for the opposite representation $\rho := \rho_N^{\mathrm{opp}}$. In other words, $F(\tau)$ is holomorphic in \mathbb{H} , vanishes at any cusps that N may have, and for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ the functional equation

$$(3) \quad F(\gamma\tau)(c\tau + d)^{-2} = \rho(\gamma)F(\tau)$$

is satisfied.

Fix a base-point $\tau_0 \in X(N)$. For each $1 \leq k \leq g$, let

$$u_k(\tau) = \int_{\tau_0}^{\tau} f_k(z) dz.$$

These functions have an important transformation property:

Theorem 5.1. $U = (u_1, \dots, u_g, 1)^t$ is a holomorphic vector-valued automorphic function for a representation $\pi_N : \Gamma \rightarrow \mathrm{GL}_{g+1}(\mathbb{C})$ arising from a non-trivial extension of the form

$$0 \rightarrow \rho \rightarrow \pi_N \rightarrow 1 \rightarrow 0.$$

Proof. Writing $U(\tau) = \left(\int_{\tau_0}^{\tau} F(z) dz, 1 \right)^t$, we first note that the functional equation (3) for F may be written as

$$F(\gamma\tau) d(\gamma\tau) = \rho(\gamma)F(\tau) d\tau.$$

For each $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, set

$$(4) \quad \Omega(\gamma) = \int_{\tau_0}^{\gamma\tau_0} F(z) dz.$$

Then for any $\gamma \in \Gamma$ we have

$$\begin{aligned} U(\gamma\tau) &= \begin{pmatrix} \int_{\tau_0}^{\gamma\tau} F(z) dz \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \int_{\tau_0}^{\gamma\tau_0} F(z) dz + \int_{\gamma\tau_0}^{\gamma\tau} F(z) dz \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \Omega(\gamma) + \int_{\tau_0}^{\tau} F(\gamma z) d(\gamma z) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \Omega(\gamma) + \rho(\gamma) \int_{\tau_0}^{\tau} F(z) dz \\ 1 \end{pmatrix} \\ &= \pi_N(\gamma)U(\tau) \end{aligned}$$

where we define

$$\pi_N(\gamma) = \begin{pmatrix} \rho(\gamma) & \Omega(\gamma) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{g+1}(\mathbb{C}).$$

One sees now that π_N is a representation of $\mathrm{PSL}_2(\mathbb{Z})$ if and only if the relation

$$\Omega(\gamma_1\gamma_2) = \rho(\gamma_1)\Omega(\gamma_2) + \Omega(\gamma_1)$$

obtains for any $\gamma_1, \gamma_2 \in \mathrm{PSL}_2(\mathbb{Z})$. Similar to the above computation, we have

$$\begin{aligned} \Omega(\gamma_1\gamma_2) &= \int_{\tau_0}^{(\gamma_1\gamma_2)\tau_0} F(z) dz \\ &= \int_{\tau_0}^{\gamma_1\tau_0} F(z) dz + \int_{\gamma_1\tau_0}^{\gamma_1(\gamma_2\tau_0)} F(z) dz \\ &= \Omega(\gamma_1) + \int_{\tau_0}^{\gamma_2\tau_0} F(\gamma_1 z) d(\gamma_1 z) \\ &= \Omega(\gamma_1) + \rho(\gamma_1) \int_{\tau_0}^{\gamma_2\tau_0} F(z) dz \\ &= \Omega(\gamma_1) + \rho(\gamma_1)\Omega(\gamma_2) \end{aligned}$$

so π_N is indeed a representation. To see that this extension of ρ by the trivial character of Γ does not split, we simply need to observe that if it did, then $\ker(\pi_N)$ would contain N . This would imply that the Riemann surface $X(N)$ admits nonconstant holomorphic functions, namely the u_k , but this is impossible since $X(N)$ is compact. This completes the proof of the theorem. \square

For each $\gamma \in N$, the map $\tau_0 \mapsto \gamma\tau_0$ defines a closed loop on $X(N)$. One observes that the proof of Proposition 1.4 in [Man72] remains valid for any Fuchsian group of the first kind in $\mathrm{PSL}_2(\mathbb{R})$, so there is a surjective homomorphism of groups $N \rightarrow H_1(X(N), \mathbb{Z})$ given by $\gamma \rightarrow [\tau_0 \mapsto \gamma\tau_0]$ whose kernel is generated by the commutator subgroup N' together with the elliptic and parabolic elements of N . Thus there is some set γ_j ,

$1 \leq j \leq 2g$, of hyperbolic elements of N that surjects onto a basis of $H_1(X(N), \mathbb{Z})$, and using the above notation we find that the vectors $\Omega(\gamma_j) \in \mathbb{C}^g$ generate the period lattice Λ that defines the Jacobian of $X(N)$. On the other hand, from the proof of Theorem 5.1 we see for each j we have

$$\pi_N(\gamma) = \begin{pmatrix} \rho(\gamma) & \Omega(\gamma) \\ 0 & 1 \end{pmatrix},$$

so in fact these vectors appear in the matrices defining the extension representation π_N . Thus, when these periods are algebraic Theorem 5.1 allows one to compute them explicitly, as was already demonstrated for modular curves in [CHMY18]. Even when they cannot be computed explicitly (since in general at least some of them are transcendental), the existence and analysis of π_N yields important information about the periods, as we demonstrate below in Section 7.

6. GROUP COHOMOLOGY

Let N be a normal subgroup of finite index in a Fuchsian group Γ of the first kind, and let ρ_N be the canonical representation associated to the cover $X(N) \rightarrow X(\Gamma)$. Theorem 5.1 shows that the periods of the curve $X(N)$ are the matrix entries of a non-trivial extension π_N of the trivial representation by ρ_N . In this section we briefly recall how such extensions are classified by *group cohomology*.

Let $\rho : \Gamma \rightarrow \mathrm{GL}(V)$ be a finite-dimensional complex representation of Γ . For any integer $n \geq 1$, denote by $C^n(\Gamma, \rho)$ the group of functions $\Gamma^n \rightarrow V$ (the *n-cochains*) and let $d^{n+1} : C^n \rightarrow C^{n+1}$ be the *coboundary* homomorphism given by

$$\begin{aligned} d^{n+1}\kappa(g_1, \dots, g_{n+1}) &= \rho(g_1)\kappa(g_2, \dots, g_n) + \sum_{i=1}^n (-1)^i \kappa(g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_{n+1}) + \\ &\quad + (-1)^{n+1} \kappa(g_1, \dots, g_n) \end{aligned}$$

For $n \geq 0$ let $Z^n(\Gamma, \rho) := \ker d^{n+1}$ be the group of *n-cocycles* and for $n \geq 1$ let $B^n(\Gamma, \rho) := \mathrm{im} d^n$ be the group of *n-coboundaries*. Since $d^{n+1} \circ d^n = 0$, we have $B^n \subset Z^n$ for all $n \geq 1$ (so (C^\bullet, d^\bullet) forms a *cochain complex*) and we can define the *n-th group cohomology* by

$$\begin{aligned} H^0(\Gamma, \rho) &:= V^\rho = \{v \in V : \rho(v) = v\} \\ H^n(\Gamma, \rho) &:= \frac{Z^n(\Gamma, \rho)}{B^n(\Gamma, \rho)}, \quad n \geq 1. \end{aligned}$$

In particular, for $n = 1$ we have:

$$\begin{aligned} Z^1(\Gamma, \rho) &= \{\kappa : \Gamma \rightarrow V \mid \kappa(g_1 g_2) = \rho(g_1)\kappa(g_2) + \kappa(g_1)\} \\ B^1(\Gamma, \rho) &= \{\kappa : \Gamma \rightarrow V \mid \kappa(g) = \rho(g)v - v \text{ for some } v \in V\}. \end{aligned}$$

Let now $\mathrm{Ext}_\Gamma^1(\rho, 1)$ be the set of isomorphism classes of extensions of Γ -representations $\rho \rightarrow \pi \rightarrow 1$, where an isomorphism $\pi \simeq \pi'$ is an isomorphism of Γ -representations which restricts to an isomorphism $\rho \simeq \rho'$. Given any $\pi \in \mathrm{Ext}_\Gamma^1(\rho, 1)$, we can find a basis for π such that

$$\pi(g) = \begin{pmatrix} \rho(g) & \kappa_\pi(g) \\ 0 & 1 \end{pmatrix},$$

as in the proof of Theorem 5.1. An easy computation shows that $\kappa_\pi \in Z^1(\Gamma, \rho)$ and that its class in H^1 completely classifies the extension π :

Proposition 6.1. *The map $\pi \mapsto \kappa_\pi$ gives a bijection*

$$\mathrm{Ext}_\Gamma^1(\rho, 1) \longleftrightarrow H^1(\Gamma, \rho).$$

□

Note that the group structure on $H^1(\Gamma, \rho)$ goes over naturally to give the *Baer sum* of extensions on $\mathrm{Ext}_\Gamma^1(\rho, 1)$, and therefore the bijection of Proposition 6.1 can be promoted to a group isomorphism, and even to an isomorphism of \mathbb{C} -vector spaces. The zero vector in $\mathrm{Ext}_\Gamma^1(\rho, 1)$, corresponding to 1-coboundaries in $B^1(\Gamma, \rho)$, is given by those representations π which are completely reducible, $\pi \simeq \rho \oplus 1$. In particular, note that if Γ is a finite group then $H^1(\Gamma, \rho) = 0$ always, since in this case the category of Γ -representations is semi-simple. Going back to Theorem 5.1, for example, we see that the canonical representation ρ_N cannot have any non-trivial extensions π_N of it by 1 if viewed as a representation of $G := \Gamma/N$. However, when lifted to a representation of Γ , Theorem 5.1 says that a *canonical* non-trivial extension class

$$\pi_N \in H^1(\Gamma, \rho_N)$$

becomes available, and is determined by the periods of $X(N)$. In the next section we study the properties of this class and connect it to the endomorphism algebra of the Jacobian of $X(N)$.

7. A SHARPER BOUND ON THE DIMENSION OF THE PERIOD $\bar{\mathbb{Q}}$ -SPAN OF RIEMANN SURFACES WITH MANY AUTOMORPHISMS

We now go back to the case of triangle groups $\Delta = \Delta(p, q, r)$. Let $N \triangleleft \Delta$ and let ρ_N be the canonical representation of $X(N) \rightarrow X(\Delta)$, lifted to a representation of Δ as in (2). As explained in Section 6, Theorem 5.1 gives a canonical class $\pi_N \neq 0 \in H^1(\Delta, \rho_N)$ encoding the periods of $X(N)$. Using this new invariant of the curve $X(N)$, we obtain the following bound:

Theorem 7.1. *Suppose $X(N) \rightarrow X(\Delta)$ is a Riemann surface of genus $g > 0$ with many automorphisms, and let $\rho_N : \Delta^{\mathrm{op}} \rightarrow \mathrm{GL}_g(\mathbb{C})$ be its canonical representation. Let $V_{X(N)} := V_{\mathrm{Jac}(X(N))}$ be the span of $\bar{\mathbb{Q}}$ -periods of $\mathrm{Jac}(X(N))$. Then*

$$\dim_{\bar{\mathbb{Q}}} V_{X(N)} \leq g - d_0 - d_1 - d_\infty$$

where $d_x = \dim_{\mathbb{C}}(\ker(\rho_N(\delta_x) - I_g))$ is the geometric multiplicity of the eigenvalue 1 in $\rho_N(\delta_x)$, $x \in \{0, 1, \infty\}$.

Proof. First, let

$$\tilde{\Delta} := \Delta(p, q, \infty) = \{\delta_0, \delta_1 : \delta_0^p = 1, \delta_1^q = 1\} \simeq \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}.$$

Let $\delta_\infty := \delta_1^{-1}\delta_0^{-1} \in \tilde{\Delta}$ and let K denote the smallest normal subgroup of $\tilde{\Delta}$ containing δ_∞^r . Taking the quotient by K determines a surjective homomorphism $\tilde{\Delta} \rightarrow \Delta$, and we can lift ρ_N to a representation of $\tilde{\Delta}$ (we will abuse notation and denote both ρ_N and its lift to $\tilde{\Delta}$ by the same symbol). We need the following lemma about the group cohomology of $\tilde{\Delta}$.

Lemma 7.2. *Let $\rho : \tilde{\Delta} \rightarrow \mathrm{GL}(V)$ be any g -dimensional complex representation with $V^\rho = \{0\}$. Let $A = \rho(\delta_0), B = \rho(\delta_1)$ and let V^A, V^B be the eigenspaces of eigenvalue 1 of A, B , respectively. Then there is canonical isomorphism of complex vector spaces*

$$\frac{V}{V^A + V^B} \xrightarrow{\delta} H^1(\tilde{\Delta}, \rho)$$

given as follows: write $v = v_1 - v_2 \in \frac{V}{V^A + V^B}$ and let $\delta(v) = \kappa_v$ be the 1-cocycle in $Z^1(\tilde{\Delta}, \rho)$ determined by

$$\kappa_v(A) = \rho(A)v_1 - v_1, \quad \kappa_v(B) = \rho(B)v_2 - v_2.$$

In particular, $\dim H^1(\tilde{\Delta}, \rho) = g - d_0 - d_1$.

Proof. The group cohomology of a free product with two factors can be computed via the Mayer-Vietoris sequence, which in the case of $\tilde{\Delta} \simeq \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ gives the exact sequence of vector spaces

$$(5) \quad V^\rho \rightarrow V^A \oplus V^B \xrightarrow{(v^A, v^B) \mapsto v^A - v^B} V \xrightarrow{\delta} H^1(\tilde{\Delta}, \rho) \rightarrow H^1(\mathbb{Z}/p\mathbb{Z}, \rho) \oplus H^1(\mathbb{Z}/q\mathbb{Z}, \rho),$$

where δ is the map given in the statement of the Lemma. We want to show that δ is an isomorphism. By assumption $V^\rho = \{0\}$. We also have that $H^1(\mathbb{Z}/N\mathbb{Z}, \rho) = 0$ for any $N \geq 1$. Indeed, if the group $\mathbb{Z}/N\mathbb{Z}$ is generated by γ , then any extension $\rho \rightarrow \pi \rightarrow 1$ of $\mathbb{Z}/N\mathbb{Z}$ -representations is determined by the matrix

$$\pi(\gamma) = \begin{pmatrix} \rho(\gamma) & \kappa(\gamma) \\ 0 & 1 \end{pmatrix}.$$

But $\pi(\gamma)$ has finite order (dividing N) which means it is diagonalizable, and $\kappa(\gamma)$ differs from 0 by a coboundary. Therefore $H^1(\mathbb{Z}/p\mathbb{Z}, \rho) = H^1(\mathbb{Z}/q\mathbb{Z}, \rho) = 0$ and δ is an isomorphism. \square

Since the fixed vectors of ρ_N are the differential forms on $X(\Delta)$, and since $X(\Delta)$ has genus zero, ρ_N satisfies the hypothesis of Lemma 7.2. Using this Lemma, we may define a canonical basis for $H^1(\tilde{\Delta}, \rho_N)$ as follows: let $a = \dim V^A = d_0, b = \dim V^B = d_1$ and choose bases $\underline{v}^A = \{v_1^A, \dots, v_a^A\}, \underline{v}^B = \{v_1^B, \dots, v_b^B\}$ for V^A, V^B , respectively. Since $V^A \cap V^B = V^{\rho_N} = \{0\}$, we may extend $\underline{v}^A \cup \underline{v}^B$ to a basis $\{v_1, \dots, v_d\} \cup \underline{v}^A \cup \underline{v}^B$ of V , where $d := g - a - b$. With respect to this choice of basis, the ‘fundamental cocycles’ $\kappa_1, \dots, \kappa_d$, determined by

$$\kappa_1(\delta_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1, \quad \kappa_1(\delta_1) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \kappa_d(\delta_0) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_d, \quad \kappa_d(\delta_1) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

form a canonical basis for $H^1(\tilde{\Delta}, \rho_N)$. We next need a lemma which determines the ‘field of definition’ of the κ_i ’s.

Lemma 7.3. *Let $\rho : \tilde{\Delta} \rightarrow \mathrm{GL}(V)$ be any finite-dimensional complex representation of finite image and with $V^\rho = \{0\}$. Suppose $\dim H^1(\tilde{\Delta}, \rho) = d$. Then there is a choice of basis for V such that the fundamental cocycles κ_i have entries in $\mathbb{Q}(\zeta_e)$, where ζ_e is a primitive e -th root of unity and e is the exponent of the finite group $\tilde{\Delta}/\ker \rho$.*

Proof. Since ρ is of finite image, a basis β can be chosen so that the matrix entries of ρ belong to $\mathbb{Q}(\zeta_e)$, where ζ_e is a primitive e -th root of unity and e is the exponent of the finite group $\tilde{\Delta}/\ker \rho$ [Bra45]. From this basis β we may compute a basis for $V^A + V^B$ and extend it to a basis of V to get fundamental cocycles $\kappa_i(A), \kappa_i(B)$, without changing the field of definition $\mathbb{Q}(\zeta_e)$ of the entries of ρ , by basic linear algebra. Since $\kappa_i(A), \kappa_i(B) \in \mathbb{Z}^g$, the entries of a general vector $\kappa_i(\gamma)$, $\gamma \in \tilde{\Delta}$, are polynomial in the matrix entries of ρ , thus they are also contained in $\mathbb{Q}(\zeta_e)$. \square

Let now π_N be the representation of Theorem 5.1, viewed as a representation of $\tilde{\Delta} \rightarrow \Delta$, and let κ_{π_N} be Ω , the 1-cocycle (4) determined by π_N as shown in the proof of that theorem. Since ρ_N has no fixed vectors, we can write Ω in terms of the fundamental cocycles κ_i ,

$$\Omega = \sum_{i=1}^d \lambda_i \kappa_i, \quad \lambda_i \in \mathbb{C}$$

where $d = \dim H^1(\tilde{\Delta}, \rho_N)$. As discussed after the proof of Theorem 5.1, the periods of the curve $X(N)$ are precisely the entries of $\Omega(\gamma)$, where γ runs through the hyperbolic elements of N . The periods of $X(N)$ are thus the entries of the vectors

$$\Omega(\gamma) = \sum_{i=1}^d \lambda_i \kappa_i(\gamma).$$

Since ρ_N has finite image, a basis can be chosen so that the entries of $\kappa_i(\gamma_i)$ are contained in $\mathbb{Q}(\zeta_e)$, where e is the exponent of the finite group $G = \Delta/N$, by Lemma 7.3. Therefore, the periods can be expressed as $\overline{\mathbb{Q}}$ -linear combinations of complex numbers $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ (which are potentially linearly dependent over $\overline{\mathbb{Q}}$). By the result of Lemma 7.2, we deduce that

$$(6) \quad \dim_{\overline{\mathbb{Q}}} V_{X(N)} \leq d = \dim H^1(\tilde{\Delta}, \rho_N) = g - d_0 - d_1$$

which proves Theorem 7.1 for the larger group $\tilde{\Delta}$. In order to deduce the sharper result for Δ , we use the fact that π_N is actually a representation of Δ , and therefore

$\pi_N(\delta_\infty^{-r}) = I_{g+1}$. In particular,

$$\begin{aligned}
 0 &= \Omega(\delta_\infty^{-r}) \\
 &= \sum_{j=0}^{r-1} \rho_N(\delta_\infty^j) \Omega(\delta_\infty^{-1}) \\
 &= \sum_{j=0}^{r-1} \rho_N(\delta_\infty^j) \Omega(\delta_0 \delta_1) \\
 &= \sum_{j=0}^{r-1} \rho_N(\delta_\infty^j) \Omega(\delta_0) \quad \left(\text{since } \Omega(\delta_1) = \sum_{i=1}^d \lambda_i \kappa_i(\delta_1) = 0 \right) \\
 &= \sum_{i=1}^d \lambda_i T \kappa_i(\delta_0)
 \end{aligned}$$

where for ease of notation we let $T := \sum_{j=0}^{r-1} \rho_N(\delta_\infty^j)$. Since ρ_N is a representation of Δ , $\rho_N(\delta_\infty)$ is of order dividing r . It is therefore diagonalizable as $\rho_N(\delta_\infty) \sim \text{diag}(\alpha_1, \dots, \alpha_g)$, where each α_i is an r -th root of unity. By a standard computation with roots of unity it follows that T has the block decomposition

$$T \sim \begin{pmatrix} r I_{d_\infty} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to a basis of eigenvectors for $\rho_N(\delta_\infty)$, where the basis for the eigenspace of eigenvalue $\alpha = 1$ is placed first. Let M be the change-of-basis matrix corresponding to this choice of basis. Then the entries of M can be chosen to be in $\overline{\mathbb{Q}}$, since the entries of $\rho_N(\delta_\infty)$ are. We thus have the vector equation

$$0 = \sum_{i=1}^d \lambda_i M^{-1} \begin{pmatrix} r I_{d_\infty} & 0 \\ 0 & 0 \end{pmatrix} M \kappa_i(\delta_0)$$

which shows that we have an additional d_∞ -many independent $\overline{\mathbb{Q}}$ -linear relations among the λ_i 's. Combined with (6), we get the sharper bound

$$\dim_{\overline{\mathbb{Q}}} V_{X(N)} \leq g - d_0 - d_1 - d_\infty,$$

which proves Theorem 7.1. □

Remark 7.4. Consider the inflation-restriction exact sequence of vector spaces

$$0 \rightarrow H^1(\Delta, \rho_N) \rightarrow H^1(\tilde{\Delta}, \rho_N) \xrightarrow{\text{Res}} H^1(K, \rho_N)^\Delta \rightarrow 0$$

associated to the group homomorphisms $K \hookrightarrow \tilde{\Delta} \rightarrow \Delta$, where $\text{Res}(\kappa) = [\kappa(\delta_\infty)^r]$. Then by the proof of Theorem 7.1 it is clear that $\dim H^1(\Delta, \rho_N) = g - d_0 - d_1 - d_\infty$ so the bound of Theorem 7.1 can be stated more concisely as

$$\dim_{\overline{\mathbb{Q}}} V_{X(N)} \leq \dim H^1(\Delta, \rho_N).$$

Remark 7.5. The bound of Theorem 7.1 tends to be *much* smaller than that of Theorem 4.1. Indeed, for $\Delta = \Delta(p, q, r)$ ‘on average’ we expect that

$$\dim_{\mathbb{C}} H^1(\Delta, \rho_N) \sim g - g/p - g/q - g/r \ll g.$$

The numbers d_0, d_1, d_∞ are easy to compute. Indeed, the character χ_N of ρ_N can be computed explicitly using the Chevalley-Weil formula [CW34] (see [Can18] for a modern account), and the numbers d_0, d_1, d_∞ can then be computed from the knowledge of χ_N and of the character table of the finite group $G = \Delta/N$. These computations can easily be implemented in GAP or MAGMA, as we show in the next section.

8. EXAMPLES

Example 8.1. (Bolza surface) In genus two, the Riemann surface with largest automorphism group is known as the *Bolza surface*. This Riemann surface has many automorphisms, and it can be uniformized as $X(N)$, by an index 48 normal subgroup $N \triangleleft \Delta(2, 3, 8)$. The automorphism group is $G \simeq \mathrm{GL}(2, 3)$. The canonical representation ρ_N in this case is irreducible, which implies that

$$J(X) \sim A^k$$

for a simple abelian variety A [Wol02]. The only possibilities are either $\dim A = 1$, $k = 2$ or $\dim A = 2$, $k = 1$. We have

$$\dim H^1(\Delta(2, 3, 8), \rho_N) = 1$$

and so by Theorems 7.1 and 3.2

$$\dim_{\overline{\mathbb{Q}}} V_X = \frac{2 \dim A^2}{\dim_{\mathbb{Q}} \mathrm{End}_0(A)} = 1.$$

If $\dim A = 2$, then this can only happen if $\dim_{\mathbb{Q}} \mathrm{End}_0(A) = 8$, which is impossible since $\dim_{\mathbb{Q}} \mathrm{End}_0(A) \leq 4$ for any abelian surface A . Therefore $\mathrm{Jac}(X) \sim E^2$ for some elliptic curve E . Moreover, by Example 3.3 we deduce that E has complex multiplication, so that X has complex multiplication as well. The period matrix of X can be computed explicitly using our methods, as in [CHMY18].

Example 8.2 (Klein quartic). There is a unique normal subgroup $N \triangleleft \Delta(2, 3, 7) = \Delta$ of index 168. The corresponding curve $X = X(N)$ is the famous ‘Klein quartic’, of genus $g = 3$. By Chevalley-Weil, the canonical representation ρ_N of this curve is irreducible, and

$$\dim H^1(\Delta, \rho_N) = 1.$$

Since ρ_N is irreducible, this means that $\mathrm{Jac}(X) \sim A^k$, where A is a simple abelian variety [Wol02]. Since $\dim \mathrm{Jac}(X) = g = 3$, either $\dim A = 1$ and $k = 3$ or $\dim A = 3$ and $k = 1$. We rule out the latter as follows. Using Thm. 3.2 and Thm. 7.1 we deduce that

$$\dim_{\overline{\mathbb{Q}}} V_X = \frac{2 \dim A^2}{\dim_{\mathbb{Q}} \mathrm{End}_0(A)} = 1.$$

If $\dim A = 3$, we get $\dim_{\mathbb{Q}} \mathrm{End}_0(A) = 18$. This is impossible, since for any abelian threefold, $\dim_{\mathbb{Q}} \mathrm{End}_0(A) \leq 6$. Therefore $\dim A = 1$, $k = 3$ and $\mathrm{Jac}(X) \sim A^3$ is totally

decomposable and $\dim_{\mathbb{Q}} \text{End}_0(A) = 2$, i.e., A is a CM elliptic curve. Thus $J(X)$ is also CM, and again it has totally decomposable Jacobian.

Example 8.3 (Macbeath curve). There is a unique normal subgroup $N \triangleleft \Delta(2, 3, 7) = \Delta$ of index 504. The corresponding curve $X = X(N)$ is the famous ‘Macbeath curve’, of genus $g = 7$. Again by Chevalley-Weil, this curve has an irreducible canonical representation ρ_N with

$$\dim H^1(\Delta, \rho_N) = 2.$$

Since ρ_N is irreducible, this means again that $\text{Jac}(X) \sim A^k$, with A simple. Since $\dim \text{Jac}(X) = g = 7$, either $\dim A = 1$ and $k = 7$ or $\dim A = 7$ and $k = 1$. Using Theorem 3.2 and Theorem 7.1 we deduce that

$$\frac{\dim A^2}{\dim_{\mathbb{Q}} \text{End}_0(A)} \leq 1.$$

If $\dim A = 7$, we get $\dim_{\mathbb{Q}} \text{End}_0(A) \geq 49$, which is impossible. Therefore $\text{Jac}(X) \sim A^7$ with $A = E$ an elliptic curve, so that $\text{Jac}(X)$ is totally decomposable. The question of whether E has complex multiplication or not was posed by Berry and Tretkoff in [BT92]. Later Wolfart [Wol02] proved that E does not have CM by finding an explicit Weierstrass equation for E . Therefore

$$(7) \quad \frac{2}{\dim A^2} \dim_{\mathbb{Q}} \text{End}_0(A) = 2 = \dim H^1(\Delta, \rho_N).$$

Example 8.4. In genus 14, there are three Hurwitz curves with automorphism group isomorphic to $\text{PSL}(2, 13)$, of order 1092. They can all be uniformized by three distinct normal subgroups of index 1092 in $\Delta(2, 3, 7)$. The canonical representations are all irreducible, with

$$\dim H^1(X(N), \rho_N) = 2.$$

Writing $\text{Jac}(X) \sim A^k$, we must again have

$$(8) \quad \frac{\dim A^2}{\dim_{\mathbb{Q}} \text{End}_0(A)} \leq 1,$$

as for the Macbeath curve. This equality rules out $\dim A = 7, 14$ but both $\dim A = 1, 2$ are possible. It is known in fact that $\text{Jac}(X) \sim E^{14}$ for $A = E$ an elliptic curve, using the *group algebra decomposition* of $\text{Jac}(X)$ [BL04, 13.6], [Pau11]. By (8) we cannot conclude whether E has complex multiplication or not. It would be interesting to determine whether or not each of the three genus 14 Hurwitz curves has complex multiplication or, equivalently, whether or not the equality (7) is attained for these Hurwitz curves.

Remark 8.5. As shown in the above examples, it is possible to use our Theorem 7.1 to give a sufficient criterion for the Riemann surface X to have *complex multiplication*. This criterion is similar in spirit to that of Streit [Str01]. It would be interesting to work out the exact relation between the two methods to detect complex multiplication.

Remark 8.6. Using the group algebra decomposition of $\text{Jac}(X)$ [BL04, 13.6] it is possible to give bounds for $\dim V_X$ via the Shiga-Wolfart Theorem 3.2. It would be interesting to determine precisely which part of the group algebra decomposition is determined by the bounds given in Theorem 7.1.

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