

# AN EXTREMAL PROBLEM FOR FUNCTIONS ANNIHILATED BY A TOEPLITZ OPERATOR

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ABSTRACT. For a bounded function  $\varphi$  on the unit circle  $\mathbb{T}$ , let  $T_\varphi$  be the associated Toeplitz operator on the Hardy space  $H^2$ . Assume that the kernel

$$K_2(\varphi) := \{f \in H^2 : T_\varphi f = 0\}$$

is nontrivial. Given a unit-norm function  $f$  in  $K_2(\varphi)$ , we ask whether an identity of the form  $|f|^2 = \frac{1}{2}(|f_1|^2 + |f_2|^2)$  may hold a.e. on  $\mathbb{T}$  for some  $f_1, f_2 \in K_2(\varphi)$ , both of norm 1 and such that  $|f_1| \neq |f_2|$  on a set of positive measure. We then show that such a decomposition is possible if and only if either  $f$  or  $\overline{z\varphi f}$  has a nonconstant inner factor. The proof relies on an intrinsic characterization of the moduli of functions in  $K_2(\varphi)$ , a result which we also extend to  $K_p(\varphi)$  (the kernel of  $T_\varphi$  in  $H^p$ ) with  $1 \leq p \leq \infty$ .

## 1. INTRODUCTION AND RESULTS

Let  $\mathbb{T}$  stand for the circle  $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$  and let  $m$  be the normalized Lebesgue measure on  $\mathbb{T}$ . For  $0 < p \leq \infty$ , the space  $L^p := L^p(\mathbb{T}, m)$  will be endowed with the usual norm  $\|\cdot\|_p$  (the term “quasinorm” should actually be used when  $0 < p < 1$ ). We also need the *Hardy space*  $H^p := H^p(\mathbb{T})$ , viewed as a subspace of  $L^p$ . The functions in  $H^p$  are thus the boundary traces (in the sense of nontangential convergence almost everywhere) of those in  $H^p(\mathbb{D})$ , the classical Hardy space on the disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ . The latter space consists, by definition, of all holomorphic functions  $f$  on  $\mathbb{D}$  that satisfy

$$\sup \{\|f_r\|_p : 0 < r < 1\} < \infty,$$

where  $f_r(\zeta) := f(r\zeta)$  for  $\zeta \in \mathbb{T}$ .

Our starting point is the following observation: Given any  $f \in H^2$  with  $\|f\|_2 = 1$ , one can find unit-norm functions  $f_1, f_2 \in H^2$  such that

$$(1.1) \quad |f|^2 = \frac{1}{2}(|f_1|^2 + |f_2|^2) \quad \text{a.e. on } \mathbb{T}$$

and

$$(1.2) \quad |f_1| \neq |f_2| \quad \text{on a set of positive measure.}$$

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This fact (to be explained in a moment) is akin to the classical result that the unit ball of  $L^1$  has no extreme points, even though the ball should currently be replaced by a suitable convex subset thereof. Namely, consider the set

$$V_0 := \{|f|^2 : f \in H^2, 0 < \|f\|_2 \leq 1\},$$

i.e., the collection of all functions  $g$  on  $\mathbb{T}$  that have the form  $g = |f|^2$  for some non-null  $f$  from the unit ball of  $H^2$ . We know from basic  $H^p$  theory (see [9, Chapter II]) that the elements  $g$  of  $V_0$  are characterized by the conditions

$$(1.3) \quad g \geq 0 \text{ a.e. on } \mathbb{T}, \quad g \in L^1, \quad \int_{\mathbb{T}} \log g \, dm > -\infty,$$

and  $\|g\|_1 \leq 1$ . (The fact that every function  $g$  satisfying (1.3) is writable as  $|f|^2$ , for some  $f \in H^2$ , was proved by Szegö in [15]. He then used this representation to study the asymptotic behavior of the polynomials that are orthogonal with respect to such a weight  $g$  on  $\mathbb{T}$ ; see [16, Chapter 12].) Clearly, the functions  $g$  that obey (1.3) form a convex cone in  $L^1$ . The portion of that cone lying in the (closed) unit ball of  $L^1$  is precisely  $V_0$ , so this last set is again convex.

We need to show that every function  $g \in V_0$  with  $\|g\|_1 = 1$  is a non-extreme point of  $V_0$ . (By the way, this will imply that  $V_0$  has no extreme points at all.) In fact, given such a  $g$ , we can always find a non-null real-valued function  $\tau \in L^\infty$  with the properties that  $\int_{\mathbb{T}} g\tau \, dm = 0$  and  $\|\tau\|_\infty \leq \frac{1}{2}$ . This done, we put

$$g_1 := g(1 + \tau), \quad g_2 := g(1 - \tau)$$

and note that

$$(1.4) \quad g = \frac{1}{2}(g_1 + g_2) \quad \text{a.e. on } \mathbb{T},$$

while  $g_1$  and  $g_2$  are both in  $V_0$ . Indeed, for  $j = 1, 2$  we have

$$\frac{1}{2}g \leq g_j \leq \frac{3}{2}g \quad \text{a.e. on } \mathbb{T},$$

which makes (1.3) true for  $g_j$  in place of  $g$ ; also,

$$\|g_j\|_1 = \int_{\mathbb{T}} g(1 \pm \tau) \, dm = \int_{\mathbb{T}} g \, dm = 1.$$

Thus (1.4) tells us that  $g$  is the midpoint of a (nondegenerate) segment whose endpoints  $g_j$  ( $j = 1, 2$ ) lie in  $V_0$  and satisfy  $\|g_j\|_1 = 1$ . Equivalently, a nontrivial decomposition (1.1) with required properties is always possible.

The purpose of this note is to study the equation (1.1) when  $f$  lies in a certain subspace of  $H^2$ , namely, in the kernel of a given Toeplitz operator. The unknowns  $f_1$  and  $f_2$  are then required to belong to the same subspace and obey (1.2). Besides, all the functions involved are supposed to be of norm 1, as before.

For an essentially bounded function  $\varphi$  on  $\mathbb{T}$ , we consider the associated operator  $T_\varphi$  (called the *Toeplitz operator with symbol*  $\varphi$ ) which acts on  $H^2$  by the rule

$$T_\varphi f := P_+(\varphi f),$$

where  $P_+$  is the orthogonal projection from  $L^2$  onto  $H^2$ . Assuming that the kernel

$$K_2(\varphi) := \{f \in H^2 : T_\varphi f = 0\}$$

is nontrivial, we look at the set

$$(1.5) \quad V_\varphi := \{|f|^2 : f \in K_2(\varphi), 0 < \|f\|_2 \leq 1\},$$

i.e., the collection of all functions  $g$  on  $\mathbb{T}$  that have the form  $g = |f|^2$  for some non-null  $f$  from the unit ball of  $K_2(\varphi)$ . This set is convex, as we shall soon see, and we are concerned with its (non-)extreme points; once these are determined, we shall arrive at the sought-after information on the solvability of (1.1) for unit-norm functions in  $K_2(\varphi)$ . We are only interested in the case where  $\varphi \in L^\infty \setminus \{0\}$  (i.e.,  $\varphi$  is non-null), since otherwise  $K_2(\varphi) = H^2$  and  $V_\varphi$  reduces to  $V_0$ , a situation we have already discussed.

Before moving further ahead, we need to gain a better understanding of the moduli of functions in  $K_2(\varphi)$ . This will be achieved by means of Theorem 1.1 below, generalizing an earlier result from [4]. When dealing with this issue, we temporarily extend our attention to the subspaces

$$(1.6) \quad K_p(\varphi) := \{f \in H^p : T_\varphi f = 0\}$$

with  $1 \leq p \leq \infty$ , not just with  $p = 2$ .

It should be noted that the operator  $P_+$ , which kills the function's negative-indexed Fourier coefficients, admits a natural extension to  $L^1$ , even though  $P_+(L^1) \not\subset L^1$ . This allows us to define the Toeplitz operator  $T_\varphi$  on  $H^1$ , whenever  $\varphi \in L^\infty$ , the range of any such operator being contained in  $P_+(L^1)$  and hence in every  $H^s$  with  $0 < s < 1$ . Consequently, the definition (1.6) is meaningful for  $p = 1$ , as well as for all bigger values of  $p$ .

For a function  $f \in H^p$  to be in  $K_p(\varphi)$ , it is necessary and sufficient that the product  $\varphi f$  be anti-analytic, which in turn amounts to saying that the ‘‘companion function’’

$$\tilde{f} := \overline{z\varphi f}$$

is in  $H^p$ . Thus,

$$K_p(\varphi) = \{f \in H^p : \tilde{f} \in H^p\}, \quad 1 \leq p \leq \infty.$$

(As a byproduct of this characterization, we mention the fact that every Toeplitz kernel  $K_p(\varphi)$  enjoys the  $F$ -property of Havin; see [10].)

A bit more terminology and notation will be needed. Given a nonnegative function  $w$  on  $\mathbb{T}$  with  $\log w \in L^1$ , the corresponding *outer function*  $\mathcal{O}_w$  is defined a.e. on  $\mathbb{T}$  by

$$\mathcal{O}_w := \exp \{\log w + i\mathcal{H}(\log w)\},$$

where  $\mathcal{H}$  stands for the harmonic conjugation operator. Functions of the form  $\lambda\mathcal{O}_w$ , with  $w$  as above and  $\lambda$  a unimodular constant, will also be referred to as outer. It is well known (and easy to check) that  $\mathcal{O}_w$  extends analytically into  $\mathbb{D}$  and has modulus  $w$  a.e. on  $\mathbb{T}$ . Furthermore,  $\mathcal{O}_w \in H^p$  if and only if  $w \in L^p$ . Finally, we recall that an  $H^\infty$  function is said to be *inner* if its modulus equals 1 a.e. on  $\mathbb{T}$ . See,

e.g., [9, Chapter II] for a systematic treatment of these concepts and of the basic facts related to them.

**Theorem 1.1.** *Let  $1 \leq p \leq \infty$ , and suppose  $\varphi$  is a non-null function in  $L^\infty$  for which  $K_p(\varphi) \neq \{0\}$ . Also, let  $g$  be a nonnegative function in  $L^{p/2}$ . The following conditions are then equivalent.*

(i.1) *There is an  $f \in K_p(\varphi)$  such that  $|f|^2 = g$  a.e. on  $\mathbb{T}$ .*

(ii.1)  *$\overline{z\varphi}g \in H^{p/2}$ .*

*Moreover, if (ii.1) holds (with  $g$  non-null) and if  $I$  is the inner factor of  $\overline{z\varphi}g$ , then the general form of a function  $f \in K_p(\varphi)$  with  $|f|^2 = g$  is given by  $f = \mathcal{O}_{\sqrt{g}}J$ , where  $J$  is an inner divisor of  $I$ .*

Among the possible symbols  $\varphi$  of our Toeplitz operators, we may single out those which are complex conjugates of inner functions. For such  $\varphi$ 's (i.e., for  $\varphi = \overline{\theta}$  with  $\theta$  inner), the corresponding Toeplitz kernels  $K_p(\varphi)$  take the form

$$H^p \cap \overline{z\theta}H^p =: K_\theta^p$$

and are known as *star-invariant* or *model* subspaces. When  $1 \leq p < \infty$ , these are precisely the invariant subspaces of the backward shift operator  $f \mapsto (f - f(0))/z$  in  $H^p$ ; see [2, 12]. We mention in passing that there are deeper connections between the two types of spaces,  $K_p(\varphi)$  and  $K_\theta^p$ . Namely, in a way, generic Toeplitz kernels can be cooked up from model subspaces; see [6, 11, 13] for details.

It was in the  $K_\theta^p$  setting that Theorem 1.1 originally appeared in [4]; see also [8, Lemma 5]. In the (tiny) special case where  $\overline{\varphi} = \theta = z^{n+1}$ , the subspace in question is populated by polynomials of degree at most  $n$ , and the equivalence between (i.1) and (ii.1) above reduces to the classical Fejér–Riesz theorem that describes the moduli of such polynomials on  $\mathbb{T}$  (see, e.g., [14, p. 26]).

The role of Theorem 1.1 in the present context consists in providing a useful – and usable – description of the set  $V_\varphi$ , as defined by (1.5). Specifically, it tells us that a nonnegative function  $g \in L^1 \setminus \{0\}$  is in  $V_\varphi$  if and only if it satisfies  $\overline{z\varphi}g \in H^1$  and  $\|g\|_1 \leq 1$ . This criterion (which implies the convexity of  $V_\varphi$ , among other things) will be repeatedly used hereafter.

Our main result, to be stated next, characterizes the extreme points of  $V_\varphi$ . Of course, every function  $g \in V_\varphi$  with  $\|g\|_1 < 1$  is non-extreme, since

$$g = \frac{1}{2}(1 + \varepsilon)g + \frac{1}{2}(1 - \varepsilon)g$$

and  $(1 \pm \varepsilon)g \in V_\varphi$  for suitably small  $\varepsilon > 0$ . Therefore, we only need to consider the case where  $\|g\|_1 = 1$ .

**Theorem 1.2.** *Let  $\varphi \in L^\infty \setminus \{0\}$  be such that  $K_2(\varphi) \neq \{0\}$ , and let  $g \in V_\varphi$  be a function with  $\|g\|_1 = 1$ . The following are equivalent.*

(i.2)  *$g$  is an extreme point of  $V_\varphi$ .*

(ii.2)  *$\overline{z\varphi}g$  is an outer function in  $H^1$ .*

This characterization is reminiscent of de Leeuw and Rudin's theorem (see [1] or [9, Chapter IV]) which identifies the extreme points of the unit ball of  $H^1$  as unit-norm outer functions. We also mention a related result from [5] that describes

the extreme points of the unit ball in  $K_1(\varphi)$ . (Namely, these are shown to be the unit-norm functions  $f \in K_1(\varphi)$  with the property that the inner factors of  $f$  and  $\tilde{f}$  are relatively prime.) In this connection, see also [3] and [7].

We now state a consequence of Theorem 1.2 which provides an additional piece of information on the geometry of  $V_\varphi$ .

**Corollary 1.3.** *Let  $\varphi \in L^\infty \setminus \{0\}$  be such that  $K_2(\varphi) \neq \{0\}$ . Then every function  $g \in V_\varphi$  with  $\|g\|_1 = 1$  has the form  $g = \frac{1}{2}(g_1 + g_2)$ , where  $g_1$  and  $g_2$  are extreme points of  $V_\varphi$ .*

Going back to Theorem 1.2, we remark that for  $g \in V_\varphi$ , the function  $\overline{z\varphi}g$  can be written as  $f\tilde{f}$ , where  $f$  is some (any) function in  $K_2(\varphi)$  with  $|f|^2 = g$  and  $\tilde{f} := \overline{z\varphi}f (\in H^2)$ . Consequently, we may rephrase condition (ii.2) above by saying that both  $f$  and  $\tilde{f}$  are outer functions. This in turn leads us to a reformulation of Theorem 1.2, which is perhaps better suited for answering our original question.

**Theorem 1.4.** *Suppose that  $\varphi \in L^\infty \setminus \{0\}$ ,  $f \in K_2(\varphi)$  and  $\|f\|_2 = 1$ . In order that there exist a decomposition of the form (1.1) with some unit-norm functions  $f_1, f_2 \in K_2(\varphi)$  satisfying (1.2), it is necessary and sufficient that either  $f$  or  $\tilde{f}$  have a nonconstant inner factor.*

Finally, we take yet another look at condition (ii.2) of Theorem 1.2. Assuming that (ii.2) holds, we know from Theorem 1.1 that the only functions  $f \in K_2(\varphi)$  with  $|f|^2 = g$  are constant multiples of  $\mathcal{O}_{\sqrt{g}}$ . It turns out that a similar conclusion is valid, under condition (ii.2), for those functions  $f \in K_2(\varphi)$  which are merely dominated by  $\sqrt{g}$  in the sense that

$$(1.7) \quad \int_{\mathbb{T}} \frac{|f|}{\sqrt{g}} dm < \infty.$$

(Clearly, this holds in particular when  $|f|^2 \leq \text{const} \cdot g$  on  $\mathbb{T}$ , let alone when  $|f|^2 = g$ .) In the more general context of  $H^p$  spaces with  $p \geq 1$ , the underlying rigidity phenomenon manifests itself in essentially the same way.

**Proposition 1.5.** *Let  $1 \leq p \leq \infty$  and let  $\varphi \in L^\infty \setminus \{0\}$  be such that  $K_p(\varphi) \neq \{0\}$ . Suppose  $g$  is a nonnegative function in  $L^{p/2} \setminus \{0\}$  for which  $\overline{z\varphi}g$  is an outer function in  $H^{p/2}$ . Then every function  $f \in K_p(\varphi)$  satisfying (1.7) is of the form  $f = c\mathcal{O}_{\sqrt{g}}$  for some constant  $c \in \mathbb{C}$ .*

In the special case where  $\varphi$  is the conjugate of an inner function, a similar rigidity result can be found (in a somewhat weaker form) as Theorem 5 in [4].

Now let us turn to the proofs of our current results.

## 2. PROOF OF THEOREM 1.1

If (i.1) holds, then

$$\overline{z\varphi}g = \overline{z\varphi}|f|^2 = f \cdot \overline{z\varphi}f = f\tilde{f}.$$

This last product is in  $H^{p/2}$ , because  $f$  and  $\tilde{f}$  are both in  $H^p$ , and we arrive at (ii.1).

Before proceeding to prove the converse, we pause to observe that

$$(2.1) \quad \log |\varphi| \in L^1,$$

thanks to our hypotheses on  $\varphi$ . Indeed, let  $f_0$  be a non-null function in  $K_p(\varphi)$ . Then  $\tilde{f}_0 := \overline{z\varphi}f_0$  is in  $H^p \setminus \{0\}$  (recall that  $|\varphi| > 0$  on a set of positive measure, while  $|f_0| > 0$  a.e.), and so

$$\log |\varphi| = \log |\tilde{f}_0| - \log |f_0| \in L^1.$$

Now suppose that (ii.1) holds, so that  $\overline{z\varphi}g =: \mathcal{G}$  is in  $H^{p/2}$ . The case of  $g \equiv 0$  being trivial, we shall henceforth assume that  $g$  is non-null; the same is then true for  $\mathcal{G}$ . (The latter conclusion relies on (2.1), which guarantees that  $|\varphi| > 0$  a.e. on  $\mathbb{T}$ .) It follows that  $\log |\mathcal{G}| \in L^1$ , and hence

$$\log g = \log |\mathcal{G}| - \log |\varphi| \in L^1$$

(where (2.1) has been used again). We may then consider the outer function  $\mathcal{O}_{\sqrt{g}} =: F$ , so that  $F \in H^p$  and  $|F| = \sqrt{g}$ , and we further claim that  $F \in K_p(\varphi)$ .

To see why, note that  $|\mathcal{G}| = |\varphi|g$ , and consequently, the outer factor of  $\mathcal{G}$  equals

$$\mathcal{O}_{|\mathcal{G}|} = \mathcal{O}_{|\varphi|}\mathcal{O}_g = \Phi F^2,$$

where  $\Phi := \mathcal{O}_{|\varphi|}(\in H^\infty)$ . Therefore, letting  $I$  denote the inner factor of  $\mathcal{G}$ , we have

$$(2.2) \quad \mathcal{G} = \Phi F^2 I.$$

Using (2.2) and the fact that

$$(2.3) \quad g = |F|^2 (= \overline{F}F),$$

we now rewrite the identity  $\overline{z\varphi}g = \mathcal{G}$  in the form

$$\overline{z\varphi}\overline{F}F = \Phi F^2 I,$$

or equivalently,

$$(2.4) \quad \overline{z\varphi}\overline{F} = \Phi F I.$$

Thus,  $\tilde{F} := \overline{z\varphi}\overline{F}$  is in  $H^p$ , which means that  $F \in K_p(\varphi)$ , as claimed above. Finally, we recall (2.3) to arrive at (i.1), with  $f = F$ . The equivalence of (i.1) and (ii.1) is thereby verified.

To prove the last assertion of the theorem, assume that  $g$  satisfies (ii.1) and that  $f$  is an  $H^p$  function with  $|f|^2 = g$ . The outer factor of  $f$  must then agree with  $F$ , defined as above, so  $f = FJ$  for some inner function  $J$ . Now, in order that  $f$  be in  $K_p(\varphi)$ , it is necessary and sufficient that

$$(2.5) \quad \tilde{f} (:= \overline{z\varphi}\overline{f}) \in H^p.$$

On the other hand, multiplying both sides of (2.4) by  $\overline{J}$  yields

$$\tilde{f} = \overline{z\varphi}\overline{F}\overline{J} = \Phi F I \overline{J} = \Phi F I / J.$$

It follows that (2.5) holds if and only if  $J$  divides  $I$ , and the proof is complete.

## 3. PROOFS OF THEOREM 1.2 AND COROLLARY 1.3

*Proof of Theorem 1.2.* We begin by showing that (i.2) implies (ii.2). Suppose that (ii.2) fails, so that the function  $\mathcal{G} := \overline{z\varphi}g \in H^1$  has a nontrivial inner factor, say  $u$ . Multiplying  $u$  by a suitable unimodular constant, if necessary, we may assume that the number  $\int_{\mathbb{T}} gu \, dm$  is purely imaginary (i.e., belongs to  $i\mathbb{R}$ ). Clearly,  $\psi := \operatorname{Re} u$  is then a nonconstant real-valued  $L^\infty$  function with  $\|\psi\|_\infty \leq 1$ ; moreover,

$$\int_{\mathbb{T}} g\psi \, dm = \operatorname{Re} \int_{\mathbb{T}} gu \, dm = 0.$$

Next, we put

$$g_1 := g(1 + \psi), \quad g_2 := g(1 - \psi)$$

and we are going to check that

$$(3.1) \quad g_j \in V_\varphi \quad \text{for } j = 1, 2.$$

Indeed, the above-mentioned properties of  $\psi$  imply that

$$g(1 \pm \psi) \geq 0 \quad \text{a.e. on } \mathbb{T},$$

while

$$\int_{\mathbb{T}} g(1 \pm \psi) \, dm = \int_{\mathbb{T}} g \, dm = 1.$$

Thus,  $g_1$  and  $g_2$  are nonnegative  $L^1$  functions, both of norm 1. We also claim that

$$(3.2) \quad \overline{z\varphi}g_j \in H^1 \quad \text{for } j = 1, 2.$$

To see why, write  $G$  for the outer factor of  $\mathcal{G}$  (so that  $\mathcal{G} = Gu$ ) and note that

$$(3.3) \quad \begin{aligned} \overline{z\varphi}g_1 &= \overline{z\varphi}g(1 + \psi) = \mathcal{G}(1 + \psi) \\ &= Gu \left( 1 + \frac{1}{2}u + \frac{1}{2}\overline{u} \right) = \frac{1}{2}G(1 + u)^2. \end{aligned}$$

A similar calculation yields

$$(3.4) \quad \overline{z\varphi}g_2 = -\frac{1}{2}G(1 - u)^2.$$

Because  $G \in H^1$  and  $(1 \pm u)^2 \in H^\infty$ , the right-hand sides of (3.3) and (3.4) are both in  $H^1$ . The claim (3.2) is thereby established, and so is (3.1).

Finally,  $g_1 \not\equiv g_2$  because  $g > 0$  a.e. and  $\psi$  is non-null. The representation

$$(3.5) \quad g = \frac{1}{2}(g_1 + g_2)$$

now allows us to conclude that  $g$  is a non-extreme point of  $V_\varphi$ , in contradiction with (i.2).

Conversely, suppose that (ii.2) is fulfilled. Thus,  $\mathcal{G} := \overline{z\varphi}g$  is an outer function in  $H^1$ . Now assume that (3.5) holds with some  $g_1$  and  $g_2$  in  $V_\varphi$ ; hence, in particular,

$$(3.6) \quad \|g_1\|_1 = \|g_2\|_1 = 1.$$

Setting  $h := g_1 - g$  and  $\mathcal{H} := \overline{z\varphi}h$ , we further observe that

$$(3.7) \quad \mathcal{H} = \overline{z\varphi}g_1 - \overline{z\varphi}g \in H^1.$$

Also, we have  $g_1 = g + h$  and  $g_2 = g - h$ , so (3.6) takes the form

$$(3.8) \quad \|g + h\|_1 = \|g - h\|_1 = 1.$$

Therefore,

$$\int_{\mathbb{T}} (|g + h| + |g - h|) dm = 2,$$

or equivalently,

$$(3.9) \quad \int_{\mathbb{T}} (|1 + \Psi| + |1 - \Psi|) d\mu = 2,$$

where  $\Psi := h/g$  and  $d\mu := g dm$ . Because  $\mu$  is a probability measure on  $\mathbb{T}$  which has the same null-sets as  $m$ , we may couple (3.9) with the obvious inequality

$$|1 + \Psi| + |1 - \Psi| \geq 2$$

to deduce that we actually have

$$|1 + \Psi| + |1 - \Psi| = 2$$

a.e. on  $\mathbb{T}$ . This in turn means that  $\Psi$  takes its values in the (real) interval  $[-1, 1]$ .

On the other hand,

$$(3.10) \quad \Psi := \frac{h}{g} = \frac{\overline{z\varphi}h}{\overline{z\varphi}g} = \frac{\mathcal{H}}{\mathcal{G}}.$$

Recalling that  $\mathcal{H} \in H^1$  (as (3.7) tells us), while  $\mathcal{G}$  is outer, we deduce from (3.10) that  $\Psi$  belongs to the Smirnov class  $N^+$  (see [9, Chapter II]). We also know that  $\Psi$  is bounded, whence

$$\Psi \in N^+ \cap L^\infty = H^\infty;$$

and since the only real-valued functions in  $H^\infty$  are constants, it follows that  $\Psi \equiv c$  for some constant  $c \in [-1, 1]$ . Consequently,  $h = cg$  and

$$\|g \pm h\|_1 = (1 \pm c)\|g\|_1 = 1 \pm c.$$

Comparing this with (3.8), we finally conclude that  $c = 0$ . Thus,  $h \equiv 0$  and  $g_1 = g_2 = g$ , so that the only decomposition of the form (3.5) is the trivial one. This brings us to (i.2) and completes the proof.  $\square$

*Proof of Corollary 1.3.* If  $g$  is an extreme point of  $V_\varphi$ , then it suffices to take  $g_1 = g_2 = g$ . Now, if  $g$  is non-extreme (so that  $\overline{z\varphi}g$  is non-outer), then we may use the representation (3.5) from the proof of the (i.2)  $\implies$  (ii.2) part above. To check that the functions  $g_1$  and  $g_2$  constructed there are actually extreme points of  $V_\varphi$ , we invoke the (ii.2)  $\implies$  (i.2) part of the theorem, coupled with the fact that the functions  $\overline{z\varphi}g_j$  ( $j = 1, 2$ ) are both outer. The latter is readily seen from (3.3) and (3.4), since each of these identities has an outer function, namely  $\pm \frac{1}{2}G(1 \pm u)^2$ , for the right-hand side.  $\square$



## 4. PROOF OF PROPOSITION 1.5

Let  $f \in K_p(\varphi)$  be a function satisfying (1.7). Then

$$(4.1) \quad \frac{|f|^2}{g} = \frac{\overline{z\varphi} \tilde{f} \cdot f}{\overline{z\varphi} g} = \frac{\tilde{f} f}{\mathcal{G}},$$

where we write  $\tilde{f} := \overline{z\varphi} f$  and  $\mathcal{G} := \overline{z\varphi} g$ , as before. Since  $\tilde{f}$  (as well as  $f$ ) is in  $H^p$ , while  $\mathcal{G}$  is an *outer* function in  $H^{p/2}$ , it follows that the quotient on the right-hand side of (4.1) lies in the Smirnov class  $N^+$ . The same is therefore true for the left-hand side of (4.1), that is, for  $|f|^2/g$ . This last ratio also belongs to  $L^{1/2}$ , as (1.7) tells us, and so

$$\frac{|f|^2}{g} \in N^+ \cap L^{1/2} = H^{1/2}.$$

Because the only nonnegative  $H^{1/2}$  functions are constants (see [9, p. 92]), we infer that

$$|f|^2 = \lambda g$$

for some constant  $\lambda \geq 0$ .

Now, if  $\lambda = 0$ , then  $f \equiv 0$  and we are done. Otherwise, since  $\mathcal{G}$  has no inner part, Theorem 1.1 (or rather its final assertion, applied with  $\lambda g$  in place of  $g$ ) allows us to conclude that  $f$  agrees, up to a constant factor of modulus 1, with the outer function  $\mathcal{O}_{\sqrt{\lambda g}}$  ( $= \sqrt{\lambda} \mathcal{O}_{\sqrt{g}}$ ). The proof is complete.

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