

# The Bergman kernel in constant curvature

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## Abstract

We present an elementary proof for an approximate expression of the Bergman kernel on homogeneous spaces, and products of them. The error term is exponentially small with respect to the inverse semiclassical parameter.

## 1 Introduction

### 1.1 Bergman kernels

This article is devoted to the study of the Bergman kernel on homogeneous spaces, that is, Kähler manifolds with constant curvature (see Definition 1.2). This class of manifolds contain complex projective spaces (on which the Bergman kernel is explicit), as well as tori and hyperbolic manifolds (on which it is not). This kernel encodes the holomorphic sections of a suitable line bundle over  $M$ .

The study of the Bergman kernel is mainly motivated by *Berezin-Toeplitz quantization*, which associates to a function  $f$  on  $M$  a sequence  $(T_N(f))_{N \in \mathbb{N}}$  of linear operators on holomorphic sections over  $M$ . Toeplitz operators allow to tackle problems arising from representation theory [5], semiclassical analysis [10] and quantum spin systems [6]. The Bergman kernel is also associated with determinantal processes [1], sampling theory [2], and nodal sets [11].

#### Definition 1.1.

- A Kähler manifold  $(M, J, \omega)$  is *quantizable* when there exists a Hermitian line bundle  $(L, h)$  over  $M$  with curvature  $-2i\pi\omega$ . The bundle  $(L, h)$  is then called prequantum line bundle over  $M$ .
- Let  $(M, J, \omega)$  be a quantizable Kähler manifold with  $(L, h)$  a prequantum bundle and let  $N \in \mathbb{N}$ .
  - The Hardy space  $H_0(M, L^{\otimes N})$  is the space of holomorphic sections of  $L^{\otimes N}$ . It is a closed subspace of  $L^2(M, L^{\otimes N})$  which consists of all square-integrable sections of the same line bundle.
  - The Bergman projector  $S_N$  is the orthogonal projector from  $L^2(M, L^{\otimes N})$  to  $H_0(M, L^{\otimes N})$ .

The simplest example of a quantizable compact Kähler manifold is the one-dimensional projective space  $\mathbb{C}\mathbb{P}^1$ , endowed with the natural complex structure  $J_{st}$  and the Fubini-Study form  $\omega_{FS}$ . A natural bundle over  $\mathbb{C}\mathbb{P}^1$  is the tautological bundle (the fibre over one point is the corresponding complex line in  $\mathbb{C}^2$ ). Then

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$L$  is the *dual* of the tautological bundle. One can show that  $H_0(\mathbb{C}\mathbb{P}^1, L^{\otimes N})$  is isomorphic to the space of homogeneous polynomials of degree  $N$  in two variables (with scalar product given by the volume form on  $\mathbb{S}^3$ ).

The space  $H_0(M, L^{\otimes N})$  is always finite-dimensional if  $M$  is compact. Indeed, since  $\Delta = -\partial\bar{\partial}$ , one has

$$H_0(M, L^{\otimes N}) = \ker \bar{\partial}_{C^\infty(M, L^{\otimes N}) \rightarrow \Omega_1(M, L^{\otimes N})} \subset \ker \Delta_{C^\infty(M, L^{\otimes N}) \rightarrow \Omega_1(M, L^{\otimes N})}.$$

The Laplace operator  $\Delta$  is elliptic on the compact manifold  $M$ , so that its kernel is finite-dimensional.

## 1.2 Kernels of linear operators between sections of line bundles

The Bergman projector  $S_N$  is a linear operator mapping the space of sections  $H_0(M, L^{\otimes N})$ , to itself. Here we describe what it means for such an operator to have an integral kernel, and the nature of this kernel.

If  $E$  and  $F$  are finite-dimensional vector spaces, then it is well known that the space  $L(F, E)$  of linear operators from  $E$  to  $F$  can be identified with  $F \otimes E^*$  where  $E^*$  is the dual of  $E$ . Using this, let us construct, for any two line bundles  $E_1 \xrightarrow{\pi_1} M_1$  and  $E_2 \xrightarrow{\pi_2} M_2$  over Riemannian manifolds, a space of kernels  $E_1 \boxtimes E_2^*$  for linear operators which associate, to a section of  $E_2$ , a section of  $E_1$ .

The space  $E_1 \boxtimes E_2^*$  will be constructed as a vector bundle over  $M_1 \times M_2$ . An informal definition is that the fiber  $(E_1 \boxtimes E_2^*)_{(x,y)}$  over a point  $(x, y) \in M_1 \times M_2$  is defined as the tensor product  $(E_1)_x \otimes (E_2)_y^*$ .

One can formally build  $E_1 \boxtimes E_2^*$  in two steps. The first step is to associate to  $E_1 \xrightarrow{\pi_1} M_1$  a bundle  $E_1' \xrightarrow{\pi_1'} M_1 \times M_2$  as follows:

$$\begin{aligned} E_1' &= E_1 \times M_2 \\ \pi_1'(e, y) &= (\pi_1(e), y). \end{aligned}$$

Then  $(E_1')_{(x,y)} = (\pi_1')^{-1}((x, y)) = \pi_1^{-1}(x) \times \{y\} \simeq (E_1)_x$ . Similarly, from the dual bundle  $E_2^*$  of  $E_2$ , one can build  $E_2^{*'} \xrightarrow{\pi_2'} M_1 \times M_2$ . Then, the second step is to define

$$E_1 \boxtimes E_2^* = E_1' \otimes E_2^{*'}.$$

Then the fibre over one point reads

$$(E_1 \boxtimes E_2^*)_{(x,y)} \simeq (E_1')_{(x,y)} \otimes (E_2^{*'})_{(x,y)} \simeq (E_1)_x \otimes (E_2)_y^*,$$

as prescribed.

A smooth section of  $E_1 \boxtimes E_2^*$  gives a linear operator between compactly supported, smooth sections of  $E_2$  and sections of  $E_1$ . Indeed, if  $K_A$  is a smooth section of  $E_1 \boxtimes E_2^*$ , then for any compactly supported, smooth section  $s$  of  $E_2$  one can define the section  $As$  of  $E_1$  as

$$(As)(x) = \int_{M_2} K_A(x, y) s(y) dVol(y).$$

Indeed,  $K_A(x, y) \in (E_1)_x \otimes (E_2)_y^*$  is a linear operator from  $(E_2)_y$  (to which  $s(y)$  belongs) and  $(E_1)_x$ . Then the integral makes sense as taking values in  $(E_1)_x$ , so that  $As$  is well-defined as a section of  $E_1$ .

In particular, in our setting the Bergman projector  $S_N$  admits a kernel as an element of  $L^{\otimes N} \boxtimes \bar{L}^{\otimes N}$ . Indeed, since  $H_0(M, L^{\otimes N})$  is finite-dimensional, it is spanned by a Hilbert base  $s_1, \dots, s_{d_N}$  of holomorphic sections of  $L^{\otimes N}$ . Then the kernel of  $S_N$  is

$$S_N(x, y) = \sum_{i=1}^{d_N} s_i(x) \otimes \overline{s_i(y)}.$$

### 1.3 Statement of the main results

**Definition 1.2.** A Kähler manifold  $(M, \omega, J)$  is called *homogeneous* under the two following conditions:

- For every two points  $x, y \in M$ , there exist an open set  $U \in M$  containing  $x$ , an open set  $V \in M$  containing  $y$ , and a biholomorphism  $\rho : U \mapsto V$  which preserves  $\omega$ .
- For every point  $x \in M$ , there exist an open set  $U \in M$  containing  $x$  and an action of  $U(d)$  by  $\omega$ -preserving biholomorphisms on  $U$ , with  $x$  as only common fixed point, such that the induced linear action on  $T_x M$  is conjugated to the tautological action of  $U(d)$  on  $\mathbb{C}^d$ .

There is a one-parameter family of local models for homogeneous manifolds of dimension  $d$ : for positive curvature  $c > 0$ , the rescaled complex projective space  $\mathbb{C}P^d$ ; for zero curvature  $c = 0$ , the vector space  $\mathbb{C}^d$ ; for negative curvature  $c > 0$ , the rescaled hyperbolic space  $\mathbb{H}^{2d}$ . In particular, on a homogeneous Kähler manifold  $(M, \omega, J)$ , in the real-analytic structure given by  $(M, J)$ , the symplectic form  $\omega$  is real-analytic.

Using the standard notion of holomorphic extensions of real-analytic functions on totally real submanifolds, let us define what will be the kernel of the Bergman projector, up to a constant multiplicative factor and an exponentially small error.

**Definition 1.3** (A particular section of  $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ ). The bundle  $L \boxtimes \overline{L}$ , when restricted to the diagonal  $M_\Delta = \{(x, y) \in M \times M, x = y\}$ , is the trivial line bundle  $M \times \mathbb{C} \rightarrow M$ . Moreover, if the first component of  $M \times M$  is endowed with the complex structure on  $M$ , and the second component with the opposite complex structure (we informally call  $M \times \overline{M}$  this complex manifold), then  $M_\Delta$  is a totally real submanifold of  $M \times \overline{M}$ .

Over a small neighbourhood of  $M_\Delta$  in  $M \times M$ , one can then uniquely define  $\Psi^1$  as the unique holomorphic section of  $L \boxtimes \overline{L}$  which is equal to 1 on  $M_\Delta$ .

This section is locally described at follows: let  $s$  be a non-vanishing holomorphic section of  $L$  over a small open set  $U \subset M$ . Let  $\phi = -\frac{1}{2} \log |s|_h$ . Then  $\phi$  is real-analytic, so that it admits a holomorphic extension  $\tilde{\phi}$ , defined on  $U \times \overline{U}$  (again, the diagonal copy of  $U$  is totally real in  $U \times \overline{U}$ ). Then

$$\Psi^1(x, y) = e^{2\tilde{\phi}(x, y)} s(x) \otimes \overline{s(y)}.$$

We then define  $\Psi^N$  as  $(\Psi^1)^{\otimes N}$ , which is a section of  $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ .

**Theorem A.** *Let  $M$  be a quantizable Kähler manifold of complex dimension  $d$  and suppose  $M$  is a product of compact homogeneous Kähler manifolds.*

*Then the Bergman projector  $S_N$  on  $M$  has an approximate kernel: there is a sequence of real coefficients  $(a_i)_{0 \leq i \leq d}$ , and positive constants  $c, C$  such that, for all  $(x, y) \in M \times M$  and for all  $N \geq 1$ , one has*

$$\left\| S_N(x, y) - \Psi^N(x, y) \sum_{k=0}^d N^{d-k} a_k \right\|_h \leq C e^{-cN}.$$

*If  $M$  is homogeneous, with curvature  $\kappa$ , then*

$$\sum_{k=0}^d N^{d-k} a_k = \frac{1}{\pi^d} (N - \kappa)(N - 2\kappa) \dots (N - d\kappa).$$

A proof of Theorem A using advanced microlocal analysis (local Bergman kernels) was first hinted in [3] and detailed in [7], where the coefficients  $a_k$  are explicitly computed through an explicit expression of the Kähler potential  $\phi$  in a chart. We propose to prove Theorem A without semiclassical tools, and to recover the coefficients  $a_k$  from an elementary observation of the case of positive curvature.

Theorem A implies exponential approximation in the  $L^2$  operator sense. Indeed, if  $K$  is a section of  $L^{\otimes N} \boxtimes \bar{L}^{\otimes N}$  with  $\|K(x, y)\|_h \leq C$  for all  $(x, y) \in M^2$ , then for  $u \in L^2(M, L^{\otimes N})$  one has

$$\begin{aligned} \int_M \left\| \int_M \langle K(x, y), u(y) \rangle_h dy \right\|_h^2 dx &\leq \int_M \int_M \|\langle K(x, y), u(y) \rangle_h\|_h^2 dy dx \\ &\leq \int_M \int_M \|K(x, y)\|_h^2 \|u(y)\|_h^2 dx dy \\ &\leq C^2 \text{Vol}(M) \|u\|_{L^2}^2. \end{aligned}$$

Expressions for the Bergman kernel such as the one appearing in Theorem A were first obtained by Charles [4] in the smooth setting; in this weaker case the section  $\Psi^N$  is only defined at every order on the diagonal, which yields an  $O(N^{-\infty})$  remainder.

Our proof of Theorem A, does not rely on microlocal analysis; the only partial differential operator involved is the Cauchy-Riemann operator  $\bar{\partial}$  acting on  $L^2(M, L^{\otimes N})$ . We use the following estimate on this operator: if  $M$  is compact, there exists  $C > 0$  such that, for every  $N \geq 1$  and  $u \in L^2(M, L^{\otimes N})$ , one has:

$$\|\bar{\partial}u\|_{L^2} \geq C \|u - S_N u\|_{L^2}. \quad (1)$$

This estimate follows from the work of Kohn [8, 9], which relies only on the basic theory of unbounded operators on Hilbert spaces; it is widely used in the asymptotic study of the Bergman kernel, where it is sometimes named after Hörmander or Kodaira.

The rest of this article is devoted to the proof of Theorem A. The plan is the following: we build an approximation  $\tilde{S}_N$ , up to exponential precision, for the Bergman kernel on compact homogeneous spaces. The method consists in constructing candidates  $\tilde{\psi}_{x,v}^N$  for the coherent states, using the local symmetries. These states are almost holomorphic and satisfy a reproducing condition; from these properties, we deduce that the associated reproducing kernel is exponentially close to the Bergman kernel.

**Remark 1.4** (Non-compact homogeneous spaces). Since Kohn's estimate (1) is valid for general homogeneous manifolds, the method of approximation of the Bergman kernel which we provide in this paper adapts to non-compact homogeneous spaces under the condition that the radius of injectivity is bounded from below. In the simple picture of hyperbolic surfaces of finite genus, we allow for the presence of funnels but not cusps (more specifically, the behaviour of the Bergman kernel far away in a cusp, where the diameter is smaller than  $N^{-\frac{1}{2}}$ , is unknown to us). The exact statement of Theorem A is valid in this context, however we cannot conclude that  $S_N$  is controlled in the  $L^2$  operator norm.

## 2 Radial holomorphic charts

Kähler potentials on a Kähler manifold  $(M, J, \omega)$  are characterised by the following property. If  $\rho$  is a local holomorphic chart for  $M$ , the pulled-back symplectic form  $\rho^*\omega$  can be seen as a function of  $\mathbb{C}^d$  into anti-Hermitian matrices of size  $2d$ . The closure condition  $d\omega = 0$  is then equivalent to the existence of a real-valued function  $\phi$  on the chart such that  $i\partial\bar{\partial}\phi = \rho^*\omega$ . Such a  $\phi$  is a Kähler potential.

From now on,  $(M, J, \omega)$  denotes a compact quantizable homogeneous Kähler manifold, of complex dimension  $d$ , and  $(L, h)$  is the prequantum bundle over  $M$ .

Near every point  $P_0 \in M$ , we will build a *radial holomorphic chart* using the local homogeneity. This chart is the main ingredient in the construction of the approximate coherent states.

**Proposition 2.1.** *For every  $P_0 \in M$ , there is an open set  $U \subset M$  with  $P_0 \in U$ , an open set  $V \subset \mathbb{C}^d$  invariant under  $U(d)$ , and a biholomorphism  $\rho : V \rightarrow U$ , such that  $\rho^*\omega$  is invariant under  $U(n)$ .*

*In particular, in this chart, there exists a Kähler potential  $\phi$  which depends only on the distance to the origin, with real-analytic regularity.*

*Proof.* Let  $\rho_0 : V_0 \mapsto U_0$  be any local holomorphic chart to a neighbourhood of  $P_0$ , with  $\rho_0(0) = P_0$ .

Since  $M$  is homogeneous, there exists an open set  $P_0 \in U_1 \subset U_0$  and an action of  $U(n)$  on  $U_1$  such that, for any  $g \in U(d)$ , one has

$$\begin{aligned} D(x \mapsto \rho_0^{-1}(g \cdot \rho_0(x)))(0) &= g \\ (g \cdot)^* J &= J \\ (g \cdot)^* \omega &= \omega. \end{aligned}$$

In particular, for  $g \in U(d)$ , the map  $\rho_g : x \mapsto g \cdot \rho_0(g^{-1}x)$  is a biholomorphism from  $V_2 = \bigcap_{g \in U(d)} g \circ \rho_0^{-1}(U_1)$  onto its image  $U_2(g)$ .

For  $x \in \bigcap_{g \in U(d)} U_2(g)$ , let us define

$$\sigma(x) = \int_{U(d)} \rho_g^{-1}(x) d\mu_{Haar}(g).$$

Then  $D(\sigma \circ \rho_0)(0) = I$ . Hence,  $\sigma$  is a biholomorphism, from a small  $U(d)$  invariant open set  $U \ni P_0$  into a small  $U(d)$  invariant open set  $V \ni 0$ . By construction  $\sigma$  is  $g$ -equivariant, in the sense that  $\sigma(gx) = g \cdot \sigma(x)$ . Then  $\sigma^{-1}$  is the requested chart since  $\omega$  is invariant under the action of  $U(d)$  on  $U$ .

Let us proceed to the second part of the Proposition. We first let  $\phi_1$  be any real-analytic Kähler potential in the chart  $\sigma^{-1}$ . We then define

$$\phi(x) = \int_{g \in U(n)} \phi_1(gx) d\mu_{Haar}(g).$$

Then  $\phi$  is a radial function since  $U(d)$  acts transitively on the unit sphere. Moreover, since  $\sigma_*\omega$  is  $U(d)$ -invariant then  $x \mapsto \phi_1(gx)$  is a Kähler potential, so that the mean value  $\phi$  is a Kähler potential.  $\square$

**Remark 2.2.** There is exactly one degree of freedom in the choice of the chart  $\rho$  in Proposition 2.1: the precomposition by a scaling  $z \mapsto \lambda z$  preserves all requested properties. In general, the metric  $\sigma_*\mathfrak{K}(\omega)$ , at zero, is a constant times the standard metric. This constant can be modified by the scaling above. Hence, without loss of generality, one can choose the chart so that the Kähler potential has the following Taylor expansion at zero:

$$\phi(x) = \frac{|x|^2}{2} + O(|x|^3),$$

so that the metric  $\sigma^*g$ , at zero, is the standard metric.

**Definition 2.3.** A chart satisfying the conditions of Proposition 2.1, such that the radial Kähler potential has the following Taylor expansion at zero:

$$\phi(x) = \frac{|x|^2}{2} + O(|x|^3),$$

is called a *radial holomorphic chart*.

The following elementary fact will be used extensively:

**Proposition 2.4.** *The radial Kähler potential  $\phi$  of a radial holomorphic chart is strongly convex. In particular, for all  $x \neq 0$  in the domain of  $\phi$  one has  $\phi(x) > 0$ .*

*Proof.* From the Taylor expansion  $\phi(x) = \frac{|x|^2}{2} + O(|x|^3)$ , one deduces that the real Hessian matrix of  $\phi$  is positive definite at zero. Near any point  $x \neq 0$  which belongs to the domain of  $\phi$ , in spherical coordinates the function  $\phi$  depends only on the distance  $r$  to the origin. The Levi form  $\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(x)$ , which is Hermitian positive definite (since  $\phi$  is strongly pseudo-convex), is then equal to  $\frac{\partial^2 \phi}{\partial r^2}(r) Id$ . In particular,  $\frac{\partial^2 \phi}{\partial r^2} > 0$  everywhere, so that  $\phi$  is strongly convex at  $x$ .  $\square$

### 3 Approximate coherent states

We first recall the notion of coherent states in Berezin-Toeplitz quantization.

**Definition 3.1.** Let  $(P_0, v) \in L$ . We define the associated coherent state, which is a section of  $L^{\otimes N}$ , as follows:

$$\psi_{P_0, v}^N = (u \mapsto \langle u(P_0), v \rangle_h)^{*_{H_0(M, L^{\otimes N})}}.$$

That is, the evaluation map  $u \mapsto \langle u(P_0), v \rangle_h$  is a linear operator on  $H_0(M, L^{\otimes N})$ , and by the Riesz representation theorem, there exists  $\psi_{P_0, v}^N$  such that linear map is  $\langle \psi_{P_0, v}^N, \cdot \rangle$ .

Let us use the radial charts above to build an approximation for coherent states on a homogeneous Kähler manifold.

**Proposition 3.2.** *There exists  $r > 0$  such that the following is true.*

1. Let  $P_0 \in M$ . There exists a radial holomorphic chart near  $P_0$ , whose domain contains  $B(0, r)$ .
2. Let  $\phi$  denote the radial Kähler potential near  $P_0$ . For all  $N \geq 1$  the quantity

$$a(N) = \int_{B(0, r)} \exp(-N\phi(|z|)) dz d\bar{z}$$

is well-defined and does not depend on  $P_0$ .

*Proof.*

1. Let  $P_1 \in M$ . By Proposition 2.1 there exists a radial holomorphic chart near  $P_1$ . Since  $M$  is homogeneous, a small neighbourhood of any  $P_0 \in M$ , of size independent of  $P_0$  since  $M$  is compact, can be mapped into a neighbourhood of  $P_1 \in M$ . By restriction of the radial holomorphic chart of Proposition 2.1 to this neighbourhood, whose preimage contains a small ball around zero, this defines a chart around  $P_0$ . Since  $M$  is compact, there is a radius  $r$  such that, for every  $P_0 \in M$ , the closed ball  $\overline{B(P_0, r)}$  is contained in the domain of the chart around  $P_0$ .
2. By construction of the chart above, the Kähler potential  $\phi$  does not depend on  $P_0$ . Moreover,  $\phi$  is a smooth function on  $\overline{B(0, r)}$ , hence the claim..

□

**Remark 3.3.** We will see at the end of the proof of Theorem A that  $a(N)^{-1}$  is exponentially close to a polynomial in  $N$ .

From now on,  $r$  is as in the claim of Proposition 3.2.

**Proposition 3.4.** *Let  $(P_0, v) \in L$ . The action of  $U(n)$  on a neighbourhood  $U$  of  $P_0$  in  $M$  can be lifted in an action on  $L_U$ .*

*Proof.* By definition of  $L$ , if  $V$  is the preimage of  $U$  by a radial holomorphic chart, the bundle  $(L_U, h)$  is isomorphic to

$$(V \times \mathbb{C}, \exp(-\phi(z))|u|^2).$$

Since  $\phi$  is invariant under  $U(n)$ , the linear action of  $U(n)$  on  $V$  can be trivially extended to  $V \times \mathbb{C}$  and preserves the metric. □

In order to treat local holomorphic sections of a prequantum bundle over a quantizable compact homogeneous Kähler manifold, let us define the Ancillary space and the approximate coherent states:

**Definition 3.5.** Let  $\phi$  be the radial Kähler potential on  $M$  and  $r$  be as in Proposition 3.2. Let  $N \in \mathbb{N}$ . The ancillary space is defined as

$$A_N = \left\{ u \text{ holomorphic on } B(0, r), \int_{B(0, r)} e^{-N\phi(z)} |u|^2 \leq +\infty \right\}.$$

It is a Hilbert space with the scalar product

$$\langle u, v \rangle_{A_N} = \int_{B(0, r)} e^{-N\phi(z)} u(z) \overline{v(z)} dz.$$

The set  $A_N$  consists of functions belonging to the usual Hardy space of the unit ball, but the scalar product is twisted by the Kähler potential  $\phi$ .

Since the function  $\phi$  appearing in the definition of  $A_N$  is a universal local Kähler potential on  $M$ , for each  $(P_0, v) \in L^*$  there is a natural isomorphism (up to multiplication of all norms by  $\|v\|_h$ )  $\mathfrak{S}_{P_0, v}^N$  between  $A_N$  and the space of  $L^2$  local holomorphic sections  $H_0(U, L^{\otimes N})$  where  $U = \sigma_{P_0}^{-1}(B(0, r))$ . We define  $\tilde{\psi}_{P_0, v}^N$  as the element of  $H_0(U, L^{\otimes N})$  associated with the constant function  $a(N)^{-1} \in A_N$ .

We set  $\tilde{\psi}_{P_0, v}^N$  to be zero outside  $\sigma^{-1}(B(0, r))$  so that  $\tilde{\psi}_{P_0, v}^N \in L^2(M, L^{\otimes N})$ . The function  $\tilde{\psi}_{P_0, v}^N$  is equivariant with respect to  $v$ : one has

$$\tilde{\psi}_{P_0, v}^N = \left( \bar{v}/v^T \right)^N \tilde{\psi}_{P_0, v^T}^N.$$

This allows us to define the approximate normalized coherent state  $\tilde{\psi}_{P_0}$  as an element of  $L^2(M, L^{\otimes N}) \otimes \bar{L}_{P_0}^{\otimes N}$ .

Let us prove that the approximate coherent states are very close to  $H_N(M, L)$ :

**Proposition 3.6.** *There exists  $c > 0$  and  $C > 0$  such that, for all  $P_0 \in M$ ,*

$$\|S_N \tilde{\psi}_{P_0}^N - \tilde{\psi}_{P_0}\|_{L^2} \leq C e^{-cN}.$$

*Proof.* Let  $\chi$  denote a test function on  $\mathbb{R}$  which is smooth and such that  $\chi = 1$  on  $[0, \frac{r}{2}]$  and  $\chi = 0$  on  $[r, +\infty)$ .

The section  $(\chi \circ |\sigma|) \tilde{\psi}_{P_0}^N$  is smooth; since  $\tilde{\psi}_{P_0}^N$  is holomorphic on  $\sigma^{-1}(B(0, r))$  and decays exponentially fast far from  $P_0$ , one has

$$\|\bar{\partial}(\chi \circ |\sigma|) \tilde{\psi}_{P_0}^N\|_{L^2} \leq C e^{-cN}.$$

From Kohn's estimate (1) we deduce that

$$\|S_N(\chi \circ |\sigma|) \tilde{\psi}_{P_0}^N - (\chi \circ |\sigma|) \tilde{\psi}_{P_0}\|_{L^2} \leq C e^{-cN}.$$

In addition, since  $\phi > c$  on  $B(0, r) \setminus B(0, r/2)$ , one has

$$\|(\chi \circ |\sigma|) \tilde{\psi}_{P_0}^N - \tilde{\psi}_{P_0}^N\|_{L^2} \leq C e^{-cN}.$$

Since  $S_N$  is an orthogonal projector, its operator norm is bounded by 1, so that the previous estimates implies

$$\|S_N(\chi \circ |\sigma|) \tilde{\psi}_{P_0}^N - S_N \tilde{\psi}_{P_0}^N\|_{L^2} \leq C e^{-cN}.$$

This ends the proof. □

To show that our approximate coherent states are indeed exponentially close to the actual coherent states we will use the following lemma.

**Lemma 3.7.** *Any continuous linear form on  $A_N$  invariant by linear unitary changes of variables is proportional to the continuous linear form  $v \mapsto \langle v, 1 \rangle$ .*

*In particular, the continuous linear form  $A_N \ni u \mapsto u(0)$  is equal to the scalar product with the constant function  $a(N)^{-1}$*

*Proof.* A Hilbert basis of  $A_N$  is given by the normalised monomials  $e_\nu z \mapsto c_\nu z^\nu$  for  $\nu \in \mathbb{N}^d$ , for some  $c_\nu > 0$ . Special elements of  $U(n)$  are the diagonal matrices  $\text{diag}(\exp(i\theta_1), \dots, \exp(i\theta_d))$  which send  $e_\nu$  into  $\exp(i\theta \cdot \nu)e_\nu$ .

A linear form  $\eta$  invariant under  $U(d)$  must be such that  $\eta(e_\nu) = \exp(i\theta \cdot \nu)\eta(e_\nu)$  for every  $\theta, \nu$ . In particular,  $\nu \neq 0 \Rightarrow \eta(e_\nu) = 0$ . Since  $\eta$  is continuous we deduce that  $\eta$  is proportional to the scalar product with  $e_0 = c_0 1$ .

For the second part of the Proposition we only need to prove that the multiplicative factor between the two continuous  $U(d)$ -invariant linear forms of  $A_N$ , evaluation at 0 on one side, scalar product with  $a(N)^{-1}$  on the other side, is 1. By Definition of  $a(N)$ , the scalar product in  $A_N$  of  $a(N)^{-1}$  with  $a(N)^{-1}$  is  $a(N)^{-1}$ , moreover the evaluation at zero of  $a(N)^{-1}$  is  $a(N)^{-1}$ , hence the claim.  $\square$

The functions  $\tilde{\psi}_{P_0, v}^N$  mimic the definition of coherent states.

**Proposition 3.8.** *There exists  $c > 0$  such that, for any  $(P_0, v_0), (P_1, v_1) \in L^*$ ,*

- *If  $\text{dist}(P_0, P_1) \leq \frac{r}{2}$ , then*

$$\left| \langle \tilde{\psi}_{P_1, v_1}^N, \tilde{\psi}_{P_0, v_0}^N \rangle - \langle \tilde{\psi}_{P_1, v_1}(P_0), v_0^{\otimes N} \rangle_h \right| = O(e^{-cN}).$$

- *In general, one has*

$$|\langle \tilde{\psi}_{P_1, v_1}^N, \tilde{\psi}_{P_0, v_0}^N \rangle| \leq C e^{-cN \text{dist}(P_0, P_1)^2}.$$

*Proof.*

- The continuous linear functional on  $A_N$  which sends  $u$  to  $u(0)$  is invariant under the action of  $U(n)$  (since 0 is a fixed point), so that, by Lemma 3.7, it is proportional to the scalar product with a constant. This property, read in the map  $\mathfrak{S}_{P_0, v_0}^N$ , means that, for every  $(P_1, v_1) \in L$  the scalar product

$$\langle \tilde{\psi}_{P_0, v_0}^N, \tilde{\psi}_{P_1, v_1}^N \rangle$$

is a constant (independent of  $P_1$ ) times  $\langle S_N \tilde{\psi}_{P_1, v_1}(P_0), v_0^{\otimes N} \rangle_h$ . The normalizing factor  $a(N)$  is such that both sides are equal to 1 if  $P_1 = P_0$ . This ends the proof since  $S_N$  is almost identity on the almost coherent states.

- If  $\text{dist}(P_0, P_1) \geq 2r$  then  $\tilde{\psi}_{P_0, v_0}^N$  and  $\tilde{\psi}_{P_1, v_1}^N$  have disjoint support so that the scalar product is zero.  
If  $r/2 \leq \text{dist}(P_0, P_1) \leq 2r$  then  $\tilde{\psi}_{P_1, v_1}^N$  is exponentially small on  $B(P_0, r/4)$  and  $\tilde{\psi}_{P_0, v_0}^N$  is exponentially small outside this ball so that the scalar product is smaller than  $C e^{-cN(4r)^2}$  for some  $c > 0$ .  
If  $P_1 \in B(P_0, r/2)$ , one can apply the previous point; the claim follows from the fact that  $\phi(|x|) \geq c|x|^2$  on  $B(P_0, r/2)$ .

$\square$



## 4 Approximate Bergman projector

We can now define the approximate Bergman projector by its kernel:  $\tilde{S}_N$  is a function on  $\overline{L}^{\otimes N} \boxtimes L^{\otimes N}$  which is linear in the fibres (or, equivalently, a section of  $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ ) defined by the formula:

$$\tilde{S}_N((x, v), (y, v')) = \langle \tilde{\psi}_{x,v}^N, \tilde{\psi}_{y,v'}^N \rangle.$$

We wish to prove that this operator is very close to the actual Bergman projector, defined by the actual coherent states  $\psi_{P_0,v}^N$ :

**Proposition 4.1.** *Let  $(P_0, v) \in L$ . Then  $S_N \tilde{\psi}_{P_0,v}^N = \psi_{P_0,v}^N$ .*

*Proof.* Let  $U = B(P_0, r)$ . By construction, the scalar product of  $\tilde{\psi}_{P_0,v}^N$  with any element of  $H_N(U, L^{\otimes N})$  is the value at  $P_0$  of this element, taken in scalar product with  $v$ . As  $H_N(M, L^{\otimes N}) \subset H_N(U, L^{\otimes N})$  in a way which preserves the scalar product with  $\tilde{\psi}_{P_0,v}^N$ , from Definition 3.1 one has  $S_N \tilde{\psi}_{P_0,v}^N = \psi_{P_0,v}^N$ .  $\square$

From Propositions 3.6 and 4.1 we deduce that approximate coherent states are, indeed, close to coherent states. In particular,

**Proposition 4.2.** *Uniformly on  $(x, y) \in M \times M$ , there holds*

$$\|\tilde{S}_N(x, y) - S_N(x, y)\|_h = O(e^{-cN}).$$

*Proof.* The exact Bergman kernel is expressed in terms of the coherent states as:

$$S_N((x, v), (y, v')) = \langle \psi_{x,v}^N, \psi_{y,v'}^N \rangle.$$

From this and the Definition of  $\tilde{S}_N$ , since

$$S_N \tilde{\psi}_{x,v}^N = \psi_{x,v}^N = \tilde{\psi}_{x,v}^N + O(e^{-cN}),$$

the kernels of  $S_N$  and  $\tilde{S}_N$  are exponentially close.  $\square$

## 5 Approximate projector in a normal chart

To conclude the proof of Theorem A in the homogeneous case, it only remains to compute an approximate expression for  $\tilde{S}_N(x, y) = \langle \tilde{\psi}_x^N, \tilde{\psi}_y^N \rangle$ . At first sight, this looks easy. Indeed, on the diagonal,  $\tilde{S}_N(x, x) = a(N)^{-1}$ . Moreover  $\tilde{S}_N$  is  $O(e^{-cN})$ -close from the Bergman kernel  $S_N$ , which is holomorphic in the first variable and anti-holomorphic in the second variable. However, one cannot conclude that  $\tilde{S}_N$  is exponentially close to the holomorphic extension of  $a(N)^{-1}$  (that is,  $a(N)^{-1} \Psi^N$ ). Indeed,  $S_N(x, x) - a(N)^{-1}$ , while exponentially small, might oscillate very fast, so that its holomorphic extension is not uniformly controlled.

By studying change of charts between radial holomorphic charts, one can prove the following Proposition.

**Proposition 5.1.** *There exists  $c > 0$  and  $C > 0$  such that, for all  $(x, y) \in M \times M$ , there holds*

$$\left\| \tilde{S}_N(x, y) - \Psi^N(x, y) a(N)^{-1} \right\|_h \leq C e^{-cN}.$$

*Proof.* It is sufficient to prove the claim for  $x, y$  close enough from each other.

We first need to understand how to change from the radial holomorphic chart around  $x$  to the radial holomorphic chart around  $y$ . By hypothesis, if  $x$  and  $y$  are two points in  $M$  at distance less than  $\frac{r}{2}$ , if  $\rho$  denotes a radial chart at  $x$ , there is a map  $\sigma : B(0, \frac{r}{2}) \rightarrow B(0, r)$ , which is biholomorphic on its image and which preserves the metric  $\rho^*g$ , and such that  $\sigma(0) = \rho(y)$ . The associated holomorphic map on  $B(0, \frac{r}{2}) \times \mathbb{C}$  which preserves the Hermitian metric pulled back by  $\rho$  on the fibre is of the form:

$$(z, v) \mapsto \left( \sigma(z), \exp \left( \frac{1}{2}(\phi(|z|^2) - \phi(|\sigma(z)|^2)) + if_\sigma(z) \right) v \right), \quad (2)$$

where  $f_\sigma$  is such that the function

$$m \mapsto \phi(|z|^2) - \phi(|\sigma(z)|^2) + if_\sigma(z)$$

is holomorphic. Such a  $f_\sigma$  exists and is unique up to an additive constant: indeed, since  $\sigma$  preserves the metric  $g$ ,  $z \mapsto \phi(|\sigma(z)|^2)$  is a Kähler potential on  $B(0, \frac{r}{2})$ . Hence, the map

$$z \mapsto \phi(|z|^2) - \phi(|\sigma(z)|^2)$$

is harmonic, so that it is the real part of a holomorphic function.

Then, by (2), in a radial holomorphic chart around  $x$ , the almost coherent state  $\tilde{\psi}_{y, v'}^N$  is written as

$$z \mapsto a(N)^{-1} \mathbb{1}_{V(y)} \overline{v'} \exp \left( -\frac{N}{2} \phi(|\sigma(z)|^2) + if_\sigma(z) \right).$$

By Proposition 3.8, the scalar product with  $\tilde{\psi}_{x, v}^N$ , with  $y$  close to  $x$ , is

$$\langle \tilde{\psi}_{y, v'}^N, \tilde{\psi}_{x, v}^N \rangle = a(N)^{-1} (v \overline{v'}) \exp \left( -\frac{N}{2} \phi(|\rho(y)|^2) + iN f_\sigma(0) \right) + O(e^{-cN}).$$

In particular, in a radial holomorphic chart  $\rho$  around  $x$ , the approximate Bergman kernel evaluated at  $x$  has the following form for  $z$  small:

$$\tilde{S}_N(\rho(z), \rho(0)) = a(N)^{-1} \exp(Ng(z)) \psi_x^N(\rho(z)) \overline{\psi_x^N(\rho(0))} + O(e^{-cN}),$$

where  $g$  is holomorphic. Using another change of charts given by (2), the form of the approximate Bergman kernel, near the diagonal, is

$$\tilde{S}_N(\rho(z), \rho(w)) = a(N)^{-1} \exp(NF(z, w)) \tilde{\psi}_x^N(\rho(z)) \overline{\tilde{\psi}_x^N(\rho(w))} + O(e^{-cN}),$$

where  $F$  is holomorphic in the first variable and anti-holomorphic in the second variable.

Moreover,  $\tilde{S}_N(z, z) = \tilde{S}_N(0, 0) = a(N)^{-1}$ , hence  $F(z, \bar{w}) = \tilde{\phi}(z \cdot \bar{w})$ .

The expression of the phase in coordinates coincides with the section  $\Psi^N$  of Definition 1.3 (the non-vanishing section  $s$  here is  $\tilde{\psi}_x^1$ ). Thus the Bergman kernel can be written as

$$\tilde{S}_N(x, y) = \Psi^N(x, y) a(N)^{-1} + O(e^{-cN}).$$

□

We will compute explicitly  $a(N)^{-1}$  in Section 6. Up to this computation, the proof of Theorem A is complete in the case of a single homogeneous manifold.

It remains to prove how to pass from homogeneous manifolds to direct products of such. This relies on the following Proposition.

**Proposition 5.2.** *Let  $M_1, M_2$  be compact quantizable Kähler manifolds and  $L_1, L_2$  be the associated pre-quantum line bundles. Then  $L_1 \boxtimes L_2$  is the prequantum line bundle over  $M_1 \times M_2$ , and*

$$H_0(M_1 \times M_2, (L_1 \boxtimes L_2)^{\otimes N}) \simeq H_0(M_1, L_1^{\otimes N}) \otimes H_0(M_2, L_2^{\otimes N}).$$

*Proof.* There is a tautological, isometric injection

$$\iota : H_0(M_1, L_1^{\otimes N}) \otimes H_0(M_2, L_2^{\otimes N}) \hookrightarrow H_0(M_1 \times M_2, (L_1 \boxtimes L_2)^{\otimes N})$$

which is such that, for  $(s_1, s_2) \in H_0(M_1, L_1^{\otimes N}) \times H_0(M_2, L_2^{\otimes N})$  and  $(x, y) \in M_1 \times M_2$ , one has

$$\iota(s_1 \otimes s_2)(x, y) = s_1(x) \otimes s_2(y).$$

It remains to prove that any element of  $H_0(M_1 \times M_2, (L_1 \boxtimes L_2)^{\otimes N})$  belongs to the image of the element above. To this end, let us prove that, for any  $(x_1, v_1), (x_2, v_2) \in L_1 \times L_2$ , the coherent state at  $((x_1, x_2), v_1 \otimes v_2)$  is given by

$$\psi_{(x_1, x_2), v_1 \otimes v_2}^N = \iota(\psi_{x_1, v_1}^N \otimes \psi_{x_2, v_2}^N).$$

Indeed, for any  $s \in H_0(M_1 \times M_2, (L_1 \boxtimes L_2)^{\otimes N})$ , one has

$$\begin{aligned} \langle s, \iota(\psi_{x_1, v_1}^N \otimes \psi_{x_2, v_2}^N) \rangle &= \int_{M_1} \left\langle \int_{M_2} \langle s(y_1, y_2), \psi_{x_2, v_2}^N(y_2) \rangle_{(L_2)_{y_2}^{\otimes N}} dy_2, \psi_{x_1, v_1}^N(y_1) \right\rangle_{(L_1)_{y_1}^{\otimes N}} dx_1 \\ &= \int_{M_1} \langle s(y_1, x_2), \psi_{x_1, v_1}^{\otimes N} \otimes v_2 \rangle_{(L_1)_{y_1}^{\otimes N} \otimes (L_2)_{x_2}^{\otimes N}} dx_1 \\ &= \langle s(x_1, x_2), v_1 \otimes v_2 \rangle_{(L_1)_{x_1}^{\otimes N} \otimes (L_2)_{x_2}^{\otimes N}} = \langle s, \psi_{(x_1, x_2), v_1 \otimes v_2}^N \rangle. \end{aligned}$$

The image of  $\iota$  thus contains all coherent states on  $M_1 \times M_2$ . Hence, the orthogonal of the range of  $\iota$  in  $H_0(M_1 \times M_2, (L_1 \boxtimes L_2)^{\otimes N})$  is zero, which concludes the proof.  $\square$

In particular, the Bergman kernel on a product  $M_1 \times M_2$  is given by

$$S_N^{M_1 \times M_2}(x_1, x_2, y_1, y_2) = S_N^{M_1}(x_1, y_1) \otimes S_N^{M_2}(x_2, y_2).$$

This, along with Propositions 4.1 and 5.1, concludes the proof of Theorem A up to the study of  $a(N)^{-1}$ , which we perform in the next section.

## 6 The coefficients of the Bergman kernel

Since, for all  $x \in M$ , one has  $\Psi^N(x, x) = 1$ , then the trace of the Bergman kernel is given by

$$\mathrm{tr}(S_N) = \sum_{i=1}^{d_N} 1 = \int_M \sum_{i=1}^{d_N} s_i(x) \overline{s_i(x)} dx = \int_M S_N(x, x) dz = a(N)^{-1} \mathrm{Vol}(M) + O(e^{-cN}).$$

In particular,  $a(N)^{-1}$  is exponentially close to an integer divided by  $\mathrm{Vol}(M)$ . Let

$$P(N) = \frac{\mathrm{tr}(S_N)}{\mathrm{Vol}(M)}.$$

In this section we compute  $P(N)$  in the case of a homogeneous manifold of dimension  $d$ . Since

$$P(N)^{-1} = \int_{B(0, r)} \exp(-N\phi(|z|)) dz d\bar{z} + O(e^{-cN}),$$

and there is a universal local model for  $M$  which depends only on its curvature  $\kappa$ , then  $P(N)$  depends only on  $\kappa$  and the dimension  $d$ . Moreover,  $P(N)^{-1}$  has real-analytic dependence on  $\kappa$ . We will give an expression for  $P(N)$  which is valid on  $\kappa \in \{\frac{1}{k}, k \in \mathbb{N}\}$ . Since  $P(N)$  is real-analytic in  $\kappa$ , it will follow that this expression is valid for all curvatures. From now on we write  $P_\kappa(N)$  to indicate that  $P(N)$  depends on  $N$  and  $\kappa$ , and only on them.

Let us consider the case of the rescaled projective space:

$$(M_k, \omega_k, J) = (\mathbb{C}\mathbb{P}^d, k\omega_{FS}, J_{st}).$$

This space is quantizable; the prequantum bundle is simply

$$L_k = (L_1)^{\otimes k},$$

so that

$$S_{N,k}(x, y) = S_{Nk,1}(x, y).$$

Moreover, the curvature of  $(M_k, \omega_k)$  is  $\frac{1}{k}$ . In other terms,

$$P_{\frac{1}{k}}(N) = \frac{\text{Vol}(M_1)}{\text{Vol}(M_k)} P_1(kN) = k^{-d} P_1(kN).$$

It remains to compute  $P_1$ . On  $\mathbb{C}\mathbb{P}^d$ , the prequantum bundle  $L_1$  is explicit: it is  $O(1)$ , the dual of the tautological line bundle. In this setting,

$$H_0(M, L^{\otimes N}) \simeq \mathbb{C}_N[X_1, \dots, X_d].$$

Hence,

$$P_1(N) = \frac{1}{\text{Vol}(\mathbb{C}\mathbb{P}^d)} \dim(\mathbb{C}_N[X_1, \dots, X_d]) = \frac{d!}{\pi^d} \binom{N+d}{d} = \frac{1}{\pi^d} (N+1) \dots (N+d).$$

Hence, for any  $\kappa$  of the form  $\frac{1}{k}$  with  $k \in \mathbb{N}$  there holds

$$P_\kappa(N) = \frac{1}{\pi^d} (N + \kappa)(N + 2\kappa) \dots (N + d\kappa).$$

Since  $P_\kappa$  has real-analytic dependence on  $\kappa$ , the formula above is true for any  $\kappa \in \mathbb{R}$ , which concludes the proof.

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