EXISTENCE RESULT FOR GENERALIZED VARIATIONAL EQUALITY

ALLAHKARAM SHAFIE, FARID BOZORGNIA

ABSTRACT. In this paper we prove the existence of solution to the Stampachia variational inequality under weakened assumptions on the given operator. As a consequence, we provide some sufficient conditions that under them the generalized equation $0 \in T(x)$ has a solution. Furthermore, by using generalized results of continuity and monotonicity, we extend the related existence results and we answer an open problem proposed by Kassay and Miholka (J Optim Theory Appl 159 (2013) 721-740).

Keywords: Variational inequality; generalized monotonicity; generalized continuity; existence results.

1. Introduction

The theory of variational inequality has been investigated extensively as methodology to study of equilibrium problems. Equilibrium is a central concept in numerous disciplines including economics, management science, operations research, and engineering, see [5, 8, 11].

In 1966, Hartman and Stampacchia introduced the variational inequality as a tool for the study of partial differential equations with applications principally drawn from mechanics, see [1].

In [2] existence result for variational inequalities is given by generalized monotone operators. As a consequence, the authors conclude the subjectivity for some classes of set-valued operators. By strengthening the continuity assumptions, they show similar subjectivity results without any monotonicity assumption.

Finding the zeroes of a set-valued map $T(x)$ are particularly important. Indeed, zeroes of the subdifferential operator of a function defined on the same space are precisely the minimum points of this function. Hence, there is an important link between the theory of (generalized) monotone operators and optimization theory, see for instance [4, 7, 12].

2. Preliminarily and Mathematical background

Throughout this paper, X is Banach space, X^* denotes its topological dual and $\langle \cdot, \cdot \rangle$ the duality pairing. For a nonempty set $A \subset X$, $\mathrm{cor} A, \mathrm{cl} A, \mathrm{cl}_{\mathrm{w}} A$,

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and conv$A$, stand for the algebraic interior, closure, weakly closure, and convex hull of the set A, respectively. Also for $x^* \in X^*$ we denote $\mathbb{R}_{++}x^* = \{tx^* : \ t > 0\}$.

Let us recall the classical terminology of generalized monotonicity of set-valued maps that we will use in the sequel. A set valued map $T : X \rightrightarrows X^*$ is said to be

- Quasimonotone on a subset K, provided that for all $x, y \in K$,
$$\exists x^* \in T(x) : \langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0 \quad \forall y^* \in T(y);$$
- Properly quasimonotone on a subset K, provided that for all
$$\{x_1, x_2, \cdots, x_n\} \subseteq K, \text{ and for all } x \in \text{conv}\{x_1, x_2, ..., x_n\},$$
there exists $i \in \{1, 2, \cdots, n\}$ such that
$$\langle x_i^*, x_i - x\rangle \geq 0 \ \forall x_i^* \in T(x_i);$$
- Pseudomonotone on a subset K, provided that for all $x, y \in K$,
$$\exists x^* \in T(x) : \langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0 \quad \forall y^* \in T(y).$$

A set-valued operator $T : X \rightrightarrows X^*$ is said to be upper sign-continuous on a convex subset K, if for any $x, y \in K$, the following implication holds:

$$\forall t \in (0, 1) \quad \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0 \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0, \tag{2.1}$$

where $x_t = tx + (1 - t)y$.

Accordingly, T is called lower sign-continuous on a convex subset K if, for or any $x, y \in K$, the following implication holds:

$$\forall t \in (0, 1) \quad \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0 \Rightarrow \inf_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0. \tag{2.2}$$

By these definitions it is clear that any lower sign-continuous map is also upper sign-continuous. Furthermore, if $T, S : X \rightrightarrows X^*$ be set-valued maps and $T \subseteq S$ and T be lower sign-continuous, then S is lower sign-continuous.

By the following example we underline that this implication is not true for lower semi-continuous mappings.

Example 2.1. Consider the following set-valued map $T : \mathbb{R} \rightrightarrows \mathbb{R}$ as

$$T(x) = \begin{cases} \{-1, 0\} & x \neq 0, \\ \{0\} & x = 0. \end{cases}$$

It is easy to cheek that conv$T$ is lower semi-continuous, but T is not lower semi-continuous.

Algebraic interior is defined as

$$\text{cor}K = \{y \in K : \ \forall x \in X, \exists \lambda > 0\,, \ \forall t \in [0, \lambda) \quad y + tx \in K\}\,.$$

Note that always int$K \subseteq$ cor$K \subseteq K$ and algebraic interior is weaker than of topological interior by the following example.

Example 2.2. Let $X = l^p, p \geq 1$ and consider the convex set K defined by $K = \left\{x = (x(n))_{n\in\mathbb{N}} \in l^p, \ \forall n \in \mathbb{N}, \ \ x(n) \geq 0\right\}$. Then int$K = \emptyset$ and

$$cor K = \left\{x = (x(n))_{n\in\mathbb{N}} \in l^p, \ \ x(n) > 0 \ \ \forall n \in \mathbb{N}\right\}.$$

Also, we say that the map $T : X \rightrightarrows X^*$ is locally upper sign-continuous at x, if there exists a convex neighbourhood U of x and an upper sign-continuous submap $\Phi_x : U \rightrightarrows X^*$ with nonempty convex w^*-compact values, satisfying

$$\Phi_x(u) \subset T(u) \setminus \{0\}, \text{ for any, } u \in U.$$

In the sequel, given a set-valued map $T : X \rightrightarrows X^*$ we consider its convex hull map $\mathrm{conv}T : X \rightrightarrows X^*$ defined by $\mathrm{conv}T(x) = \mathrm{conv}(T(x))$.

The proof of the following proposition is straightforward.

Proposition 2.1. *Let $T : X \rightrightarrows X^*$ be a set-valued map and $x \in \mathrm{dom}T$. If T is locally upper sign-continuous at x, then $\mathrm{conv}T$ is locally upper sign-continuous at x.*

The following example shows that the reverse of Proposition 2.1 is not true.

Example 2.3. Let the set-valued map $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$T(x) = \begin{cases} \{-1, 1\} & x = 0, \\ [-1, 1] & x \neq 0. \end{cases} \tag{2.3}$$

Note that

$$\mathrm{conv}T(x) = [-1, 1].$$

One can check that $\mathrm{conv}T$ is locally upper sign-continuous but T is not locally upper sign-continuous at $x = 0$.

Definition 2.1. A map $T : X \rightrightarrows X^*$ is said to be weakly dually lower semicontinuous on a subset K if for any $x \in K$ and for any net $(y_\alpha)_\alpha \subseteq K$ such that $y_\alpha \rightharpoonup y$, the following implication holds:

$$\limsup_{\alpha} \inf_{y^*_\alpha \in T(y_\alpha)} \langle y^*_\alpha, y_\alpha - x\rangle \geq 0 \Rightarrow \inf_{y^* \in T(y)} \langle y^*, y - x\rangle \geq 0\,.$$

It is worth to mention that any weakly lower semicontinuous map on K is weakly dually lower semicontinuous on K but this concept is strictly weaker than the lower semicontinuouity. For example choose $K = [-1, 1]$, and define the set-valued map $T : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$T(x) = \begin{cases} [-1, 0], & \text{if } x = 0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

T is dually lower semicontinuous on K but it is not lower semicontinuous at $x = 0$. Also note that if T is a dually lower semicontinuous, then $\mathrm{conv}T$ is so. However, the reciprocal is not true in general. For instance the set-valued map $T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$T(x) = \begin{cases} \mathbb{Q}, & \text{if } x \geq 0, \\ \mathbb{Q}^c, & \text{if } x < 0, \end{cases}$$

is not dually lower semicontinuous but $\mathrm{conv}T = \mathbb{R}$ is dually lower semicontinuous.

The variational inequality problem which we consider in this paper can be formulated as follows. Given a nonempty and convex subset K of X, find an element $\bar{x} \in K$ such that

$$\sup_{x^* \in T(\bar{x})} \langle x^*, y - \bar{x}\rangle \geq 0 \quad \forall y \in K. \text{ (VI)}$$

We will consider the following concepts of solutions of the Stampacchia variational inequality.

- Stampacchia solutions:

$$S(T,K) = \{x \in K : \ \exists x^* \in T(x) \ \text{with} \ \langle x^*, y - x \rangle \geq 0, \ \ \forall y \in K\}\,.$$

- Star Stampacchia solutions:

$$S^*(T,K) = \{x \in K : \exists x^* \in T(x) \setminus \{0\}, \text{with} \ \ \langle x^*, y - x \rangle \geq 0, \ \forall y \in K\}\,.$$

- Weak Stampacchia solutions:

$$S^w(T,K) = \{x \in K : \forall y \in K \ \exists x^* \in T(x) \, \text{with} \, \langle x^*, y - x \rangle \geq 0\,\}\,. \tag{2.4}$$

- Minty solutions

$$M(T,K) = \{x \in K : \ \forall y^* \in T(y)\,, \ \forall y \in K, \ \langle y^*, y - x \rangle \geq 0\,\}\,.$$

We recall that $x \in K$ is a local solution of the Minty variational inequality if there exists a neighborhood U of x such that $x \in M(T, K \cap U)$. The set of all local solution is denoted by $LM(T,K)$. It is obvious that $M(T,K) \subseteq LM(T,K)$. The following example illustrates that converse is not necessarily true.

Example 2.4. Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be set-valued map as

$$T(x) = \begin{cases} \{1\} & x \in (-1,1), \\ \{2\} & \ \ x \notin (-1,1). \end{cases}$$

Clearly $0 \in LM(T,K)$ but $0 \notin M(T,K)$.

Remark 2.1. One can see that the solution of Stampacchia variational inequality problem is also the solution of the problem (VI).

The topic of variational inequality appears in the calculus of variations in minimizing a functional over a convex set of constraints. The Euler equation must be replaced by a set of inequalities. Here, we briefly mention the classical obstacle problem. Consider the following functional, $I(u)$, defined

$$I(u) = \int_\Omega L(x,u,\nabla u)dx.$$

The Lagrangian $L(x,u,z)$ is assumed to be jointly convex in (u,z), proper, and lower semi-continuous. The obstacle problem is formulated as a constrained minimization:

$$u^* = \operatorname*{argmin}_{u \in K} I(u);$$

where the convex constraint set K is given by

$$K = \{u \in H, u \geq \varphi \text{ in } \Omega, \quad u = g \text{ on the boundary}\}.$$

Let DI be the derivative associated with the Gâteaux differentiable functional I, i.e.

$$\frac{d}{d\varepsilon} I(u + \varepsilon v)|_{\varepsilon=0} = \langle DI(u), v \rangle\,.$$

Then the minimization problem is equivalent to finding $u^* \in K$ such that:

$$\langle DI(u^*), u^* - v \rangle > 0, \quad \forall v \in K.$$

3. Existence results

In this section, we present our results.

Lemma 3.1. *Let A be a subset of X and $x^* \in X^*$ be nonzero and $y \in X$ be given. If $\langle x^*, x - y\rangle \geq 0$ for all $x \in A$, then $\langle x^*, x - y\rangle > 0$ for all $x \in \mathrm{cor}A$.*

Proof. Suppose on the contrary, there exists $x_0 \in \mathrm{cor}A$ with $\langle x^*, x - y\rangle \leq 0$ this gives $\langle x^*, x - y\rangle = 0$. Consider $z \in X$ so there is a positive net $t_\alpha \subset \mathbb{R}$ such that $x_0 + t_\alpha z \longrightarrow x_0$. For $x_0 \in \mathrm{cor}A$ there exists β such that

$$x + t_\beta z \in \mathrm{cor}A \subseteq A,$$

which by assumption implies that $\langle x^*, x_0 + t_\beta z - y\rangle \geq 0$. From the last relation, we obtain

$$\langle x^*, x_0 + t_\beta z - y\rangle = \langle x^*, x_0 - y\rangle + t_\beta\langle x^*, z\rangle = t_\beta\langle x^*, z\rangle \geq 0,$$

hence $\langle x^*, z\rangle \geq 0$. Next, since $z \in X$ is arbitrary, we conclude that $x^* = 0$ which is contradiction. □

Notice that if A is convex set, then the reverse of Lemma 3.1 holds.
We need the following lemma in the sequel.

Lemma 3.2. *Let A be a convex subset of X and $\mathrm{cor}A \neq \emptyset$. Then*

$$\mathrm{cl_w}A = \mathrm{cl_w}(\mathrm{cor}A).$$

Proof. Clearly, we have $\mathrm{cl_w}(\mathrm{cor}A) \subseteq \mathrm{cl_w}A$. To see the reverse inclusion, let $x \in \mathrm{cl_w}A, a \in \mathrm{cor}A$, then there exists $x_\alpha \in A$ such that $x_\alpha \rightharpoonup x$. Thanks to Lemma 1.9 in [6], one has $[a, x_\alpha) \subset \mathrm{cor}A$. Next choose $0 < t_\alpha < 1$ such that $t_\alpha \longrightarrow 0$. Hence $t_\alpha a + (1 - t_\alpha)x_\alpha \in cor A$. On the other hand

$$t_\alpha a + (1 - t_\alpha)x_\alpha \rightharpoonup x,$$

which implies that $x \in \mathrm{cl_w}(\mathrm{cor}A)$. □

In the next Lemma, we provide conditions on the map T that relate the LM solutions and S^w solutions.

Lemma 3.3. *Let K be nonempty convex subset of X and $T : X \rightrightarrows X^*$ be a set-valued map. If $\mathrm{conv}T$ is locally upper sign-continuous, then*

$$LM(T, K) \subseteq S^w(T, K).$$

Proof. First assume that $x \in LM(T, K)$, then there exists a convex neighborhood U_x of x such that $x \in M(T, K \cap U_x)$. On the other hand, by locally upper sign-continuity of $\mathrm{conv}T$ there exists a convex neighborhood V_x of x, and an upper sign-continuous submap $\Phi_x : V_x \rightrightarrows X^*$ with non-empty convex w^*-compact values satisfying

$$\Phi_x(v) \subseteq \mathrm{conv}T(v) \setminus \{0\}, \quad \text{for any } \; v \in V_x.$$

Hence, $x \in M(T, K \cap U_x \cap V_x)$. Now, let y be an element of K, since $K \cap U_x \cap V_x$ is convex then there exists $y_1 \in [x, y] \cap U_x \cap V_x$ such that

$$[x, y_1] \subseteq K \cap U_x \cap V_x.$$

Thus one has

$$\langle z^*, z - x\rangle \geq 0, \quad \text{for all } \; z \in [x, y_1] \; \text{ and } \; z^* \in \Phi_x(z).$$

Hence

$$\inf_{z\in[x,y_1]}\inf_{z^*\in\Phi_x(z)}\langle z^*,z-x\rangle\geq 0.$$

Upper sign-continuity of Φ_x implies that

$$\sup_{x^*\in\Phi_x(x)}\langle x^*,z-x\rangle\geq 0,\qquad \forall z\in[x,y_1].$$

Now, for $z=y_1$ one can obtain

$$\sup_{x^*\in\Phi_x(x)}\langle x^*,y_1-x\rangle\geq 0.$$

Since $\Phi_x(x)$ is compact, there exists $x_y^*\in\Phi_x(x)\subseteq \text{conv}T(x)\setminus\{0\}$ such that

$$\left\langle x_y^*,y_1-x\right\rangle\geq 0.$$

On the other hand, there exists $0<t<1$ such that $y_1=tx+(1-t)y$ and therefore $\left\langle x_y^*,y-x\right\rangle\geq 0$. Now since $x_y^*\in\text{conv}T(x)$, then there exists $0\leq t_i\leq 1$ such that

$$x_y^*=\sum_{i=1}^n t_i x_{iy}^*,\quad \sum_{i=1}^n t_i=1,\quad x_{iy}^*\in T(x).$$

This implies

$$\left\langle \sum_{i=1}^n t_i x_{iy}^*,y-x\right\rangle=\sum_{i=1}^n t_i\left\langle x_{iy}^*,y-x\right\rangle\geq 0.$$

Therefore there exists $0\leq j\leq 1$ such that $\left\langle x_{jy}^*,y-x\right\rangle\geq 0$ and so one has $x\in S^w(T,K)$. □

Proposition 3.4. *[3] Let K be a nonempty, convex subset of the topological vector space X and let $T:X\rightrightarrows X^*$ be quasimonotone and is not properly quasimonotone. Then one has $LM(T,K)\neq\emptyset$.*

We need the following Lemma in the sequel.

Lemma 3.5. *Let K be a weakly compact subset of X. If $T:X\rightrightarrows X^*$ is quasimonotone, then $LM(T,K)\neq\emptyset$.*

Proof. The proof is straightforward by Theorem 5.1 in [9] and Proposition 3.4. □

The following Theorem is an extension of Theorem 3.1 in [4] without coercivity, locally bounded and hemiclosed conditions on T and reflexivity of Banach space X.

Theorem 3.6. *Let K be a nonempty convex subset of X. Assume that $T:X\rightrightarrows X^*$ be a quasimonotone operator that is not properly quasimonotone. If* conv$T$ *is locally upper sign-continuous, then the variational inequality* (VI) *has a solution. If moreover, $K=X$ and for all $x\in K$, $T(x)$ is weakly compact, then the generalized equation $0\in T(x)$ admits a solution.*

Proof. By Lemma 3.3 and Proposition 3.4 and Remark 2.1 it is easy to cheek the existence of solution (VI). Now, let $\bar{x}$ be a solution of variational inequality (VI). Since $T(\bar{x})$ is weakly compact for $y \in K$ there exists $x_y^* \in T(\bar{x})$ such that

$$0 \leq \sup_{x^* \in T(\overline{x})} \langle x^*, y - \overline{x} \rangle = \left\langle x_y^*, y - \overline{x} \right\rangle .$$

This means that $\{0\}$ cannot be strongly separated from the closed convex set $T(x)$ and therefore, $0 \in T(x)$. □

In the case that operator T is properly quasimonotone, we can not use the Proposition 3.6. In order to overcome this flaw, under the weaker condition of Theorem 2.1 of [3] one can get the following result without any coercivity condition.

Theorem 3.7. *Let K, U be nonempty convex subsets of X and $K \cap \mathrm{cor}U$ be nonempty and weakly compact. Further, let $T : X \rightrightarrows X^*$ be a quasimonotone operator on K. If T is locally upper sign-continuous, then the variational inequality* (VI) *has a solution.*

Proof. Suppose that T be properly quasimonotone, hence by Lemma 3.5 one has $LM(T, K \cap \mathrm{cor}U) \neq \emptyset$. Choose $x_0 \in LM(T, K \cap \mathrm{cor}U)$, then

$$\exists\, x_0^* \in T(x_0), \quad \forall y \in K \cap corU; \quad \langle x_0^*, y - x_0 \rangle \geq 0.$$

Now for every $z \in K$ there exists $t > 0$ such that

$$x_0 + t(z - x_0) \in K \cap \mathrm{cor}U,$$

which implies that $\langle x_0^*, z - x_0 \rangle \geq 0$. Therefore $x_0 \in S(T, K)$ which completes the proof. □

Lemma 3.8. *Let K be a convex subset of X with $\mathrm{cor}K \neq \emptyset$ and the set valued map $T : X \rightrightarrows X^*$ be quasimonotone and weakly dually lower semicontinuous on K. If $S(T, K) \nsubseteq M(T, K)$, then the generalized equation $0 \in T(x)$ admits a solution.*

Proof. Suppose that for all $x \in X$ we have $0 \notin T(x)$, and $x \in S(T, K)$ be given. Hence, there exists $x^* \in T(x)$ such that

$$\langle x^*, y - x \rangle \geq 0, \quad \text{for all}\, y \in K, \quad \text{and} \quad x^* \neq 0.$$

By Lemma 3.1 it follows that

$$\langle x^*, z - x \rangle > 0, \qquad \forall z \in \mathrm{cor}K.$$

For any $y \in K$ by Lemma 3.2 there exists net $y_\alpha \in \mathrm{cor}K$ such that $y_\alpha \rightharpoonup y$. Consequently, for any α, $\langle x^*, y_\alpha - x \rangle > 0$ and thus by quasimonotonicity,

$$\langle y_\alpha^*, y_\alpha - x \rangle \geq 0, \quad \text{for all } y_\alpha^* \in T(y_\alpha).$$

Finally, by weakly dually lower semicontinuity at T one has $\langle y^*, y - x \rangle \geq 0$ for each $y^* \in T(y)$. The later indicates that $x \in M(T, K)$, therefore

$$S(T, K) \nsubseteq M(T, K).$$

□

Remark 3.1. It is worth to note that the condition (D) or (4.1) in [7] on the set $K \subseteq X$ is equivalent to $M(T, K) \neq \emptyset$. Also if $\mathrm{int}K \neq \emptyset$ in condition D then $LM(T, K) \neq \emptyset$.

The proof of the following Propositions (3.9) and (3.10) are straightforward.

Proposition 3.9. *Assume that $T : X \rightrightarrows X^*$ is an upper sign-continuous set-valued on K whose values are convex and compact sets. If $M(T,K) \neq \emptyset$, then the variational inequality (VI) has a solution.*

Proposition 3.10. *Assume that $T : X \rightrightarrows X^*$ is a locally upper sign-continuous set-valued on K whose values are convex and w^*-compact sets. If $M(T,K) \neq \emptyset$, then the generalization $0 \in T(x)$ has a solution.*

Lemma 3.11. *Let K be an algebraically open set in X. Then one has $S(T,K) \subseteq Z_T$. Here Z_T is the set all zeros of T,* i.e $Z_T = \{x \in X : \ 0 \in T(x)\}$.

Proof. Suppose that $\bar{x} \in S(T,K)$, then there exists $x^* \in T(\bar{x})$ such that for all $y \in K$ one has $\langle x^*, y - \bar{x}\rangle \geq 0$. Since $\bar{x} \in \text{cor}K$ hence for given $x \in X$ one have $\bar{x} + tx \in K$ for some $t > 0$. So we have $\langle x^*, (\bar{x} + tx) - \bar{x}\rangle \geq 0$ which implies that $\langle x^*, x\rangle \geq 0$ and this means that $x^* = 0$ thus $0 \in T(\bar{x})$. □

Proposition 3.12. *Let T be a quasimonotone operator, which* conv$T$ *is lower semi-continuous at $x \in \text{cor}K$. Then*

$$(\forall x^* \in X^* \setminus \{0\},\ T(x) \not\subseteq \mathbb{R}_{++}x^*) \Longleftrightarrow 0 \in T(x).$$

Proof. Suppose in the contrary, $0 \notin T(x)$ then there exists $x^* \in T(x)$ and $y^* \in T(x) \setminus \{0\}$ such that for every $\lambda > 0$, one has $x^* \neq \lambda y^*$. Hence there exists $w \in X$ such that

$$\langle x^*, w\rangle > 0 > \langle y^*, w\rangle.$$

Obviously $x^* \in \text{conv}T(x)$ and $x + \frac{1}{n}w \longrightarrow x$. By using the lower semi-continuity of conv$T$, there exists $y_n^* \in \text{conv}T(x + \frac{1}{n}w)$ such that $y_n^* \longrightarrow y^*$. On the other hand, one can have

$$\langle x^*, (x + \frac{1}{n}w) - x\rangle > 0.$$

Since T is quasimonotone,then for $y_n^* \in \text{conv}T(x + \frac{1}{n}w)$ it holds

$$\langle y_n^*, (x + \frac{1}{n}w) - x\rangle \geq 0,$$

and thus $\langle y^*, w\rangle \geq 0$, which is contradiction. □

By similar argument in previous proposition one can prove the following.

Proposition 3.13. *Let T be a quasimonotone operator, which is lower sin continuous on K, and $x \in \text{cor}K$. Then*

$$(\forall x^* \in X^* \setminus \{0\},\ T(x) \not\subseteq \mathbb{R}_{++}x^*) \Longleftrightarrow 0 \in T(x).$$

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Department of Mathematics, Instituto Superior Técnico, Lisbon.
E-mail address: bozorg@math.ist.utl.pt