

Q -graded Hopf quasigroups

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Abstract. Firstly, we introduce a class of new algebraic systems which generalize Hopf quasigroups and Hopf π -algebras called Q -graded Hopf quasigroups, and research some properties of them. Secondly, we define the representations of Q -graded Hopf quasigroups, i.e Q -graded Hopf quasimodules, research the construction method and fundamental theorem of them. Thirdly, we research the smash products of Q -graded Hopf quasigroups.

Keywords: Q -graded Hopf quasigroups; Q -graded Hopf quasimodules; Smash products.

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Introduction

Hopf quasigroups[5] were introduced by Klim and Majid, there are generalization of Hopf algebras[7] that are not required to be associative. In publications we can find some results on Hopf quasigroup : Breziński research the Hopf modules and the fundamental theorem[1] on Hopf quasigroups, Breziński with Jiao show the smash (co)products[2] and R -smash products[3] on Hopf quasigroups, Jiao and Wang give the smash biproducts[4] on quasigroups, Klim and Majid introduce the bicrossproduct[6] on Hopf quasigroups.

On the other hand, Hopf π -coalgebras[9] were introduced by Virelizier, and then continued by Zunino and Wang. It turns out that many of the classical results in Hopf algebras can be generalized to the Hopf π -coalgebras, Virelizier gives a generalized version of the Fundamental theorem for Hopf algebras and introduced π -integrals, Zunino introduced Yetter-Drinfeld modules[16], the Drinfeld double, and a generalization of the center construction of a monoidal category[15], Wang introduces Doi-Hopf modules[11], entwined modules[14], Drinfeld'codouble[12] and coalgebra Galois theory[13] for Hopf π -coalgebras,

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and he proves a version of Maschke's theorem [10]. Hopf π -algebras are dual of Hopf π -coalgebras [8].

In this paper, we introduce Q -graded Hopf quasigroups, they generalize the Hopf quasigroups like Hopf π -algebras generalize Hopf algebras and they generalize Hopf π -algebras like Hopf quasigroups generalize Hopf algebras. The paper is organized as follows: In section 1, we recall the definitions of quasigroups, Hopf π -algebras and Hopf quasigroups. In section 2, we give the definition of Q -graded Hopf quasigroups and generalize many classical theorems of Hopf quasigroups to Q -graded Hopf quasigroups. In section 3, we define Q -graded Hopf quasimodules and show a construct method and fundamental theorem of Hopf module on it. In section 4, we show the smash products on Q -graded Hopf quasigroups.

Throughout this article, π is a group, Q is a quasigroup and all the vector spaces, tensor products and maps are over a fixed field k . For a coalgebra C , we will use the Heyneman-Sweedler's notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$ (summation omitted).

1 Preliminaries

Definition 1.1. A quasigroup is a set Q with a product, identity e and with the property that for each $p \in Q$ there is $p^{-1} \in Q$ satisfying

$$p^{-1}(pq) = q, \quad (qp)p^{-1} = q, \quad \forall q \in Q.$$

It is easy to see that in any quasigroup Q , one has unique inverses and

$$(p^{-1})^{-1} = p, \quad (pq)^{-1} = q^{-1}p^{-1}, \quad \forall p, q \in Q.$$

A quasigroup is *flexile* if $p(qp) = (pq)p$ for all $p, q \in Q$ and *alternative* if also $p(pq) = (pp)q$, $p(qq) = (pq)q$ for all $p, q \in Q$. It is called *Moufang* if $p(q(pr)) = ((pq)p)r$ for all $p, q, r \in Q$ or *commutative* if $pq = qp$ for all $p, q \in Q$ or *associative* if $p(qr) = (pq)r$, for all $p, q, r \in Q$. Obviously, associative quasigroup is group.

Remark 1.2. Let Q be a quasigroup, then the following identities are equivalent, $\forall p, q, r \in Q$, (1) $p(q(pr)) = ((pq)p)r$, (2) $((pq)r)q = p(q(rq))$, (3) $(pq)(rp) = (p(qr))p$.

Definition 1.3. A Hopf quasigroup is a possibly-noassociative but unital algebra H equipped with algebra homomorphisms $\Delta : H \rightarrow H \otimes H$, $\epsilon : H \rightarrow k$ forming a coassociative coalgebra and a map $S : H \rightarrow H$ such that

$$S(h_1)(h_2g) = \epsilon(h)g = h_1(S(h_2)g), \quad (gh_1)S(h_2) = g\epsilon(h) = (g(S(h_1))h_2), \quad \forall h, g \in H.$$

Definition 1.4. A π -graded algebra (π -algebra) is a family $A = \{A_p\}_{p \in \pi}$ of k -spaces endowed with a family $m = \{m_{p,q} : A_p \otimes A_q \rightarrow A_{pq}\}_{p,q \in \pi}$ of k -maps (the Q -graded

multiplication) and a k -linear map $\mu : k \rightarrow A_e$ (the unit) such that (we denote $m_{p,q}(a \otimes b) \equiv ab$, $\mu(1_k) \equiv 1$)

$$a(bc) = (ab)c, \quad a1 = a = 1a, \quad \forall p, q, r \in \pi, \quad a \in A_p, \quad b \in A_q, \quad c \in A_r.$$

Definition 1.5. Let A be a π -graded algebra, a π -graded left A -module is a family $M = \{M_p\}_{p \in \pi}$ of k -spaces endowed with a family $\varphi = \{\varphi_{p,q} : A_p \otimes M_q \rightarrow M_{pq}\}_{p,q \in \pi}$ of k -maps such that (we denote $\varphi(a \otimes m) \equiv a \cdot m$)

$$(ab) \cdot m = a \cdot (b \cdot m), \quad 1 \cdot m = m, \quad \forall p, q, r \in \pi, \quad a \in A_p, \quad b \in A_q, \quad m \in M_r.$$

A map of π -graded left A -modules is a family $f = \{f_p : M_p \rightarrow M'_p\}_{p \in \pi}$ of k -maps such that

$$f_{pq}(a \cdot m) = a \cdot f_q(m), \quad \forall p, q \in \pi, \quad a \in A_p, \quad m \in M_q.$$

Definition 1.6. A π -graded Hopf algebra (Hopf π -algebra) is a π -graded algebra $H = \{H_p, m_{p,q}, \mu\}_{p,q \in \pi}$ endowed with a family $S = \{S_p : H_p \rightarrow H_{p^{-1}}\}_{p \in \pi}$ of k -linear maps (the antipode) such that each H_p is a coassociative counitary coalgebra with comultiplication Δ_p and counit map ϵ_p , and $\{m_{p,q}\}_{p,q \in \pi}$ and μ are coalgebra maps and (we denote $\Delta_p(h) \equiv h_{(1,p)} \otimes h_{(2,p)}$)

$$S_p(h_{(1,p)})h_{(2,p)} = \epsilon_p(h)1 = h_{(1,p)}S_p(h_{(2,p)}), \quad \forall p \in \pi, \quad h \in H_p.$$

Definition 1.7. Given a k -space M , a k -map $R : M \otimes M \rightarrow M \otimes M$ is said to be a solution of the Long-equation if

$$R^{12}R^{23} = R^{23}R^{12},$$

where $R^{12} = R \otimes I$, $R^{23} = I \otimes R : M \otimes M \otimes M \rightarrow M \otimes M \otimes M$.

2 Q -graded Hopf quasigroups

Definition 2.1. A Q -graded Hopf quasigroup H is

(1) A Q -graded algebra, i.e a family $H = \{H_p\}_{p \in Q}$ of k -spaces endowed with a family $m = \{m_{p,q} : H_p \otimes H_q \rightarrow H_{pq}\}_{p,q \in Q}$ of k -maps (the Q -graded multiplication) and a k -linear map $\mu : k \rightarrow H_e$ (the unit) such that $h1 = h = 1h$, $\forall p \in Q$, $h \in H_p$.

(2) Each H_p is a coassociative counitary coalgebra with comultiplication Δ_p and counit ϵ_p ,

(3) $\{m_{p,q}\}$ are coalgebra maps, i.e

$$(hg)_{(1,pq)} \otimes (hg)_{(2,pq)} = h_{(1,p)}g_{(1,q)} \otimes h_{(2,p)}g_{(2,p)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \quad (2.1)$$

$$\epsilon_{pq}(hg) = \epsilon_p(h)\epsilon_q(g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.2)$$

(4) μ is coalgebra map, i.e

$$\Delta_e(1) = 1 \otimes 1, \quad \epsilon_e(1) = 1_k.$$

(5) A family $S = \{S_p : H_p \rightarrow H_{p^{-1}}\}_{p \in Q}$ of k -linear maps (the antipode) such that

$$S_p(h_{(1,p)})(h_{(2,p)}g) = \epsilon_p(h)g = h_{(1,p)}(S_p(h_{(2,p)}g)), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.3)$$

$$(gS_p(h_{(1,p)}))h_{(2,p)} = g\epsilon_p(h) = (gh_{(1,p)})S_p(h_{(2,p)}), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.4)$$

H is a Q -graded Hopf quasigroup, then

(1) H is flexible if Q is flexible and

$$h_{(1,p)}(gh_{(2,p)}) = (h_{(1,p)}g)h_{(2,p)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.5)$$

(2) H is alternative if H is flexible, Q is alternative and

$$h_{(1,p)}(h_{(2,p)}g) = (h_{(1,p)}h_{(2,p)})g, \quad h(g_{(1,p)}g_{(2,p)}) = (hg_{(1,p)})g_{(2,p)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.6)$$

(3) H is Moufang if Q is Moufang and

$$h_{(1,p)}(g(h_{(2,p)}f)) = ((h_{(1,p)}g)h_{(2,p)})f, \quad \forall p, q, r \in Q, \quad h \in H_p, \quad g \in H_q, \quad f \in H_r. \quad (2.7)$$

(4) H is commutative if Q is commutative and

$$hg = gh, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.8)$$

(5) H is cocommutative if

$$\Delta_p(h) = h_{(1,p)} \otimes h_{(2,p)} = h_{(2,p)} \otimes h_{(1,p)}, \quad \forall p \in Q, \quad h \in H_p. \quad (2.9)$$

Now we show an example of Q -graded Hopf quasigroups.

Example 2.2. For any quasigroup Q , we can construct a Q -graded Hopf quasigroup as follows: a family $kQ = \{H_p = (kp)\}_{p \in Q}$ of k -spaces, the multiplication $m_{p,q}(k_1p \otimes k_2q) = k_1k_2pq$, the unitary element $e \in Q$, comultiplication $\Delta_p(p) = p \otimes p$, counit $\epsilon_p(p) = 1_k$, antipode $S_p(p) = p^{-1}$.

Proposition 2.3. H is a Q -graded Hopf quasigroup, then

(1)

$$S_p(h_{(1,p)})h_{(2,p)} = \epsilon_p(h)1 = h_{(1,p)}S_p(h_{(2,p)}), \quad \forall p \in Q, \quad h \in H_p. \quad (2.10)$$

(2)

$$S_{pq}(hg) = S_q(g)S_p(h), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \quad (2.11)$$

(3)

$$S_p(h)_{(1,p^{-1})} \otimes S_p(h)_{(2,p^{-1})} = \Delta_{p^{-1}}(S_p(h)) = S_p(h_{(2,p)}) \otimes S_p(h_{(1,p)}), \quad \forall p \in Q, \quad h \in H_p. \quad (2.12)$$

(4)

$$S_\epsilon(1) = 1. \quad (2.13)$$

(5)

$$\epsilon_{p^{-1}} S_p(h) = \epsilon_p(h), \quad \forall p \in Q, \quad h \in H_p. \quad (2.14)$$

Proof. (1) and (4) are obtained by the definition of Q -graded Hopf quasigroup.

(2)

$$\begin{aligned} S_q(g)S_p(h) &= S_q(g_{(1,q)}\epsilon_q(g_{(2,q)}))S_p(h_{(1,p)}\epsilon_p(h_{(2,p)})) \\ &\stackrel{(2.2)}{=} S_q(g_{(1,q)})S_p(h_{(1,p)})\epsilon_{pq}(h_{(2,p)}g_{(2,q)}) \\ &= S_q(g_{(1,q)})((S_p(h_{(1,p)})(h_{(2,p)}g_{(2,q)})_{(1,pq)})S_{pq}((h_{(2,p)}g_{(2,p)})_{(2,pq)}) \\ &= S_q(g_{(1,q)})((S_p(h_{(1,p)})(h_{(2,p)}g_{(2,q)}))S_{pq}(h_{(3,p)}g_{(3,q)}) \\ &= S_q(g_{(1,q)}(1,q))((S_p(h_{(1,p)}(1,p))(h_{(1,p)}(2,p)g_{(1,q)}(2,q)))S_{pq}(h_{(2,p)}g_{(2,q)}) \\ &\stackrel{(2.3)}{=} S_q(g_{(1,q)}(1,q))(\epsilon_p(h_{(1,p)})g_{(1,q)}(2,q)S_{pq}(h_{(2,p)}g_{(2,q)})) \\ &= S_q(g_{(1,q)}(1,q))(g_{(1,q)}(2,q)S_{pq}(\epsilon_p(h_{(1,p)})h_{(2,p)}g_{(2,q)})) \\ &= S_q(g_{(1,q)}(1,q))(g_{(1,q)}(2,q)S_{pq}(hg_{(2,q)})) \\ &\stackrel{(2.3)}{=} \epsilon_q(g_{(1,q)})S_{pq}(hg_{(2,q)}) \\ &= S_{pq}(h\epsilon_q(g_{(1,q)})g_{(2,q)}) = S_{pq}(hg). \end{aligned}$$

(3)

$$\begin{aligned} \Delta_{p^{-1}}(S_p(h)) &= S_p(h_{(2,p)})_{(1,p^{-1})} \otimes \epsilon_p(h_{(1,p)})S_p(h_{(2,p)})_{(2,p^{-1})} \\ &\stackrel{(2.3)}{=} S_p(h_{(2,p)})_{(1,p^{-1})} \otimes S_p(h_{(1,p)}(1,p))(h_{(1,p)}(2,p)S_p(h_{(2,p)})_{(2,p^{-1})}) \\ &= \epsilon_p(h_{(1,p)}(1,p)(2,p))S(h_{(2,p)})_{(1,p^{-1})} \\ &\otimes S_p(h_{(1,p)}(1,p)(1,p))(h_{(1,p)}(2,p)S_p(h_{(2,p)})_{(2,p^{-1})}) \\ &\stackrel{(2.3)}{=} S_p(h_{(1,p)}(1,p)(2,p)(1,p))(h_{(1,p)}(1,p)(2,p)(2,p)S_p(h_{(2,p)})_{(1,p^{-1})}) \\ &\otimes S_p(h_{(1,p)}(1,p)(1,p))(h_{(1,p)}(2,p)S_p(h_{(2,p)})_{(2,p^{-1})}) \\ &= S_p(h_{(2,p)})(h_{(3,p)}S_p(h_{(5,p)})_{(1,p^{-1})}) \otimes S_p(h_{(1,p)})(h_{(4,p)}S_p(h_{(5,p)})_{(2,p^{-1})}) \\ &= S_p(h_{(1,p)}(2,p))(h_{(2,p)}(1,p)(1,p)S_p(h_{(2,p)}(2,p))_{(1,p^{-1})}) \\ &\otimes S_p(h_{(1,p)}(1,p))(h_{(2,p)}(1,p)(2,p)S_p(h_{(2,p)}(2,p))_{(2,p^{-1})}) \\ &\stackrel{(2.1)}{=} S_p(h_{(1,p)}(2,p))(h_{(2,p)}(1,p)S_p(h_{(2,p)}(2,p)))_{(1,e)} \otimes S_p(h_{(1,p)}(1,p))(h_{(2,p)}(1,p)S_p(h_{(2,p)}(2,p)))_{(2,e)} \\ &\stackrel{(2.10)}{=} S_p(h_{(1,p)}(2,p))\epsilon_p(h_{(2,p)}) \otimes S_p(h_{(1,p)}(1,p)) \\ &= S_p(h_{(2,p)}) \otimes S_p(h_{(1,p)}). \end{aligned}$$

(5)

$$\begin{aligned}
& (\epsilon_{p^{-1}}S_p)(h) \\
&= \epsilon_{p^{-1}}(S_p(h_{(1,p)}))\epsilon_p(h_{(2,p)}) \\
&\stackrel{(2.2)}{=} \epsilon_e(S_p(h_{(1,p)})h_{(2,p)}) \\
&= \epsilon_e(1\epsilon_p(h)) \\
&= \epsilon_p(h).
\end{aligned}$$

□

Proposition 2.4. *Let H be a Q -graded Hopf quasigroup, if H is commutative or cocommutative, then $S_{p^{-1}}S_p(h) = h$, $\forall p \in Q$, $h \in H_p$.*

Proof. If H is commutative,

$$\begin{aligned}
S_{p^{-1}}S_p(h) &= S_{p^{-1}}(S_p(h_{(1,p)}))\epsilon_p(h_{(2,p)}) \\
&\stackrel{(2.10)}{=} S_{p^{-1}}(S_p(h_{(1,p)}))(S_p(h_{(2,p)})h_{(3,p)}) \\
&= S_{p^{-1}}(S_p(h_{(1,p)(1,p)}))(S_p(h_{(1,p)(2,p)})h_{(2,p)}) \\
&\stackrel{(2.12)}{=} S_{p^{-1}}(S_p(h_{(1,p)})(_{2,p^{-1}}))(S_p(h_{(1,p)})(_{1,p^{-1}})h_{(2,p)}) \\
&\stackrel{(2.8)}{=} (h_{(2,p)}S_p(h_{(1,p)})(_{1,p^{-1}}))S_{p^{-1}}(S_p(h_{(1,p)})(_{2,p^{-1}})) \\
&\stackrel{(2.4)}{=} h_{(2,p)}\epsilon_{p^{-1}}(S_p(h_{(1,p)})) \\
&\stackrel{(2.14)}{=} h_{(2,p)}\epsilon_p(h_{(1,p)}) = h,
\end{aligned}$$

If H is cocommutative,

$$\begin{aligned}
S_{p^{-1}}S_p(h) &= S_{p^{-1}}(S_p(h_{(1,p)}))\epsilon_p(h_{(2,p)}) \\
&\stackrel{(2.10)}{=} S_{p^{-1}}(S_p(h_{(1,p)}))(S_p(h_{(2,p)})h_{(3,p)}) \\
&= S_{p^{-1}}(S_p(h_{(1,p)(1,p)}))(S_p(h_{(1,p)(2,p)})h_{(2,p)}) \\
&\stackrel{(2.12)}{=} S_{p^{-1}}(S_p(h_{(1,p)})(_{2,p^{-1}}))(S_p(h_{(1,p)})(_{1,p^{-1}})h_{(2,p)}) \\
&\stackrel{(2.9)}{=} S_{p^{-1}}(S_p(h_{(1,p)})(_{1,p^{-1}}))(S_p(h_{(1,p)})(_{2,p^{-1}})h_{(2,p)}) \\
&\stackrel{(2.3)}{=} \epsilon_{p^{-1}}(S_p(h_{(1,p)}))h_{(2,p)} \\
&\stackrel{(2.14)}{=} \epsilon_p(h_{(1,p)})h_{(2,p)} = h.
\end{aligned}$$

□

Proposition 2.5. *Let H be a Moufang Q -graded Hopf quasigroup such that S_p is invertible for any $p \in Q$, then $\forall p, q, r \in Q$, $h \in H_p$, $g \in H_q$, $f \in H_r$, the following three conditions are equivalent :*

- (1) $h_{(1,p)}(g(h_{(2,p)}f)) = ((h_{(1,p)}g)h_{(2,p)})f$,
(2) $((hg_{(1,q)})f)g_{(2,q)} = h(g_{(1,p)}(fg_{(2,q)}))$,
(3) $(h_{(1,p)}g)(fh_{(2,p)}) = (h_{(1,p)}(gf))h_{(2,p)}$.

Proof. (1) \Rightarrow (2)

$$\begin{aligned}
& ((S_r(f)S_p(h)_{(1,p^{-1})})S_q(g))S_p(h)_{(2,p^{-1})} \\
& \stackrel{(2.12)}{=} ((S_r(f)S_p(h_{(2,p)}))S_q(g))S_p(h_{(1,p)}) \\
& \stackrel{(2.11)}{=} S_{p(qpr)}(h_{(1,p)}(g(h_{(2,p)}f))) \\
& = S_{((pq)p)r}((h_{(1,p)}g)h_{(2,p)})f \\
& \stackrel{(2.11)}{=} S_r(f)(S_q(h_{(2,p)})(S_q(g)S_p(h_{(1,p)}))) \\
& \stackrel{(2.12)}{=} S_r(f)(S_q(h)_{(1,p^{-1})}(S_q(g)S_p(h)_{(2,p^{-1})}))
\end{aligned}$$

using S_p are invertible and H is *Moufang*, the above equation is equivalent to

$$((hg_{(1,q)})f)g_{(2,q)} = h(g_{(1,p)}(fg_{(2,q)})), \quad \forall p, q, r \in Q, \quad h \in H_p, \quad g \in H_q, \quad f \in H_r.$$

(2) \Rightarrow (1) The proof is similarly to (1) \Rightarrow (2).

(1) \Rightarrow (3)

$$\begin{aligned}
h(gf) &= h_{(1,p)}\epsilon_p(h_{(2,p)})(gf) \\
&= h_{(1,p)}(g\epsilon_p(h_{(2,p)}))f \\
& \stackrel{(2.3)}{=} h_{(1,p)}(g(h_{(2,p)}(1,p))(S_p(h_{(2,p)}(2,p))f)) \\
&= h_{(1,p)(1,p)}(g(h_{(1,p)}(2,p))(S_p(h_{(2,p)}))f)) \\
&= ((h_{(1,p)}(1,p)g)h_{(1,p)}(2,p))(S_p(h_{(2,p)}))f,
\end{aligned}$$

replacing g with $gf_{(1,r)}$ and f with $S_r(f_{(2,r)})$, we have:

$$hg\epsilon_r(f) \stackrel{(2.4)}{=} h((gf_{(1,r)})S_r(f_{(2,r)})) \stackrel{(2.11)}{=} ((h_{(1,p)}(1,p)(gf_{(1,r)}))h_{(1,p)}(2,p))S_{rp}(f_{(2,r)}h_{(2,p)}),$$

replacing h with $h_{(1,p)}$ and f with $f_{(1,r)}$, multiply on the right by $f_{(2,r)}h_{(2,p)}$, we have:

$$\begin{aligned}
& \epsilon_r(f_{(1,r)})(h_{(1,p)}g)(f_{(2,r)}h_{(2,p)}) \\
&= (((h_{(1,p)}(1,p)(1,p)(gf_{(1,r)}(1,r)))h_{(1,p)}(1,p)(2,p))S_{rp}(f_{(1,r)}(2,r)h_{(1,p)}(2,p)))(f_{(2,r)}h_{(2,p)}),
\end{aligned}$$

the left side of the above equation equals $(h_{(1,p)}g)(fh_{(2,p)})$, now consider the right side of the equation: by the coassociativity and Δ_p is algebra map, the right side equals:

$$\begin{aligned}
&= (((h_{(1,p)}(gf_{(1,r)}))h_{(2,p)})S_{rp}(f_{(2,r)}h_{(3,p)}))(f_{(3,r)}h_{(4,p)}) \\
& \stackrel{(2.1)}{=} (((h_{(1,p)}(gf_{(1,r)}))h_{(2,p)})S_{rp}((f_{(2,r)}h_{(3,p)})(1,rp))((f_{(2,r)}h_{(3,p)})(2,rp))) \\
& \stackrel{(2.4)}{=} ((h_{(1,p)}(gf_{(1,r)}))h_{(2,p)})\epsilon_{rp}(f_{(2,r)}h_{(3,p)}) \\
& \stackrel{(2.2)}{=} ((h_{(1,p)}(gf_{(1,r)}))h_{(2,p)})\epsilon_p(h_{(3,p)})\epsilon_r(f_{(2,r)}) \\
&= (h_{(1,p)}(gf))h_{(2,p)},
\end{aligned}$$

so,

$$(h_{(1,p)}g)(fh_{(2,p)}) = (h_{(1,p)}(gf))h_{(2,p)}, \quad \forall p, q, r \in Q, \quad h \in H_p, \quad g \in H_q, \quad f \in H_r.$$

(3) \Rightarrow (1) Assume (3) holds, then

$$\begin{aligned} & ((h_{(1,p)(1,p)}g)(f_{(1,r)}h_{(1,p)(2,p)}))S_{rp}(f_{(2,r)}h_{(2,p)}) \\ &= ((h_{(1,p)(1,p)}(gf_{(1,r)}))h_{(1,p)(2,p)})S_{rp}(f_{(2,r)}h_{(2,p)}), \end{aligned}$$

the left side of above equation equals:

$$\begin{aligned} &= ((h_{(1,p)}g)(f_{(1,r)}h_{(2,p)}))S_{rp}(f_{(2,r)}h_{(2,p)}) \\ &\stackrel{(2.1)}{=} ((h_{(1,p)}g)(fh_{(2,p)})_{(1,rp)})S_{rp}((fh_{(2,p)})_{(2,rp)}) \\ &\stackrel{(2.4)}{=} (h_{(1,p)}g)\epsilon_{rp}(fh_{(2,p)}) \\ &\stackrel{(2.2)}{=} (h_{(1,p)}g)\epsilon_r(f)\epsilon_p(h_{(2,p)}) \\ &= (hg)\epsilon_r(f), \end{aligned}$$

so, we have:

$$hg\epsilon_r(f) = ((h_{(1,p)(1,p)}(gf_{(1,r)}))h_{(1,p)(2,p)})(S_p(h_{(2,p)})S_r(f_{(2,r)})),$$

replacing g with $gS_r(f_{(1,r)})$ and f with $f_{(2,r)}$,

$$h(gS_r(f_{(1,r)}))\epsilon_r(f_{(2,r)}) = ((h_{(1,p)(1,p)}((gS_r(f_{(1,r)}))f_{(2,r)(1,r)}))h_{(1,p)(2,p)})(S_p(h_{(2,p)})S_r(f_{(2,r)(2,r)})),$$

so,

$$h(gS_r(f)) = ((h_{(1,p)(1,p)}g)h_{(1,p)(2,p)})(S_p(h_{(2,p)})S_r(f)),$$

replacing h with $h_{(1,p)}$ and f with $S_{pr}^{-1}(h_{(2,p)}f)$, we have:

$$h_{(1,p)}(g(h_{(2,p)}f)) = ((h_{(1,p)(1,p)(1,p)}g)h_{(1,p)(1,p)(2,p)})(S_p(h_{(1,p)(2,p)})(h_{(2,p)}f)) \stackrel{(2.3)}{=} ((h_{(1,p)}g)h_{(2,p)})f.$$

□

Lemma 2.6. *Let H be a cocommutative and flexible Q -graded Hopf quasigroup, then*

$$h_{(1,p)}(gS_p(h_{(2,p)})) = (h_{(1,p)}g)S_p(h_{(2,p)}), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

Proof.

$$\begin{aligned}
& (h_{(1,p)}(gS_p(h_{(1,p)(2,p)})))h_{(2,p)} \\
& \stackrel{(2.9)}{=} (h_{(1,p)}(gS_p(h_{(3,p)})))h_{(2,p)} \\
& \stackrel{(2.5)}{=} h_{(1,p)}((gS_p(h_{(3,p)})))h_{(2,p)} \\
& \stackrel{(2.9)}{=} h_{(1,p)}((gS_p(h_{(2,p)(1,p)})))h_{(2,p)(2,p)} \\
& \stackrel{(2.4)}{=} h_{(1,p)}(g\epsilon_p(h_{(2,p)})) \\
& = (h_{(1,p)}g)\epsilon_p(h_{(2,p)}) \\
& \stackrel{(2.4)}{=} ((h_{(1,p)}g)S_p(h_{(2,p)(1,p)}))h_{(2,p)(2,p)} \\
& = (h_{(1,p)(1,p)}g)S_p(h_{(1,p)(2,p)})h_{(2,p)},
\end{aligned}$$

So,

$$h_{(1,p)}(gS_p(h_{(2,p)})) = (h_{(1,p)}g)S_p(h_{(2,p)}), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

□

Therefore we have a notion of left adjoint action of a Q -graded Hopf quasigroup H when it is cocommutative and flexible.

Definition 2.7. Let Q be an associative quasigroup and H be a Q -graded Hopf quasigroup, an associator δ be defined by

$$(hg)f = \delta(h_{(1,p)}, g_{(1,p)}, f_{(1,r)})(h_{(2,p)}(g_{(2,p)}f_{(2,r)})), \quad \forall p, q, r \in Q, \quad h \in H_p, \quad g \in H_q, \quad f \in H_r.$$

Theorem 2.8. Let Q be an associative quasigroup and H be a Q -graded Hopf quasigroup, (1) the associator δ exists and is uniquely determined as

$$\delta(h, g, f) = ((h_{(1,p)}g_{(1,q)})f_{(1,r)})S_{pqr}(h_{(2,p)}(g_{(2,q)}f_{(2,r)})), \quad \forall p, q, r \in Q, \quad h \in H_p, \quad g \in H_q, \quad f \in H_r.$$

$$(2) \quad \delta(1, h, g) = \delta(h, 1, g) = \delta(h, g, 1) = \epsilon_p(h)\epsilon_q(g)1.$$

$$(3) \quad \delta(h_{(1,p)}, S_p(h_{(2,p)}), g) = \delta(S_p(h_{(1,p)}), h_{(2,p)}, g) = \epsilon_p(h)\epsilon_q(g)1,$$

$$\delta(h, g_{(1,q)}, S_q(g_{(2,q)})) = \delta(h, S_q(g_{(1,q)}), g_{(2,q)}) = \epsilon_p(h)\epsilon_q(g)1,$$

$$\delta(h_{(1,p)}g_{(1,q)}, S_q(g_{(2,q)}), S_p(h_{(2,p)})) = \delta(S_p(h_{(1,p)})S_q(g_{(1,q)}), g_{(2,q)}, h_{(2,p)}) = \epsilon_p(h)\epsilon_q(g)1,$$

$$\delta(S_p(h_{(1,p)}), S_q(g_{(1,q)}), g_{(2,q)}h_{(2,p)}) = \delta(h_{(1,p)}, g_{(1,p)}, S_q(g_{(2,q)})S_p(h_{(2,p)})) = \epsilon_p(h)\epsilon_q(g)1,$$

$$\delta(S_p(h_{(1,p)}), h_{(2,p)}S_q(g_{(1,q)}), g_{(2,q)}) = \delta(h_{(1,p)}, S_p(h_{(2,p)})g_{(1,q)}, S_q(g_{(2,q)})) = \epsilon_p(h)\epsilon_q(g)1.$$

Proof. (1) We suppose δ exists then applying it to the rebracket, then

$$\begin{aligned}
& ((h_{(1,p)}g_{(1,q)})f_{(1,r)})S_{pqr}(h_{(2,p)}(g_{(2,q)}f_{(2,r)})) \\
& = (\delta(h_{(1,p)(1,p)}, g_{(1,q)(1,q)}, f_{(1,r)(1,r)})(h_{(1,p)(2,p)}(g_{(1,q)(2,q)}f_{(1,r)(2,r)})))S_{pqr}(h_{(2,p)}(g_{(2,q)}f_{(2,r)})) \\
& \stackrel{(2.1)}{=} (\delta(h_{(1,p)}, g_{(1,q)}, f_{(1,r)})(h_{(2,p)}(g_{(2,q)}f_{(2,r)}))_{(1,pqr)})S_{pqr}((h_{(2,p)}(g_{(2,q)}f_{(2,r)}))_{(2,pqr)}) \\
& \stackrel{(2.4)}{=} \delta(h, g, f),
\end{aligned}$$

we verify similarly,

$$\begin{aligned}
& \delta(h_{(1,p)}, g_{(1,q)}, f_{(1,r)})(h_{(2,p)}(g_{(2,q)}f_{(2,r)})) \\
&= (((h_{(1,p)(1,p)}g_{(1,q)(1,q)})f_{(1,r)(1,r)})S_{pqr}(h_{(1,p)(2,p)}(g_{(1,q)(2,q)}f_{(1,r)(2,r)})))(h_{(2,p)}(g_{(2,q)}f_{(2,r)})) \\
&= (((h_{(1,p)}g_{(1,p)})f_{(1,r)})S_{pqr}(h_{(2,p)(1,p)}(g_{(2,q)(1,q)}f_{(2,r)(1,r)})))(h_{(2,p)(2,p)}(g_{(2,q)(2,q)}f_{(2,r)(2,r)})) \\
&= (((h_{(1,p)}g_{(1,p)})f_{(1,r)})S_{pqr}((h_{(2,p)}(g_{(2,q)}f_{(2,r)}))_{(1,pqr)}))(h_{(2,p)}(g_{(2,q)}f_{(2,r)}))_{(2,pqr)} \\
&\stackrel{(2.4)}{=} ((h_{(1,p)}g_{(1,p)})f_{(1,r)})\epsilon_{pqr}(h_{(2,p)}(g_{(2,q)}f_{(2,r)})) \stackrel{(2.2)}{=} (hg)f. \\
&(2)
\end{aligned}$$

$$\delta(1, h, g) = (h_{(1,p)}g_{(1,q)})S_{pq}(h_{(2,p)}g_{(2,q)}) = (hg)_{(1,pq)}S_{pq}((hg)_{(2,pq)}) \stackrel{(2.10)}{=} \epsilon_p(h)\epsilon_q(g)1,$$

the rest are similar.

(3)

$$\begin{aligned}
\delta(h_{(1,p)}, h_{(2,p)}; g) &= ((h_{(1,p)(1,p)}S_p(h_{(2,p)})(_{(1,p^{-1})})g_{(1,q)})S_q(h_{(1,p)(2,p)}(S_p(h_{(2,p)})(_{(2,p^{-1})})g_{(2,q)})) \\
&\stackrel{(2.12)}{=} ((h_{(1,p)(1,p)}S_p(h_{(2,p)(2,p)}))g_{(1,q)})S_q(h_{(1,p)(2,p)}(S_p(h_{(2,p)(1,p)})g_{(2,q)})) \\
&= ((h_{(1,p)(1,p)}S_p(h_{(2,p)}))g_{(1,q)})S_q(h_{(1,p)(2,p)(1,p)}(S_p(h_{(1,p)(2,p)(2,p)})g_{(2,q)})) \\
&\stackrel{(2.3)}{=} ((h_{(1,p)}S_p(h_{(2,p)}))g_{(1,q)})S_q(g_{(2,q)}) \\
&\stackrel{(2.10)}{=} \epsilon_p(h)g_{(1,q)}S_q(g_{(2,q)}) \stackrel{(2.10)}{=} \epsilon_p(h)\epsilon_q(g)1,
\end{aligned}$$

the rest are similar. \square

3 The first fundamental theorem for Q -graded Hopf quasi-groups

Definition 3.1. Let $H = \{H_p\}_{p \in Q}$ be a (not necessary associative) Q -graded algebra and for any $p \in Q$, H_p is a (not necessary coassociative) coalgebra $(H_p, \Delta_p, \epsilon_p)$ such that for any $p, q \in Q$, $m_{p,q}$, μ are maps of coalgebras, then

(1) A family $\beta = \{\beta_{p,q} : H_p \otimes H_q \rightarrow H_{pq} \otimes H_q, \beta_{p,q}(h \otimes g) = hg_{(1,q)} \otimes g_{(2,q)}\}_{p,q \in Q}$ of k -maps is called the right Galois map of H .

(2) A family $\gamma = \{\gamma_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{pq}, \gamma_{p,q}(h \otimes g) = h_{(1,p)} \otimes h_{(2,p)}g\}_{p,q \in Q}$ of k -maps is called the left Galois map of H .

(3) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_{pq} \otimes H_q\}_{p,q \in Q}$ of k -maps is almost left H -linear if

$$\phi_{p,q}(h \otimes g) = (h \otimes 1)\phi_{e,q}(1 \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(4) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_{pq} \otimes H_q\}_{p,q \in Q}$ of k -maps is almost right H -colinear if

$$\phi_{p,q}(h \otimes g) = (I \otimes \epsilon_q)\phi_{p,q}(h \otimes g_{(1,q)}) \otimes g_{(2,q)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(5) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_{pq} \otimes H_q\}_{p,q \in Q}$ of k -maps is right H -colinear if

$$\phi_{p,q}(h \otimes g_{(1,q)}) \otimes g_{(2,q)} = (I \otimes \Delta_q)\phi_{p,q}(h \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(6) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{pq}\}_{p,q \in Q}$ of k -maps is almost right H -linear if

$$\phi_{p,q}(h \otimes g) = \phi_{p,e}(h \otimes 1)(1 \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(7) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{pq}\}_{p,q \in Q}$ of k -maps is almost left H -colinear if

$$\phi_{p,q}(h \otimes g) = h_{(1,p)} \otimes (\epsilon_p \otimes I)\phi_{p,q}(h_{(2,p)} \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(8) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{pq}\}_{p,q \in Q}$ of k -maps is left H -colinear if

$$h_{(1,p)} \otimes \phi_{p,q}(h_{(2,p)} \otimes g) = (\Delta_p \otimes I)\phi_{p,q}(h \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(9) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_{pq^{-1}} \otimes H_q\}_{p,q \in Q}$ of k -maps is almost left H -linear if

$$\phi_{p,q}(h \otimes g) = (h \otimes 1)\phi_{e,q}(1 \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(10) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_{pq^{-1}} \otimes H_q\}_{p,q \in Q}$ of k -maps is almost right H -colinear if

$$\phi_{p,q}(h \otimes g) = (I \otimes \epsilon_q)\phi_{p,q}(h \otimes g_{(1,q)}) \otimes g_{(2,q)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(11) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_{pq^{-1}} \otimes H_q\}_{p,q \in Q}$ of k -maps is right H -colinear if

$$\phi_{p,q}(h \otimes g_{(1,q)}) \otimes g_{(2,q)} = (I \otimes \Delta_q)\phi_{p,q}(h \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(12) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{p^{-1}q}\}_{p,q \in Q}$ of k -maps is almost right H -linear if

$$\phi_{p,q}(h \otimes g) = \phi_{p,e}(h \otimes 1)(1 \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(13) A family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{p^{-1}q}\}_{p,q \in Q}$ of k -maps is almost left H -colinear if

$$\phi_{p,q}(h \otimes g) = h_{(1,p)} \otimes (\epsilon_p \otimes I)\phi_{p,q}(h_{(2,p)} \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(14) A family $\phi = \{\phi_{p,q} : H_p \otimes H_p \rightarrow H_p \otimes H_{p^{-1}q}\}_{p,q \in Q}$ of k -maps is left H -colinear if

$$h_{(1,p)} \otimes \phi_{p,q}(h_{(2,p)} \otimes g) = (\Delta_p \otimes I)\phi_{p,q}(h \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

Lemma 3.2. *If for any $p \in Q$, Δ_p is a coassociative, then the notion of almost H -colinearity coincides with that of H -colinearity.*

Proof. Suppose a family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{pq}\}_{p,q \in Q}$ of k -maps is left H -colinear,

$$\begin{aligned} & \phi_{p,q}(h \otimes g) \\ &= (I \otimes \epsilon_p \otimes I)(\Delta_p \otimes I)\phi_{p,q}(h \otimes g) \\ &= (I \otimes \epsilon_p \otimes I)(h_{(1,p)} \otimes \phi_{p,q}(h_{(2,p)} \otimes g)) \\ &= h_{(1,p)} \otimes (\epsilon_p \otimes I)\phi_{p,q}(h_{(2,p)} \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \end{aligned}$$

therefore ϕ is almost left H -colinear. Similarly, suppose a family $\phi = \{\phi_{p,q} : H_p \otimes H_q \rightarrow H_p \otimes H_{pq}\}_{p,q \in Q}$ of k -maps is almost left H -colinear,

$$\begin{aligned} & (\Delta_p \otimes I)\phi_{p,q}(h \otimes g) \\ &= (\Delta_p \otimes I)(h_{(1,p)} \otimes (\epsilon_p \otimes I)\phi(h_{(2,p)} \otimes g)) \\ &= h_{(1,p)(1,p)} \otimes h_{(1,p)(2,p)} \otimes (\epsilon_p \otimes I)\phi(h_{(2,p)} \otimes g) \\ &= h_{(1,p)} \otimes h_{(2,p)(1,p)} \otimes (\epsilon_p \otimes I)\phi(h_{(2,p)(2,p)} \otimes g) \\ &= h_{(1,p)} \otimes \phi_{p,q}(h_{(2,p)} \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \end{aligned}$$

therefore ϕ is left H -colinear. Analogously we can get the other results. \square

Lemma 3.3. *Let H be a Q -graded Hopf quasigroup, then the right Galois map β is almost left H -linear and almost right H -colinear, the left Galois map γ is almost right H -linear and almost left H -colinear.*

Proof. Since

$$\begin{aligned} \beta_{p,q}(h \otimes g) &= hg_{(1,p)} \otimes g_{(2,p)} = (h \otimes 1)(g_{(1,q)} \otimes g_{(2,q)}) \\ &= (h \otimes 1)\beta_{e,q}(1 \otimes g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \end{aligned}$$

and

$$\begin{aligned} \beta_{p,q}(h \otimes g) &= hg_{(1,q)} \otimes g_{(2,q)} = hg_{(1,q)}\epsilon_q(g_{(2,q)}) \otimes g_{(3,q)} = (I \otimes \epsilon_q)(hg_{(1,q)} \otimes g_{(2,q)}) \otimes g_{(3,q)} \\ &= (I \otimes \epsilon_q)\beta_{p,q}(h \otimes g_{(1,q)}) \otimes g_{(2,q)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \end{aligned}$$

β is almost left H -linear and almost H -colinear. Similarly we can get the result for γ . \square

Theorem 3.4. *Let $H = \{H_p\}_{p \in Q}$ be a (not necessary associative) Q -graded algebra and for any $p \in Q$, H_p is a (not necessary coassociative) coalgebra $(H_p, \Delta_p, \epsilon_p)$ such that for*

any $p, q \in Q$, $m_{p,q}$, μ are maps of coalgebras. Then H is a Q -graded Hopf quasigroup \iff

(1) For any $p \in Q$, Δ_p is coassociative .

(2) There is a family $\beta^* = \{\beta_{p,q}^* : H_p \otimes H_q \rightarrow H_{pq^{-1}} \otimes H_q\}_{p,q \in Q}$ of k -maps which is almost left H -linear and satisfy

$$\beta_{pq,q}^* \beta_{p,q}(h \otimes g) = h \otimes g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q,$$

$$\beta_{pq^{-1},q} \beta_{p,q}^*(h \otimes g) = h \otimes g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

(3) There is a family $\gamma^* = \{\gamma_{p,q}^* : H_p \otimes H_q \rightarrow H_p \otimes H_{p^{-1}q}\}_{p,q \in Q}$ of k -maps which is almost right H -linear and satisfy

$$\gamma_{p,pq}^* \gamma_{p,q}(h \otimes g) = h \otimes g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q,$$

$$\gamma_{p,p^{-1}q} \gamma_{p,q}^*(h \otimes g) = h \otimes g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

Proof. (\implies) H is a Q -graded Hopf quasigroup, by definition, $\forall p \in Q$, Δ_p is coassociative. We set

$$\beta^* = \{\beta_{p,q}^* : H_p \otimes H_q \rightarrow H_{pq^{-1}} \otimes H_q, \beta_{p,q}^*(h \otimes g) = hS_q(g_{(1,q)}) \otimes g_{(2,q)}\}_{p,q \in Q}$$

and

$$\gamma^* = \{\gamma_{p,q}^* : H_p \otimes H_q \rightarrow H_p \otimes H_{p^{-1}q}, \gamma_{p,q}^*(h \otimes g) = h_{(1,p)} \otimes S_p(h_{(2,p)})g\}_{p,q \in Q}.$$

Since

$$\begin{aligned} \beta_{p,q}^*(h \otimes g) &= hS_q(g_{(1,q)}) \otimes g_{(2,q)} = (h \otimes 1)(S_q(g_{(1,q)}) \otimes g_{(2,q)}) \\ &= (h \otimes 1)\beta_{e,q}^*(1 \otimes g), \quad \forall p, q \in Q, \quad h \in H_{pq}, \quad g \in H_q, \end{aligned}$$

β^{-1} is almost left H -linear .

And we have

$$\begin{aligned} \beta_{pq,q}^* \beta_{p,q}(h \otimes g) &= \beta_{pq,q}^*(hg_{(1,q)} \otimes g_{(2,q)}) = (hg_{(1,q)})S_q(g_{(2,q)}) \otimes g_{(3,q)} \\ &\stackrel{(2.4)}{=} h \otimes g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \end{aligned}$$

and

$$\begin{aligned} \beta_{pq^{-1},q} \beta_{p,q}^*(h \otimes g) &= \beta_{pq^{-1},q}(hS_q(g_{(1,q)}) \otimes g_{(2,q)}) = (hS_q(g_{(1,q)}))g_{(2,q)} \otimes g_{(3,q)} \\ &\stackrel{(2.4)}{=} h \otimes g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q. \end{aligned}$$

Analogous computation yields the result for γ^* .

(\Leftarrow) We denote

$$\beta_{e,q}^*(1 \otimes g) \equiv g^{[1,q^{-1}]} \otimes g^{[2,q]}, \quad \forall q \in Q, \quad g \in H_q,$$

$$\gamma_{p,e}^*(h \otimes 1) \equiv h^{(1,p)} \otimes h^{(2,p^{-1})}, \quad \forall p \in Q, \quad h \in H_p.$$

Since β^* and γ^* are almost H -linear, we have

$$\beta_{p,q}^*(h \otimes g) = (h \otimes 1)\beta_{e,q}^*(1 \otimes g) = hg^{[1,q^{-1}]} \otimes g^{[2,q]}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q,$$

$$\gamma_{p,q}^*(h \otimes g) = \gamma_{p,e}^*(h \otimes 1)(1 \otimes g) = h^{(1,p)} \otimes h^{(2,p^{-1})}g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

Define two families of k -linear maps:

$$S = \{S_q : H_q \rightarrow H_{q^{-1}}, S_q(g) = g^{[1,q^{-1}]} \epsilon_q(g^{[2,q]})\}_{q \in Q}$$

$$S^* = \{S_p^* : H_p \rightarrow H_{p^{-1}}, S_p^*(h) = \epsilon_p(h^{(1,p)})h^{(2,p^{-1})}\}_{p \in Q}.$$

Since for any $p \in Q$, Δ_p is coassociative and β is almost right H -colinear, by lemma 2.2, β is right H -colinear, and hence β^{-1} is right H -colinear. So

$$\begin{aligned} g_{(1,q)}^{[1,q^{-1}]} \otimes g_{(1,q)}^{[2,q]} \otimes g_{(2,q)} &= \beta_{e,q}^*(1 \otimes g_{(1,q)}) \otimes g_{(2,q)} = (I \otimes \Delta_q)\beta_{e,q}^*(1 \otimes g) = (I \otimes \Delta_q)(g^{[1,q^{-1}]} \otimes g^{[2,q]}) \\ &= g^{[1,q^{-1}]} \otimes g_{(1,q)}^{[2,q]} \otimes g_{(2,q)}^{[2,q]}, \quad \forall q \in Q, \quad g \in H_q. \end{aligned}$$

Applying $I \otimes \epsilon_q \otimes I$ to both sides of above equation, we have

$$g^{[1,q^{-1}]} \otimes g^{[2,q]} = g^{[1,q^{-1}]} \otimes \epsilon_q(g_{(1,q)}^{[2,q]})g_{(2,q)}^{[2,q]} = g_{(1,q)}^{[1,q^{-1}]} \epsilon_q(g_{(1,q)}^{[2,q]}) \otimes g_{(2,q)} = S_q(g_{(1,q)}) \otimes g_{(2,q)}.$$

So, we have

$$\begin{aligned} h \otimes g &= \beta_{pq,q}^* \beta_{p,q}(h \otimes g) = \beta_{pq,q}^*(hg_{(1,q)} \otimes g_{(2,q)}) = (hg_{(1,q)} \otimes 1)\beta_{e,q}^*(1 \otimes g_{(2,q)}) \\ &= (hg_{(1,q)})S_q(g_{(2,q)}) \otimes g_{(3,q)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \end{aligned}$$

and

$$\begin{aligned} h \otimes g &= \beta_{pq^{-1},q} \beta_{p,q}^*(h \otimes g) = \beta_{pq^{-1},q}((h \otimes 1)\beta_{e,q}^*(1 \otimes g)) = \beta_{pq^{-1},q}(hS_q(g_{(1,q)}) \otimes g_{(2,q)}) \\ &= (hS_q(g_{(1,q)}))g_{(2,q)} \otimes g_{(3,q)} \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \end{aligned}$$

so we get

$$(hg_{(1,q)})S_q(g_{(2,q)}) \otimes g_{(3,q)} = h \otimes g = (hS_q(g_{(1,q)}))g_{(2,q)} \otimes g_{(3,q)}.$$

Applying $I \otimes \epsilon_q$ to the above equation, we have

$$(hg_{(1,q)})S_q(h_{(2,q)}) = h\epsilon_q(g) = (hS_q(g_{(1,q)}))h_{(2,q)}, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

Following similar chain of arguments we can conclude that the the map S^* satisfy

$$S_p^*(h_{(1,p)})(h_{(2,p)}g) = \epsilon_p(h)g = h_{(1,p)}(S_p^*(h_{(2,p)})g), \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q.$$

Finally, we can compute

$$\begin{aligned} S_p^*(h) &= S_p^*(h_{(1,p)})\epsilon_p(h_{(2,p)}) = (S_p^*(h_{(1,p)})h_{(2,p)})h_{(3,p)} \\ &= \epsilon_p(h_{(1,p)})S_p(h_{(2,p)}) = S_p(h), \quad \forall p \in Q, \quad h \in H_p. \end{aligned}$$

Thus the maps S and S^* coincide, H is a Q -graded Hopf quasigroup. \square

4 Q -graded quasimodules and Q -graded Hopf quasimodules

Definition 4.1. Let H be a Q -graded Hopf quasigroup, a Q -graded H -quasimodule is a pair (M, φ) , where $M = \{M_q\}_{q \in Q}$ is a family of k -spaces and $\varphi = \{\varphi_{p,q} : H_p \otimes M_q \rightarrow M_{pq}\}_{p,q \in Q}$ (Q -graded action) is a family of k -maps such that

$$1 \cdot m = m, \quad \forall q \in Q, \quad m \in M_q,$$

$$h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot m) = S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot m) = \epsilon_p(h)m, \quad \forall p, q \in Q, \quad h \in H_p, \quad m \in M_q. \quad (4.1)$$

A map of Q -graded H -quasimodules is a family $f = \{f_q : M_q \rightarrow N_q\}_{q \in Q}$ of k -maps such that

$$f_q(h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot m)) = h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot f_q(m)), \quad \forall p, q \in Q, \quad h \in H_p, \quad m \in M_q. \quad (4.2)$$

Remark 4.2. H is a Q -graded Hopf quasimodule with the Q -graded multiplication as Q -graded action.

Example 4.3. Let Q be a quasigroup and $X = \{X_q\}_{q \in Q}$ is a set, then we can define a Q -graded kQ -quasimodule on a family $kX = \{(kX_q)\}_{q \in Q}$ of k -spaces as follows:

$$(k_1p) \cdot (k_2X_q) = k_1k_2X_{pq}, \quad \forall p, q \in Q.$$

Proposition 4.4. Suppose M, N are two Q -graded H -quasimodules, then $M \otimes N = \{M_q \otimes N_q\}_{q \in Q}$ is a Q -graded H -quasimodule with the diagonal action:

$$h \cdot (m \otimes n) = h_{(1,p)} \cdot m \otimes h_{(2,p)} \cdot n, \quad \forall p, q \in Q, \quad h \in H_p, \quad m \otimes n \in M_q \otimes N_q.$$

Proof.

$$\begin{aligned}
& h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (m \otimes n)) \\
&= h_{(1,p)} \cdot (S_p(h_{(2,p)})_{(1,p^{-1})} \cdot m \otimes S_p(h_{(2,p)})_{(2,p^{-1})} \cdot n) \\
&\stackrel{(2.12)}{=} h_{(1,p)} \cdot (S_p(h_{(3,p)}) \cdot m \otimes S_p(h_{(2,p)}) \cdot n) \\
&= h_{(1,p)} \cdot (S_p(h_{(4,p)}) \cdot m) \otimes h_{(2,p)} \cdot (S_p(h_{(3,p)}) \cdot n) \\
&\stackrel{(4.1)}{=} \epsilon_p(h)(m \otimes n).
\end{aligned}$$

Similarly, we can proof $S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot (m \otimes n)) = \epsilon_p(h)(m \otimes n)$, $M \otimes N$ with the diagonal action is a Q -graded H -quasimodule. \square

Definition 4.5. Let H be a Q -graded Hopf quasigroup, a Q -graded H -Hopf quasimodule is a Q -graded H -quasimodule $M = \{M_q, \varphi_{p,q}\}_{p,q \in Q}$ and $\forall q \in Q$, M_q is coassociative counitary left H_q -comodule (M_q, ρ_q) such that (we denote $\rho_q(m) = m_{(-1,q)} \otimes m_{(0,q)}$)

$$(h \cdot m)_{(-1,pq)} \otimes (h \cdot m)_{(0,pq)} = h_{(1,p)} m_{(-1,q)} \otimes h_{(2,p)} \cdot m_{(0,q)} \quad \forall p, q \in Q, h \in H_p, m \in M_q. \quad (4.3)$$

A map of Q -graded H -Hopf quasimodules is a map of Q -graded H -quasimodules $f = \{f_q : M_q \rightarrow N_q\}_{q \in Q}$ such that

$$m_{(-1,q)} \otimes f_q(m_{(0,q)}) = f_q(m)_{(-1,q)} \otimes f_q(m)_{(0,q)}, \quad \forall q \in Q, m \in M_q. \quad (4.4)$$

Example 4.6. Let Q be a quasigroup, we can define a Q -graded kQ -Hopf quasimodule on Q -graded kQ -quasimodule $kX = \{(kX_q)\}_{q \in Q}$ by

$$\rho_q(X_q) = q \otimes X_q, \quad \forall q \in Q.$$

Lemma 4.7. Let M be a Q -graded H -Hopf quasimodule, then

$$\rho_e(S_q(m_{(-1,q)}) \cdot m_{(0,q)}) = 1 \otimes S_q(m_{(-1,q)}) \cdot m_{(0,q)}, \quad \forall q \in Q, m \in M_q. \quad (4.5)$$

$$\rho_q(h \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)})) = h_{(1,p)} \otimes h_{(2,p)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}), \quad \forall p, q \in Q, h \in H_p, m \in M_q. \quad (4.6)$$

Proof.

$$\begin{aligned}
& \rho_e(S_q(m_{(-1,q)}) \cdot m_{(0,q)}) \\
&= (S_q(m_{(-1,q)}) \cdot m_{(0,q)})_{(-1,e)} \otimes (S_q(m_{(-1,q)}) \cdot m_{(0,q)})_{(0,e)} \\
&\stackrel{(4.3)}{=} (S_q(m_{(-2,q)}))_{(1,q^{-1})} m_{(-1,q)} \otimes (S_q(m_{(-2,q)}))_{(2,q^{-1})} \cdot m_{(0,q)} \\
&\stackrel{(2.12)}{=} S_q(m_{(-2,q)}) m_{(-1,q)} \otimes S_q(m_{(-3,q)}) \cdot m_{(0,q)} \\
&\stackrel{(2.10)}{=} \epsilon_q(m_{(-1,q)}) 1 \otimes S_q(m_{(-2,q)}) \cdot m_{(0,q)} \\
&= 1 \otimes S_q(m_{(-1,q)}) \cdot m_{(0,q)},
\end{aligned}$$

$$\begin{aligned}
& \rho_q(h \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)})) \\
& \stackrel{(4.3)}{=} h_{(1,p)}(S_q(m_{(-1,q)}) \cdot m_{(0,q)})_{(-1,q)} \otimes h_{(2,p)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)})_{(0,q)} \\
& \stackrel{(4.5)}{=} h_{(1,p)} \otimes h_{(2,p)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}).
\end{aligned}$$

□

Theorem 4.8. *Let (M, φ, ρ) be a Q -graded H -Hopf quasimodule, we replace the Q -graded action φ by $\triangleright = \{\triangleright_{p,q}: H_p \otimes M_q \rightarrow M_{pq}, \triangleright_{p,q}(h \otimes m) = (hm_{(-2,q)}) \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)})\}_{p,q \in Q}$ (we denote $\triangleright_{p,q}(h \otimes m) \equiv h \triangleright m$), then $(M, \triangleright, \rho)$ is a Q -graded H -Hopf quasimodule.*

Proof. Obviously,

$$1 \triangleright m = m_{(-2,q)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}) = \epsilon_q(m_{(-1,q)})m_{(0,q)} = m, \quad \forall q \in Q, m \in M_q.$$

Verifying the conditon (4.1) :ince

$$g \triangleright m = (gm_{(-2,r)}) \cdot (S_r(m_{(-1,r)}) \cdot m_{(0,r)}), \quad \forall q, r \in Q, g \in H_q, r \in M_r,$$

we have

$$\rho(g \triangleright m) \stackrel{(4.6)}{=} (gm_{(-2,r)})_{(1,qr)} \otimes (gm_{(-2,r)})_{(2,qr)} \cdot (S_r(m_{(-1,r)})m_{(0,r)}),$$

and

$$\begin{aligned}
& (g \triangleright m)_{(-2,qr)} \otimes (g \triangleright m)_{(-1,qr)} \otimes (g \triangleright m)_{(0,qr)} \\
& = (gm_{(-2,r)})_{(1,qr)} \otimes (gm_{(-2,r)})_{(2,qr)} \otimes (gm_{(-2,r)})_{(3,qr)} \cdot (S_r(m_{(-1,r)}) \cdot m_{(0,r)}),
\end{aligned}$$

using the above equation,

$$h \triangleright (g \triangleright m)$$

$$\begin{aligned}
& = (h(g \triangleright m)_{(-2,qr)}) \cdot (S_{qr}((g \triangleright m)_{(-1,qr)}) \cdot (g \triangleright m)_{(0,qr)}) \\
& = (h(gm_{(-2,r)})_{(1,qr)}) \cdot (S_{qr}((gm_{(-2,r)})_{(2,qr)}) \cdot ((gm_{(-2,r)})_{(3,qr)} \cdot (S_r(m_{(-1,r)}) \cdot m_{(0,r)}))) \\
& = (h(gm_{(-2,r)})) \cdot (S_r(m_{(-1,r)}) \cdot m_{(0,r)}), \quad \forall p, q, r \in Q, h \in H_p, g \in H_q, m \in M_r,
\end{aligned}$$

replacing h with $S_p(h_{(1,p)})$ and g with $h_{(2,p)}$,

$$\begin{aligned}
& S_p(h_{(1,p)}) \triangleright (h_{(2,p)} \triangleright m) \\
& = (S_p(h_{(1,p)})(h_{(2,p)}m_{(-2,q)})) \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}) \\
& = \epsilon_p(h)m_{(-2,q)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}) \\
& = \epsilon_p(h)m, \quad \forall p, q \in Q, h \in H_p, m \in M_q.
\end{aligned}$$

Similarly, we can get

$$h_{(1,p)} \triangleright (S_p(h_{(2,p)}) \triangleright m) = \epsilon_p(h)m, \forall p, q \in Q, h \in H_p, m \in M_q.$$

So, (M, φ) is a Q -graded H -quasimodule. Now we verify the condition (4.3) as follows:

$$\begin{aligned} \rho(h \triangleright m) &= \rho((hm_{(-2,q)}) \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)})) \\ &\stackrel{(4.6)}{=} (hm_{(-2,q)})_{(1,pq)} \otimes (hm_{(-2,q)})_{(2,pq)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}) \\ &\stackrel{(2.1)}{=} h_{(1,p)}m_{(-3,q)} \otimes (h_{(2,p)}m_{(-2,q)}) \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}) \\ &= h_{(1,p)}m_{(-1,q)} \otimes h_{(2,p)} \triangleright m_{(0,q)}. \end{aligned}$$

□

Definition 4.9. Let M be a Q -graded H -Hopf quasimodule, the coinvariant of M on H is a vector space

$$M^{coH} = \{m \in M_e \mid \rho_e(m) = 1 \otimes m\}.$$

Lemma 4.10. Let $M = \{M_q\}_{q \in Q}$ be a Q -graded H -Hopf quasimodule, then $H \otimes M^{coH} = \{H_q \otimes M^{coH}\}_{q \in Q}$ is a Q -graded H -Hopf quasimodule.

Proof. Defining the Q -graded H -module action φ on $H \otimes M^{coH}$:

$$h \cdot (g \otimes m) = hg \otimes m, \forall p, q \in Q, h \in H_p, g \otimes m \in H_q \otimes M^{coH}.$$

Since

$$1 \cdot (g \otimes m) = g \otimes m, \forall q \in Q, g \otimes m \in H_q \otimes M^{coH},$$

and

$$\begin{aligned} &S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot (g \otimes m)) \\ &= S_p(h_{(1,p)})(h_{(2,p)}g) \otimes m \\ &\stackrel{(2.3)}{=} \epsilon_p(h)(g \otimes m) \\ &\stackrel{(2.3)}{=} h_{(1,p)}(S_p(h_{(2,p)})g) \otimes m \\ &= h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (g \otimes m)), \\ &\forall p, q \in Q, h \in H_p, g \otimes m \in H_q \otimes M^{coH}, \end{aligned}$$

$(H \otimes M^{coH}, \varphi)$ is a Q -graded H -quasimodule.

For any $q \in Q$, we define H_q -comodule coaction on $H_q \otimes M^{coH}$:

$$\rho_q(g \otimes m) = g_{(1,q)} \otimes g_{(2,q)} \otimes m, \forall q \in Q, g \otimes m \in H_q \otimes M^{coH}.$$

Since

$$\begin{aligned}
& (\Delta_q \otimes I \otimes I)(\rho_q(g \otimes m)) \\
&= (\Delta_q \otimes I \otimes I)(g_{(1,q)} \otimes g_{(2,q)} \otimes m) \\
&= g_{(1,q)} \otimes g_{(2,q)} \otimes g_{(3,q)} \otimes m \\
&= (I \otimes \rho_q)(g_{(1,q)} \otimes g_{(2,q)} \otimes m) \\
&= (I \otimes \rho_q)\rho_q(g \otimes m), \quad \forall q \in Q, \quad g \otimes m \in H_q \otimes M^{coH},
\end{aligned}$$

and

$$(\epsilon_q \otimes I \otimes I)\rho_q(g \otimes m) = (\epsilon_q \otimes I \otimes I)(g_{(1,q)} \otimes g_{(2,q)} \otimes m) = g \otimes m, \quad \forall q \in Q, \quad g \otimes m \in H_q \otimes M^{coH},$$

$(H_q \otimes M^{coH}, \rho_q)$ is a coassociative counitary H_q -comodule.

Now we verify the condition (4.3) for $H \otimes M^{coH}$,

$$\begin{aligned}
& (h \cdot (g \otimes m))_{(-1,pq)} \otimes (h \cdot (g \otimes m))_{(0,pq)} \\
&= (hg \otimes m)_{(-1,pq)} \otimes (hg \otimes m)_{(0,pq)} \\
&= (hg)_{(-1,pq)} \otimes (hg)_{(2,pq)} \otimes m \\
&\stackrel{(2.1)}{=} h_{(1,p)}g_{(1,p)} \otimes h_{(2,p)}g_{(2,p)} \otimes m \\
&= h_{(1,p)}(g \otimes m)_{(-1,q)} \otimes h_{(2,p)} \cdot (g \otimes m)_{(0,q)}, \\
&\quad \forall p, q \in Q, \quad h \in H_p, \quad g \otimes m \in H_q \otimes M^{coH}.
\end{aligned}$$

So, $H \otimes M^{coH}$ is a Q -graded H -Hopf quasimodule. \square

Theorem 4.11. *Let $M = \{M_q\}_{q \in Q}$ be a Q -graded H -Hopf quasimodule, then (as Q -graded H -Hopf quasimodule)*

$$H \otimes M^{coH} \cong M.$$

Proof. For any $q \in Q$, we define

$$\sigma_q : H_q \otimes M^{coH} \rightarrow M_q, \quad g \otimes m \rightarrow g \cdot m, \quad \forall q \in Q, \quad g \otimes m \in H_q \otimes M^{coH}.$$

Verify $\sigma = \{\sigma_q\}_{q \in Q}$ satisfy the condition (4.2):

$$\begin{aligned}
& \sigma_q(h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (g \otimes m))) \\
&= \sigma_q(h_{(1,p)}(S_p(h_{(2,p)})g) \otimes m) \\
&\stackrel{(4.1)}{=} \sigma_q(\epsilon_p(h)(g \otimes m)) \\
&= \epsilon_p(h)g \cdot m \\
&\stackrel{(4.1)}{=} h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (g \cdot m)) \\
&= h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot \sigma_q(g \otimes m)), \\
&\quad \forall p, q \in Q, \quad h \in H_p, \quad g \otimes m \in H_q \otimes M^{coH}.
\end{aligned}$$

Verify $\sigma = \{\sigma_q\}_{q \in Q}$ satisfy the condition (4.4)

$$\begin{aligned}
& (g \otimes m)_{(-1,q)} \otimes \sigma_q((g \otimes m)_{(0,q)}) \\
&= g_{(1,q)} \otimes \sigma_q(g_{(2,q)} \otimes m) \\
&= g_{(1,q)} \otimes g_{(2,q)} \cdot m \\
&\stackrel{(4.3)}{=} (g \cdot m)_{(-1,q)} \otimes (g \cdot m)_{(0,q)} \\
&= \sigma_q(g \otimes m)_{(-1,q)} \otimes \sigma_q(g \otimes m)_{(0,q)}, \\
&\forall q \in Q, g \otimes m \in H_q \otimes M^{coH}.
\end{aligned}$$

So, σ is a map of Q -graded H -quasimodules.

For any $q \in Q$, we define

$$\sigma_q^{-1} : M_q \rightarrow H_q \otimes M^{coH}, m \rightarrow m_{(-2,q)} \otimes S_q(m_{(-1,q)} \cdot m_{(0,q)}),$$

noticing by (4.5), $\sigma_q^{-1}(m) \in H_q \otimes M^{coH}$, $\forall q \in Q$, $m \in M_q$, so σ^{-1} is well defined.

Since

$$\begin{aligned}
& \sigma_q(\sigma_q^{-1}(m)) \\
&= m_{(-2,q)} \cdot (S_q(m_{(-1,q)}) \cdot m_{(0,q)}) \\
&\stackrel{(4.1)}{=} \epsilon_q(m_{(-1,q)})m_{(0,q)} \\
&= m, \forall q \in Q, m \in M_q,
\end{aligned}$$

and

$$\begin{aligned}
& \sigma_{q^{-1}}\sigma_q(g \otimes m) \\
&= (g \cdot m)_{(-2,q)} \otimes S_q((g \cdot m)_{(-1,q)}) \cdot (g \cdot m)_{(0,q)} \\
&\stackrel{(4.3)}{=} g_{(1,q)}m_{(-2,q)} \otimes S_q(g_{(2,q)}m_{(-1,q)}) \cdot (g_{(3,q)} \cdot m_{(0,q)}) \\
&= g_{(1,q)} \otimes S_q(g_{(2,q)}) \cdot (g_{(3,q)} \cdot m) \\
&\stackrel{(4.1)}{=} g \otimes m, \forall q \in Q, g \otimes m \in H_q \otimes M^{coH},
\end{aligned}$$

σ is an isomorphism of Q -graded H -Hopf quasimodules, then $M \cong H \otimes M^{coH}$. □

5 Q -graded Long dimodules

Definition 5.1. Let H be a Q -graded Hopf quasigroups, a Q -graded Long H -dimodule is a triple (M, φ, ρ) , where (M, φ) is a Q -graded left H -quasimodule, $(M, \rho) = \{(M_q, \rho_{q,e}) :$

$\rho_{q,e} : M_q \rightarrow M_q \otimes H_e, \rho_{q,e}(m) \equiv m_{(0,q) \otimes m_{(1,e)}}\}_{q \in Q}$ is a family of coassociative counitary right H_e -comodules such that

$$(h \cdot m)_{(0,pq)} \otimes (h \cdot m)_{(1,e)} = h \cdot m_{(0,q)} \otimes m_{(1,e)}, \quad \forall p, q \in Q, h \in H_p, m \in M_q. \quad (5.1)$$

Example 5.2. Let Q be a quasigroup, we can define a Q -graded Long kQ -dimodule on Q -graded kQ -quasimodule kX by

$$\rho_{q,e}(X_q) = X_q \otimes e, \quad \forall q \in Q.$$

Lemma 5.3. Let (M, φ, ρ) be a Q -graded Long H -dimodule, then we have

$$\rho_{q,e}(S_e(m_{(1,e)}) \cdot m_{(0,q)}) = S_e(m_{(1,e)(2,e)}) \cdot m_{(0,q)} \otimes m_{(1,e)(1,e)}, \quad \forall q \in Q, m \in M_q, \quad (5.2)$$

and

$$\rho_{pq,e}(h \cdot (S_e(m_{(1,e)}) \cdot m_{(0,q)})) = h \cdot (S_e(m_{(1,e)(2,e)}) \cdot m_{(0,q)}) \otimes m_{(1,e)(1,e)}, \quad \forall p, q \in Q, h \in H_p, m \in M_q. \quad (5.3)$$

Proof.

$$\begin{aligned} & \rho_{q,e}(S_e(m_{(1,e)}) \cdot m_{(0,q)}) \\ &= (S_e(m_{(1,e)}) \cdot m_{(0,q)})_{(0,q)} \otimes (S_e(m_{(1,e)}) \cdot m_{(0,q)})_{(1,e)} \\ &\stackrel{(5.1)}{=} S_e(m_{(2,e)}) \cdot m_{(0,q)} \otimes m_{(1,e)} \\ &= S_e(m_{(1,e)(2,e)}) \cdot m_{(0,q)} \otimes m_{(1,e)(1,e)}, \end{aligned}$$

and

$$\begin{aligned} & \rho_{pq,e}(h \cdot (S_e(m_{(1,e)}) \cdot m_{(0,q)})) \\ &= (h \cdot (S_e(m_{(1,e)}) \cdot m_{(0,q)}))_{(0,pq)} \otimes (h \cdot (S_e(m_{(1,e)}) \cdot m_{(0,q)}))_{(1,e)} \\ &\stackrel{(5.1)}{=} h \cdot (S_e(m_{(1,e)}) \cdot m_{(0,q)})_{(0,q)} \otimes (S_e(m_{(1,e)}) \cdot m_{(0,q)})_{(1,e)} \\ &\stackrel{(5.2)}{=} h \cdot S_e(m_{(1,e)(2,e)}) \cdot m_{(0,q)} \otimes m_{(1,e)(1,e)}. \end{aligned}$$

□

Proposition 5.4. Let M be a Q -graded H -quasimodule, then $M \otimes H_e = \{M_q \otimes H_e\}_{q \in Q}$ is a Q -graded Long H -dimodule by :

$$h \cdot (m \otimes g) = (h \cdot m) \otimes g, \quad \forall p, q \in Q, h \in H_p, m \in M_q, g \in H_e,$$

$$\rho_{q,e}(m \otimes g) = m \otimes g_{(1,e)} \otimes g_{(2,e)}, \quad \forall q \in Q, m \in M_q, g \in H_e.$$

Proof. Since

$$1 \cdot (m \otimes g) = m \otimes g,$$

and

$$\begin{aligned}
& S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot (m \otimes g)) \\
&= S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot m) \otimes g \\
&= \epsilon_p(h)(m \otimes g) \\
&= h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot m) \otimes g \\
&= h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (m \otimes g)),
\end{aligned}$$

therefore $M \otimes H_e$ is a Q -graded H -quasimodule. By the coassociativity of Δ_e and the counitality of ϵ_e , it is easy to check $M \otimes H_e$ is a right H_e -comodule. Now we proof the condition (5.1) as follows:

$$\begin{aligned}
& \rho_{pq,e}(h \cdot (m \otimes g)) \\
&= \rho_{pq,e}(h \cdot m \otimes g) \\
&= (h \cdot m \otimes g)_{(0,pq)} \otimes (h \cdot m \otimes g)_{(1,e)} \\
&= h \cdot m \otimes g_{(1,e)} \otimes g_{(2,e)} \\
&= h \cdot (m \otimes g)_{(0,q)} \otimes (m \otimes g)_{(1,e)},
\end{aligned}$$

so, $M \otimes H_e$ is a Q -graded Long H -dimodule. □

Proposition 5.5. *Let H be a Q -graded Hopf quasigroup and M is a right H_e -comodule, then $H \otimes M = \{H_q \otimes M\}_{q \in Q}$ is a Q -graded Long H -dimodule by*

$$\begin{aligned}
h \cdot (g \otimes m) &= hg \otimes m, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \quad m \in M. \\
\rho(g \otimes m) &= g \otimes m_0 \otimes m_1, \quad \forall q \in Q, \quad g \in H_q, \quad m \in M.
\end{aligned}$$

Proof. Since

$$\begin{aligned}
& h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (g \otimes m)) \\
&= h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot g) \otimes m \\
&= \epsilon_p(h)(g \otimes m) \\
&= S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot g) \otimes m \\
&= S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot (g \otimes m)),
\end{aligned}$$

$H \otimes M$ is a Q -graded left H -quasimodule.

Verify $H \otimes M$ is a right H_e -comodule is easy and be left to reader, we verify condtion (5.1) as follows:

$$\begin{aligned}
& (h \cdot (g \otimes m))_{(0,pq)} \otimes (h \cdot (g \otimes m))_{(1,e)} \\
&= (hg \otimes m)_{(0,pq)} \otimes (hg \otimes m)_{(1,e)} \\
&= hg \otimes m_0 \otimes m_1 \\
&= h \cdot (g \otimes m_0) \otimes m_1 \\
&= h \cdot (g \otimes m)_{(0,q)} \otimes (g \otimes m)_{(1,e)}.
\end{aligned}$$

Then, $H \otimes M$ is a Q -graded Long H -dimodule. □

Corollary 5.6. *Let (M, φ) be a Q -graded left H -quasimodule, then (M, φ, ρ) , then M is a Q -graded Long H -dimodule by*

$$\rho_{q,e}(m) = m \otimes 1.$$

Proposition 5.7. *Suppose (M, φ_M, ρ_M) and (N, φ_N, ρ_N) are two Q -graded Long H -dimodules, then $\{M_q \otimes N_q\}_{q \in Q}$ is a Q -graded Long H -dimodule by*

$$h \cdot (m \otimes n) = h_{(1,p)} \cdot m \otimes h_{(2,p)} \cdot n, \quad \forall p, q \in Q, \quad h \in H_p, \quad m \otimes n \in M_q \otimes N_q,$$

$$\rho_{q,e}(m \otimes n) = m_{(0,q)} \otimes n_{(0,q)} \otimes m_{(1,e)} n_{(1,e)}, \quad \forall q \in Q, \quad m \otimes n \in M_q \otimes N_q.$$

Proof. Since

$$1 \cdot (m \otimes n) = 1 \cdot m \otimes 1 \cdot n = m \otimes n.$$

and

$$\begin{aligned} & h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (m \otimes n)) \\ &= h_{(1,p)} \cdot (S_p(h_{(3,p)}) \cdot m \otimes S_p(h_{(2,p)}) \cdot m) \\ &= h_{(1,p)} \cdot (S_p(h_{(4,p)}) \cdot m) \otimes h_{(2,p)} \cdot (S_p(h_{(3,p)}) \cdot n) \\ &= h_{(1,p)} \cdot S_p(h_{(3,p)}) \cdot m \otimes \epsilon_p(h_{(2,p)}) n \\ &= \epsilon_p(h)(m \otimes n), \end{aligned}$$

similarly, we can get $S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot (m \otimes n)) = \epsilon_p(h)(m \otimes n)$, so $\{M_q \otimes N_q\}_{q \in Q}$ is a Q -graded H -quasimodule.

It is easy to check $\{M_p \otimes N_p\}_{p \in Q}$ is a family of right H_e -comodules, now we verify the condition (5.1):

$$\begin{aligned} \rho_{pq,e}(h \cdot (m \otimes n)) &= \rho_{pq,e}(h_{(1,p)} \cdot m \otimes h_{(2,p)} \cdot n) \\ &= (h_{(1,p)} \cdot m)_{(0,pq)} \otimes (h_{(2,p)} \cdot n)_{(0,pq)} \otimes (h_{(1,p)} \cdot m)_{(1,e)} (h_{(2,p)} \cdot n)_{(1,e)} \\ &\stackrel{(5.1)}{=} h_{(1,p)} \cdot m_{(0,q)} \otimes h_{(2,p)} \cdot n_{(0,q)} \otimes m_{(1,e)} n_{(1,e)} \\ &= h \cdot (m \otimes n)_{(0,q)} \otimes (m \otimes n)_{(1,e)}, \end{aligned}$$

therefore $\{M_q \otimes N_q\}_{q \in Q}$ is a Q -graded Long H -dimodule. □

Proposition 5.8. *Let (M, φ, ρ) be a Long H -dimodule. Then, for any $q \in Q$, the map*

$$R_q : M_q \otimes M_q \rightarrow M_q \otimes M_q, \quad R_q(m \otimes n) = n_{(1,e)} \cdot m \otimes n_{(0,q)},$$

is a solution of Long-equation for M_q .

Proof.

$$\begin{aligned}
& R_q^{12} R_q^{23} (l \otimes m \otimes n) \\
&= R_q^{12} (l \otimes n_{(1,e)} \cdot m \otimes n_{(0,q)}) \\
&= (n_{(1,e)} \cdot m)_{(1,e)} l \otimes (n_{(1,e)} \cdot m)_{(0,q)} \otimes n_{(0,q)} \\
&\stackrel{(5.1)}{=} m_{(1,e)} \cdot l \otimes n_{(1,e)} \cdot m_{(0,q)} \otimes n_{(0,q)} \\
&= R_q^{23} (m_{(1,e)} \cdot l \otimes m_{(0,q)} \otimes n) \\
&= R_q^{23} R_q^{12} (l \otimes m \otimes n), \quad \forall q \in Q, \quad l, m, n \in M_q.
\end{aligned}$$

we have $R_q^{12} R_q^{23} = R_q^{23} R_q^{12}$, $\forall q \in Q$.

□

By a Q -graded Long H -dimodule, we can construct a family of solution to Long-equation.

6 The smash product for Q -graded Hopf quasigroups

Definition 6.1. Let H be a Q -graded Hopf quasigroup, a H -quasimodule is a pair (M, φ) , where M is a k -space, $\varphi = \{\varphi_p : H_p \otimes M \rightarrow M\}_{p \in Q}$ is a family of k -maps such that,

$$1 \cdot m = m, \quad \forall m \in M,$$

$$S_p(h_{(1,p)}) \cdot (h_{(2,p)} \cdot m) = \epsilon_p(h)m = h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot m), \quad \forall p \in Q, \quad h \in H_p, \quad m \in M. \quad (6.1)$$

Definition 6.2. Let H be a Q -graded Hopf quasigroup, an (not necessarily associative) algebra A is called a H -quasimodule algebra if A is a H -quasimodule and

$$(h_{(1,p)} \cdot a)(h_{(2,p)} \cdot b) = h \cdot (ab), \quad \forall p \in Q, \quad h \in H_p, \quad a, b \in A, \quad (6.2)$$

$$h \cdot 1 = \epsilon_p(h)1, \quad \forall p \in Q, \quad h \in H_p. \quad (6.3)$$

A coalgebra C is a H -quasimodule coalgebra if C is a H -quasimodule and

$$\Delta(h \cdot c) = h_{(1,p)} \cdot c_1 \otimes h_{(2,p)} \cdot c_2, \quad \forall p \in Q, \quad h \in H_p, \quad c \in C, \quad (6.4)$$

$$\epsilon(h \cdot c) = \epsilon_p(h)\epsilon(c), \quad \forall p \in Q, \quad h \in H_p, \quad c \in C. \quad (6.5)$$

A Hopf quasigroup A is called a H -quasimodule Hopf quasigroup if it is both a H -quasimodule algebra and a H -quasimodule coalgebra.

Lemma 6.3. Let H is a Q -graded Hopf quasigroup, A is a H -quasimodule Hopf quasigroup, then

$$h \cdot S(a) = S(h \cdot a), \quad \forall p \in Q, \quad h \in H_p, \quad a \in A. \quad (6.6)$$

Proof. Since

$$(h_{(1,p)} \cdot a_1)(h_{(2,p)} \cdot S(a_2)) \stackrel{(6.1)}{=} h \cdot (a_1 S(a_2)) \stackrel{(6.3)}{=} \epsilon_p(h)\epsilon(a)1,$$

and

$$(h_{(1,p)} \cdot a_1)S(h_{(2,p)} \cdot a_2) = (h \cdot a)_1 S((h \cdot a)_2) = \epsilon(h \cdot a)1 \stackrel{(6.5)}{=} \epsilon_p(h)\epsilon(a)1,$$

we have

$$(h_{(1,p)} \cdot a_1)(h_{(2,p)} \cdot S(a_2)) = (h_{(1,p)} \cdot a_1)S(h_{(2,p)} \cdot a_2),$$

replacing h with $h_{(2,p)}$ and a with a_2 , and multiplying by $S(h_{(1,p)} \cdot a_1)$ on the left, we have

$$S(h_{(1,p)} \cdot a_1)((h_{(2,p)} \cdot a_2)S(h_{(3,p)} \cdot a_3)) = S(h_{(1,p)} \cdot a_1)((h_{(2,p)} \cdot a_2)(h_{(3,p)} \cdot S(a_3))), \quad (6.7)$$

then we get

$$\begin{aligned} S(h \cdot a) &= \epsilon(h_{(1,p)} \cdot a_1)S(h_{(2,p)} \cdot a_2) \\ &\stackrel{(2.3)}{=} S((h_{(1,p)} \cdot a_1)_1)((h_{(1,p)} \cdot a_1)_2 S(h_{(2,p)} \cdot a_2)) \\ &\stackrel{(2.1)}{=} S(h_{(1,p)} \cdot a_1)((h_{(2,p)} \cdot a_2)S(h_{(3,p)} \cdot a_3)) \\ &\stackrel{(6.6)}{=} S(h_{(1,p)} \cdot a_1)((h_{(2,p)} \cdot a_2)(h_{(3,p)} \cdot S(a_3))) \\ &\stackrel{(2.1)}{=} S((h_{(1,p)} \cdot a_1)_1)((h_{(1,p)} \cdot a_1)_2 (h_{(2,p)} \cdot S(a_2))) \\ &\stackrel{(2.3)}{=} \epsilon(h_{(1,p)} \cdot a_1)(h_{(2,p)} \cdot S(a_2)) \\ &= \epsilon_p(h_{(1,p)})\epsilon(a_1)(h_{(2,p)} \cdot S(a_2)) = h \cdot S(a). \end{aligned}$$

□

Theorem 6.4. *Suppose H be a Q -graded Hopf quasigroup and A is a H -quasimodule Hopf quasigroup such that*

$$h_{(1,p)} \otimes h_{(2,p)} \cdot a = h_{(2,p)} \otimes h_{(1,p)} \cdot a, \quad \forall p \in Q, \quad h \in H_p, \quad a \in A, \quad (6.8)$$

then the following statements are equivalent:

(1) *There is a smash product Q -graded Hopf quasigroup $A \rtimes H$ built on $A \otimes H = \{A \otimes H_p\}_{p \in Q}$ with*
(smash product)

$$(a \rtimes h)(b \rtimes g) = a(h_{(1,p)} \cdot b) \otimes h_{(2,p)}g, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \quad a, b \in A. \quad (6.9)$$

(unit)

$$1_{A \rtimes H} = 1 \otimes 1,$$

(Coproduct)

$$\Delta_p^*(a \otimes h) = (a_1 \otimes h_{(1,p)}) \otimes (a_2 \otimes h_{(2,p)}), \quad (6.10)$$

(counit)

$$\epsilon_p^*(a \otimes h) = \epsilon(a)\epsilon_p(h), \quad (6.11)$$

(antipode)

$$S_p^*(a \otimes h) = S_p(h_{(2,p)} \cdot S(a)) \otimes S_p(h_{(1,p)}), \quad (6.12)$$

(2)

$$g \cdot (S_p(h) \cdot a) = (gS_p(h)) \cdot a, \quad \forall p, q \in Q, \quad h \in H_p, \quad g \in H_q, \quad a \in A. \quad (6.13)$$

Proof. (1) \implies (2)

$$\epsilon(a)\epsilon_p(h)b \otimes g$$

$$\stackrel{(6.9)}{=} (b \otimes g)\epsilon_p^*(a \otimes h)$$

$$\stackrel{(2.4)}{=} ((b \rtimes g)S_p^*((a \rtimes h)_{(1,p)}))(a \rtimes h)_{(2,p)}$$

$$\stackrel{(6.4)}{=} ((b \rtimes g)S_p^*(a_1 \rtimes h_{(1,p)}))(a_2 \rtimes h_{(2,p)})$$

$$\stackrel{(6.12)}{=} ((b \rtimes g)(S_p(h_{(2,p)} \cdot S(a_1)) \rtimes S_p(h_{(1,p)})))(a_2 \rtimes h_{(3,p)})$$

$$\stackrel{(2.12)}{=} ((b \rtimes g)(S_p(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1) \rtimes S_p(h_{(1,p)})_{(2,p^{-1})}))(a_2 \rtimes h_{(2,p)})$$

$$\stackrel{(6.9)}{=} (b(g_{(1,q)} \cdot (S_p(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1)))) \rtimes g_{(2,q)}S_p(h_{(1,p)})_{(2,p^{-1})}(a_2 \rtimes h_{(3,p)})$$

$$\stackrel{(6.9)}{=} (b(g_{(1,q)} \cdot (S(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1))))((g_{(2,q)}S_p(h_{(1,p)})_{(2,p^{-1})}) \cdot a_2) \otimes (g_{(3,q)}S(h_{(1,p)})_{(3,p^{-1})})h_{(2,p)}$$

$$\stackrel{(6.6)}{=} (bS(g_{(1,q)} \cdot (S_p(h_{(1,p)})_{(1,p^{-1})} \cdot a_1))((g_{(2,q)}S_p(h_{(1,p)})_{(2,p^{-1})}) \cdot a_2) \otimes (g_{(3,q)}S_p(h_{(1,p)})_{(3,p^{-1})})h_{(2,p)}.$$

Applying $I \otimes \epsilon_q$ to both sides of this equation, set $b = g_{(1,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1)$ and replace a, g, h by $a_2, g_{(2,q)}, h_{(2,q)}$, we obtain

$$\begin{aligned} & g \cdot (S_p(h) \cdot a) \\ &= ((g_{(1,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1))S(g_{(2,q)} \cdot (S_p(h)_{(2,p^{-1})} \cdot a_2))((g_{(3,q)}S_p(h)_{(3,p^{-1})}) \cdot a_3) \\ &= ((g_{(1,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1)_1)S(g_{(2,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1)_2))((g_{(3,q)}S_p(h)_{(3,p^{-1})}) \cdot a_3) \\ &= ((g_{(1,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1))_1S((g_{(1,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1))_2)((g_{(2,q)}S_p(h)_{(2,p^{-1})}) \cdot a_2) \\ &= \epsilon^*(g_{(1,q)} \cdot (S_p(h)_{(1,p^{-1})} \cdot a_1))((g_{(2,q)}S_p(h)_{(2,p^{-1})}) \cdot a_2) \\ &= \epsilon_q(g_{(1,q)})\epsilon_{p^{-1}}(S_p(h)_{(1,p^{-1})})\epsilon(a_1)((g_{(2,q)}S_p(h)_{(2,p^{-1})}) \cdot a_2) \\ &= (gS_p(h)) \cdot a. \end{aligned}$$

□

(2) \implies (1)

It is easy to check $A \rtimes H$ is a Q -graded algebra and $\forall p \in Q$, $A \rtimes H_p$ is a coassociative counitary coalgebra and μ is a coalgebra map,

Since

$$\begin{aligned}
& ((a \rtimes h)(b \rtimes g))_{(1,pq)} \otimes ((a \rtimes h)(b \rtimes g))_{(2,pq)} = \Delta_{pq}^*((a \rtimes h)(b \rtimes g)) \\
& \stackrel{(6.9)}{=} \Delta_{pq}^*(a(h_{(1,p)} \cdot b) \otimes h_{(2,p)}g) \\
& \stackrel{(6.4)}{=} (a(h_{(1,p)} \cdot b))_1 \otimes (h_{(2,p)}g)_{(1,pq)} \otimes (a(h_{(1,p)} \cdot b))_2 \otimes (h_{(2,p)}g)_{(2,pq)} \\
& = (a_1(h_{(1,p)} \cdot b)_1) \otimes (h_{(2,p)}g)_{(1,pq)} \otimes (a_2(h_{(1,p)} \cdot b)_2) \otimes (h_{(2,p)}g)_{(2,pq)} \\
& \stackrel{(6.4)}{=} (a_1(h_{(1,p)} \cdot b_1)) \otimes (h_{(3,p)}g)_{(1,pq)} \otimes (a_2(h_{(2,p)} \cdot b_2)) \otimes (h_{(4,p)}g)_{(2,pq)} \\
& \stackrel{(2.1)}{=} (a_1(h_{(1,p)} \cdot b_1)) \otimes (h_{(3,p)}g_{(1,q)}) \otimes (a_2(h_{(2,p)} \cdot b_2)) \otimes (h_{(4,p)}g_{(2,q)}) \\
& \stackrel{(6.8)}{=} (a_1(h_{(1,p)} \cdot b_1)) \otimes (h_{(2,p)}g_{(1,q)}) \otimes (a_2(h_{(3,p)} \cdot b_2)) \otimes (h_{(4,p)}g_{(2,q)}) \\
& \stackrel{(6.9)}{=} (a_1 \rtimes h_{(1,p)})(b_1 \rtimes g_{(1,q)}) \rtimes (a_2 \rtimes h_{(2,p)})(b_2 \rtimes g_{(2,q)}) \\
& \stackrel{(6.4)}{=} (a \rtimes h)_{(1,p)}(b \rtimes g)_{(1,q)} \rtimes (a \rtimes h)_{(2,p)}(b \rtimes g)_{(2,q)},
\end{aligned}$$

and

$$\begin{aligned}
& \epsilon_{pq}^*((a \rtimes h)(b \rtimes g)) \\
& = \epsilon_{pq}^*(a(h_{(1,p)} \cdot b) \otimes h_{(2,p)}g) \\
& = \epsilon(a(h_{(1,p)} \cdot b))\epsilon_{pq}(h_{(2,p)}g) \\
& = \epsilon(a)\epsilon_p(h_{(1,p)})\epsilon(b)\epsilon_p(h_{(2,p)})\epsilon_q(g) \\
& = \epsilon(a)\epsilon_p(b)\epsilon(b)\epsilon_q(g) \\
& = \epsilon_p^*(a \otimes b)\epsilon_q^*(b \otimes g),
\end{aligned}$$

the condition (2.1) is verified.

Since

$$\begin{aligned}
& S_p^*((a \rtimes h)_{(1,p)})((a \rtimes h)_{(2,p)}(b \rtimes g)) \\
& = S_p^*(a_1 \rtimes h_{(1,p)})(a_2 \rtimes h_{(2,p)}(b \rtimes g)) \\
& = (S_p(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1) \rtimes S_p(h_{(1,p)})_{(2,p^{-1})})(a_2(h_{(2,p)} \cdot b) \rtimes h_{(3,p)}g) \\
& = (S_p(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1))(S_p(h_{(1,p)})_{(2,p^{-1})} \cdot (a_2(h_{(2,p)} \cdot b))) \\
& \otimes S_p(h_{(1,p)})_{(3,p^{-1})}(h_{(3,p)}g) \\
& = S_p(h_{(1,p)})_{(1,p^{-1})} \cdot (S(a_1)(a_2(h_{(2,p)} \cdot b))) \otimes S_p(h_{(1,p)})_{(2,p^{-1})}(h_{(3,p)}g) \\
& = \epsilon(a)S_p(h_{(2,p)}) \cdot (h_{(3,p)} \cdot b) \otimes (S_p(h_{(1,p)})(h_{(4,p)}g)) = \epsilon(a)\epsilon_p(h)b \otimes g,
\end{aligned}$$

and

$$\begin{aligned}
& (a \rtimes h)_{(1,p)}(S_p^*((a \rtimes h)_{(2,p)})(b \rtimes g)) \\
&= (a_1 \rtimes h_{(1,p)})(S_p^*(a_2 \rtimes h_{(2,p)})(b \rtimes g)) \\
&= (a_1 \rtimes h_{(1,p)})((S_p(h_{(3,p)}) \cdot S(a_2) \rtimes S_p(h_{(2,p)}))(b \rtimes g)) \\
&= a_1(h_{(1,p)} \cdot ((S_p(h_{(5,p)}) \cdot S(a)) \rtimes (S_p(h_{(4,p)}) \cdot b))) \otimes h_{(2,p)}(S_p(h_{(3,p)})g) \\
&= a_1(h_{(1,p)} \cdot ((S_p(h_{(3,p)}) \cdot S(a_2)) \rtimes (S_p(h_{(2,p)}) \cdot b))) \otimes g \\
&= a_1(h_{(1,p)} \cdot (S_p(h_{(2,p)}) \cdot (S(a_2)b))) \otimes g \\
&= \epsilon_p(h)a_1(S(a_2)b) \otimes g = \epsilon(a)\epsilon_p(h)b \otimes g,
\end{aligned}$$

the equation (2.3) is verified.

Similarly, since

$$\begin{aligned}
& ((b \rtimes g)(a \rtimes h)_{(1,p)})S_p^*((a \rtimes h)_{(2,p)}) \\
&= ((b \rtimes g)(a_1 \rtimes h_{(1,p)}))S_p^*(a_2 \rtimes h_{(2,p)}) \\
&= (b(g_{(1,q)} \cdot a_1) \rtimes g_{(2,q)}h_{(1,p)})(S_p(h_{(3,p)}) \cdot S(a_2) \rtimes S_p(h_{(2,p)})) \\
&= (b(g_{(1,q)} \cdot a_1))((g_{(2,q)}h_{(1,p)}) \cdot (S_p(h_{(4,p)}) \cdot S(a_2))) \otimes (g_{(3,q)}h_{(2,p)})S_p(h_{(3,p)}) \\
&= (b(g_{(1,q)} \cdot a_1))((g_{(2,q)}h_{(1,p)}) \cdot (S_p(h_{(2,p)}) \cdot S(a_2))) \otimes g_{(3,q)} \\
&\stackrel{(6.13)}{=} (b(g_{(1,q)} \cdot a_1))(((g_{(2,q)}h_{(1,p)})S_p(h_{(2,p)})) \cdot S(a_2)) \otimes g_{(3,q)} \\
&= \epsilon_p(h)(b(g_{(1,q)} \cdot a_1))(g_{(2,q)} \cdot S(a_2)) \otimes g_{(3,q)} \\
&\stackrel{(6.6)}{=} \epsilon_p(h)(b(g_{(1,q)} \cdot a_1))S(g_{(2,q)} \cdot a_2) \otimes g_{(3,q)} \\
&= \epsilon_p(h)(b(g_{(1,q)} \cdot a_1))S((g_{(1,q)} \cdot a)_2) \otimes g_{(2,q)} \\
&= \epsilon(a)\epsilon_p(h)b \otimes g,
\end{aligned}$$

and

$$\begin{aligned}
& ((b \rtimes g)S^*((a \rtimes h)_{(1,p)}))(a \rtimes h)_{(2,p)} \\
&= ((b \rtimes g)S_p^*(a_1 \rtimes h_{(1,p)}))(a_2 \rtimes h_{(2,p)}) \\
&= ((b \rtimes g)(S_p(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1) \rtimes S_p(h_{(1,p)})_{(2,p^{-1})}))(a_2 \rtimes h_{(2,p)}) \\
&= (b(g_{(1,q)} \cdot (S_p(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1)))) \rtimes g_{(2,q)}S_p(h_{(1,p)})_{(2,p^{-1})}(a_2 \rtimes h_{(2,p)}) \\
&= (b(g_{(1,q)} \cdot (S(h_{(1,p)})_{(1,p^{-1})} \cdot S(a_1))))((g_{(2,q)}S_p(h_{(1,p)})_{(2,p^{-1})} \cdot a_2) \otimes (g_{(3,q)}S_p(h_{(1,p)})_{(3,p^{-1})}h_{(2,p)}) \\
&= (b((g_{(1,p)}S_p(h_{(1,p)})_{(1,p^{-1})} \cdot a)_1)(g_{(1,q)}S_p(h_{(1,p)})_{(1,p^{-1})} \cdot a)_2 \otimes (g_{(2,q)}S_p(h_{(1,p)})_{(2,p^{-1})}h_{(2,p)}) \\
&= \epsilon(a)b \otimes (gS_p(h_{(1,p)}))h_{(2,p)} = \epsilon(a)\epsilon(h)b \otimes g,
\end{aligned}$$

the equation (2.4) is verified, therefore $A \otimes H$ is a Q -graded Hopf quasigroup.

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