The graded algebra of Steenrod qth powers

Grant Walker

email: grant.walker@manchester.ac.uk 2010 Mathematics Subject Classification: 55S10

Abstract

The algebra A_q of Steenrod qth powers, where $q = p^e$ is a power of a prime p , is isomorphic to a subalgebra A'_{q} of the algebra of Steenrod pth powers A_p . The filtration of A_p by powers of its augmentation ideal was studied by J. P. May in his Princeton thesis of 1964. We extend some of May's results to A_q and obtain a convenient set of defining relations for the graded algebra $E^0(\mathsf{A}_q)$. In the case $q = p$, we recover the observation of S. B. Priddy that the subalgebra $E^0(\mathsf{A}_p(n-2))$ of $E^0(\mathsf{A}_p)$ generated by the elements P^{p^j} for $0 \leq j \leq n-2$ is isomorphic to the graded algebra associated to the augmentation ideal filtration of the group algebra $\mathbb{F}_p \mathsf{U}(n)$, where $\mathsf{U}(n)$ is the group of upper unitriangular matrices over \mathbb{F}_p .

The Arnon A basis of A_p is given by monomials which are minimal in the left lexicographic order on formal monomials in the Steenrod powers. K. G. Monks (for $p = 2$) and D. Yu. Emelyanov and Th. Yu. Popelensky (for $p > 2$) have found a triangular relation between this basis and the Milnor basis using a certain ordering on the Milnor basis. We introduce a variant of the Arnon A basis which is minimal for the right order, and show that this basis and Arnon's original A basis are also triangularly related to the Milnor basis of A_q using the right order on the Arnon A basis.

1 Introduction

Given a prime p, we denote by A_p the algebra of Steenrod pth powers. As an algebra over the field of p elements \mathbb{F}_p , A_p may be regarded as the subalgebra of the mod p Steenrod algebra \mathcal{A}_p which is generated by the elements P^r , $r \geq 0$, subject to the relation $P^0 = 1$ and the Adem relations

$$
P^{a}P^{b} = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^{j}, \ a < pb,
$$

where the binomial coefficients are taken mod p , or alternatively as the quotient algebra $A_p/A_p\beta A_p$, where β is the Bockstein. As a subalgebra of A_p , the element

P^r is given the degree $2r(p-1)$, but for simplicity we regrade A_p by giving P^r the 'reduced' degree r. Thus when $p = 2$, P^r will mean Sq^r , and not Sq^{2r} .

For a prime power $q = p^e$, where $e \geq 1$, the algebra A_q of Steenrod qth powers [\[12,](#page-28-0) Chapter 11] can be defined as an algebra over the Galois field \mathbb{F}_q by generators P^r , $r \geq 0$, subject to the relation $P^0 = 1$ and the Adem relations

$$
P^{a}P^{b} = \sum_{j=0}^{[a/q]} (-1)^{a+j} \binom{(q-1)(b-j)-1}{a-qj} P^{a+b-j} P^{j}, \ a < qb,
$$
 (1)

where, as before, the binomial coefficients are taken mod p . As the coefficients lie in the prime subfield \mathbb{F}_p , we have an algebra A'_q defined over \mathbb{F}_p by the same generators and relations, and an isomorphism $\rho : A_q^{\dagger} \to A_q^{\dagger} \otimes_{\mathbb{F}_p} \mathbb{F}_q$ of Hopf algebras.

We introduce the algebras A_q and A'_q in Section [2.](#page-2-0) The algebra A'_q may be identified with the subalgebra of A_p with Milnor basis given by the elements $P(R) = P(r_1, r_2, \ldots)$ such that $r_j = 0$ when $j \neq 0$ mod e [\[9,](#page-28-1) Section 12.3]. As for A_p , we grade A_q by assigning degree r to P^r. With this choice of gradings, the element $P^r \in A_q$ corresponds to $P(0,\ldots,0,r) \in A'_q$, with r in position e, and the map ρ multiplies the grading by $(q-1)/(p-1)$. It can be verified as in [\[9,](#page-28-1) Proposition 3.2.1 or 12.3.3] that the relations [\(1\)](#page-1-0) are satisfied in A'_q . As in [\[9,](#page-28-1) Chapter 3], it follows from the Adem relations and the action of A'_q on the polynomial algebra $\mathbb{F}_p[x]$ that the elements $P(0,\ldots,0,p^s)$ for $s\geq 0$ are indecomposable in A'_{q} , and form a minimal set of generators.

In Section [3](#page-8-0) we discuss the filtration of A_p by powers of the augmentation ideal A_p^+ . This was studied by J. P. May in his Princeton thesis of 1964 [\[5\]](#page-28-2), and is known as the **May filtration**. Thus an element $\theta \in A_p$ has May filtration $M(\theta) = m$ if $\theta \in (\mathsf{A}_p^+)^m$ but $\theta \notin (\mathsf{A}_p^+)^{m+1}$. Since the ideal A_p^+ is generated by the indecomposable elements $P^{p^j}, j \geq 0$, the elements of May filtration 1 in degree p^j are the elements whose expansion in any basis containing P^{p^j} contains P^{p^j} as a term.

Example 1.1. Since $Sq^2 Sq^2 = Sq^3Sq^1 = Sq^1Sq^2Sq^1$, $M(Sq^2Sq^2) = 3$. Similarly $M(Sq^{2}Sq^{1}Sq^{2}) = 3$ and $M(Sq^{2}Sq^{1}Sq^{2}Sq^{1}) = 4$.

Definition 1.2. For an integer $a \geq 0$, let $a = \sum_{i=0}^{r} a_i p^i$ be the base p expansion of a, where $0 \le a_i \le p-1$ for all i. We define $\alpha(a) = \sum_{i=0}^r a_i$.

The May filtration has some elementary properties.

Proposition 1.3. (i) If $\theta \in A_p^d$, then $M(\theta) \ge \alpha(d)$.

- (ii) For all $\theta_1, \theta_2 \in A_p$, $M(\hat{\theta}_1 \theta_2) \geq M(\theta_1) + M(\theta_2)$.
- (iii) For all $\theta \in A_p$, $M(\chi(\theta)) = M(\theta)$, where χ is the antipode of A_p .

Proof. For (i), an element $\theta \in A_p^d$ cannot be the product of $\langle \alpha(d) \rangle$ elements of the form P^{p^j} . For (ii), we observe that $(A_p^+)^r(A_p^+)^s \subseteq (A_p^+)^{r+s}$. For (iii), we observe that $\chi(\mathsf{A}_p^+) = \mathsf{A}_p^+$. ◻

May determined the filtration on A_p by evaluating it on the Milnor basis, and showing that for a general element $\theta \in A_p$, $M(\theta)$ is the minimum of the filtrations of the terms in the expansion of θ in the Milnor basis.

Theorem 1.4. (May) For any sequence $R = (r_1, r_2, \ldots, r_\ell)$ of integers ≥ 0 , $M(P(R)) = \sum_{i=1}^{\ell} i \alpha(r_i).$

We give a proof of Theorem [1.4](#page-2-1) in Section [3,](#page-8-0) and extend the result to the prime power case.

Given a filtered algebra A, we denote the associated graded algebra by $E^0(A)$. The following observation of S. B. Priddy relates the Steenrod algebra to the group algebra of the group $\mathsf{U}(n)$ of $n \times n$ upper unitriangular matrices over \mathbb{F}_p . A proof of this result, based on [\[5\]](#page-28-2), is given in [\[7\]](#page-28-3). The subalgebra $A_p(n-2)$ of A_p is generated by the elements P^{p^i} , $0 \le i \le n-2$, and has dimension $p^{n(n-1)/2}$ as a vector space over \mathbb{F}_p .

Theorem 1.5. (Priddy) The algebras $E^0(\mathbb{F}_p\mathsf{U}(n))$ and $E^0(\mathsf{A}_p(n-2))$ are isomorphic as graded Hopf algebras.

In Section [4](#page-11-0) we give a convenient set of generators and relations for $E^0(\mathsf{A}_q)$, and in Section [5](#page-15-0) we use a theorem of Quillen [\[11\]](#page-28-4) to prove Theorem [1.5.](#page-2-2) In Sections [6,](#page-19-0) [7](#page-21-0) and [8](#page-24-0) we extend the work of K. G. Monks (for $p = 2$) and of D. Yu. Emelyanov and Th. Yu. Popelensky (for $p > 2$) on the relation between various bases of A_p [\[3,](#page-28-5) [8\]](#page-28-6). In Section [6](#page-19-0) we show that some of the bases known as P_t^s -bases coincide with the Milnor basis up to higher May filtration, and hence give the same basis of $E^0(A_q)$. In Sections [7](#page-21-0) and [8,](#page-24-0) we introduce a variant of Arnon's A basis, and show that this basis and the original A basis are triangular with respect to the Milnor basis of A_q , using the right lexicographic order on the A basis. This is a different ordering from the one used in [\[3,](#page-28-5) [8\]](#page-28-6).

2 The algebras A_q and A_q' q

Let p be a prime number and let $q = p^e, e \ge 1$, be a power of p. The Milnor basis elements $P_e^r = P(0, \ldots, 0, r)$, with $r \ge 1$ in position e, generate a Hopf subalgebra A'_{q} of A_{p} . This subalgebra has an additive basis given by Milnor basis elements $P(R)$, where $R = (r_1, r_2, \ldots)$ is a finite sequence of integers ≥ 0 such that $r_j = 0$ if j is not a multiple of e . This is clear from Milnor's product formula, since a Milnor matrix $X = (x_{i,j})$ which arises when two such elements are multiplied must have entries $x_{i,j} = 0$ unless i and j are both divisible by e, so that $i + j$ is divisible by e. Hence we make the following definition.

Definition 2.1. For $e \geq 1$ and $R = (r_1, \ldots, r_\ell)$, let R_e be the sequence whose jth term is r_k if $j = ke$ for $k \geq 1$, and is 0 otherwise, and let $P_e(R) = P(R_e)$ be the corresponding Milnor basis element of A_p . The algebra A'_q is the subalgebra of A_p with \mathbb{F}_p -basis elements $P_e(R)$ for all finite sequences R of integers ≥ 0 .

Proposition 2.2. The Poincaré series $\Pi(\mathsf{A}'_q, \tau) = \prod_{j \geq 1} 1/(1 - \tau^{(q^j-1)/(p-1)})$.

Proof. Since we use the reduced grading on A_p in which P^r has degree r, $P(R)$ has degree $\sum_{j\geq 1} r_j(p^j-1)/(p-1)$, and $P_e(R)$ has degree $\sum_{j\geq 1} r_j(q^j-1)/(p-1)$. Thus $\dim(A'_q)^{\bar{d}}$ is the number of solutions $R = (r_1, r_2, \ldots)$ of the equation

$$
(p-1)d = (q-1)r_1 + (q^2 - 1)r_2 + \dots + (q^j - 1)r_j + \dots, \tag{2}
$$

where $r_j \geq 0$ for $j \geq 1$. A solution of [\(2\)](#page-3-0) gives an expression for $(p-1)d$ as the sum of $|R| = \sum_j r_j$ terms, of which r_j are equal to $q^j - 1$ for $j \ge 1$. Since $1/(1 - \tau^{(q^j-1)/(p-1)}) = \sum_{i \geq 0} \tau^{i(q^j-1)/(p-1)}$, this corresponds to a term of degree d in the power series expansion of $\prod_{j\geq 1} 1/(1-\tau^{(q^j-1)/(p-1)})$. \Box

We show that A'_{q} satisfies the Adem relations [\(1\)](#page-1-0).

Proposition 2.3. Let $q = p^e$ where $e \ge 1$, and for $r \ge 0$, let $P_e^r = P(0, \ldots, 0, r)$ in A_p , where r is in position e. Then for $a < qb$

$$
P_e^a P_e^b = \sum_{j=0}^{[a/q]} (-1)^{a+j} \binom{(q-1)(b-j)-1}{a-qj} P_e^{a+b-j} P_e^j
$$

in A_p , where the binomial coefficient is taken mod p.

Proof. By the Milnor product formula

$$
P_e^a P_e^b = \sum_{k=0}^b {a+b-(q+1)k \choose b-k} P_e(a+b-(q+1)k, k).
$$

Using this to expand both sides of the required relation in the Milnor basis, and equating coefficients of $P_e(a+b-(q+1)k, k)$, we see that the relation is equivalent to the identity

$$
\binom{a+b-(q+1)k}{b-k} = \sum_{j=k}^{[a/q]} (-1)^{a+j} \binom{(q-1)(b-j)-1}{a-qj} \binom{a+b-(q+1)k}{j-k},
$$

of mod p binomial coefficients, where $0 \leq k \leq b$. The change of variables $a' =$ $a - qk$, $b' = b - k$ and $j' = j - k$ reduces this to the case $k = 0$, namely

$$
\binom{a'+b'}{b'} = \sum_{j'=0}^{[a'/q]} (-1)^{a'+j'} \binom{(q-1)(b'-j')-1}{a'-qj'} \binom{a'+b'}{j'}.
$$
 (3)

Replacing a', b' by a, b , we prove [\(3\)](#page-3-1) by induction on $a+b$. The base case $(a, b) =$ $(0, 0)$ holds since $\binom{c}{0}$ $\binom{c}{0} = 1$ for all integers c. We assume that the cases $(a - 1, b)$ and $(a, b - 1)$ hold, and prove the case (a, b) . For this we use the identity

$$
\begin{pmatrix} c-1 \ d \end{pmatrix} + \begin{pmatrix} c-1 \ d-1 \end{pmatrix} = \begin{pmatrix} c \ d \end{pmatrix} = \begin{pmatrix} c-q \ d \end{pmatrix} + \begin{pmatrix} c-q \ d-q \end{pmatrix}
$$
 (4)

for mod p binomial coefficients, which follows from $(1+x)^c = (1+x)^q (1+x)^{c-q}$, since $(1+x)^q = 1+x^q$ mod p. Writing $c = (q-1)(b-j)$ and $d = a - qj$, we can expand each term on the right of [\(3\)](#page-3-1) in the form

$$
\begin{pmatrix} c-1 \ d \end{pmatrix} \begin{pmatrix} a+b \ j \end{pmatrix} = \begin{pmatrix} c-1 \ d \end{pmatrix} \begin{pmatrix} a+b-1 \ j \end{pmatrix} + \begin{pmatrix} c-1 \ d \end{pmatrix} \begin{pmatrix} a+b-1 \ j-1 \end{pmatrix}
$$

$$
= -\begin{pmatrix} c-1 \ d-1 \end{pmatrix} \begin{pmatrix} a+b-1 \ j \end{pmatrix} + \begin{pmatrix} c-q \ d \end{pmatrix} \begin{pmatrix} a+b-1 \ j \end{pmatrix} + \begin{pmatrix} c-q \ d-q \end{pmatrix} \begin{pmatrix} a+b-1 \ j \end{pmatrix} + \begin{pmatrix} c-1 \ d \end{pmatrix} \begin{pmatrix} a+b-1 \ j-1 \end{pmatrix}.
$$

The alternating sum over j of the first term on the right is $\binom{(a-1)+b}{b}$ and that b of the second term is $\binom{a+(b-1)}{b-1}$ $(b-1)(b-1)$, by the inductive hypothesis. Since $c - q =$ $(q-1)(b-j-1) - 1$ and $d - q = a - q(j+1)$, the third and fourth terms cancel on taking the alternating sums over j. Since $\binom{a+b-1}{b}$ $\binom{b-1}{b} + \binom{a+b-1}{b-1}$ $\binom{+b-1}{b-1} = \binom{a+b}{b}$ $\binom{+b}{b},$ this completes the inductive step.

The Adem relations lead to a basis of admissible monomials for A'_q .

Definition 2.4. For $A = (a_1, a_2, \ldots, a_\ell)$, where $a_i \geq 0$ for $1 \leq i \leq \ell$, $P_e^A =$ $P_e^{a_1}P_e^{a_2}\cdots P_e^{a_\ell}$ is a **monomial** in A'_q . The sequence $A = (a_1, \ldots, a_\ell)$ and the monomial P_e^A are **admissible** if $a_i \ge qa_{i+1}$ for $1 \le i < l$.

If $a_i = 0$ for all i then $P_e^0 = 1$ is admissible. Otherwise it suffices to consider finite sequences A of positive integers. If $A = (a_1, a_2, \ldots, a_\ell)$, with $a_i > 0$ for $1 \leq i \leq \ell$, we define the **length** len(A) = ℓ and the **modulus** $|A| = \sum_{i=1}^{\ell} a_i$. By adding trailing zeros, we may identify A with the infinite sequence $(a_1, a_2, \ldots),$ where $a_i = 0$ for $i > \ell$. We introduce two linear orders on such sequences, the left order \leq_l and the right order \leq_r .

Definition 2.5. Let $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$ be sequences of integers ≥ 0 . Then $A \leq_l B$ if and only if, for some k, $a_j = b_j$ for $1 \leq j \leq k$ and $a_k \leq b_k$, and $A \leq r B$ if and only if, for some k, $a_j = b_j$ for $j > k$ and $a_k > b_k$.

Thus \lt_l is the usual left lexicographic order, but \lt_r is the reversal of the usual right lexicographic order.

Proposition 2.6. Every element of A'_q is a sum of admissible monomials.

Proof. For $d \geq 0$, let S^d denote the set of sequences $A = (a_1, a_2, \ldots)$ such that $a_i > 0$ for $i \leq \text{len}(A)$ and $|A| = d$. Each element of S^d gives a corresponding monomial $P_e^A \in (A_q')^d$. If A is not admissible, then $0 < a_k < qa_{k+1}$ for some k.

Using the Adem relation [\(1\)](#page-1-0) with $a = a_k$ and $b = a_{k+1}$, we may write P_e^A as a sum of monomials P_e^B , where B is obtained from A by replacing (a, b) by $(a+b-j, j)$ for some j such that $0 \leq j \leq [a/q] < b$. In the case $j = 0$, we omit the corresponding term of B: this does not affect P^B since $P_e^0 = 1$. Then $B >_{l,r} A$. Hence P_e^A can be written as a sum of monomials P_e^B which are greater than P_e^A in both the left and right orders. Iteration of this procedure must stop, since S^d is a finite set. Hence P_e^A can be expressed as a sum of admissible monomials. \Box

Definition 2.7. Let $A = (a_1, \ldots, a_\ell)$ be an admissible sequence of length ℓ . The Milnor sequence of A, or of Sq^A , is $R = (r_1, \ldots, r_\ell)$, where $r_j = a_j - qa_{j+1}$ for $1 \leq j < \ell$ and $r_{\ell} = a_{\ell}$.

Proposition 2.8. The map which sends an admissible sequence A to its Milnor sequence R is a bijection from the set of admissible sequences to the set of finite sequences of integers ≥ 0 , and it preserves the length ℓ and the right order \lt_r . If $P_e^A \in (\mathsf{A}'_q)^d$, then $|R| = qa_1 - d$ and $d = \sum_{j=1}^{\ell} (q^j - 1)r_j$.

Proof. Since $r_{\ell} = a_{\ell}$, the map $A \mapsto R$ preserves length. Given $R = (r_1, \ldots, r_{\ell})$ of length ℓ , the linear equations $r_j = a_j - q a_{j+1}$ can be solved recursively for $j = \ell, \ell - 1, \ldots, 1$ to give $a_j = \sum_{i=j}^{\ell} q^{i-j} r_i$. In particular, $a_1 = \sum_{i=1}^{\ell} q^{i-1} r_i$. Since $A = (a_1, a_2, \ldots)$ is admissible, these equations give an inverse map $R \mapsto A$.

For the right order, it suffices to consider sequences of the same length ℓ , since $A \leq r B$ when len(A) > len(B). Let $A = (a_1, \ldots, a_\ell)$ and $B = (b_1, \ldots, b_\ell)$ be admissible sequences of length ℓ , with corresponding Milnor sequences $R =$ (r_1, \ldots, r_ℓ) and $S = (s_1, \ldots, s_\ell)$. If $a_j = b_j$ for $j > k$ and $a_k > b_k$, then $r_j = s_j$ for $j > k$ and $r_k > s_k$. Hence $R <_r S$ if $A <_r B$. The sum of the equations $r_j = a_j - qa_{j+1}, 1 \le j < \ell$, gives $|R| = a_1 - (a_2 + \cdots + a_s) = qa_1 - d$. With the jth equation weighted by q^{j-1} , $d = \sum_{j=1}^{\ell} (q^j - 1)r_j$. ப

Proposition 2.9. The set of admissible monomials is a \mathbb{F}_p -basis for A'_q .

Proof. By Proposition [2.6,](#page-4-0) the admissible monomials span A'_q , and it follows from Proposition [2.8](#page-5-0) that there is a bijection between admissible monomials and Milnor basis elements in A'_q . \Box

Following [\[9\]](#page-28-1), we denote by $\sin(a)$ the set of 2-powers in the base 2 decomposition of an integer $a \geq 0$. For an odd prime p, this becomes a multiset.

Definition 2.10. For an integer $a \ge 0$, let $a = \sum_{i=0}^{r} a_i p^i$, where $0 \le a_i \le p-1$ for all i. We denote by $\text{pin}(a)$ the multiset of $\alpha(d)$ powers of p whose sum is d.

When $p = 3$, for example, $\pi(25) = \{1, 3, 3, 9, 9\}$ and $\alpha(25) = 5$. Thus $\alpha(a)$ is the number of elements in pin(a), counted with multiplicity $\leq p-1$. The binomial coefficient $\binom{a}{b}$ $\binom{a}{b} \neq 0$ mod p if and only if $\text{pin}(b)$ is a sub-multiset of $\text{pin}(a)$.

Proposition 2.11. The set of elements $P_e^{p^j}$ $e^{p p^j}, j \geq 0$, is a minimal generating set for A'_q as an algebra over \mathbb{F}_p .

Proof. Since $P_e^0 = 1$, A'_q is generated by the elements P_e^k , $k \geq 1$. If k is not a power of p, then $k = p^r s$, where $r \ge 0$ and $s \ge 1$. Let $a = p^r$ and $b = p^r (s - 1)$ in the Adem relation [\(1\)](#page-1-0). We claim that $p^r \in \text{pin}((p-1)b-1)$. Since $(p-1)b-1 = -1$ mod p^r, this is equivalent to proving that $(p-1)b-1 \neq p^r-1 \mod p^{r+1}$. Since $b = p^r(s-1)$, $(p-1)b-1 = p^r-1 \mod p^{r+1}$ if and only if $(s-1)(p-1) = 1 \mod p$ p, and this is equivalent to $s(p-1) = 0 \text{ mod } p$. Since $s \neq 0 \text{ mod } p$, this proves the claim.

It follows that $\binom{(p-1)b-1}{a}$ $\binom{a}{a}$ = 0 mod p, and so the last term $P_e^{a+b} = P_e^k$ appears in the Adem relation. Hence P_e^k is in the subalgebra generated by the elements P_e^i for $i < k$. By iterating the argument, it follows that A'_q is generated by the elements $P_e^{p^j}$ $e^{p^j}, j \geq 0.$

If $P_e^{p^j}$ e^{p^p} could be omitted from this generating set, then by considering the grading on A_q it would follow that $P_e^{p^j}$ e^{p^p} is in the subalgebra generated by the elements P_e^i for $i < p^j$. But this is false, since for the action of A_p on $\mathbb{F}_p[x]$, we have $P_e^i(x^{p^j}) = 0$ for $0 < i < p^j$, while $P_e^{p^j}$ $e^{p^j}(x^{p^j}) = x^{qp^j}$. Hence the generating set is minimal. \Box

The next result shows that A'_q is a Hopf subalgebra of A_p . The proof follows from the corresponding formulae in A_p .

Proposition 2.12. The coproduct
$$
\phi : A'_q \to A'_q \otimes A'_q
$$
 satisfies
\n(i) $\phi(P_e^k) = \sum_{i+j=k} P_e^i \otimes P_e^j$ for all $k \ge 0$,
\n(ii) $\phi(P_e(R)) = \sum_{S+T=R} P_e(S) \otimes P_e(T)$ for all sequences R.

The dual Hopf algebra A_p^* of A_p is a polynomial algebra over \mathbb{F}_p with generators ξ_j , $j \geq 1$, of degree $p^j - 1$ [\[6\]](#page-28-7). The dual algebra $(A'_q)^*$ is a quotient of A_p^* , and may be described as follows.

Proposition 2.13. For $e \geq 1$ and $q = p^e$, $(A'_q)^* = A_p^*/I_q$, where I_q is the Hopf ideal in A_p^* generated by the elements ξ_j such that e does not divide j. The algebra $(A'_q)^*$ is a polynomial algebra over \mathbb{F}_p with generators ξ_{ke} , $k \geq 1$ of degree $q^k - 1$. The coproduct in $(A'_q)^*$ is defined by $\phi(\xi_{ke}) = \sum_{i=0}^k \xi_{(k-i)e}^{q^i} \otimes \xi_{ie}$, and the antipode by $\chi(\xi_{ke}) = \sum_A \xi(A)$, where the sum is over compositions $A = (a_1, \ldots, a_s)$ of k, and $\xi(A) = \xi_{a_1} \xi_{a_2}^{q^{a_1}}$ $a_{2}^{qa_{1}} \xi_{a_{3}e}^{q^{a_{1}+a_{2}}}$ $\epsilon_{a_3e}^{q^{a_1+a_2}}\cdots \xi_{a_se}^{q^{a_1+\cdots+a_{s-1}}},$ $a_s e^{a_1+\cdots+a_{s-1}}$.

We next introduce the algebra A_q of Steenrod qth powers [\[12,](#page-28-0) Chapter 11].

Definition 2.14. Let $q = p^e$ be a prime power, where $e \geq 1$. The algebra A_q of Steenrod qth powers is the algebra over the Galois field \mathbb{F}_q generated by elements P^r , $r \geq 0$, subject to the relation $P^0 = 1$ and the Adem relations [\(1\)](#page-1-0).

Note that the coefficients in the Adem relations lie in the prime subfield \mathbb{F}_p , so that we also have an algebra defined over \mathbb{F}_p with the same generators and relations. The preceding results allow us to identify this algebra with A'_q .

Proposition 2.15. There is an isomorphism $\rho : A_q \to A'_q \otimes_{\mathbb{F}_p} \mathbb{F}_q$ of Hopf algebras defined by $\rho(P^r) = P_e^r$, $r \ge 0$. \Box

As for A_p , we grade A_q by assigning degree r to P^r for $r \geq 0$. Since $P_e^r =$ $P(0, \ldots, 0, r) \in A_p$, with r in position e, has degree $r(q-1)/(p-1)$, the map $ρ$ multiplies the grading by $(q-1)/(p-1)$. The algebra A_q has a Milnor basis $\sum_i r_i(q^i-1)/(q-1)$. Then Propositions [2.2](#page-3-2) and [2.11](#page-5-1) give the following result. given by elements $P(R)$, where $\rho(P(R)) = P_e(R)$. The degree of $P(R)$ in A_q is

Proposition 2.16. (i) The elements P^{p^j} , $j \geq 0$, form a minimal generating set for A_q as an algebra over \mathbb{F}_q .

(ii) The Poincaré series of A_q is $\Pi(A_q, \tau) = \prod_{j \geq 1} 1/(1 - \tau^{(q^j-1)/(q-1)})$. \Box

The Milnor product formula holds in A_q as in A_p , with the row sum condition $r_i = \sum_{j\geq 0} p^j x_{i,j}$ on Milnor matrices replaced by $r_i = \sum_{j\geq 0} q^j x_{i,j}$, the column and diagonal sum conditions unchanged, and the multinomial coefficients taken in \mathbb{F}_p .

The standard action of A_p on the polynomial algebra $\mathbb{F}_p[x]$ is determined by the formula $P^{k}(x^d) = \binom{d}{k}$ $\kappa^{d}(x^{d+ (p-1)k})$, so that $P^{k}(x^{d})$ is a nonzero multiple of x^{d+k} if $\text{pin}(k) \subseteq \text{pin}(d)$, and $P^k(x^d) = 0$ otherwise. It is usual to grade $\mathbb{F}_p[x]$ by giving x degree 2 if $p > 2$, but we assign grading 1 to x in all cases, as we deal only with A_p and not the full Steenrod algebra A_p . Thus the action of P^k on a power of x raises the degree by $k(p-1)$. The Milnor basis element $P(0,\ldots,0,k) \in A'_q$, with k in position e, acts on $\mathbb{F}_p[x]$ by $P(0,\ldots,0,k)(x^d) = {d \choose k}$ $\binom{d}{k} x^{d+(q-1)k}$. We use the isomorphism ρ of Proposition [2.15](#page-7-0) to transfer this action to A_q , so that (with notation in A_q) the corresponding statement is $P^k(x^d) = \binom{d}{k}$ $\binom{d}{k} x^{d+(q-1)k}$. This action of A_q in $\mathbb{F}_p[x]$ can alternatively be defined directly, as in [\[9,](#page-28-1) Chapter 1], by defining a 'total Steenrod qth power' $P = \sum_{k\geq 0} P^k$ acting on the polynomial algebra $\mathbb{F}_p[x_1,\ldots,x_n]$ by the formulae $P(1) = 1, \mathbb{P}(x_i) = x_i + x_i^q$ $_i^q$ for $1 \leq i \leq n$ and the Cartan formula $P(fg) = P(f)P(g)$ for polynomials f and g.

For $p = 2$ and $k > 0$, the combinatorial description of the formula $Sq^k(x^d) =$ x^{d+k} given in [\[9,](#page-28-1) Proposition 6.1.2] shows that the (reverse) base 2 expansion of $d+k$ is obtained from that of d by replacing at least one subsequence of the form $1 \cdots 1$ 0 by a subsequence $0 \cdots 0$ 1 of the same length. Hence the number of digits 1 in the sequence may be decreased, but not increased, and so $\alpha(d+k) \leq \alpha(d)$. A similar argument holds for an odd prime p , using the reversed base p expansions of d and $d+k(p-1)$. We find that the expansion of $d+k(p-1)$ is obtained from that of d by replacing at least one subsequence $a_1 \cdots a_{r-1}$ 0, where $0 \le a_i \le p-1$, by a subsequence $0 \, b_2 \cdots b_r$, with $\sum_i b_i \leq \sum_i a_i$. The same is true for the action of a general element $\theta \in A_p$, since θ is a sum of compositions of the elements P^k . The generalization to A_q behaves in the same way, using the mod p expansions of d and $d + k(p-1)$, since the action of A_q on $\mathbb{F}_p[x_1, \ldots, x_n]$ is the same as that of A'_q .

3 The May filtration of A_{q}

The May filtration of A_q is defined as the filtration by powers of the augmentation ideal A_q^+ . The following result extends Theorem [1.4](#page-2-1) to A_q .

Proposition 3.1. For $R = (r_1, r_2, \ldots, r_\ell)$, the Milnor basis element $P(R) \in A_q$ has May filtration $M(P(R)) = \sum_{i=1}^{\ell} i \alpha(r_i)$. In particular, for $s \geq 0$ and $t \geq 1$ the element $P_t^s = P(0, \ldots, 0, p^s)$, where p^s is in position t, has May filtration t.

The numerical function α is defined for A_q as for A_p (Definition [2.10\)](#page-5-2). Thus if $a = \sum_{j\geq 0} a_j p^j$ where $0 \leq a_j \leq p-1$ for all j, then $\alpha(a) = \sum_{j\geq 0} a_j$, so that $\alpha(a)$ is the cardinality of the multiset $\text{pin}(a)$ given by the base p digits of a. We note that the element $P_e(R) \in A'_q \subset A_p$ has May filtration $M(P_e(R)) = e \cdot M(P(R)).$

We shall establish Proposition [3.1](#page-8-1) by proving inequalities in each direction.

Proposition 3.2. For $R = (r_1, r_2, \ldots, r_\ell)$, the Milnor basis element $P(R) \in A_q$ has May filtration $M(P(R)) \geq \sum_{i=1}^{\ell} i \alpha(r_i)$.

Proof. We argue by induction on $|R| = \sum_{i=1}^{\ell} r_i$. In the base case $|R| = 1$, $R = (0, \ldots, 0, 1)$ with 1 in position t, say. Thus the degree of $P(R)$ in A_q is $d = (q^t - 1)/(q - 1) = 1 + q + q² + \cdots + q^{t-1}$. Since an element of degree d in A_q must have May filtration $\geq \alpha(d)$, $M(P(R)) \geq t$. Thus we assume, as main induction hypothesis, that for $r \geq 2$ the inequality holds for all Milnor basis elements $P(S)$ in A_q with $|S| < r$, and we consider $P(R)$ with $|R| = r$.

For the inductive step, we first deal with the case where R has only one nonzero term r, using a subsidiary induction on $\alpha(r)$. If $\alpha(r) = 1$, then $R =$ $(0, \ldots, 0, p^s)$ with p^s in position t, so that $P(R) = P_t^s$. Thus $P(R)$ has degree $d = p^{s}(q^{t}-1)/(q-1)$ and $\alpha(d) = t$, so as before $M(P(R)) \geq t$. For the inductive step on $\alpha(r)$, let $R = (0, \ldots, 0, r)$ where $\alpha(r) > 1$. Let $r = a + b$, where $a, b > 0$ are chosen so that $\text{pin}(r)$ is the disjoint union of $\text{pin}(a)$ and $\text{pin}(b)$. Then in the Milnor product formula for $P(0, \ldots, 0, a)P(0, \ldots, 0, b)$ (both of length t) the initial Milnor matrix gives a nonzero multiple of $P(R)$. Any other term $P(S)$ in the product arises from a Milnor matrix of the form

$$
X = \begin{array}{c|ccccc}\n & 0 & \cdots & 0 & b - c \\
\hline\n0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
a - q^t c & 0 & \cdots & 0 & c\n\end{array}
$$
, where $c > 0$.

Thus $S = (0, \ldots, 0, (a - q^t c) + (b - c), 0, \ldots, 0, c)$, where $\text{pin}(a - q^t c)$ and $\text{pin}(b - c)$ are disjoint, and the nonzero terms are in positions t and 2t. Then $|S| = r - q^t c$ r, and so by the main induction hypothesis on $|S|$, $M(P(S)) \geq t\alpha((a - q^t c) +$ $(b-c)+2t\alpha(c)$. Since $\alpha(a) \leq \alpha(a-q^tc) + \alpha(q^tc) = \alpha(a-q^tc) + \alpha(c)$ and $\alpha(b) \leq$ $\alpha(b-c) + \alpha(c)$, $M(P(S)) \ge t(\alpha(a) + \alpha(b)) = t\alpha(r)$. By the induction hypothesis on $\alpha(r)$, we also have $M(P(0, \ldots, 0, a)) \geq t\alpha(a)$ and $M(P(0, \ldots, 0, b)) \geq t\alpha(b)$, so $M(P(0, \ldots, 0, a)P(0, \ldots, 0, b)) \geq t\alpha(r)$ also. Hence the Milnor product formula implies that $M(P(R)) \geq t\alpha(r)$. This completes the induction on $\alpha(r)$, and the proof in the case $R = (0, \ldots, 0, r)$.

We may now assume that $R = (0, \ldots, 0, r_1, r_2, \ldots, r_\ell)$, where $r_1 > 0$ is in position k, and where $r_i > 0$ for some $i > 1$. Applying the Milnor product formula to $P(0, \ldots, 0, r_2, \ldots, r_\ell) P(0, \ldots, 0, r_1)$, the initial Milnor matrix gives a nonzero multiple of $P(R)$. Any other term $P(S)$ in the product arises from a Milnor matrix of the form

where the nonzero column is column k, the nonzero rows are rows $k + 1, \ldots, k +$ $\ell-1$, and some $c_i > 0$. Thus $|S| = |R| - q^k \sum_{i=2}^{\ell} c_i < |R|$, and so by the induction hypothesis on $|S|$

$$
M(P(S)) \geq k\alpha(r_1 - \sum_{i=2}^{\ell} c_i) + \sum_{i=2}^{\ell} (k+i-1)\alpha(r_i - q^k c_i) + \sum_{i=2}^{\ell} (2k+i-1)\alpha(c_i),
$$

where we have used the assumption that the coefficient $b(X) \neq 0$ to deal with diagonals in X with two nonzero entries.

We use the inequalities

$$
\alpha(r_1) \leq \alpha(r_1 - \sum_{i=2}^{\ell} c_i) + \alpha(\sum_{i=2}^{\ell} c_i),
$$

$$
\alpha(r_i) \leq \alpha(r_i - q^k c_i) + \alpha(c_i), \ 2 \leq i \leq \ell.
$$

Weighting and adding these inequalities, we obtain

$$
k\alpha(r_1) + \sum_{i=2}^{\ell} (k+i-1)\alpha(r_i) \le k\alpha(r_1 - \sum_{i=2}^{\ell} c_i) + k\alpha(\sum_{i=2}^{\ell} c_i)
$$

+
$$
\sum_{i=2}^{\ell} (k+i-1)\alpha(r_i - q^k c_i) + \sum_{i=2}^{\ell} (k+i-1)\alpha(c_i)
$$

$$
\le M(P(S)).
$$

We conclude that if the term $P(S)$ appears in the expansion of

$$
P(0, \ldots, 0, r_2, \ldots, r_\ell) P(0, \ldots, 0, r_1)
$$

then $M(P(S)) \geq \sum_{i=1}^{\ell} (k+i-1)\alpha(r_i)$. By the induction hypothesis, the product itself has May filtration $\geq \sum_{i=2}^{\ell} (k+i-1)\alpha(r_i) + k\alpha(r_1) = \sum_{i=1}^{\ell} (k+i-1)\alpha(r_i)$, and so the same is true for the initial term $P(R)$. This completes the induction.

To complete the proof of Theorem [1.4,](#page-2-1) we use the standard action of Milnor basis elements on polynomials to prove the reverse inequality.

Proposition 3.3. For $R = (r_1, r_2, \ldots, r_\ell)$, the Milnor basis element $P(R) \in A_q$ has May filtration $M(P(R)) \leq \sum_{i=1}^{\ell} i \alpha(r_i)$.

Proof. We first consider two special cases. For the first case, let $q = p$ and $P(R) = P_t^s$, so that $P_t^s(x^{p^s}) = x^{p^{s+t}} \in \mathbb{F}_p[x]$. There is only one digit 1 in the (reversed) base p expansion of p^s , which is moved from position s to position $s + t$. This move cannot be broken down into more than t steps by the action of Steenrod operations in A_p of positive degree, and so P_t^s cannot be expressed as an element of $(A_p^+)^{t+1}$. Hence $M(P_t^s) \leq t$. For a general $q = p^e$, the element $P_t^s \in A_q$ corresponds via ρ to $P_{et}^s \in A'_q$, and the same argument holds with t replaced by et.

For the second case, consider the equation $P^r(x^r) = x^{pr}$. Each of the $\alpha(r)$ nonzero digits of the base p expansion of r is moved one place to the right to give the base p expansion of pr. This cannot be done by a sequence of more than $\alpha(r)$ moves, and so $M(P^r) \leq \alpha(r)$. For a general $q = p^e$, we have $P_t^s(x^{q^s}) = x^{q^{s+t}}$, where $P_t^s = P(0, \ldots, 0, p^s)$ with p^s is in position t. Replacing the equation $P^{r}(x^{r}) = x^{pr}$ by $P^{r}(x^{r}) = x^{qr}$, the effect on the base p expansion is to move all $\alpha(r)$ base p digits of r through e places to the right to give the base p expansion of qr.

For the general case, we combine these two examples, using the action of A_q on polynomials over \mathbb{F}_p in variables x_1, x_2, \ldots , and representing monomials by arrays called 'blocks' as in [\[9\]](#page-28-1), where the rows of the block are formed by the reverse binary expansions of the exponents. Let $R = (r_1, r_2, \ldots, r_\ell)$. Then $P(R) \in A_q$ acts on a product $x_1x_2 \cdots x_{|R|}$ to give the sum of all the monomials obtained by raising r_i of the variables x_j to $x_j^{q_i}$ $_j^q$ for $1 \leq i \leq \ell$. By specializing the first r_1 variables to a new variable y_1 , the next r_2 variables to a new variable y_2 , and so $\frac{q r_1}{1} y_2^{q^2 r_2}$ $y_2^{q^2r_2}\cdots y_\ell^{q^\ell r_\ell}$ ther terms, since no on, we find that $P(R)(y_1^{r_1})$ $i_1^{r_1} y_2^{r_2}$ $x_2^{r_2}\cdots y_\ell^{r_\ell}$ $y_{\ell}^{r_{\ell}}$ = $y_1^{qr_1}$ other specialization of the variables leads to the same monomial. This monomial $\frac{qr_1}{1}y_2^{q^2r_2}$ $\frac{q^2r_2}{2}\cdots y_\ell^{q^\ell r_\ell}$ $y_1^{qr_1}$ $q^{\iota_{r_\ell}}$ has the same α -count as the original monomial $y_1^{r_1}$ $x_1^{r_1}y_2^{r_2}$ $y_2^{r_2}\cdots y_\ell^{r_\ell}$ $l_{\ell}^{r_{\ell}},$ and its base p block is obtained by moving each of the nonzero digits in row i a total of ei places to the right, for $1 \leq i \leq \ell$. This cannot be achieved by a sequence of more than $\sum_{i=1}^{\ell} i \alpha(r_i)$ Steenrod operations of positive degree. The result follows. □

4 The graded algebra $E^0(A_q)$

In this section we consider the graded algebra $E^0(\mathsf{A}_q)$ associated to the May filtration of A_{q} . Given a filtered algebra A, We use the same notation ambiguously for elements of A and for the corresponding elements of the associated graded algebra $E^0(A)$.

Using Theorem [1.4,](#page-2-1) the structure of $E^0(A_q)$ can be described as follows. The Milnor basis elements $P(R)$, $R = (r_1, r_2, ...)$ form a \mathbb{F}_q -basis of $E^0(A_q)$, with $P(R)$ in grading $M(P(R)) = \sum_{i=1}^{\ell} i \alpha(r_i)$. The product in $E^{0}(A_q)$ is given by $P(R)P(S) = \sum_{T} P(T)$, where this sum is obtained by deleting all terms $P(T)$ in the corresponding product in A_q such that $M(P(T)) > M(P(R)) + M(P(S))$. For example, $Sq(4,2)Sq(1,2) = Sq(1,3,1) + Sq(4,2,1)$ in A_2 , but $Sq(4,2)Sq(1,2) =$ $Sq(4, 2, 1)$ in $E^0(A_2)$. By the 'degree' and 'grading' of $\theta \in E^0(A_q)$ we mean the degree of θ as an element of A_q and its filtration $M(\theta)$.

Proposition 4.1. Let $A_q(n-2)$ be the subalgebra of A_q generated by P^{p^i} , $0 \leq$ $i < (n-1)e$. The graded algebra $E^0(\mathsf{A}_q(n-2))$ associated to the May filtration of $A_q(n-2)$ can be defined by generators and relations as follows.

1. There is one generator $P_t^s = P(0, \ldots, 0, p^s)$ in each q-atomic degree

$$
a = p^s(q^t - 1)/(q - 1)
$$

with $s \geq 0$, $t \geq 1$ and $(s/e) + t < n$. There are $en(n-1)/2$ such degrees a. We denote this generator by $P[a]$, and if a and b are q-atomic we denote the commutator $P[a]P[b] - P[b]P[a]$ by $[P[a], P[b]$.

2. The defining relations are the power relations $P[a]^p = 0$ for all a, and the commutator relations: for $a > b$,

$$
[P[a], P[b]] = \begin{cases} P[a+b], & \text{if } a+b \text{ is } q\text{-atomic and is not a power of } p, \\ 0, & \text{otherwise.} \end{cases}
$$

Thus a \mathbb{F}_q -basis for $E^0(\mathsf{A}_q(n-2))$ is given by monomials with exponents $\leq p-1$ in the generators $P[a]$, taken in a fixed but arbitrary order. The dimension of the \mathbb{F}_q -algebra $E^0(\mathsf{A}_q(n-2))$ is $p^{en(n-1)/2} = q^{n(n-1)/2}$.

The q-atomic number $a = p^{s}(q^{t} - 1)/(q - 1)$ has reversed base p expansion $0 \ldots 0 1 0 \ldots 0 1 \ldots 0 \ldots 0 1$, where there are t equally spaced digits 1 in positions $s, s + e, \ldots, s + (t - 1)e$. By Proposition [3.1,](#page-8-1) $P[a]$ has grading t in $E^0(A_q)$.

Example 4.2. In the case $q = p$, $n = 4$, $E^0(A_p(2))$ has 6 generators $P[1] =$ $P(1), P[p] = P(p), P[p+1] = P(0,1), P[p^2] = P(p^2), P[p^2 + p] = P(0, p)$ and $P[p^2 + p + 1] = P(0, 0, 1)$. The nonzero commutators are $[P[p], P[1]] = P[p + 1]$, $[P[p^2], P[p]] = P[p^2 + p]$ and $[P[p^2 + p], P[1]] = [P[p^2], P[p+1]] = P[p^2 + p + 1].$

Example 4.3. In the case $q = 4$, $n = 3$, $E^0(A_4(1))$ has 6 generators $P[a]$, $a =$ $1, 2, 4, 8, 5, 10$ with relations $P[a]^2 = 0$ and nonzero commutators $[P[4], P[1]] =$ $P[5]$ and $[P[8], P[2]] = P[10]$. The elements $P[a] \in A_4$ correspond respectively to $Sq(0, 1), Sq(0, 2), Sq(0, 4), Sq(0, 8), Sq(0, 0, 0, 1)$ and $Sq(0, 0, 0, 2)$ in $\mathsf{A}'_4 \subset \mathsf{A}_2$.

These power and commutator relations do not generally hold in A_q , but only in the graded algebra $E^0(\mathsf{A}_q)$. For example, in A_4 we have $[P[8], P[1]] = P[5]P[4]$, $[P[4], P[2]] = P[5]P[1]$ and $[P[8], P[4]] = P[10]P[2]$, where in each case the right hand side has filtration 3, and $P[4]P[4] = P(3, 1)$ and $P[8]P[8] = P(6, 2)$, where in each case the right hand side has filtration 4.

The particular power relations $(P_1^s)^p = (P^{p^s})^p = 0$ in $E^0(\mathsf{A}_q(n-2))$ can be proved by observing that the expansion of $(P^{p^s})^p$ in the admissible basis does not involve $P^{p^{s+1}}$. This is because $P^{p^{s+1}}$ is indecomposable in A_q (Proposition [2.16\(](#page-7-1)i)), and since $p^{s+1} = p^s + \cdots + p^s$ (*p* terms) is the unique minimal decomposition of p^{s+1} as the sum of more than one power of p, all other admissible monomials in degree p^{s+1} have May filtration $\geq p+1$. To prove the general power relation $(P_t^s)^p = 0$ and the commutator relations, we use the Milnor product formula.

Proof of Proposition [4.1.](#page-11-1) Given a q-atomic number $a = p^{s}(q^{t} - 1)/(q - 1)$, let $P[a] = P(0, \ldots, 0, p^s)$ be the corresponding element of $A_q(n-2)$, where p^s is in position t. Similarly let $b = p^u(q^v - 1)/(q - 1)$ so that $P[b] = P(0, \ldots, 0, p^u)$ with p^u in position v. Then we wish to prove that in $E^0(A_q(n-2))$ we have $P[a]^p = 0$ for all a and $[P[a], P[b]] = P[a + b]$ when $a \geq b$ and $s = u + v$ or $u = s + t$, and $[P[a], P[b]] = 0$ otherwise. Equivalently, $P[a]^p$ has May filtration $> pt$, $[P[a], P[b]] - P[a + b]$ has May filtration $> t + v$ when $s = u + v$ or $u = s + t$, and otherwise $[P[a], P[b]]$ has May filtration $> t + v$.

The calculation of $P[a]P[b]$ and $P[b]P[a]$ by the Milnor product formula produces Milnor matrices of the form

We may assume that $t \geq v$, and that if $t = v$ then $s \geq u$.

We first consider the case $c > 0$. Since c appears in position $t + v$, by Proposition [3.2](#page-8-2) the corresponding term in $P[a]P[b]$ has May filtration $\geq t + v$, and is > t + v unless the array satisfies $\alpha(p^s - p^v c) + \alpha(c) = \alpha(p^s) = 1$ and $\alpha(p^u - c) + \alpha(c) = \alpha(p^u) = 1$. Thus $p^u - c = 0$ and $p^s - p^v c = 0$, i.e. $s = u + v$, $c = p^u$. The corresponding element is $P[a + b] = P(0, \ldots, 0, p^u)$, where p^u is in position $t + v$. The case $u = s + t$ arises similarly by considering $P[b]P[a]$.

Next we consider the initial term $c = 0$, with $a \neq b$. The $c = 0$ terms in $P[a]P[b]$ and $P[b]P[a]$ are equal (to $P(0,\ldots,0,p^u,0,\ldots,0,p^s)$ if $t > v$, and to $P(0, ..., 0, p^u + p^s)$ if $t = v, s > u$, and so they cancel in $[P[a], P[b]]$.

Finally we consider the initial term $c = 0$, with $a = b$, so that $t = v$ and $s = u$. The $c = 0$ term is $2P(0, \ldots, 0, 2p^s)$ since the binomial coefficient $\binom{2p^s}{p^s}$ $\binom{2p^s}{p^s} = 2$ mod p. This gives the result $P[a]^2 = 0$ in the case $p = 2$. For $p > 2$ we need to consider products $P(0,\ldots,0,ip^s)P(0,\ldots,0,jp^s)$ for $0 \leq i,j \leq p-1$. These produce similar Milnor matrices to those shown above. Again the case $c > 0$ gives terms of higher May filtration. The initial Milnor matrix $c = 0$ gives a multiple $\binom{(i+j)p^s}{i p^s}$ $\binom{+j)p^s}{ip^s} = \binom{i+j}{i}$ $\binom{+j}{i}$, which is nonzero mod p when $i + j < p$ and is 0 mod p when $i + j = p$. It follows that $P[a]^p = 0$ in $E^0(\mathsf{A}_q)$. \Box

Proposition [4.1](#page-11-1) allows us to determine the dimension of the kth filtration quotient $(A_q^+)^k/(A_q^+)^{k+1}$ as a vector space over \mathbb{F}_q for each degree $d \geq 0$. Since $E^0(\mathsf{A}_q)$ has a basis given by taking the products of the generators $P[a]$ in a fixed but arbitrary order, this dimension is the number of multisets $A = \{a_1, a_2, \ldots, a_\ell\}$ of q-atomic numbers with multiplicities $\leq p-1$, sum $|A| = \sum_{i=1}^{\ell} a_i = d$ and filtration $\sum_{i=1}^{\ell} M(P[a_i]) = k$. Thus we have the following result.

Proposition 4.4. The Poincaré series of the graded algebra $E^0(A_p(n-2))$ is

$$
\Pi(E^0(\mathsf{A}_p(n-2)),\tau) = \left(\frac{1-\tau^p}{1-\tau}\right)^{n-1} \left(\frac{1-\tau^{2p}}{1-\tau^2}\right)^{n-2} \cdots \left(\frac{1-\tau^{(n-1)p}}{1-\tau^{n-1}}\right).
$$

For $q = p^e$, the Poincaré series of $E^0(A_q(n-2))$ is the eth power of this series. \Box

Example 4.5. The basis for $E^0(A_2(1))$ given by Proposition [4.1](#page-11-1) is shown below, together with the grading, using the order 1, 3, 2 on 2-atomic numbers. Since $Sq[1] = Sq(1), Sq[2] = Sq(2)$ and $Sq[3] = Sq(0, 1)$, this basis is the Milnor basis. The Poincaré series is $(1 + \tau)^2 (1 + \tau^2) = 1 + 2\tau + 2\tau^2 + 2\tau^3 + \tau^4$.

The corresponding series for $E^0(A_4(1))$ is $(1+\tau)^4(1+\tau^2)^2 = 1+4\tau+8\tau^2+12\tau^3+$ $14\tau^4 + 12\tau^5 + 8\tau^6 + 4\tau^7 + \tau^8$. It can be obtained by applying Proposition [3.1](#page-8-1) to the Milnor basis elements $P(r_1, r_2)$ with $0 \le r_1 \le 15$ and $0 \le r_2 \le 3$.

For $M \geq 0$, there is a natural bijection between monomials $P[A]$ in the generators $P[a]$ with exponents $\leq p-1$, degree d and grading M in $E^0(\mathsf{A}_q)$, and Milnor basis elements $P(R) = P(r_1, r_2, \ldots)$ with degree d and May filtration M in A_q . This has been described, by K. G. Monks [\[8\]](#page-28-6) for $p = 2$ and by D. Yu. Emelyanov and Th. Yu. Popelensky [\[3\]](#page-28-5) for $p > 2$, as follows: if $P[a] = P_t^s$, so that $a = p^s(q^t - 1)/(q - 1)$, then $P[a]$ appears with exponent k in $P[A]$ if and only if p^s has coefficient k in the base p expansion of r_t . Hence $d = \sum_j k_j a_j = \sum_i r_i (q^i - 1)/(q - 1)$, since if $a_j = p^{s_j} (q^{t_j} - 1)/(q - 1)$ then $r_i = \sum_{t_j=i}^{s} k_j p^{s_j}$ and $\alpha(r_i) = \sum_{t_j=i}^{s} k_j$, and $M = \sum_j k_j t_j = \sum_i i \alpha(r_i)$.

Example 4.6. The first few 3-atomic numbers are shown below. The monomial $P[1]P[4]^2P[3]^2$ corresponds to the Milnor basis element $P(7, 2)$, with $d = 15$ and $M = 7$.

. . . ²⁷ . . . 9 36 . . . 3 12 39 . . . $1 \quad 4 \quad 13 \quad 40 \quad \cdots$

Example 4.7. The first few 4-atomic numbers are shown below. The monomial $P[1]P[10]P[8]$ corresponds to the Milnor basis element $P(9, 2)$, with $d = 19$ and $M = 4$. .

$$
\begin{array}{c}\n\vdots \\
8 & \vdots \\
4 & 20 & \vdots \\
2 & 10 & 42 & \vdots \\
1 & 5 & 21 & 85 & \cdots\n\end{array}
$$

A sum of elements of filtration M in a filtered algebra can have filtration $> M$, as the next example shows.

Example 4.8. The following table shows the grading of the Milnor basis in $E^0(A_2)$ for elements of degree 12.

grading				
Milnor basis $\lceil Sq(0,4) \rceil$		Sq(6,2)	$\mid Sq(5,0,1) \mid Sq(2,1,1)$	
	Sq(12)	Sq(9,1)		Sq(3,3)

Thus for $q = 2$ and degree $d = 12$, $(A_2^+)^6 / (A_2^+)^7$ is spanned by the elements

$$
Sq1Sq2Sq4Sq2Sq1Sq2 = Sq(3,3) = Sq9Sq3,Sq2Sq4Sq2Sq1Sq2Sq1 = Sq(2,1,1) + Sq(3,3) = Sq8Sq3Sq1 + Sq9Sq2Sq1.
$$

But $Sq^8Sq^3Sq^1$ and $Sq^9Sq^2Sq^1$ have filtration 4, since their Milnor basis expansion involves $Sq(9, 1)$ in each case. It follows that there is no 'admissible basis' of the filtration quotients of A_2 .

If a similar situation were to occur for a sum of Milnor basis elements of the same May filtration in A_q , then, in view of the bijection described above, some filtration quotient of A_q would have dimension greater than that given by Proposition [4.4.](#page-13-0) Thus, by working down from the highest filtration in a given degree, we obtain the following result [\[5\]](#page-28-2).

Proposition 4.9. For any element $\theta \in A_q$, the May filtration of θ is the minimum of the May filtrations of the terms in the expansion of θ in the Milnor basis. \Box

5 The graded algebra $E^0(\mathbb{F}_p \mathsf{U}(n))$

Given a group G and a field F , the augmentation ideal A of the group algebra FG is the kernel of the homomorphism $FG \rightarrow F$ which maps $g \mapsto 1$ for all $g \in G$. We filter FG by the powers A^i of A, and define the associated graded algebra $E^0(FG) = \bigoplus_{i \geq 0} A^i / A^{i+1}$, where $A^0 = FG$ and the product of $a \in A^i$ and $b \in A^j$ is defined by $\overline{a} \overline{b} = \overline{ab} \in A^{i+j}$, where \overline{a} and \overline{b} are the cosets $a + A^{i+1}$ and $b + A^{j+1}$. In this section, we consider $E^0(\mathbb{F}_p \mathsf{U}(n))$ where p is prime and $U(n) \subset GL(n, \mathbb{F}_p)$ is the subgroup of upper unitriangular matrices, and we prove Theorem [1.5](#page-2-2) by finding generators and defining relations for $E^0(\mathbb{F}_p\mathsf{U}(n))$ which correspond to those of Proposition [4.1](#page-11-1) for $E^0(\mathsf{A}_p(n-2))$.

Example 5.1. In the case $p = 2$, the group $U(3)$ is dihedral of order 8. It is generated by e_1, e_2, e_3 where

$$
e_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

with defining relations $e_1^2 = e_2^2 = e_3^2 = I$, $(e_1, e_3) = (e_2, e_3) = I$ and $(e_1, e_2) = e_3$, where I is the identity matrix and $(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$ is the commutator. The augmentation ideal A of $\mathbb{F}_2\mathsf{U}(3)$ is given by formal sums of an even number of matrices, and is generated by $f_1 = 1 + e_1$, $f_2 = 1 + e_2$ and $f_3 = 1 + e_3$, where $1 = I$. These satisfy the relations $f_1^2 = f_2^2 = f_3^2 = 0$, $[f_1, f_3] = [f_2, f_3] = 0$ and $[f_1, f_2] = f_3 + f_1f_3 + f_2f_3 + f_1f_2f_3$, where $[a_1, a_2] = a_1a_2 + a_2a_1$ is the commutator. The last relation shows that $f_3 \in A^2$ and that $[f_1, f_2] = f_3$ in the graded algebra $E^0(\mathbb{F}_2 \cup (3))$. This algebra has dimension 8 over \mathbb{F}_2 , with basis $\{1, f_1, f_2, f_3, f_1f_2, f_1f_3, f_2f_3, f_1f_2f_3\}$ and defining relations $f_1^2 = f_2^2 = f_3^2 = 0$, $[f_1, f_3] = [f_2, f_3] = 0$ and $[f_1, f_2] = f_3$. The grading is shown by the diagram

	13	f_1f_3	$f_1 f_2 f_3$	
		J_2J_3		

In this case, the corresponding basis for $E^0(A_2(1))$ obtained using $f_1 \leftrightarrow Sq^1$ and $f_2 \leftrightarrow Sq^2$ is the Milnor basis

A theorem of Quillen [\[11\]](#page-28-4) describes the structure of $E^0(FG)$ as the universal enveloping algebra of a Lie algebra $L(G)$ associated to G. If F has prime characteristic p, then $L(G)$ is a 'restricted' Lie algebra, as it has an additional structure map called a 'pth power map'. The universal enveloping algebra is also taken in the restricted sense.

The construction of the Lie algebra $L(G)$ is based on the dimension series of the group G. The kth dimension subgroup $D_k(G)$ is the normal subgroup of G which consists of elements $g \in G$ such that $g - 1 \in A^k$. In particular, $D_1(G) = G$. The quotients $D_k(G)/D_{k+1}(G)$ are Abelian, and, for the purpose of constructing $L(G)$ as their direct sum, these quotients are written additively. Thus $L(G) = \bigoplus_{k \geq 1} D_k(G)/D_{k+1}(G)$ is a graded Abelian group, and we define a Lie product on $L(G)$ by $[\overline{g_1}, \overline{g_2}] = \overline{(g_1, g_2)}$, where $(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$ is the commutator in G . Various identities for commutators in G then translate into bilinearity, anti-commutativity and the Jacobi identity in $L(G)$ [\[10\]](#page-28-8).

We embed $L(G)$ additively into $E^0(FG)$ by the map $\theta : \overline{g} \mapsto g - 1 + A^{k+1}$, where g has grading k in $L(G)$. As it is an associative algebra, $E^0(FG)$ also has a Lie product defined by $[\overline{g_1}, \overline{g_2}] = \overline{g_1} \overline{g_2} - \overline{g_2} \overline{g_1}$, and the map θ is a homomorphism of Lie rings. Finally we take coefficients in F, to get a map $\theta : L(G) \otimes_{\mathbb{Z}} F \to E^0(FG)$ of Lie algebras over F. The pth power map on $E^0(FG)$ is defined by $\overline{g}^{[p]} = \overline{g}^p$.

Example 5.2. In Example [5.1,](#page-15-1) $D_2 = \{1, e_3\}$ is the centre of $\mathsf{U}(3)$, and $D_3 = \{1\}$. Regarded as elements of $L(\mathsf{U}(3))$, the cosets $1 + D_2$, $e_1 + D_2$, $e_2 + D_2$, $e_1e_2 + D_2$ are written as 0, $\overline{e_1}$, $\overline{e_2}$ and $\overline{e_1} + \overline{e_2}$ respectively, and the cosets $1 + D_3$, $e_3 + D_3$ as $0, \overline{e_3}$. The commutator relations in Example [5.1](#page-15-1) show that the Lie product in $L(\mathsf{U}(3))$ is given by $\overline{[e_1, e_3]} = [\overline{e_2}, \overline{e_3}] = 0$ and $\overline{[e_1, e_2]} = \overline{e_3}$. The elements $\overline{e_1}, \overline{e_2}$ of $L(\mathsf{U}(3))$ have grading 1, while $\overline{e_3}$ has grading 2.

The embedding $\theta: L(\mathsf{U}(3)) \to E^0(\mathbb{F}_2 \mathsf{U}(3))$ maps $\overline{e_i}$ to f_i for $i = 1, 2, 3$. The relations in $E^0(\mathbb{F}_2 \mathsf{U}(3))$ of Example [5.1](#page-15-1) show that θ is a map of Lie algebras, and that $\bar{g} \mapsto \bar{g}^{[2]}$ is the zero map. The (ungraded) Lie algebra $L(\mathsf{U}(3))$ is isomorphic to the Lie algebra of nilpotent upper triangular 3×3 matrices over \mathbb{F}_2 [\[7\]](#page-28-3).

Given a restricted Lie algebra L over a field F of characteristic p , the restricted universal enveloping algebra $U(L)$ is the quotient of the tensor algebra of L by the relations $[a, b] = a \otimes b - b \otimes a$ and $a^{[p]} = a \otimes a \otimes \cdots \otimes a$ (p factors), for all $a, b \in L$. Then $U(L)$ is an associative algebra, and has a Lie product defined by commutators. The natural map $L \to U(L)$ is then an embedding of restricted Lie algebras. By the Poincaré-Birkhoff-Witt theorem, if x_1, x_2, \ldots, x_m is an ordered F-basis for L, then a F-basis for $U(L)$ is given by the monomials $x_{i_1}^{\alpha_1}$ $\frac{\alpha_1}{i_1} x_{i_2}^{\alpha_2}$ $\frac{\alpha_2}{i_2} \cdots x_{i_r}^{\alpha_r}$ i_r with $i_1 < i_2 < \cdots < i_r$ and $0 \leq \alpha_i < p$.

Theorem 5.3. (Quillen, [\[11\]](#page-28-4)) The map $\theta : L(G) \otimes_{\mathbb{Z}} F \to E^0(FG)$ extends to an isomorphism

$$
\theta: U(L(G) \otimes_{\mathbb{Z}} F) \to E^0(FG)
$$

of graded algebras over F. If F has prime characteristic p, then $U(L(G) \otimes_{\mathbb{Z}} F)$ is taken in the restricted sense.

When $G = U(n)$, the group of upper unitriangular $n \times n$ matrices over \mathbb{F}_p , the dimension subgroups are the terms of its lower central series: $D_k(\mathsf{U}(n))$ consists of all matrices $U = (u_{i,j})$ with $k-1$ diagonals of zeros above the main diagonal, so that $u_{i,j} = 0$ for $1 \leq j - i < k$. Thus $D_k(\mathsf{U}(n))/D_{k+1}(\mathsf{U}(n))$ is an elementary Abelian group of order p^{n-k} . To construct the Lie algebra $L(\mathsf{U}(n))$, we write the group $D_k(\mathsf{U}(n))/D_{k+1}(\mathsf{U}(n))$ additively, so as to regard it as vector space of dimension $n - k$ over \mathbb{F}_p .

For $1 \leq i \leq j \leq n$, let $E_{i,j} \in U(n)$ be the elementary $n \times n$ matrix over \mathbb{F}_p obtained by changing the (i, j) th entry of the identity matrix I from 0 to 1. The commutator $(E_{i,j}, E_{k,\ell}) = E_{i,j}^{-1} E_{k,\ell}^{-1} E_{i,j} E_{k,\ell}$ is $E_{i,\ell}$ if $j = k$, $E_{k,j}$ if $i = \ell$, and I otherwise.

We write $E_{i,j} = e_a$, where $a = (p^{j-1} - p^{i-1})/(p-1)$ is a p-atomic number. The conditions $1 \leq i < j \leq n$ on i and j then correspond to the conditions $1 \leq$ $a \le (p^{n-1}-1)/(p-1)$, and for p-atomic numbers a and $b = (p^{k-1}-p^{\ell-1})/(p-1)$ the commutator $(e_a, e_b) = e_c$ where $c = a + b = (p^{\ell-1} - p^{i-1})/(p-1)$ if $j = k$, $(e_a, e_b) = e_{c'}$ where $c' = a + b = (p^{j-1} - p^{k-1})/(p-1)$ if $i = \ell$, and $(e_a, e_b) = I$ otherwise. Thus, in terms of the (reverse) base p expansions $0 \cdots 0$ $1 \cdots 1$ of a and b, $(e_a, e_b) = e_{a+b}$ if $a > b$ and the blocks of 1s in the expansions of a and b abut, so as to form a single block of 1s under base p addition, as shown below,

$$
\begin{array}{cccc}\n0 \cdots 0 & 1 \cdots 1 \\
0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 \\
\hline\n0 \cdots 0 & 1 \cdots 1 & 1 \cdots 1\n\end{array}
$$

and otherwise e_a and e_b commute. Thus $(e_a, e_b) = e_{a+b}$ if $a > b$ and $a+b$ is p-atomic and is not a power of p, and otherwise $(e_a, e_b) = I$.

The quotient group $D_k(\mathsf{U}(n))/D_{k+1}(\mathsf{U}(n))$ is generated by the cosets $\overline{e_a}$ = $e_a + D_{k+1}(\mathsf{U}(n))$ of the elements e_a with an entry 1 on the kth superdiagonal $j-i = k$. As a vector space over \mathbb{F}_p , $L(\mathsf{U}(n))$ is the direct sum of these quotients, and has dimension $n(n-1)/2$. Thus $L(\mathsf{U}(n))$ is the graded elementary Abelian p-group generated by the elements $\overline{e_a}$ for $a \leq (p^{n-1} - 1)/(p - 1)$, and $\overline{e_a}$ has grading $k = j - i$. Thus the elements of $L(\mathsf{U}(n))$ are formal sums of elements $\overline{e_a}$ corresponding to the matrices $e_a \in \mathsf{U}(n)$.

The Lie product in $L(\mathsf{U}(n))$ is defined by $[\overline{e_a}, \overline{e_b}] = \overline{(e_a, e_b)}$. It follows from the above discussion of commutators in $\mathsf{U}(n)$ that $[\overline{e_a}, \overline{e_b}] = \overline{e_{a+b}}$ if $a > b, a+b$ is p-atomic and is not a power of p, and otherwise $[\overline{e_a}, \overline{e_b}] = 0$. Since $\overline{e_{a+b}}$ has grading $k+\ell$ if $\overline{e_a}$ has grading k and $\overline{e_b}$ has grading ℓ , this product makes $L(\mathsf{U}(n))$ into a graded Lie algebra.

For each $e_a \in \mathsf{U}(n)$ as above, we define $f_a = 1 - e_a$ in $\mathbb{F}_p\mathsf{U}(n)$. If e_a has a 1 on the kth superdiagonal, then $f_a \in A^k$, and so $f_a + A^{k+1}$ is a well-defined element of the graded algebra $E^0(\mathbb{F}_p \mathsf{U}(n))$. Since the elements $\overline{e_a}$ form a \mathbb{F}_p basis for $L(\mathsf{U}(n))$, we can define a map $\theta : L(\mathsf{U}(n)) \to E^0(\mathbb{F}_p\mathsf{U}(n))$ of graded algebras by $\theta(\overline{e_a}) = f_a + A^{k+1}$, where $\overline{e_a}$ has grading $k = j - i$ in $L(\mathsf{U}(n))$ and $a = (p^{j-1} - p^{i-1})/(p-1).$

We shall show that θ is an embedding of Lie algebras, where the Lie product of $x, y \in E^0(\mathbb{F}_p \mathsf{U}(n))$ is defined by $[x, y] = xy - yx$. Thus we wish to prove

$$
\begin{array}{rcl}\n\theta([\overline{e_a}, \overline{e_b}]) & = & \theta(\overline{e_a})\theta(\overline{e_b}) - \theta(\overline{e_b})\theta(\overline{e_a}) \\
& = & (f_a + A^{k+1})(f_b + A^{\ell+1}) - (f_b + A^{\ell+1})(f_a + A^{k+1})\n\end{array}
$$

where $\overline{e_a}$ has grading k and $\overline{e_b}$ has grading ℓ . Since products such as $f_a \cdot A^{\ell+1}$ are in $A^{k+\ell+1}$, this simplifies to

$$
\theta([\overline{e_a}, \overline{e_b}]) = f_a f_b - f_b f_a + A^{k+\ell+1} = e_a e_b - e_b e_a + A^{k+\ell+1}.
$$

Since $\overline{[e_a, e_b]} = \overline{e_{a+b}}$ if $a+b$ is p-atomic, and $\overline{[e_a, e_b]} = 0$ otherwise, $\theta(\overline{[e_a, e_b]}) =$ $1-e_{a+b}+A^{k+\ell+1}$ if $a+b$ is p-atomic, and $\theta([\overline{e_a}, \overline{e_b}])=0+A^{k+\ell+1}$ otherwise. The result holds in the second case since e_a and e_b commute in $\mathsf{U}(n)$. In the first case, $e_a^{-1}e_b^{-1}$ $e_b^{-1}e_a e_b = e_{a+b}$, so $e_b e_a = e_a e_b e_{a+b}$. Hence $e_a e_b - e_b e_a = e_a e_b (1 - e_{a+b})$. Since $1 - e_{a+b} \in A^{k+\ell}$ and $e_a e_b \in 1+A$, $e_a e_b - e_b e_a + A^{k+\ell+1} = 1 + e_{a+b} + A^{k+\ell+1}$. Thus the result holds in the first case. It follows that θ is a Lie algebra homomorphism.

We next apply Theorem [5.3.](#page-16-0) The restricted universal enveloping algebra $U(L(\mathsf{U}(n)))$ is an associative algebra over \mathbb{F}_p of dimension $p^{n(n-1)/2}$. It has a basis given by products of the elements $\overline{e_a}^k$, where $0 \leq k \leq p-1$ and $a =$ $(p^{j-1}-p^{i-1})/(p-1)$ for $1 \leq i < j \leq n$, the *p*-atomic numbers a being taken in a fixed but arbitrary order. The pth power map $x \mapsto x^{[p]}$ is trivial since $(e_a+1)^p = 0$ in $\mathbb{F}_p(\mathsf{U}(n))$. By Quillen's theorem, the map $\theta: L(\mathsf{U}(n)) \to E^0(\mathbb{F}_p(\mathsf{U}(n)))$ extends to an isomorphism $\theta: U(L(\mathsf{U}(n))) \to E^0(\mathbb{F}_p\mathsf{U}(n))$ of graded algebras over \mathbb{F}_p . On basis elements we have

$$
\theta(\overline{e_{a_1}} \otimes \overline{e_{a_2}} \otimes \cdots \otimes \overline{e_{a_r}}) = f_{a_1} f_{a_2} \cdots f_{a_r} + A^{k+1}
$$

where a_1, a_2, \ldots, a_r are taken in the preferred order, and k is the sum of the gradings of the elements $\overline{e_{a_i}}$.

In the associative algebra $U(L(\mathsf{U}(n)))$, the Lie bracket $[x, y] = xy - yx$, and so the relations $\overline{e_a}^p = 0$ and $\overline{e_a} \overline{e_b} - \overline{e_a} \overline{e_b} = [\overline{e_a}, \overline{e_b}] = \overline{e_{a+b}}$ or 0 can be regarded as power-commutator relations defining $U(L(\mathsf{U}(n)))$ as an algebra. Thus Quillen's theorem provides us with a corresponding definition of the algebra $E^0(\mathbb{F}_p \mathsf{U}(n))$ by generators and relations. We write $f_a = 1 - e_a \in \mathbb{F}_p(\mathsf{U}(n))$ where a is p-atomic, and translate these relations as $f_a^p = 0$ and $f_a f_b - f_b f_a = f_{a+b}$ if $a > b$ and $a+b$ is *p*-atomic, and $f_a f_b = f_b f_a$ otherwise.

Proof of Theorem [1.5.](#page-2-2) The graded algebras $E^0(\mathbb{F}_p \mathsf{U}(n))$ and $E^0(\mathsf{A}_p(n-2))$ have the same dimension $p^{n(n-1)/2}$ as vector spaces over \mathbb{F}_p . The discussion above shows that the elements f_a corresponding to the $n(n-1)/2$ p-atomic numbers a such that $1 \le a \le (p^{n-1}-1)/(p-1)$ generate $E^0(\mathbb{F}_p \mathsf{U}(n))$, and satisfy relations which correspond to those of Proposition [4.1](#page-11-1) for the generators P_t^s of $E^0(\mathsf{A}_p(n-2))$. Hence the map $E^0(\mathbb{F}_p \mathsf{U}(n)) \longrightarrow E^0(\mathsf{A}_p(n-2))$ which sends f_a to P_t^s , where $a = p^{s}(p^{t}-1)/(p-1)$, is an isomorphism of graded algebras.

Further, $E^0(\mathbb{F}_p \mathsf{U}(n))$ and $E^0(\mathsf{A}_p(n-2))$ are isomorphic as Hopf algebras. The generators $P[a]$ of $E^0(A_p(n-2))$ and f_a of $E^0(\mathbb{F}_p\mathsf{U}(n))$ are coproduct primitives. Since $P[a] = P_t^s = P(0, \ldots, 0, p^s), \ \phi(P(0, \ldots, 0, k)) = \sum_{i+j=k} P(0, \ldots, 0, i) \otimes$ $P(0, \ldots, 0, j)$. With the nonzero elements in position t, the May filtration of $P(0,\ldots,0,k)$ is $t\alpha(k)$, and that of $P(0,\ldots,0,i) \otimes P(0,\ldots,0,j)$ is $t(\alpha(i) + \alpha(j)),$ so in $E^0(\mathsf{A}_q)$ the sum is over i, j such that $i + j = k$ and $\alpha(i) + \alpha(j) = \alpha(k)$. Since $\alpha(k) = 1$ when $k = p^s$, the only terms which survive are given by $i = 0$, $j = k$ and $i = k, j = 0$.

The coproduct in $\mathbb{F}_p\mathsf{U}(n)$ is given by $\phi(g) = g \otimes g$ for all $g \in \mathsf{U}(n)$. Hence $\phi(f_a) = \phi(1 - e_a) = \phi(1) - \phi(e_a) = 1 \otimes 1 - e_a \otimes e_a = 1 \otimes 1 - (1 - f_a) \otimes (1 - f_a) =$ $f_a \otimes 1 + 1 \otimes f_a - f_a \otimes f_a$ for the coproduct in $\mathbb{F}_p \mathsf{U}(n)$. Hence in $E^0(\mathbb{F}_p \mathsf{U}(n))$ we have $\phi(f_a) = f_a \otimes 1 + 1 \otimes f_a$. It follows that the isomorphism $f_a \leftrightarrow P[a]$ preserves coproducts. The antipodes are also preserved: since $P[a]$ and f_a are primitive, $\chi(P[a]) = -P[a]$ and $\chi(f_a) = -f_a$. □

Remark 5.4. By Definition [2.1,](#page-2-3) for $q = p^e$ the subalgebra $\mathsf{A}'_q(n-2)$ of A'_q is contained in the subalgebra $A_p(ne-2)$ of A_p . The isomorphism $E^0(\mathbb{F}_p\mathsf{U}(n)) \longrightarrow$ $E^0(\mathsf{A}_p(n-2))$ of Theorem [1.5](#page-2-2) maps $E^0(\mathsf{A}'_q(n-2))$ to the subalgebra of $E^0(\mathbb{F}_p\mathsf{U}(n))$ generated by matrices $U = (u_{i,j})$ in $\mathsf{U}(n)$ with non-zero entries only on every eth superdiagonal, so that $u_{i,j} = 0$ if e does not divide $j - i$. Using Proposition [2.15,](#page-7-0) we have a corresponding description of $E^0(\mathsf{A}_q(n-2))$.

6 P_t^s bases for $E^0(A_q)$

Let p be a prime and let $q = p^e$ be a power of p. Following Wood [\[13\]](#page-28-9), we define the Y- and Z- orders on p - and q -atomic numbers as follows.

Definition 6.1. The Y-order on *p*-atomic numbers $p^{s}(p^{t}-1)/(p-1)$ is the left lexicographic order on pairs (s, t) , while the Z-order is the left lexicographic order on pairs $(s + t, s)$, where $s \geq 0$ and $t \geq 1$. More generally, the Y- and Z- orders on q-atomic numbers $p^{s}(q^{t}-1)/(q-1)$ are the left lexicographic orders on the triples (s', t, s'') and $(s' + t, s', s'')$ respectively, where $s = s'e + s''$, $0 \le s'' < e$.

Example 6.2. Example [4.6](#page-14-0) shows the first few 3-atomic numbers. The Y-order $1, 4, 13, 40, \ldots, 3, 12, 39, \ldots, 9, 36, \ldots, 27, \ldots$ takes rows left to right and upwards. The Z-order 1, 4, 3, 13, 12, 9, 40, 39, 36, 27, . . . takes diagonals right to left and upwards.

Example 6.3. Example [4.7](#page-14-1) shows the first few 4-atomic numbers. Since $e =$ 2, we apply the previous recipes to rows taken in pairs. The Y-order is thus $1, 2, 5, 10, 21, 42, 85, \ldots, 4, 8, 20, 40, \ldots, 16, 32, \ldots$, while the Z-order is $1, 2, 5, 10,$ $4, 8, 21, 42, 20, 40, 16, 32, 85, \ldots$

The P_t^s bases of A_p defined by Monks [\[8\]](#page-28-6) and Emelyanov-Popelensky [\[2\]](#page-28-10) are obtained by fixing an arbitrary order on the set of p-atomic numbers, and then taking products of the elements P_t^s in weakly increasing order, with each such element being repeated up to $p - 1$ times. The bijection described in Section [4](#page-11-0) gives a triangular relation between these products and the Milnor basis. This relationship is sharpened in Proposition [6.5.](#page-21-1) For the Y- and $Z-P_t^s$ bases, which are defined using the Y- and Z- orders).

Given such a basis for A_p , the basis elements which are products of elements P_t^s such that e divides t give a basis for the subalgebra A'_q , where $q = p^e$. The corresponding elements of A_q give a P_t^s -basis for A_q , whose elements are products of the elements $P[a] = P_t^s = P(0, \ldots, 0, p^s)$, with one such element in each qatomic degree $a = p^{s}(q^{t} - 1)/(q - 1)$, each being repeated up to $p - 1$ times.

Example 6.4. Let $p = 2$, so that $P_t^s = Sq(0, ..., 0, 2^s)$ has degree $a = 2^s(2^t - 1)$. We denote this element alternatively by $Sq[a]$, and use the Z-order. The ta-ble below augments that of Example [4.8](#page-14-2) by showing the $Z-P_t^s$ elements corresponding to the Milnor basis, abbreviating a product $Sq[a_1]Sq[a_2] \cdots Sq[a_r]$ as $Sq[a_1, a_2, \ldots, a_r]$, with the a_i in increasing Z-order $1, 3, 2, 7, 6, 4, 15, 14, 12, 8, \ldots$

In this example, the Y- P_t^s basis is obtained from the Z- P_t^s basis by replacing $Sq[2]Sq[7]$ by $Sq[7]Sq[2]$. Since $Sq(2)Sq(0, 0, 1) = Sq(2, 0, 1) = Sq(0, 0, 1)Sq(2)$, these bases coincide in degree 12. (They differ in degree 18, as $Sq(4)Sq(0, 0, 2) \neq$ $Sq(0, 0, 2)Sq(4)$, but the 'error term' $Sq(3, 0, 0, 1)$ has higher May filtration.) The transition matrix from either P_t^s basis to the Milnor basis in degree 12 is shown below. Note that the diagonal submatrices corresponding to filtrations 2, 4, 5 and 6 are identity matrices. We show that this situation holds more generally.

Proposition 6.5. The Y- and Z- P_t^s bases of A_q coincide with the Milnor basis up to elements of higher May filtration, so that all three bases give the same basis of $E^0(A_q)$.

Proof. Let a and b be q-atomic numbers. In the cases where $P[a]$ and $P[b]$ do not commute in $E^0(A_q)$, a and b occur in the same order in the Y- and Zorderings. Hence the Y- and Z- P_t^s bases of A_q coincide up to elements of higher May filtration.

To prove that the Milnor basis of $E^0(\mathsf{A}_q)$ also coincides with these, we consider a Z-basis element $P[a_1]P[a_2] \cdots P[a_k]$, where a_1, a_2, \ldots, a_k is a weakly increasing sequence of q-atomic numbers with $a_{i+p-1} > z$ a_i for all i. We assume by induction on k that $P[a_2] \cdots P[a_k] = P(R)$ in $E^0(\mathsf{A}_q)$, where R is the sequence corresponding to a_2, \ldots, a_k as in Section [4](#page-11-0) (see Examples [4.6](#page-14-0) and [4.7\)](#page-14-1). For the inductive step, let $a = p^{s}(q^{t} - 1)/(q - 1)$. We use the Milnor product formula to show that $P[a]P(R) = P(R')$ in $E^0(\mathsf{A}_q)$ where $R' = R + (0, \ldots, 0, p^s)$, with p^s in position t. We have to consider Milnor matrices

where $\sum_i p^i c_i = p^s$.

We argue as in the proof of Proposition [3.2.](#page-8-2) By considering row t , all such matrices except the initial matrix $c_0 = p^s$, $c_i = 0$ for $i > 0$ and the final matrix $c_s = 1, c_i = 0$ for $i \neq s$ give elements of higher May filtration. By considering column s, the same is true for the final matrix. □

7 The Arnon A basis of A_p

Monks $[8]$ has compared a number of bases for A_2 with the Milnor basis $[6]$, and has shown that several of them are triangularly related to the Milnor basis for a suitable bijection between the two bases and for suitable orderings on them. His results have been generalized to A_p by Emelyanov and Popelensky [\[3\]](#page-28-5).

For example, the admissible basis of A_p is triangularly related to the Milnor basis as follows. Recall that admissible monomials in A_p are elements P^A = $P^{a_1}P^{a_2}\cdots P^{a_\ell}$, where $a_i\geq pa_{i+1}$ for $1\leq i\leq \ell$, and that they form a \mathbb{F}_p -basis for A_p . A triangular relation between this basis and the Milnor basis is obtained by associating to A the sequence $R = (r_1, r_2, \ldots, r_\ell)$, where $r_i = a_i - pa_{i+1}$, and using the right lexicographic order on the sequences A and R . The table below

shows the result in degree 9 when $p = 2$.

	Sq(2,0,1)	Sq(0,3)	Sq(3,2)	Sq(6, 1)	Sq(9)
$\overline{Sq}^{6}Sq^{2}Sq^{1}$					
Sq^6Sq^3					
Sq^7Sq^2					
Sq^8Sq^1					
Sq^9					

When comparing two bases in this way, we need only specify the bijection and the ordering on one of them, as the ordering on the other is defined by the bijection. Alternatively, we could use orderings on both bases to define the bijection.

The Arnon A basis of A_p [\[1,](#page-28-11) [2\]](#page-28-10) is defined as follows. We begin with the admissible monomials $X_k^n = P^{p^n} P^{p^{n-1}} \cdots P^{p^k}, n \ge k \ge 0$, one in each p-atomic degree $p^{k}(p^{n-k+1}-1)/(p-1)$. A general Arnon A basis element is a product of these elements taken in Z-order, with each factor X_k^n repeated no more than $p-1$ times. Thus the Arnon A basis is constructed in the same way as the Z- P_t^s basis, but using the elements X_k^n instead of the elements P_t^s . The basis elements have the form $X_{k_1}^{n_1}$ $\frac{n_1}{k_1} \cdot X_{k_2}^{n_2}$ $\frac{n_2}{k_2}\cdots X^{n_\ell}_{k_\ell}$ $\frac{n_{\ell}}{k_{\ell}}$, where $(n_i, k_i) <_{l} (n_{i+p-1}, k_{i+p-1})$ for all *i*. In particular, the factors are distinct when $p = 2$.

In the case $p = 2$, the bijection defined by Monks [\[8\]](#page-28-6) between the Arnon A basis and the Milnor basis can be described as follows. Given a Milnor basis element $Sq(R) = Sq(r_1, r_2, \ldots)$, the corresponding Arnon A basis element has one factor X_k^n corresponding to each element in the binary decompositions of the terms of the sequence R. For each i and j such that $2^i \in \text{bin}(r_j)$ we take X_k^n , where $k = i$ and $n = i + j - 1$, and multiply these elements in Z-order to form the required Arnon A basis element. Conversely, given an Arnon A basis element $X_{k_1}^{n_1}$ $x_{k_1}^{n_1} \cdot X_{k_2}^{n_2}$ $\chi_{k_2}^{n_2}\cdots\chi_{k_\ell}^{n_\ell}$ $\frac{n_{\ell}}{k_{\ell}}$, let r_j be the sum of the 2-powers 2^i where $j = n_t - k_t + 1$ is the length of one of the elements $X_{k_t}^{n_t}$ $k_t^{n_t}$ and $i = k_t$, and form the Milnor basis element $Sq(R) = Sq(r_1, r_2, \ldots)$. For a general prime p, bin(r) is replaced by the multiset pin(r) (Definition [2.10\)](#page-5-2). For example if $p = 3$ then the Milnor basis element $P(7,2)$ corresponds to the Arnon A basis element $P^1 \cdot P^3 P^1 \cdot P^3 P^1 \cdot P^3 \cdot P^3$, since $7 = 1 + 3 + 3$ and $2 = 1 + 1$ give the weakly Z-increasing sequence $(1, 4, 4, 3, 3)$.

Monks (for $p = 2$) and Emelyanov-Popelensky (for $p > 2$) show that the Arnon A basis and the Milnor basis are triangularly related using this bijection between the two bases and the following ordering on the Arnon A basis. Each Arnon A basis element is defined by a non-decreasing sequence of p-atomic numbers. These sequences are placed in decreasing left lexicographic order \lt_Z , taken with respect to the Z-order and not the usual increasing order on p-atomic numbers.

For example, for $p = 2$ and degree $d = 9$, the change of basis matrix is as follows.

	Sq(2,0,1)	Sq(0,3)	Sq(6,1)	Sq(9)	Sq(3,2)
$Sq^2 \cdot Sq^4Sq^2Sq^1$					
$Sq^2Sq^1 \cdot Sq^4Sq^2$					
$Sq^2Sq^1 \cdot Sq^2 \cdot Sq^4$					
$Sq^1 \cdot Sq^8$					
\cdot $Sq^2 \cdot Sq^4Sq^2$ Sq^1					

Here the 2-atomic sequences giving the Arnon A elements are ordered as follows:

 $(2, 7)$ > $\mathcal{Z}(3, 6)$ > $\mathcal{Z}(3, 2, 4)$ > $\mathcal{Z}(1, 8)$ > $\mathcal{Z}(1, 2, 6)$.

Our main result, Theorem [8.1,](#page-24-1) states that the Arnon A basis and the Milnor basis are also triangularly related using the same bijection, but with a different choice of orderings. The ordering on the Arnon A basis is the reversed right lexicographic order \leq_r , where these elements are treated simply as monomials in the generators P^{p^j} , $j \geq 0$ of A_p . This ordering ignores the factorization of Arnon A basis elements as products of p-atomic basis elements. For example, for $p = 2$ and $d = 9$ the change of basis matrix is as follows.

	Sq(9)	$Sq(2,0,1)$ $Sq(0,3)$ $Sq(3,2)$		Sq(6,1)
$Sq^1 \cdot Sq^8$				
$Sq^2 \cdot Sq^4Sq^2Sq^1$				
$Sq^2Sq^1 \cdot Sq^4Sq^2$				
$Sq^1 \cdot Sq^2 \cdot Sq^4Sq^2$				
$Sq^2Sq^1 \cdot Sq^2 \cdot Sq^4$				

Remark 7.1. The ordering \leq_R which appears in [\[1,](#page-28-11) [2\]](#page-28-10) is not the same as \leq_r : it is obtained by first right-justifying the exponent sequences and then taking lexicographic order from the right. For example, $(1,1,2) >_R (3,1) = (0,3,1)$, since $2 > 1$. However, $(3, 1) = (3, 1, 0) >r (1, 1, 2)$ since $0 < 2$.

In order to establish the A basis of A_2 , Arnon [\[1\]](#page-28-11) considers the set of all formal monomials in the elements Sq^{2^j} , $j \geq 0$, by which we mean elements of the free algebra A generated by these symbols. Thus A_2 is the quotient algebra of A by the two-sided ideal generated by the Adem relations and $Sq^0 = 1$. As for A_2 , A is graded by giving Sq^{2^j} degree 2^j . When the formal monomials of a given degree are taken in increasing left order \leq_l , Arnon shows that the minimal monomials form the A basis $[1,$ Theorem $5(A)$.

If the left lexicographic order \leq_l is replaced by \leq_r , then the minimal monomials do not in general give the A basis. For $p = 2$, the first counterexample is in degree 9, where the \leq_r -minimal basis is obtained from the A basis by replacing $Sq^2 \cdot Sq^4Sq^2Sq^1$ with $Sq^4Sq^2Sq^1Sq^2$, which is lower in the (reversed) right order \leq_r . The relation

$$
Sq^{2} \cdot Sq^{4}Sq^{2}Sq^{1} = Sq^{4}Sq^{2}Sq^{1} \cdot Sq^{2} + Sq^{1} \cdot Sq^{2} \cdot Sq^{4}Sq^{2} + Sq^{2}Sq^{1} \cdot Sq^{2} \cdot Sq^{4} \quad (5)
$$

shows that $Sq^2 \cdot Sq^4Sq^2Sq^1$ is reducible in the right order.

We define a variant of the A basis by replacing the Z-order on 2-atomic degrees by the Y-order, and we refer to the original Arnon A basis as the Z-Arnon A basis, the new basis as the Y-Arnon A basis. The Y-Arnon A basis elements have the form $X_{k_1}^{n_1}$ $\frac{n_1}{k_1} \cdot X_{k_2}^{n_2}$ $\frac{n_2}{k_2}\cdots X_{k_\ell}^{n_\ell}$ $\frac{n_{\ell}}{k_{\ell}}$, where $(n_1, k_1) <_{r} (n_2, k_2) <_{r} \cdots <_{r} (n_{\ell}, k_{\ell}).$ Thus the elementary Arnon A monomials X_k^n are multiplied in Y-order of their degrees. The Y- and Z-Arnon A bases first differ in degree $(p+1)^2$, by replacing $P^p \cdot P^{p^2} P^p P^1$ by $P^{p^2} P^p P^1 \cdot P^p$. For $p = 2$ this is illustrated by [\(5\)](#page-23-0).

8 The Arnon A basis of A_q

In this section we generalize the Y- and Z-Arnon A bases to A_q and prove our main result, Theorem [8.1.](#page-24-1) For $q = p^e$, the elementary Arnon A monomial $X_k^n =$ $P^{p^n}P^{p^{n-e}}\cdots P^{p^k}$ is defined when $n\geq k\geq 0$ and $n=k$ mod e, and its degree is the q-atomic number $p^{k}(q^{t+1}-1)/(q-1)$, where $n = k + te$. A general Y- or Z-Arnon A basis element is a product of these elements taken in Y- or Z-order, with each factor X_k^n repeated no more than $p-1$ times. In other words, the Y- or Z-Arnon A basis is constructed in the same way as the corresponding P_t^s basis, with the elements P_t^s replaced by the elements X_k^n . Thus the basis elements have the form $X_{k_1}^{n_1}$ $\chi_{k_1}^{n_1} \cdot X_{k_2}^{n_2}$ $\frac{n_2}{k_2}\cdots X^{n_\ell}_{k_\ell}$ $\sum_{k_{\ell}}^{n_{\ell}}$, $(n_1, k_1) \leq_l (n_2, k_2) \leq_l \cdots \leq_l (n_{\ell}, k_{\ell}),$ where $(n_i, k_i) <_l (n_{i+p-1}, k_{i+p-1})$ for all *i*.

Theorem 8.1. (i) The Y-Arnon A monomials in A_q are the minimal Steenrod monomials in increasing \leq_r order.

(ii) The Y-Arnon A monomials form a basis of A_q .

(iii) The Y- and Z-Arnon A bases of A_q are triangular with respect to the Milnor basis, using the \leq_r order on the Arnon A basis and the bijection with the Milnor basis given by the q-atomic sequences.

Statement (ii) follows at once from (i). It follows from (iii) that the Z-Arnon A basis is a basis for A_q , and that it is triangular with respect to the Y-Arnon A basis when both are in the \leq_r order.

The proof of Theorem [8.1](#page-24-1) will occupy the rest of this section. We begin by expressing the particular elements X_0^n of the Arnon A basis, regarded as elements of the graded algebra $E^0(A_q)$, in the Z- P_t^s basis. A typical Z- P_t^s basis element has the form $P[a_1]P[a_2] \cdots P[a_r]$, where (a_1, a_2, \ldots, a_r) is a sequence of q-atomic numbers in increasing Z-order, and has May filtration $\sum_{i=1}^{r} \alpha(a_i)$.

Proposition 8.2. Let $q = p^e$ where p is prime, and let $n = te + k$, where $0 \leq k < e$. Then in $E^0(\mathsf{A}_q)$

$$
X_k^n = \sum_{r=k}^t X_k^{(r-1)e+k} P[p^k(q^{t+1} - q^r)/(q-1)], \text{ where } X_k^{-e} = 1.
$$

When $k = 0$, $t = 1$ this states that $P^q P^1 = P^1 P[q] + P[q + 1]$, and when $k = 0, t = 2$ that $P^{q^2}P^qP^1 = P^qP^1P[q^2] + P^1P[q^2+q] + P[q^2+q+1]$. Recall that $P[q] = P^q$, $P[q^2] = P^{q^2}$, $P[q + 1] = P(0, 1)$, $P[q^2 + q] = P(0, q)$ and $P[q^2+q+1] = P(0,0,1).$

Proof. We first establish the case $k = 0$, namely

$$
X_0^n = \sum_{r=0}^t X_0^{(r-1)e} P[(q^{t+1} - q^r)/(q-1)],
$$

by induction on t. Thus let $t \geq 2$, and assume that the above formula holds for $t-1$ in place of t. Since $X_0^{te} = P^{q^t} X_0^{(t-1)e}$, the induction hypothesis gives 0 $X_0^{te} = P^{q^t} \sum_{r=0}^{t-1} X_0^{(r-1)e} P[(q^t - q^r)/(q-1)].$ Since P^{q^t} and P^{q^r} commute in $E^0(\mathsf{A}_q)$ for 0 ≤ $r \le t - 2$, this gives $X_0^n = \sum_{r=0}^{t-1} X_0^{(r-1)e} P^{q^t} P[(q^t - q^r)/(q-1)]$. Using the relation $[P^{q^t}, P[(q^t - q^r)/(q - 1)]] = P[(q^{t+1} - q^r)/(q - 1)]$, it follows that

$$
X_0^{te} = \sum_{r=0}^{t-1} X_0^{(r-1)e} (P[(q^t - q^r)/(q-1)]P^{q^t} + P[(q^{t+1} - q^r)/(q-1)]).
$$

Applying the induction hypothesis again, this gives

$$
X_0^{te} = X_0^{(t-1)e} P^{q^t} + \sum_{r=0}^{t-1} X_0^{(r-1)e} P[(q^{t+1} - q^r)/(q-1)]
$$

=
$$
\sum_{r=0}^{t} X_0^{(r-1)e} P[(q^{t+1} - q^r)/(q-1)].
$$

This completes the induction, and the proof for $k = 0$. The general case follows, as the relations in $E^0(A_q)$ are unchanged when all exponents are multiplied by 口 p .

As in [\[1\]](#page-28-11), to prove parts (i) and (ii) of Theorem [8.1](#page-24-1) it suffices to show that any monomial in A_q which is not of the required form is reducible in the \leq_r order, and that the number of monomials of the required form in each degree d is the dimension of A_q^d . The second statement is clear since, as for the P_t^s bases, there is one elementary Arnon A monomial in each q -atomic degree, and these are used to form monomials with exponents $\lt p$. For the first statement, we observe that a formal monomial $P^A = P^{p^{j_1}} \cdots P^{p^{j_r}}$ is not a Y-Arnon A basis element if and only if at least one of the following cases occurs:

(1) for some $k, j_k > j_{k+1}$ and $j_k - j_{k+1} \neq e;$

(2) the sequence $P[p^m]X_k^n$ with $k < m'e$, $m \le n$ and $m = m'e + m''$, $0 \le$ $m'' < e$, appears in P^A ;

(3) the sequence $X_k^n P[p^k]$ with $k < n$ appears in P^A ;

(4) the sequence $X_k^n \cdots X_k^n$ (*p* factors) appears in P^A .

To prove these statements, it is sufficient to work in $E^0(A_q)$. Case (1) is resolved immediately, since $P[p^s]$ and $P[p^t]$ commute in $E^0(\mathsf{A}_q)$ if $t \neq s + e$.

We deal next with Case (2). For $p = 2$, the minimal case $Sq^2 \cdot Sq^4Sq^2Sq^1$ is lowered in the \leq_r order by relation [\(5\)](#page-23-0) above. We use Proposition [8.2](#page-24-2) to obtain a similar reduction in general.

Proposition 8.3. In $E^0(A_q)$, $P^{p^m}X_k^n$ can be reduced in the \leq_r order for $k < m'e$, $m \leq n$ and $m = m'e + m''$, $0 \leq m'' < e$.

Proof. We can immediately reduce to $k = m - 1$, since for $k < m - 1$ we have $P^{p^m} X_k^n = P^{p^m} X_{m-1}^n \cdot X_k^{m-2}$ $\binom{m-2}{k}$, and a reduction of $P^{p^m}X_{m-1}^n$ in the right order gives a reduction of $P^{p^m} X_{m-1}^n \cdot X_k^{m-2}$ $\binom{m-2}{k}$ in the right order.

Similarly, we can reduce to $n = m$ or $n = m + e$ using Case (1). For example, suppose that we wish to reduce $P^p \cdot P^{p^3} P^{p^2} P^p P^1$ in the \leq_r order. Since $P^p P^{p^3} =$ $P^{p^3}P^p$ in $E^0(\mathsf{A}_p)$, it is sufficient to lower $P^p \cdot P^{p^2}P^pP^1$ in the \leq_r order.

Thus let $n = m$, $k = m - e$. We have $P^{p^m} \cdot P^{p^m - e} = P^{p^m} (P^{p^{m-e}} P^{p^m} + P^{p^m} P^{p^m} - P^{p^m} P^{p^m}$ $P[p^m + p^{m-e}]$ in $E^0(A_q)$. The first term is $\lt_r P^{p^m} \cdot P^{p^m-p^m-e}$. The second term reduces to $P[p^m + p^{m-e}]P^{p^m}$ in $E^0(\mathsf{A}_q)$. On expanding $P[p^m + p^{m-e}]$ as $P^{p^m}P^{p^{m-e}} - P^{p^{m-e}}P^{p^m}$, we obtain two terms, and both are $\lt_p P^{p^m} \cdot P^{p^m-p^{m-e}}$.

Finally let $n = m + e$, $k = m - e$. By Proposition [8.2,](#page-24-2) we have $P^{p^m} X_{m-e}^{m+e} =$ $P^{p^m}(P^{p^m}P^{p^{m-e}}P[p^{m+e}] + P^{p^{m-e}}P[p^{m+e}+p^m] + P[p^{m+e}+p^m+p^{m-e}]).$ The first term can be lowered in the \leq_r order by the previous case, and the second by expanding $P[p^{m+e}+p^m]$ as $P^{p^{m+e}}P^{p^m}-P^{p^m}P^{p^{m+e}}$. The third term $P^{p^m}P[p^{m+e}+p^m]$ $[p^{m} + p^{m-e}] = P[p^{m+e} + p^{m} + p^{m-e}]P^{p^{m}}$ in $E^{0}(\mathsf{A}_{q})$. The expansion $P[p^{m+e} + p^{m} + p^{m-e}]$ $[p^{m-e}] = [P^{p^{m+e}}, [P^{p^{m}}, P^{p^{m-e}}]]$ gives four terms, which are all $\lt_r P^{p^{m}} X_{m-e}^{m+e}$ $P^{p^m}\cdot P^{p^{m+e}}P^{p^m}P^{p^{m-e}}.$

We next treat case (3) by the same method.

Proposition 8.4. In $E^0(A_q)$, $X_k^n P^{p^k}$ can be reduced in the \leq_r order for $k < n$, $n = k + te.$

Proof. We can immediately reduce to the case $n = k + e$, since for $n > k + e$ we have $X_k^n P^{p^k} = X_{k+2e}^n X_k^{k+e} P^{p^k}$, and a reduction of $X_k^{k+e} P^{p^k}$ in the right order gives a reduction of $X_{k+2e}^n X_k^{k+e} P^{p^k}$ in the right order.

For $n = k + e$, we have $X_k^{k+e} P^{p^k} = P^{p^{k+e}} P^{p^k} P^{p^k} = (P^{p^k} P^{p^{k+e}} + P[p^{k+e} + p^k]) P^{p^k}$ in $E^0(\mathsf{A}_q)$. The first term is $\lt_r P^{p^k+p^{p^k}}P^{p^k}$, and the second term $P[p^{k+e} + p^{p^k}]$ $[p^k]P^{p^k} = P^{p^k}P[p^{k+e} + p^k] = P^{p^k}[P^{p^{k+e}}, P^{p^k}]$ in $E^0(\mathsf{A}_q)$, and both terms are \lt_r $P^{p^k+e}P^{p^k}P^{p^k}.$ П

The following example illustrates the proof of case (4), using the same method.

Example 8.5. Let $p = 3$ and consider $X_1^2 X_1^2 X_1^2 = P^3 P^1 P^3 P^1 P^3 P^1$. We reduce this in the right order in $E^0(A_3)$ as follows. Recalling that $P[4] = P(0, 1)$, we have $P^3P^1P^3P^1 = P^3P^1P^3P^1(P^1P^3 + P[4])$. The first term is \lt_r $P^3P^1P^3P^1P^3P^1$, and the second is equal to $P[4]P^3P^1P^3P^1 = P[4]P^3P^1(P^1P^3 + P^2P^1)$ $P[4]$. Since $P[4] = [P^3, P^1]$, the first term is again $\lt_r P^3 P^1 P^3 P^1 P^3 P^1$. The second term $P[4]P^{3}P^{1}P[4] = P[4]P[4]P^{3}P^{1} = P[4]P[4](P^{1}P^{3} + P[4])$. The first term is again $\lt_r P^3 P^1 P^3 P^1 P^3 P^1$, and the second $P[4]P[4]P[4] = 0$ in $E^0(\mathsf{A}_3)$.

Proposition 8.6. In $E^0(A_q)$, $X_k^n \cdots X_k^n$ (p factors) can be reduced in the \leq_r order for $k \leq n$, $n = k + te$.

Proof. We first expand the last factor X_k^n as $\sum_{r=k}^t X_k^{(r-1)e+k} P[p^k(q^{t+1}-q^r)/(q-1)]$ 1)] using Proposition [8.2.](#page-24-2) This gives $n - k + 1$ terms, of which all but the last term $X_k^n \cdots X_k^n P[p^n + p^{n-e} + \cdots + p^k]$ (with $p-1$ factors X_k^n) are shown to be $\langle K_k^n \cdots X_k^n$ (with p factors) by expanding $P[(p^{n+e}-p^r)/(p-1)]$ as an iterated commutator. The last term is equal to $P[p^n + p^{n-e} + \cdots + p^k]X_k^n \cdots X_k^n$ since $P[p^n + p^{n-e} + \cdots + p^k]$ commutes with X_k^n in $E^0(\mathsf{A}_q)$. We repeat this process by expanding the last factor X_k^n using Proposition [8.2,](#page-24-2) and observe that all terms but the last are $\lt_r X_k^n \cdots X_k^n$ (*p* factors), while the last term is equal to $P[p^n + p^{n-e} + \cdots + p^k]P[p^n + p^{n-e} + \cdots + p^k]X_k^n \cdots X_k^n$ $(p-2 \text{ factors } X_k^n)$. Iterating this process a further $p-2$ times, we are left with the term $P[p^n +$ $p^{n-e} + \cdots + p^k] \cdots P[p^n + p^{n-e} + \cdots + p^k]$ (*p* factors), which is 0 in $E^0(\mathsf{A}_q)$. \Box

This completes the proof of parts (i) and (ii) of Theorem [8.1.](#page-24-1) We turn to part (iii).

Proposition 8.7. The Y- and Z-Arnon A bases of A_q are triangular with respect to the Milnor basis, when the Arnon bases are taken in \leq_r order.

Proof. By Proposition [6.5,](#page-21-1) it suffices to show that the Y- and Z-Arnon A bases are triangular with respect to the corresponding P_t^s bases. Again it suffices to work in $E^0(A_q)$. We consider the expression of P_t^s basis elements in the Arnon A basis.

We begin by showing that $P_t^s = X_s^{s+(t-1)e} + \leq_r$ -lower terms in the Arnon A basis. The general case follows by taking products. For example, $Sq(0, 1)$ $Sq^2Sq^1 + Sq^1 \cdot Sq^2$ and $Sq(0, 2) = Sq^4Sq^2 + Sq^2 \cdot Sq^4$ in the Arnon A basis. Hence $Sq(0,1)Sq(0,2) = Sq²Sq¹ · Sq⁴Sq² + Sq²Sq¹ · Sq² · Sq⁴ + Sq¹ · Sq² · Sq⁴Sq² +$ $Sq^1 \cdot Sq^2 \cdot Sq^2 \cdot Sq^4$. The first term on the right is the Arnon A basis element corresponding to the P_t^s basis element $Sq(0, 1)Sq(0, 2)$, and the other terms are \leq_r -lower.

As an example, when $p = 2$ we have $P_2^s = Sq[3 \cdot 2^s] = [Sq[2^{s+1}], Sq[2^s]] =$ $Sq^{2^{s+1}}Sq^{2^s} + Sq^{2^s} \cdot Sq^{2^{s+1}}$ in the Arnon A basis for $E^0(A_2)$. In A_2 itself, there are terms of higher May filtration, for example when $s = 2$, $Sq(0, 4) = Sq^{8}Sq^{4} + Sq^{4}$. $Sq^{8} + Sq(3, 3)$, and $M(Sq(3, 3)) = 6$. In the general case, we can express P_{t}^{s} as the iterated commutator

$$
P_t^s = P[p^s(q^t - 1)/(q - 1)] = [P[p^{s + (t-1)e}], [P[p^{s + (t-2)e}], [\dots [[P[p^{s + e}], P[p^s]] \dots]]
$$

of length t in the generators P^{p^j} . The \leq_r -maximal term in the expansion of the iterated commutator is the Arnon A element X_s^{s+t-1} . ◻

References

- [1] Dan Arnon, Monomial bases in the Steenrod algebra, J. Pure Appl. Algebra 96 (1994), 215–223.
- [2] D. Yu. Emelyanov and Th. Yu. Popelensky, On monomial bases in the mod p Steenrod algebra, J. Fixed Point Theory Appl. 17 (2015) 341–353.
- [3] D. Yu. Emelyanov and Th. Yu. Popelensky, Change of bases in the Steenrod algebra mod p, Mat. Sbornik 208 (2017) No. 4, 3-16 (Russian): English translation in *Sbornik: Math.* **208** No. 4, $465-478$.
- [4] D. Yu. Emelyanov, On Wood basis for the mod p Steenrod algebra, Fundam. Prikl. Math. 20, No3, 2015), 83–90 (Russian), English translation in J. Math. *Sci.* (*N.Y.*) **225**, No. 4, (2017), 590–595.
- [5] J. P. May, The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra, Ph.D. thesis, Princeton University 1964.
- [6] John Milnor, The Steenrod algebra and its dual, Ann. of Math. 67 (1958), 150–171.
- [7] S. A. Mitchell, On the Steinberg module, representations of the symmetric groups and the Steenrod algebra, J. Pure Appl. Algebra 39 (1986), 275–281.
- [8] Kenneth G. Monks, Change of basis, monomial relations, and the P_t^s bases for the Steenrod algebra, J. Pure Appl. Algebra 125 (1998), 235–260.
- [9] Grant Walker and Reginald M. W. Wood, Polynomials and the mod 2 Steenrod algebra, London Math. Soc. Lecture Note Series 441, 442, (Cambridge University Press 2018).
- [10] Inder Bir S. Passi, Group rings and their augmentation ideals, Lecture Notes in Mathematics 715, (Springer-Verlag, New York 1979).
- [11] Daniel G. Quillen, On the associated graded ring of a group ring, J. Algebra 10 (1968), 411–418.
- [12] Larry Smith, Polynomial Invariants of Finite Groups, (A. K. Peters Ltd., Wellesley MA, 1955).
- [13] R. M. W. Wood, A note on bases and relations in the Steenrod algebra, Bull. Lond. Math. Soc. **27** (1995), 380–386.

School of Mathematics, Alan Turing Building, University of Manchester, Manchester M13 9PL, UK.