

APPROXIMATION OF FEYNMAN PATH INTEGRALS WITH NON-SMOOTH POTENTIALS

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ABSTRACT. We study the convergence in L^2 of the time slicing approximation of Feynman path integrals under low regularity assumptions on the potential. Inspired by the custom in Physics and Chemistry, the approximate propagators considered here arise from a series expansion of the action. The results are ultimately based on function spaces, tools and strategies which are typical of Harmonic and Time-frequency analysis.

1. INTRODUCTION

Consider the Schrödinger equation

$$(1) \quad i\hbar\partial_t u = -\frac{1}{2}\hbar^2\Delta u + V(t, x)u$$

where $0 < \hbar \leq 1$ and the potential $V(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, is a real-valued function. Feynman's groundbreaking intuition consisted in recasting the corresponding propagator as a formal integral on an infinite dimensional space of paths in configuration space (see [13, 14]). Since then, several approaches have been proposed to make rigorous this insight: we do not attempt to reconstruct the enormous literature on this topic, but we refer to the books [1, 7, 34, 38, 39] and the references therein.

The starting point is the formula for the propagator for the free particle ($V \equiv 0$):

$$U(t, s)f(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} \frac{|x-y|^2}{2(t-s)}} f(y) dy.$$

One can notice that the phase in this integral coincides with the corresponding classical action $S(t, s, x, y)$, which is defined in general as

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follows. First of all, one introduces the action along a path γ in \mathbb{R}^d by the formula

$$S[\gamma] = \int_s^t \mathcal{L}(\gamma(\tau), \dot{\gamma}(\tau), \tau) d\tau,$$

\mathcal{L} being the Lagrangian of the corresponding classical system. Suppose now that for $t - s$ small enough there is only one classical path γ (i.e. a path satisfying the Euler-Lagrange equations) such that the boundary condition $\gamma(s) = y$, $\gamma(t) = x$ hold. One then defines the action $S(t, s, x, y)$ along that path (alias generating function) as

$$(2) \quad S(t, s, x, y) = \int_s^t \mathcal{L}(\gamma(\tau), \dot{\gamma}(\tau), \tau) d\tau.$$

When $\mathcal{L}(x, v) = |v|^2/2$ we retrieve $S(t, s, x, y) = |x - y|^2/(2(t - s))$.

For a wide class of *smooth* potentials with at most quadratic growth at infinity it was shown in [15, 16] (see also [17, 18, 19, 24, 25, 26, 27, 28, 29, 30, 42]) that the propagator $U(t, s)$ for $t - s \neq 0$ small enough can be represented as an oscillatory integral operator (OIO), namely

$$(3) \quad U(t, s)f(x) = \frac{1}{(2\pi i(t - s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S(t,s,x,y)} b(\hbar, t, s, x, y) f(y) dy$$

for a suitable amplitude function $b(\hbar, t, s, x, y)$.

In concrete situations, except for a few cases, there is no hope to obtain the exact propagator in an explicit, closed form. Therefore, it is a common practice in Physics to consider approximate propagators (parametrics) $E^{(N)}(t, s)$ defined by

$$(4) \quad E^{(N)}(t, s)f(x) = \frac{1}{(2\pi i(t - s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S^{(N)}(t,s,x,y)} f(y) dy,$$

where

$$(5) \quad S^{(N)}(t, s, x, y) = \frac{|x - y|^2}{2(t - s)} + \sum_{k=1}^N W_k(x, y)(t - s)^k$$

is essentially a modified N -th order Taylor expansion of the action S at $t = s$; see Section 2 below for the precise construction of the functions W_k .

It is clear that the operators $E^{(N)}(t, s)$ do no longer satisfy the evolution property $U(t, s) = U(t, \tau)U(\tau, s)$. The spirit of the time slicing approach can be condensed as follows (see the monograph [20] for a comprehensive treatment instead): for any subdivision $\Omega : s = t_0 < t_1 < \dots < t_L = t$ of the interval $[s, t]$, consider the composition

$$(6) \quad E^{(N)}(\Omega, t, s) = E^{(N)}(t, t_{L-1})E^{(N)}(t_{L-1}, t_{L-2}) \dots E^{(N)}(t_1, s),$$

which has integral kernel

$$(7) \quad K^{(N)}(\Omega, t, s, x, y) \\ = \prod_{j=1}^L \frac{1}{(2\pi i(t_j - t_{j-1})\hbar)^{d/2}} \int_{\mathbb{R}^{d(L-1)}} \exp\left(\frac{i}{\hbar} \sum_{j=1}^L S^{(N)}(t_j, t_{j-1}, x_j, x_{j-1})\right) \prod_{j=1}^{L-1} dx_j,$$

with $x = x_L$ and $y = x_0$.

It is reasonable to believe that the operators $E^{(N)}(\Omega, t, s)$ converge to the actual propagator as $\omega(\Omega) := \sup\{t_j - t_{j-1} : j = 1, \dots, L\} \rightarrow 0$, in line with Feynman's insight. In order to keep the technicalities at minimum, in this note we confine our investigation to the space of bounded operators in $L^2(\mathbb{R}^d)$, endowed with the usual operator norm. Nevertheless, this basic framework leaves room for considering potentials characterized by mild regularity assumptions. A suitable reservoir of such functions is the so-called Sjöstrand's class, which can be provisionally defined as the space of tempered distributions $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |\langle \sigma, M_\omega T_x g \rangle| d\omega < \infty,$$

for any non-zero Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, where M_ω and T_x , $x, \omega \in \mathbb{R}^d$ respectively denote the modulation and translation operators:

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t), \quad T_x f(t) = f(t - x).$$

In particular, the following condition precisely defines the functions we are interested in.

Assumption (A) $V(t, x)$ is a real-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and there exists $N \in \mathbb{N}$, $N \geq 1$, such that¹

$$(8) \quad \partial_t^k \partial_x^\alpha V \in C_b^0(\mathbb{R}, M^{\infty,1}(\mathbb{R}^d)),$$

for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ satisfying

$$2k + |\alpha| \leq 2N.$$

As a rule of thumb, a function in $M^{\infty,1}(\mathbb{R}^d)$ is bounded on \mathbb{R}^d and locally enjoys the mild regularity of the Fourier transform of an L^1 function. This family of functions is largely known in the context of both pseudodifferential operators and phase space analysis - see for instance [22] and the references therein. Although it may seem an exotic symbols class at a first glance, the connections with other function spaces are manifold: for instance, it contains smooth functions all of whose derivatives are bounded and, more generally, any function whose

¹We denote by $C_b^0(\mathbb{R}, X)$ the space of continuous and bounded functions $\mathbb{R} \rightarrow X$.

Fourier transform is a finite complex measure. It is worth mentioning that the latter class of potentials has been investigated in relation to path integrals by Albeverio [1] and Itô [23]. Actually, the Sjöstrand's class reveals to coincide with a special type of modulation space. In general terms, modulation spaces are Banach spaces defined by controlling the time-frequency concentration and the decay of its elements - see the subsequent section for the details. They were introduced by Feichtinger in the '80s (cf. the pioneering papers [11, 12]) and they were soon recognized as the optimal environment for Time-frequency Analysis. They also provide a fruitful context to set problems in Harmonic Analysis and PDEs. In particular, several studies on the Schrödinger's equation have been conducted from this perspective insofar: among others we mention [3, 4, 6] and the references therein.

The aforementioned results motivate in a natural way the problems we consider in this note. We now state our main result.

Theorem 1.1. *Assume the condition in Assumption (A). For every $T > 0$, there exists a constant $C = C(T) > 0$ such that, for $0 < t - s \leq T\hbar$, $0 < \hbar \leq 1$, and any subdivision Ω of the interval $[s, t]$, we have*

$$(9) \quad \|E^{(N)}(\Omega, t, s) - U(t, s)\|_{L^2 \rightarrow L^2} \leq C\omega(\Omega)^N.$$

Notice that estimates of this type were already obtained in [16, 42] for a different type of short-time approximate propagators - the so-called Birkhoff-Maslov parametrices (see also [36, 37]). Such parametrices have the form of oscillatory integral operators as in (3), for suitable amplitudes $b(\hbar, t, s, x, y)$ constructed from the Hamiltonian flow, and satisfy better estimates, with *positive* powers of \hbar in the right-hand side of (9). In fact, they are very good approximations of the true propagator both for short-time and \hbar small. However, as already observed, the computation of the exact action S is in general a non trivial task and the study of those parametrices require fine arguments and tools from microlocal analysis. On the other hand, the rougher parametrices in (4) are often preferred by the Physics and Chemistry communities for practical purposes; see [7, 31, 32, 33].

The paper is organized as follows. In Section 2 we fix the notation and list the preliminary definitions and results. Section 3 contains the construction of a suitable short-time approximation for the action and the proof of some of its properties. In Section 4 we study the operator $E^{(N)}(t, s)$ as a parametrix. In Section 5 we present a general strategy for suitable higher-order parametrices and we prove our main result (Theorem 1.1).

2. PRELIMINARIES

Notation. We set $\mathbb{N} = \{0, 1, 2, \dots\}$ and employ the multi-index notation: in particular, given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, we write

$$|\alpha| = \alpha_1 + \dots + \alpha_d, \quad x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d},$$

$$\partial_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

We set $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ for the scalar product on \mathbb{R}^d . The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of temperate distributions by $\mathcal{S}'(\mathbb{R}^d)$. The brackets $\langle f, g \rangle$ denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbb{R}^d)$. The conjugate exponent p' of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$.

The notation $f \lesssim g$ stands for $f \leq Cg$, for a suitable constant $C > 0$.

The Fourier transform of a function f on \mathbb{R}^d is normalized as

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x \omega} f(x) dx, \quad \omega \in \mathbb{R}^d.$$

Recall the definition of translation and modulation operators: for any $x, \omega \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$,

$$(T_x f)(t) := f(t - x), \quad (M_\omega f)(t) := e^{2\pi i \omega t} f(t).$$

These operators can be extended by duality on temperate distributions: for any $x, \omega \in \mathbb{R}^d$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\langle T_x u, \varphi \rangle := \langle u, T_{-x} \varphi \rangle, \quad \langle M_\omega u, \varphi \rangle := \langle u, M_{-\omega} \varphi \rangle.$$

In accordance with a harmless improper custom, we occasionally write $(T_x u)(t) = u(t - x)$ even for $u \in \mathcal{S}'(\mathbb{R}^d)$. The composition $\pi(x, \omega) = M_\omega T_x$ constitutes a so-called time-frequency shift.

For any $\lambda \neq 0$, consider the unitary and non-normalized scalar dilation operators on $L^2(\mathbb{R}^d)$ defined by

$$(10) \quad U_\lambda f(x) := |\lambda|^{d/2} f(\lambda x), \quad D_\lambda f(x) := f(\lambda x), \quad f \in L^2(\mathbb{R}^d).$$

These definitions naturally extend to the case of an invertible matrix $A \in \text{GL}(d, \mathbb{R})$ as

$$(11) \quad U_A f(x) := |\det A|^{1/2} f(Ax), \quad D_A f(x) := f(Ax).$$

2.1. Short-time Fourier transform. The short-time Fourier transform (STFT) of a signal $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined as

(12)

$$V_g f(x, \omega) = \langle f, \pi(x, \omega)g \rangle = \mathcal{F}(f \cdot T_x g)(\omega) = \int_{\mathbb{R}^d} f(y) \overline{g(y-x)} e^{-2\pi i y \omega} dy.$$

For a thorough account on the properties of the STFT see [21]. It is important to remark that the STFT is deeply connected with other well-known phase-space transforms, such as the ambiguity distribution

$$Amb(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i \omega y} f\left(y + \frac{x}{2}\right) \overline{g\left(y - \frac{x}{2}\right)} dy,$$

and the Wigner transform

$$W(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i y \omega} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)} dy.$$

In particular, the following relations hold for any $x, \omega \in \mathbb{R}^d$ and $f \in \mathcal{S}'(\mathbb{R}^d), g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$:

$$A(f, g)(x, \omega) = e^{\pi i x \omega} V_g f(x, \omega), \quad W(f, g)(x, \omega) = 2^d e^{4\pi i x \omega} V_{\mathcal{I}g} f(2x, 2\omega),$$

where $\mathcal{I}g(t) = g(-t)$.

For this and other aspects of the connection with phase space analysis, see also [8].

2.2. Modulation spaces. Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (mixed-norm space). Equivalently, $M^{p,q}$ contains the distributions f such that

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q} < \infty,$$

with trivial adjustments if p or q is ∞ . If $p = q$, we write M^p instead of $M^{p,p}$.

It can be proved that $M^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition does not depend on the choice of the window g . For this and other properties we address the reader to [21].

The Sjöstrand's class, originally defined in [40], coincides with the choice $p = \infty, q = 1$. We have that $M^{\infty,1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and it is a Banach algebra under pointwise product. In fact, much more is true (cf. [35]): for any $1 \leq p, q \leq \infty$, the following continuous embedding holds:

$$M^{\infty,1} \cdot M^{p,q} \hookrightarrow M^{p,q}.$$

Within the broad family of (weighted) modulation spaces, we can retrieve a number of well-known classical spaces, such as Sobolev or Bessel potential spaces - see again [21]. Here, we confine ourselves to remark that $L^2 = M^2$.

For the benefit of the reader, let us recall a result on the boundedness of dilation operators on modulation spaces that will be repeatedly used hereafter - cf. [41, Theorem 3.1] and [5, Proposition 3.1].

Lemma 2.1. *Let $1 \leq p, q \leq \infty$ and $A \in \text{GL}(d, \mathbb{R})$. For any $f \in M^{p,q}(\mathbb{R}^d)$,*

$$\|D_A f\|_{M^{p,q}} \lesssim C_{p,q}(A) \|f\|_{M^{p,q}},$$

where

$$C_{p,q}(A) = |\det A|^{-(1/p-1/q+1)} (\det(I + A^\top A))^{1/2}.$$

3. SHORT-TIME ACTION AND RELATED ESTIMATES

We begin with a brief discussion devoted to explain the structure of formula (5) for the approximate action $S^{(N)}(t, s, x, y)$ appearing in (4). We refer to [7, Section 4.5] for more details.

It is well known that, for a classical Hamiltonian of physical type

$$H(x, p, t) = \frac{1}{2}p^2 + V(t, x),$$

the action $S(t, s, x, y)$ satisfies the Hamilton-Jacobi equation

$$(13) \quad \frac{\partial S}{\partial t} + \frac{1}{2}|\nabla_x S|^2 + V(t, x) = 0.$$

In order for $E^{(N)}$ to be a parametrix in a sense to be specified (cf. Remark 4.3 below), we consider the slightly modified equation

$$(14) \quad \frac{\partial S}{\partial t} + \frac{1}{2}|\nabla_x S|^2 + V(t, x) + \frac{i\hbar d}{2(t-s)} - \frac{i\hbar}{2}\Delta_x S = 0,$$

and look for a solution S in the form $S(t, s, x, y) = \frac{|x-y|^2}{2(t-s)} + R(t, s, x, y)$, $s < t$. This yields an equivalent equation for R , namely

$$\frac{\partial R}{\partial t} + \frac{1}{2}|\nabla_x R|^2 + V(t, x) + \frac{1}{t-s}(x-y) \cdot \nabla_x R - \frac{i\hbar}{2}\Delta_x R = 0.$$

Assume that

$$R(t, s, x, y) = W_0 + W_1(s, x, y)(t-s) + W_2(s, x, y)(t-s)^2 + \dots,$$

where the functions $W_k(s, x, y)$ will be briefly denoted by $W_k(x, y)$ from now on. We immediately find $W_0 = 0$ and, for $k \geq 1$, by equating to 0 the coefficient of the term $(t-s)^{k-1}$, we obtain the equations

$$(15) \quad kW_k(x, y) + (x-y) \cdot \nabla_x W_k(x, y) = F_k(x, y),$$

where we set

(16)

$$F_k(x, y) = -\frac{1}{2} \sum_{\substack{j+\ell=k-1 \\ j \geq 1, \ell \geq 1}} \nabla_x W_j \cdot \nabla_x W_\ell - \frac{1}{(k-1)!} \partial_t^{k-1} V(s, x) + \frac{i\hbar}{2} \Delta_x W_{k-1}.$$

Lemma 3.1. *For any integer $k \geq 1$ there exists a unique continuous solution of (15), namely*

$$(17) \quad W_k(x, y) = \int_0^1 \tau^{k-1} F_k(\tau x + (1-\tau)y, y) d\tau.$$

Proof. According to the methods of characteristics, along the curves of type $x_u(\lambda) = y + ue^\lambda$, where $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^d$ has unitary norm, the original PDE (15) becomes a linear ODE with respect to the variable λ :

$$\frac{d}{d\lambda} W_k(x_u(\lambda), y) + kW_k(x_u(\lambda), y) = F_k(x_u(\lambda), y),$$

whose solutions are given by

$$W_k(x_u(\lambda), y) = e^{-k\lambda} \left(\int_{-\infty}^{\lambda} e^{k\sigma} F_k(x_u(\sigma), y) d\sigma + C \right),$$

where $C \in \mathbb{R}$ is an arbitrary constant. Notice that $\lambda = \log \|x - y\|$ and the change of variable $\sigma = \log (\|x - y\| \tau)$ thus gives

$$W_k(x, y) = \int_0^1 \tau^{k-1} F_k(\tau x + (1-\tau)y, y) d\tau + \frac{C}{\|x - y\|^k}.$$

It is therefore clear that the unique continuous solution corresponds to $C = 0$. \square

For $N = 1, 2, \dots$, we now define the approximate generating functions as in (5), namely

$$(18) \quad S^{(N)}(t, s, x, y) = \frac{|x - y|^2}{2(t - s)} + R^{(N)}(t, s, x, y),$$

where

$$(19) \quad R^{(N)}(t, s, x, y) := \sum_{k=1}^N W_k(x, y) (t - s)^k$$

and $W_k(x, y)$ is defined in (17).

In particular, the first-order approximation of the action ($N = 1$) is

$$S^{(1)}(t, s, x, y) = \frac{|x - y|^2}{2(t - s)} - (t - s) \int_0^1 V(s, \tau x + (1 - \tau)y) d\tau,$$

It is worth mentioning that determining the correct short-time approximations to the action functional is an important matter, initially investigated by Makri and Miller in [31, 32, 33] - see also [9] and the recent paper [10] for its relevance to quantization issues. We remark that the authors consider power series solutions of the Hamilton-Jacobi equation (13) instead of (14), but this would provide poor approximating power for $E^{(N)}$ as a parametrix, namely a first order error in $t - s$ regardless of N . In fact, the results in [16] show that the parametrix in (4) with $S^{(N)}$ replaced by the true action S does not enjoy better estimates than a first order one, even for smooth potentials.

Concerning the regularity of the terms W_k in $S^{(N)}$, we can prove the following result.

Proposition 3.2. *If V satisfies Assumption (A), then for any $1 \leq k \leq N$ we have*

$$\|\partial_x^\alpha W_k\|_{M^{\infty,1}(\mathbb{R}^{2d})} \leq C, \quad \text{for } |\alpha| \leq 2(N - k + 1), \quad s \in \mathbb{R},$$

for some constant $C > 0$.

Proof. Let us first prove the claim for $k = 1$. For $(z, \zeta) \in \mathbb{R}^{2d}$, the STFT of $\partial_x^\alpha W_1$, $|\alpha| \leq 2N$, can be written as

$$|V_g \partial_x^\alpha W_1(z, \zeta)| = \left| \int_0^1 \tau^{|\alpha|} V_g [\partial_x^\alpha V(s, \tau x + (1 - \tau)y)](z, \zeta) d\tau \right|.$$

We now think of V as a function on \mathbb{R}^{2d} . More precisely, define

$$V'(s, x, y) := V(s, x), \quad s \in \mathbb{R}, \quad x, y \in \mathbb{R}^d$$

and notice that V' still satisfies Assumption (A) with $M^{\infty,1}(\mathbb{R}^d)$ replaced by $M^{\infty,1}(\mathbb{R}^{2d})$.

Let us introduce the parametrized matrices $M_\tau = \begin{pmatrix} \tau I & (1 - \tau) I \\ 0 & I \end{pmatrix} \in \text{GL}(2d, \mathbb{R})$, with $\tau \in (0, 1]$. We can thus write $V(s, \tau x + (1 - \tau)y) = V'(s, M_\tau(x, y))$, and from Lemma 2.1 we have $\partial_x^\alpha V'(s, M_\tau(x, y)) \in M^{\infty,1}(\mathbb{R}^{2d})$. Therefore,

$$\begin{aligned} \|\partial_x^\alpha W_1\|_{M^{\infty,1}(\mathbb{R}^{2d})} &\lesssim \int_0^1 \tau^{|\alpha|} \|\partial_x^\alpha V'(s, M_\tau \cdot)\|_{M^{\infty,1}(\mathbb{R}^{2d})} d\tau \\ &\lesssim \left(\int_0^1 \tau^{|\alpha|} C_{\infty,1}(M_\tau) d\tau \right) \|\partial_x^\alpha V'\|_{M^{\infty,1}(\mathbb{R}^{2d})} < C, \end{aligned}$$

where the last estimate follows from the fact that

$$C_{\infty,1}(M_\tau) = (\det(I + M_\tau^\top M_\tau))^{1/2}$$

is a continuous function of the parameter $\tau \in [0, 1]$.

Assume now that the claim holds for any W_j up to a certain $k \leq N-1$ and consider

$$|V_g \partial_x^\alpha W_{k+1}(z, \zeta)| = \left| \int_0^1 \tau^{k+|\alpha|} V_g [\partial_x^\alpha F_{k+1}(\tau x + (1-\tau)y, y)](z, \zeta) d\tau \right|.$$

It is easy to deduce from (16) and the hypothesis on W_1, \dots, W_k that $\partial_x^\alpha F_{k+1}(x, y) \in M^{\infty,1}(\mathbb{R}^{2d})$ whenever $|\alpha| \leq 2(N-k)$. Again from Lemma 2.1 we have $\partial_x^\alpha F_{k+1}(M_\tau(x, y)) \in M^{\infty,1}(\mathbb{R}^{2d})$, and by the same arguments as before we have

$$\begin{aligned} \|\partial_x^\alpha W_{k+1}\|_{M^{\infty,1}(\mathbb{R}^{2d})} &\lesssim \int_0^1 \tau^{k+|\alpha|} \|\partial_x^\alpha F_{k+1}(M_\tau \cdot)\|_{M^{\infty,1}(\mathbb{R}^{2d})} d\tau \\ &\lesssim \left(\int_0^1 \tau^{k+|\alpha|} C_{\infty,1}(M_\tau) d\tau \right) \|\partial_x^\alpha W_{k+1}\|_{M^{\infty,1}(\mathbb{R}^{2d})} < C. \end{aligned}$$

The claim is then proved by induction. \square

Proposition 3.3. *If the potential function V satisfies Assumption (A), then $e^{\frac{i}{\hbar}R^{(N)}} \in M^{\infty,1}(\mathbb{R}^{2d})$, with $R^{(N)}$ as in (19). More precisely,*

$$\|e^{\frac{i}{\hbar}R^{(N)}}\|_{M^{\infty,1}} \leq C(T),$$

for $0 \leq t-s \leq T\hbar$, $0 < \hbar \leq 1$.

Proof. If V satisfies Assumption (A), Proposition 3.2 holds and $\partial^\alpha W_k(x, y) \in M^{\infty,1}(\mathbb{R}^{2d})$ for any $|\alpha| \leq 2(N-k+1)$. In particular, $W_k \in M^{\infty,1}(\mathbb{R}^{2d})$ for all $k = 1, \dots, N$ and thus $R^{(N)} \in M^{\infty,1}(\mathbb{R}^{2d})$.

Given that $M^{\infty,1}(\mathbb{R}^{2d})$ is a Banach algebra for pointwise multiplication, it is enough to show the desired estimate for $e^{\frac{i}{\hbar}(t-s)^k W_k}$, for any $1 \leq k \leq N$:

$$\begin{aligned} \left\| e^{\frac{i}{\hbar}(t-s)^k W_k} \right\|_{M^{\infty,1}} &= \left\| \sum_{n=0}^{\infty} \frac{i^n (t-s)^{kn} (W_k)^n}{\hbar^n n!} \right\|_{M^{\infty,1}} \\ &\leq \sum_{n=0}^{\infty} \frac{C^{n-1} (t-s)^{kn} \|W_k\|_{M^{\infty,1}}^n}{\hbar^n n!} \\ &= C^{-1} e^{\frac{C}{\hbar}(t-s)^k \|W_k\|_{M^{\infty,1}}} \leq C(T), \end{aligned}$$

for $0 \leq t-s \leq T\hbar$, $0 < \hbar \leq 1$. \square

4. SHORT-TIME APPROXIMATE PROPAGATOR

Let us first recall that the Cauchy problem for the Schrödinger equation with bounded potentials is globally well-posed in $L^2(\mathbb{R}^d)$. This is an easy and classic result that can be stated as follows.

Proposition 4.1. *Assume that V is a real-valued function on $\mathbb{R} \times \mathbb{R}^d$ satisfying $V \in C^\infty(\mathbb{R}, L^\infty(\mathbb{R}^d))$ and let $s \in \mathbb{R}$. Then, the Cauchy problem*

$$\begin{cases} i\hbar\partial_t u = -\frac{1}{2}\hbar^2\Delta u + V(t, x)u \\ u(s, x) = u_0(x) \end{cases}$$

is (backward and) forward globally well-posed in $L^2(\mathbb{R}^d)$ and the corresponding propagator $U(t, s)$ is a unitary operator on $L^2(\mathbb{R}^d)$.

Consider the parametrix $E^{(N)}(t, s)$ in (4). We have the following result.

Proposition 4.2. *For every $T > 0$ there exists $C = C(T) > 0$ such that, for $0 < t - s \leq T\hbar$, $0 < \hbar \leq 1$, we have*

$$(20) \quad \|E^{(N)}(t, s)\|_{L^2 \rightarrow L^2} \leq C.$$

Moreover, for $f \in L^2(\mathbb{R}^d)$ we have

$$(21) \quad \lim_{t \searrow s} E^{(N)}(t, s)f = f$$

in $L^2(\mathbb{R}^d)$.

Proof. First, notice that

$$E^{(N)}(t, s)f(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\frac{|x-y|^2}{2(t-s)}} e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)} f(y) dy$$

is an OIO with the free-particle-action as phase function and amplitude $a^{(N)}(t, s, x, y) := \exp\left(\frac{i}{\hbar}R^{(N)}(t, s, x, y)\right) \in M^{\infty,1}(\mathbb{R}^{2d})$ by Proposition 3.3. There is a number of results concerning the L^2 -boundedness in this context, but in order to keep track of the short-time behaviour in the estimates a few more steps are needed.

First, notice that the dilation operators defined in (10) allow us to rephrase $E^{(N)}(t, s)$ as follows:

$$E^{(N)}(t, s) = U_{\frac{1}{\sqrt{\hbar(t-s)}}} \tilde{E}^{(N)}(t, s) U_{\sqrt{\hbar(t-s)}},$$

where

$$\tilde{E}^{(N)}(t, s)f(x) = \frac{1}{(2\pi i)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{2}|x-y|^2} \tilde{a}^{(N)}(t, s, x, y)f(y) dy$$

is an OIO whose phase function is free from time and \hbar dependence and the amplitude is

$$\tilde{a}^{(N)}(t, s, x, y) = e^{\frac{i}{\hbar} \sum_{k=1}^N W_k(\sqrt{\hbar(t-s)}x, \sqrt{\hbar(t-s)}y)} (t-s)^k = D_{\sqrt{\hbar(t-s)}} a^{(N)}(x, y).$$

In particular, $\tilde{a}^{(N)} \in M^{\infty,1}(\mathbb{R}^{2d})$ by Lemma 2.1, and

$$\|\tilde{a}^{(N)}\|_{M^{\infty,1}} \leq C(T) \|a^{(N)}\|_{M^{\infty,1}}$$

for $0 < \hbar(t-s) \leq T$.

We are then able to prove (20) by means of boundedness results for this kind of operators, such as [2, Theorem 2.1], and Proposition 3.3:

$$\|E^{(N)}(t, s)\|_{L^2 \rightarrow L^2} = \|\tilde{E}^{(N)}(t, s)\|_{L^2 \rightarrow L^2} \lesssim \|\tilde{a}^{(N)}\|_{M^{\infty,1}} \lesssim \|a^{(N)}\|_{M^{\infty,1}} \leq C(T),$$

for $0 < t-s \leq T\hbar$.

For what concerns strong convergence to the identity as $t \searrow s$, consider the operator

$$H^{(N)}(t, s) f(x) = \frac{1}{(2\pi i (t-s) \hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} \frac{|x-y|^2}{2(t-s)}} \left(e^{\frac{i}{\hbar} R^{(N)}(t,s,x,y)} - 1 \right) f(y) dy$$

and employ again the dilations in order to write

$$H^{(N)}(t, s) = U_{\frac{1}{\sqrt{\hbar(t-s)}}} \tilde{H}^{(N)}(t, s) U_{\sqrt{\hbar(t-s)}},$$

where

$$\tilde{H}^{(1)}(t, s) f(x) = \frac{1}{(2\pi i)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{2}|x-y|^2} \tilde{b}^{(N)}(t, s, x, y) f(y) dy$$

is an OIO with amplitude $\tilde{b}^{(N)}(t, s, x, y) = \tilde{a}^{(N)}(t, s, x, y) - 1 \in M^{\infty,1}(\mathbb{R}^{2d})$.

The latter can be expanded as follows:

$$\begin{aligned} \tilde{b}^{(N)}(t, s, x, y) &= e^{\frac{i}{\hbar} R^{(N)}(t,s,\sqrt{\hbar(t-s)}x,\sqrt{\hbar(t-s)}y)} - 1 \\ &= \frac{i}{\hbar} (t-s) \overline{R^{(N)}} \left(t, s, \sqrt{\hbar(t-s)}x, \sqrt{\hbar(t-s)}y \right), \end{aligned}$$

where

$$\begin{aligned} \overline{R^{(N)}}(t, s, x, y) &= \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \left(\frac{t-s}{\hbar} \right)^{n-1} \left(\sum_{k=1}^N W_k \left(\sqrt{\hbar(t-s)}x, \sqrt{\hbar(t-s)}y \right) (t-s)^{k-1} \right)^n. \end{aligned}$$

The algebra property of the Sjöstrand's class and Lemma 2.1 imply that $\overline{R^{(N)}}$ belongs to a bounded subset of $M^{\infty,1}(\mathbb{R}^{2d})$ for $0 < t-s \leq T\hbar$, $0 < \hbar \leq 1$. It is then clear that $\tilde{b}^{(N)} \rightarrow 0$ in $M^{\infty,1}(\mathbb{R}^{2d})$ for $t \searrow s$.

Therefore, the OIO with operator $\tilde{b}^{(N)}$ has operator norm converging to 0 as $t \searrow s$, and (21) follows. \square

Remark 4.3. A direct check shows that the $E^{(N)}(t, s)$ is a parametrix, meaning that

$$\left(i\hbar\partial_t + \frac{1}{2}\hbar^2\Delta - V(t, x) \right) E^{(N)}(t, s) = G^{(N)}(t, s),$$

with

(22)

$$G^{(N)}(t, s)f = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S^{(N)}(t,s,x,y)} g_N(\hbar, t, s, x, y) f(y) dy,$$

where, from the construction of $S^{(N)}$ (see in particular eqs. (5), (15) and (16)), the amplitude g_N satisfies

$$\begin{aligned} g_N(\hbar, t, s, x, y) &= -\frac{\partial S^{(N)}}{\partial t} - \frac{1}{2} |\nabla_x S^{(N)}|^2 - V(t, x) - \frac{i\hbar d}{2(t-s)} + \frac{i\hbar}{2} \Delta_x S^{(N)} \\ &= \sum_{k=N}^{2N} -\frac{1}{2} \left(\sum_{\substack{j+\ell=k \\ j, \ell \geq 1}} \nabla_x W_j \cdot \nabla_x W_\ell (t-s)^k \right) \\ &\quad + \frac{i\hbar}{2} \Delta_x W_N(x, y) (t-s)^N \\ &\quad - \frac{(t-s)^N}{(N-1)!} \int_0^1 (1-\tau)^{N-1} (\partial_t^N V) ((1-\tau)s + \tau t, x) d\tau. \end{aligned}$$

Hence, by Assumption (A) and Proposition 3.2

$$\|g_N(\hbar, t, s, \cdot, \cdot)\|_{M^{\infty,1}(\mathbb{R}^{2d})} \leq C(t-s)^N$$

for $0 < t-s \leq T$, with a constant $C = C(T) > 0$ independent of $\hbar \in (0, 1]$.

Following the path of the preceding proof, by means of suitable dilations we can conveniently recast $G^{(N)}(t, s)$ as an OIO with time and \hbar independent phase and amplitude

$$\tilde{g}^{(N)}(\hbar, t, s, x, y) = D \sqrt{\hbar(t-s)} \left[e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)} g^{(N)}(\hbar, t, s, x, y) \right].$$

We have $\tilde{g}^{(N)} \in M^{\infty,1}(\mathbb{R}^{2d})$ by Lemma 2.1 - in fact, more than this is true:

$$\|\tilde{g}^{(N)}(\hbar, t, s, \cdot, \cdot)\|_{M^{\infty,1}} \leq C(t-s)^N$$

for $0 < t-s \leq T\hbar$ and a constant $C = C(T) > 0$. Therefore, $G^{(N)}(t, s)$ extends to a bounded operator on $L^2(\mathbb{R}^d)$ (cf. again [2]) and arguments similar to those of the preceding proof lead to the estimate

$$(23) \quad \|G^{(N)} f\|_{L^2} \leq C \|\tilde{g}^{(N)}\|_{M^{\infty,1}} \|f\|_{L^2} \leq C(T) (t-s)^N \|f\|_{L^2},$$

for $0 < t-s \leq T\hbar$.

The preceding discussion is the bedrock of the following result.

Theorem 4.4. *For every $T > 0$, there exists a constant $C = C(T) > 0$ such that*

$$(24) \quad \|E^{(N)}(t, s) - U(t, s)\|_{L^2 \rightarrow L^2} \leq C\hbar^{-1}(t-s)^{N+1},$$

whenever $0 < t-s \leq T\hbar$.

Proof. The propagator $U(t, s)$ clearly satisfies the equation

$$(i\hbar\partial_t - H)U(t, s)f = 0$$

for every $f \in L^2(\mathbb{R}^d)$, where $H = -(\hbar^2/2)\Delta + V$ is the Hamiltonian operator, with V as in Assumption (A). On the other hand

$$(i\hbar\partial_t - H)E^{(N)}(t, s)f = G^{(N)}(t, s)f,$$

which can be rephrased in integral form (Duhamel's principle) as

$$E^{(N)}(t, s)f = U(t, s)f - i\hbar^{-1} \int_s^t U(t, \tau)G^{(N)}(\tau, s)f d\tau.$$

Therefore, given $f \in L^2(\mathbb{R}^d)$, by (23) we have

$$\begin{aligned} \|U(t, s)f - E^{(N)}(t, s)f\|_{L^2} &= \left\| \hbar^{-1} \int_s^t U(t, \tau)G^{(N)}(\tau, s)f d\tau \right\|_{L^2} \\ &\leq \hbar^{-1} \int_s^t \|U(t, \tau)\|_{L^2 \rightarrow L^2} \|G^{(N)}(\tau, s)f\|_{L^2} d\tau \\ &\leq C(T)\hbar^{-1} \int_s^t \|f\|_{L^2} (t-s)^N d\tau \\ &\leq C'(T)\hbar^{-1}(t-s)^{N+1} \|f\|_{L^2}, \end{aligned}$$

for $0 < t-s \leq T\hbar$. □

5. AN ABSTRACT RESULT AND PROOF OF THE MAIN RESULT (THEOREM 1.1)

We begin by presenting a convergence result for the approximate propagators in its full generality. In fact, it can be regarded as a generalization of [16, Lemma 3.2]. We also use in the proof some ingenious tricks from that paper.

Theorem 5.1. *Assume that for some $\delta > 0$ we have a family of operators $E^{(N)}(t, s)$ for $0 < t - s \leq \delta$, and $U(t, s)$, $s, t \in \mathbb{R}$, bounded in $L^2(\mathbb{R}^d)$, satisfying the following conditions:*

U enjoys the evolution property $U(t, \tau)U(\tau, s) = U(t, s)$ for every $s < \tau < t$ and for every $T > 0$ there exists a constant $C_0 \geq 1$ such that

$$(25) \quad \|U(t, s)\|_{L^2 \rightarrow L^2} \leq C_0 \text{ for } 0 < t - s \leq T.$$

Moreover, for some constant $C_1 > 0$ we have

$$(26) \quad \|E^{(N)}(t, s) - U(t, s)\|_{L^2 \rightarrow L^2} \leq C_1(t - s)^{N+1} \text{ for } t - s \leq \delta.$$

For any subdivision $\Omega : s = t_0 < t_1 < \dots < t_L = t$ of the interval $[s, t]$, with $\omega(\Omega) = \sup\{t_j - t_{j-1} : j = 1, \dots, L\} < \delta$, consider therefore the composition $E^{(N)}(\Omega, t, s)$ in (6).

Then, for every $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$(27) \quad \|E^{(N)}(\Omega, t, s) - U(t, s)\|_{L^2 \rightarrow L^2} \leq C\omega(\Omega)^N(t - s) \text{ for } 0 < t - s \leq T.$$

More precisely,

$$C = C(T) = C_0^2 C_1 \exp\left(C_0 C_1 \omega(\Omega)^N T\right).$$

Proof. Let

$$R^{(N)}(t, s) := E^{(N)}(t, s) - U(t, s)$$

so that by (26), we have

$$(28) \quad \|R^{(N)}(t, s)\| \leq C_1(t - s)^{N+1} \text{ for } 0 < t - s \leq \delta.$$

Hence we can write

$$(29) \quad \begin{aligned} & E^{(N)}(\Omega, t, s) - U(t, s) \\ &= (U(t, t_{L-1}) + R^{(N)}(t, t_{L-1})) \dots (U(t_1, s) + R^{(N)}(t_1, s)) - U(t, s). \end{aligned}$$

One expands the above product and obtains a sum of ordered products of operators, where each product has the following structure: *from right to left* we have, say, q_1 factors of type U , p_1 factors of type $R^{(N)}$, q_2 factors of type U , p_2 factors of type $R^{(N)}$, etc., up to q_k factors of type U , p_k factors of type $R^{(N)}$, to finish with q_{k+1} factors of type U . We can schematically write such a product as

$$\underbrace{U \dots U}_{q_{k+1}} \underbrace{R^{(N)} \dots R^{(N)}}_{p_k} \underbrace{U \dots U}_{q_k} \dots \underbrace{R^{(N)} \dots R^{(N)}}_{p_1} \underbrace{U \dots U}_{q_1}.$$

Here $p_1, \dots, p_k, q_1, \dots, q_k, q_{k+1}$ are non negative integers whose sum is L , with $p_j > 0$ and we can of course group together the consecutive factors of type U , using the evolution property assumed for U . Now, for $0 < t - s \leq T$ we estimate the $L^2 \rightarrow L^2$ norm of the above ordered

product using the known estimates for each factor, namely (25) and (28). In particular, by using the assumption $C_0 \geq 1$, we get

$$\begin{aligned} &\leq C_0^{k+1} \prod_{j=1}^k \prod_{i=1}^{p_j} C_1(t_{J_j+i} - t_{J_j+i-1})^{N+1} \\ &\leq C_0 \prod_{j=1}^k \prod_{i=1}^{p_j} C_0 C_1(t_{J_j+i} - t_{J_j+i-1})^{N+1} \end{aligned}$$

where $J_j = p_1 + \dots + p_{j-1} + q_1 + \dots + q_j$ for $j \geq 2$ and $J_1 = q_1$.

The sum over $p_1, \dots, p_k, q_1, \dots, q_{k+1}$ of these terms is in turn

$$\begin{aligned} &\leq C_0 \left\{ \prod_{j=1}^L (1 + C_0 C_1 (t_j - t_{j-1})^{N+1}) - 1 \right\} \\ &\leq C_0 \left\{ \exp \left(\sum_{j=1}^L C_0 C_1 (t_j - t_{j-1})^{N+1} \right) - 1 \right\} \\ &\leq C_0 \left\{ \exp (C_0 C_1 \omega(\Omega)^N (t - s)) - 1 \right\} \\ &\leq C_0^2 C_1 \omega(\Omega)^N (t - s) \exp (C_0 C_1 \omega(\Omega)^N (t - s)) \end{aligned}$$

where in the last inequality we used $e^\tau - 1 \leq \tau e^\tau$, for $\tau \geq 0$.

This gives (27) with $C = C(T)$ as in the statement and concludes the proof. \square

We can now prove our main result.

Proof of Theorem 1.1. The claim follows at once from Theorem 4.4 and Theorem 5.1 applied with T replaced by $T\hbar$, $C_0 = 1$, $C_1 = C\hbar^{-1}$, where C is the constant appearing in (24), and using $t - s \leq T\hbar$. \square

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