

HYBRID ESTIMATION FOR ERGODIC DIFFUSION PROCESSES BASED ON NOISY DISCRETE OBSERVATIONS

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ABSTRACT. We consider parametric estimation for ergodic diffusion processes with noisy sampled data based on the hybrid method, that is, the multi-step estimation with the initial Bayes type estimators. In order to select proper initial values for optimisation of the quasi likelihood function of ergodic diffusion processes with noisy observations, we construct the initial Bayes type estimator based on the local means of the noisy observations. The asymptotic properties of the initial Bayes type estimators and the hybrid multi-step estimators with the initial Bayes type estimators are shown, and a concrete example and the simulation results are given.

1. INTRODUCTION

We consider the d -dimensional ergodic diffusion process defined by the stochastic differential equation

$$dX_t = b(X_t, \beta) dt + a(X_t, \alpha) dw_t, \quad X_0 = x_0,$$

where $\{w_t\}_{t \geq 0}$ is an r -dimensional Wiener process, x_0 is a d -dimensional random vector independent of $\{w_t\}_{t \geq 0}$, $\alpha \in \Theta_1$ and $\beta \in \Theta_2$ are unknown parameters, $\Theta_i \subset \mathbf{R}^{m_i}$ is bounded, open and convex sets in \mathbf{R}^{m_i} admitting Sobolev's inequalities for embedding $W^{1,p}(\Theta_i) \hookrightarrow C(\bar{\Theta}_i)$ (see Adams and Fournier, 2003; Yoshida, 2011) for $i = 1, 2$, $\theta^* = (\alpha^*, \beta^*)$ is the true parameter vector, and $a : \mathbf{R}^d \times \Theta_1 \rightarrow \mathbf{R}^d \otimes \mathbf{R}^r$ and $b : \mathbf{R}^d \times \Theta_2 \rightarrow \mathbf{R}^d$ are known functions.

Our concern in this paper is the estimation of $\theta := (\alpha, \beta) \in \Theta := \Theta_1 \times \Theta_2$ with long-term, discrete and noisy observation defined as the sequence of the d -dimensional random vectors $\{Y_{ih_n}\}_{i=0, \dots, n}$ such that for all $i = 0, \dots, n$,

$$Y_{ih_n} := X_{ih_n} + \Lambda^{1/2} \varepsilon_{ih_n},$$

where $h_n > 0$ is the discretisation step satisfying $h_n \rightarrow 0$ and $T_n := nh_n \rightarrow \infty$ as $n \rightarrow \infty$, $\{\varepsilon_{ih_n}\}_{i=0, \dots, n}$ is the i.i.d. sequence of d -dimensional random vectors with $\mathbf{E}[\varepsilon_0] = \mathbf{0}$ and $\text{Var}(\varepsilon_0) = I_d$, the components of ε_0 are independent of each other and have symmetric distribution with respect to 0, and Λ is a $d \times d$ real matrix being positive semi-definite, defining the variance of noise term $\Lambda^{1/2} \varepsilon_{ih_n}$. Let us assume the half-vectorisation $\theta_\varepsilon := \text{vech} \Lambda$ is in the bounded, convex and open parameter space $\Theta_\varepsilon \subset \mathbf{R}^{d(d+1)/2}$, and denote $\Xi := \Theta_\varepsilon \times \Theta_1 \times \Theta_2$.

Statistical inference for ergodic diffusion processes has been researched for the last few decades, for instance, see Florens-Zmirou (1989); Yoshida (1992); Bibby and Sørensen (1995); Kessler (1995, 1997); Kutoyants (2004); Iacus (2008); De Gregorio and Iacus (2013, 2018); Iacus and Yoshida (2018) and references therein. The parametric inference for ergodic diffusion processes with discrete and noisy observations has been researched in Favetto (2014, 2016) and Nakakita and Uchida (2017, 2018a,b,c). For parametric estimation for non-ergodic diffusion processes in the presence of market microstructure noise, see Ogihara (2018). Favetto (2014) proposes a simultaneous quasi likelihood function $\mathbb{H}_n(\alpha, \beta)$ which necessitates optimisation with respect to both

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α and β and shows maximum likelihood (ML) type estimators $(\hat{\alpha}_n, \hat{\beta}_n) = \arg \max \mathbb{H}_n(\alpha, \beta)$ have consistency even if the variance of noise is unknown; Favetto (2016) discusses asymptotic normality of the estimator proposed in Favetto (2014) when the variance of noise is known; Nakakita and Uchida (2017, 2018b) suggest adaptive quasi likelihood functions $\mathbb{H}_{1,n}(\alpha)$ and $\mathbb{H}_{2,n}(\beta)$ which succeed in lessening the computational burden in comparison to Favetto (2014, 2016), and prove consistency and asymptotic normality of the adaptive ML type estimators corresponding to the quasi likelihoods; Nakakita and Uchida (2018a) use those quasi likelihood functions for likelihood-ratio-type test and show the asymptotic behaviour of test statistics under both null hypotheses and alternative ones; Nakakita and Uchida (2018c) analyse those quasi likelihood functions with the framework of quasi likelihood analysis (QLA) proposed by Yoshida (2011), and show the polynomial large deviation inequality (PLDI) for the quasi likelihood functions and consequently the convergence of moments of adaptive ML type estimators and adaptive Bayes type ones. For details of adaptive estimation for diffusion processes, see Yoshida (1992, 2011); Uchida and Yoshida (2012, 2014).

In general, however, the optimisation of quasi likelihood functions for diffusion processes, regardless of noise existence, is strongly dependent on initial values, especially in the case where the volatility function a or drift function b are nonlinear with respect to parameters. Hence, Kaino et al. (2017) and Kaino and Uchida (2018a,b) propose hybrid multi-step estimation procedure for diffusion processes where initial values in optimisation are derived from Bayes type estimation with reduced sample sizes and the sequential optimisation with these initial values is implemented, which inherits the idea of hybrid multi-step estimation for diffusion processes with full sample sizes in Kamatani and Uchida (2015) (see also Kutoyants, 2017). In this research, we also consider hybrid multi-step estimation and apply the idea into inference problem, in particular, PLDI for the quasi likelihood functions and the convergence of moments of estimators for discretely and noisily observed ergodic diffusion processes since PLDI and convergence of moment of estimators are key tools to show the mathematical validity of information criteria for model selection problems (see Uchida, 2010; Fujii and Uchida, 2014; Eguchi and Masuda, 2018).

This paper consists of the following parts: Section 2 deals with the notation; we define the initial and multi-step estimators and set the main theorem for the polynomial-type large deviation inequalities, moment estimates of the Bayes type estimators and convergences of moments in Section 3; a concrete example and simulation results are given in Section 4, the conclusions of this work are summarised in Section 5, and finally we give the proofs of the results in Section 6.

2. NOTATION AND ASSUMPTION

First of all we give the notation used throughout this paper.

- For every matrix A , A^T is the transpose of A , and $A^{\otimes 2} := AA^T$.
- For every set of matrices A and B of the same size, $A[B] := \text{tr}(AB^T)$. Moreover, for any $m \in \mathbf{N}$, $A \in \mathbf{R}^m \otimes \mathbf{R}^m$ and $u, v \in \mathbf{R}^m$, $A[u, v] := v^T Au$.
- Let us denote the ℓ -th element of any vector v as $v^{(\ell)}$ and (ℓ_1, ℓ_2) -th one of any matrix A as $A^{(\ell_1, \ell_2)}$.
- For any vector v and any matrix A , $|v| := \sqrt{\text{tr}(v^T v)}$ and $\|A\| := \sqrt{\text{tr}(A^T A)}$.
- For every $p > 0$, $\|\cdot\|_p$ is the $L^p(P_{\theta^*})$ -norm.
- $A(x, \alpha) := a(x, \alpha)^{\otimes 2}$, $a(x) := a(x, \alpha^*)$, $A(x) := A(x, \alpha^*)$ and $b(x) := b(x, \beta^*)$.
- For $i = 1, 2, 3$ and $\tau_i \in (1, 2]$, $p_{\tau_i, n} := h_n^{-1/\tau_i}$, $\Delta_{\tau_i, n} := p_{\tau_i, n} h_n$, $k_{\tau_i, n} := n/p_{\tau_i, n} = n h_n^{1/\tau_i}$.
- With respect to filtration, for all $i = 1, 2, 3$, $\mathcal{G}_t := \sigma(x_0, w_s : s \leq t)$, $\mathcal{G}_{j, i, n}^{\tau_i} := \mathcal{G}_{j \Delta_{\tau_i, n} + i h_n}$, $\mathcal{G}_{j, n}^{\tau_i} := \mathcal{G}_{j, 0, n}^{\tau_i}$, $\mathcal{A}_{j, i, n}^{\tau_i} := \sigma(\varepsilon_{\ell h_n} : \ell \leq j p_{\tau_i, n} + i - 1)$, $\mathcal{A}_{j, n}^{\tau_i} := \mathcal{A}_{j, 0, n}^{\tau_i}$, $\mathcal{H}_{j, i, n}^{\tau_i} := \mathcal{G}_{j, i, n}^{\tau_i} \vee \mathcal{A}_{j, i, n}^{\tau_i}$ and $\mathcal{H}_{j, n}^{\tau_i} := \mathcal{H}_{j, 0, n}^{\tau_i}$.

With respect to X_t , we assume the following conditions.

- [A1] (i) $\inf_{x, \alpha} \det A(x, \alpha) > 0$.

(ii) For a constant C , for all $x_1, x_2 \in \mathbf{R}^d$,

$$\sup_{\alpha \in \Theta_1} \|a(x_1, \alpha) - a(x_2, \alpha)\| + \sup_{\beta \in \Theta_2} |b(x_1, \beta) - a(x_2, \beta)| \leq C |x_1 - x_2|.$$

(iii) For all $p \geq 0$, $\sup_{t \geq 0} \mathbf{E}_{\theta^*} [|X_t|^p] < \infty$.

(iv) There exists a unique invariant measure $\nu = \nu_{\theta^*}$ on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ and for all $p \geq 1$ and $f \in L^p(\nu)$ with polynomial growth,

$$\frac{1}{T} \int_0^T f(X_t) dt \xrightarrow{P} \int_{\mathbf{R}^d} f(x) \nu(dx).$$

(v) For any polynomial growth function $g : \mathbf{R}^d \rightarrow \mathbf{R}$ satisfying $\int_{\mathbf{R}^d} g(x) \nu(dx) = 0$, there exist $G(x)$, $\partial_{x^{(i)}} G(x)$ with at most polynomial growth for $i = 1, \dots, d$ such that for all $x \in \mathbf{R}^d$,

$$L_{\theta^*} G(x) = -g(x),$$

where L_{θ^*} is the infinitesimal generator of X_t .

Remark 1. *Paradoux and Veretennikov (2001) show a sufficient condition for [A1]-(v). Uchida and Yoshida (2012) also introduce the sufficient condition for [A1]-(iii)-(v) assuming [A1]-(i)-(ii), $\sup_{x, \alpha} A(x, \alpha) < \infty$ and $\exists c_0 > 0$, $M_0 > 0$ and $\gamma \geq 0$ such that for all $\beta \in \Theta_2$ and $x \in \mathbf{R}^d$ satisfying $|x| \geq M_0$,*

$$\frac{1}{|x|} x^T b(x, \beta) \leq -c_0 |x|^\gamma.$$

[A2] There exists $C > 0$ such that $a : \mathbf{R}^d \times \Theta_1 \rightarrow \mathbf{R}^d \otimes \mathbf{R}^r$ and $b : \mathbf{R}^d \times \Theta_2 \rightarrow \mathbf{R}^d$ have continuous derivatives satisfying

$$\begin{aligned} \sup_{\alpha \in \Theta_1} |\partial_x^j \partial_\alpha^i a(x, \alpha)| &\leq C (1 + |x|)^C, \quad 0 \leq i \leq 4, \quad 0 \leq j \leq 2, \\ \sup_{\beta \in \Theta_2} |\partial_x^j \partial_\beta^i b(x, \beta)| &\leq C (1 + |x|)^C, \quad 0 \leq i \leq 4, \quad 0 \leq j \leq 2. \end{aligned}$$

With the invariant measure ν , we define

$$\begin{aligned} \mathbb{Y}_1^{\tau_3}(\alpha; \vartheta^*) &:= -\frac{1}{2} \int \left\{ \text{tr} \left(A^{\tau_3}(x, \alpha, \Lambda^*)^{-1} A^{\tau_3}(x, \alpha^*, \Lambda^*) - I_d \right) + \log \frac{\det A^{\tau_3}(x, \alpha, \Lambda^*)}{\det A^{\tau_3}(x, \alpha^*, \Lambda^*)} \right\} \nu(dx), \\ \mathbb{Y}_2(\beta; \vartheta^*) &:= -\frac{1}{2} \int A(x, \alpha^*)^{-1} \left[(b(x, \beta) - b(x, \beta^*))^{\otimes 2} \right] \nu(dx), \\ \mathbb{V}_1(\alpha; \vartheta^*) &:= -\frac{2}{9} \int_{\mathbf{R}^d} \|A^{\tau_1}(x, \alpha, \Lambda_*) - A^{\tau_1}(x, \alpha^*, \Lambda_*)\|^2 \nu(dx) = -\frac{2}{9} \int_{\mathbf{R}^d} \|A(x, \alpha) - A(x, \alpha^*)\|^2 \nu(dx), \\ \mathbb{V}_2(\beta; \vartheta^*) &:= -\frac{1}{2} \int_{\mathbf{R}^d} |b(x, \beta) - b(x, \beta^*)|^2 \nu(dx), \end{aligned}$$

where $A^\tau(x, \alpha, \Lambda) := A(x, \alpha) + 3\Lambda \mathbf{1}_{\{2\}}(\tau)$. For these functions, let us assume the following identifiability conditions hold.

[A3] There exist $\chi_1(\alpha^*) > 0$ and $\chi'_1(\beta^*) > 0$ such that for all $\alpha \in \Theta_1$ and $\beta \in \Theta_2$, $\mathbb{V}_1(\alpha; \vartheta^*) \leq -\chi_1(\vartheta^*) |\alpha - \alpha^*|^2$ and $\mathbb{V}_2(\beta; \vartheta^*) \leq -\chi'_1(\vartheta^*) |\beta - \beta^*|^2$.

[A4] For all $\tau_3 \in (1, 2]$, there exist $\chi_2(\alpha^*) > 0$ and $\chi'_2(\beta^*) > 0$ such that for all $\alpha \in \Theta_1$ and $\beta \in \Theta_2$, $\mathbb{Y}_1^{\tau_3}(\alpha; \vartheta^*) \leq -\chi_2(\vartheta^*) |\alpha - \alpha^*|^2$ and $\mathbb{Y}_2(\beta; \vartheta^*) \leq -\chi'_2(\vartheta^*) |\beta - \beta^*|^2$.

The next assumption is concerned with the moments of noise.

[A5] For any $k > 0$, ε_{ih_n} has k -th moment and the components of ε_{ih_n} are independent of the other components for all i , $\{w_t\}_{t \geq 0}$ and x_0 . In addition, for all odd integer k , $i = 0, \dots, n$, $n \in \mathbf{N}$, and $\ell = 1, \dots, d$, $\mathbf{E}_{\theta^*} \left[\left(\varepsilon_{ih_n}^{(\ell)} \right)^k \right] = 0$, and $\mathbf{E}_{\theta^*} [\varepsilon_{ih_n}^{\otimes 2}] = I_d$.

[A6] There exist $\gamma \in (2/3, 1)$ and $\gamma' \in (0, \gamma]$ such that $n^{-\gamma} \leq h_n \leq n^{-\gamma'}$ for sufficiently large n .

Remark 2. γ' should be smaller than or equal to γ such that $n^{-\gamma} \leq n^{-\gamma'}$. γ should be larger than $2/3$ and smaller than 1 ; otherwise for some $C_1, C_2 > 0$, $k_{\tau,n} \Delta_{\tau,n}^2 = np_{\tau,n} h_n^2 = nh_n^{2-1/\tau} \geq n^{1-\gamma(2-1/\tau)} \geq n^{1-3\gamma/2} > C_1 > 0$, which must converge to 0 for $\tau = \tau_3$ in Theorem 2, or $T_n = nh_n \leq n^{1-\gamma'} < C_2$ as $\gamma \geq \gamma'$, which must diverge in entire discussion. In addition, note that under [A6], $k_n = nh_n^{1/\tau_i} \geq n^{1-\gamma/\tau_i} \geq n^{1-\gamma} \rightarrow \infty$ for all $i = 1, 2, 3$.

3. MULTI-STEP ESTIMATOR AND PLDI

3.1. Setting of the initial and multi-step estimators. We define sequences of local means such that $\{\bar{Y}_{\tau,j}\}_{j=0,\dots,k_{\tau,n}-1}$, $\{\bar{X}_{\tau,j}\}_{j=0,\dots,k_{\tau,n}-1}$ and $\{\bar{\varepsilon}_{\tau,j}\}_{j=0,\dots,k_{\tau,n}-1}$, where

$$\bar{Y}_{\tau,j} = \frac{1}{p_{\tau,n}} \sum_{i=0}^{p_{\tau,n}-1} Y_{j\Delta_{\tau,n}+ih_n}, \quad \bar{X}_{\tau,j} = \frac{1}{p_{\tau,n}} \sum_{i=0}^{p_{\tau,n}-1} X_{j\Delta_{\tau,n}+ih_n}, \quad \bar{\varepsilon}_{\tau,j} = \frac{1}{p_{\tau,n}} \sum_{i=0}^{p_{\tau,n}-1} \varepsilon_{j\Delta_{\tau,n}+ih_n}$$

for $j = 0, \dots, k_{\tau,n} - 1$ and $\tau = \tau_1, \tau_2, \tau_3$. For the detailed properties of local means, see Favetto (2014, 2016); Nakakita and Uchida (2017, 2018b,c).

We set for $i = 1, 2$, $\eta_i \in (\gamma, 1]$ and \underline{n}_{η_i} such that $\underline{n}_{\eta_i} = n^{\eta_i} \leq n$ satisfying $\underline{T}_{\eta_i,n} := \underline{n}_{\eta_i} h_n \leq T_n := nh_n$ (then it holds $\underline{T}_{\eta_i,n} \rightarrow \infty$), and correspondingly $\underline{k}_{\eta_i,\tau_i,n} := \underline{n}_{\eta_i}/p_{\tau_i,n} = n^{\eta_i} h_n^{1/\tau_i}$.

Remark 3. η_1 and η_2 should be larger than γ to support the divergence $\underline{T}_{\eta_i,n} := \underline{n}_{\eta_i} h_n \rightarrow \infty$. η_1 and η_2 actually work to make $(\eta_1 - \gamma/\tau_1) / (1 - \gamma'/\tau_3) > 0$ and $(\eta_2 - \gamma) / (1 - \gamma') > 0$, which are the quantities appearing in Remark 6.

Let us set $q_1, q_2 \in (0, 1/2]$ and the following quasi likelihood functions:

$$\begin{aligned} \mathbb{W}_{1,\tau_1,n}(\alpha|\Lambda) &:= -\frac{1}{2} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left\| \Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda) \right\|^2, \\ \mathbb{W}_{2,\tau_2,n}(\beta) &:= -\frac{1}{2} \sum_{j=1}^{\underline{k}_{\eta_2,\tau_2,n}-2} \left| \Delta_{\tau_2,n}^{-1} |\bar{Y}_{\tau_2,j+1} - \bar{Y}_{\tau_2,j} - \Delta_{\tau_2,n} b(\bar{Y}_{\tau_2,j-1}, \beta)|^2. \end{aligned}$$

Using these quasi likelihood functions, we also define the next two functions such that

$$\mathbb{H}_{1,\tau_1,n}^{(0)}(\alpha|\Lambda) = \frac{1}{\underline{k}_{\eta_1,\tau_1,n}^{1-2q_1}} \mathbb{W}_{1,\tau_1,n}(\alpha|\Lambda), \quad \mathbb{H}_{2,\tau_2,n}^{(0)}(\beta) = \frac{1}{\underline{T}_{\eta_2,n}^{1-2q_2}} \mathbb{W}_{2,\tau_2,n}(\beta).$$

Then, the initial estimators are defined as follows:

$$\begin{aligned} \hat{\Lambda}_n &:= \frac{1}{2n} \sum_{i=0}^{n-1} (Y_{(i+1)h_n} - Y_{ih_n})^{\otimes 2}, \\ \tilde{\alpha}_{q_1,\tau_1,n}^{(0)} &:= \frac{\int_{\Theta_1} \alpha \exp\left(\mathbb{H}_{1,\tau_1,n}^{(0)}(\alpha|\hat{\Lambda}_n)\right) \pi_1(\alpha) d\alpha}{\int_{\Theta_1} \exp\left(\mathbb{H}_{1,\tau_1,n}^{(0)}(\alpha|\hat{\Lambda}_n)\right) \pi_1(\alpha) d\alpha}, \\ \tilde{\beta}_{q_2,\tau_2,n}^{(0)} &:= \frac{\int_{\Theta_2} \beta \exp\left(\mathbb{H}_{2,\tau_2,n}^{(0)}(\beta)\right) \pi_2(\beta) d\beta}{\int_{\Theta_2} \exp\left(\mathbb{H}_{2,\tau_2,n}^{(0)}(\beta)\right) \pi_2(\beta) d\beta}, \end{aligned}$$

where $0 < \inf_{\alpha \in \Theta_1} \pi_1(\alpha) \leq \sup_{\alpha \in \Theta_1} \pi_1(\alpha) < \infty$ and $0 < \inf_{\beta \in \Theta_2} \pi_2(\beta) \leq \sup_{\beta \in \Theta_2} \pi_2(\beta) < \infty$. Note that $\hat{\Lambda}_n$ uses the whole data.

In the next place, we define the hybrid multi-step estimators. We introduce the quasi likelihood functions in Nakakita and Uchida (2018c) such that

$$\begin{aligned}\mathbb{H}_{1,n}(\alpha|\Lambda) &:= -\frac{1}{2} \sum_{j=1}^{k_{\tau_3,n}-2} \left(\left(\frac{2}{3} \Delta_{\tau_3,n} A_n^{\tau_3}(\bar{Y}_{\tau_3,j-1}, \alpha, \Lambda) \right)^{-1} \left[(\bar{Y}_{\tau_3,j+1} - \bar{Y}_{\tau_3,j})^{\otimes 2} \right] \right. \\ &\quad \left. + \log \det A_n^{\tau_3}(\bar{Y}_{\tau_3,j-1}, \alpha, \Lambda) \right), \\ \mathbb{H}_{2,n}(\beta|\alpha) &:= -\frac{1}{2} \sum_{j=1}^{k_{\tau_3,n}-2} \left((\Delta_{\tau_3,n} A(\bar{Y}_{\tau_3,j-1}, \alpha))^{-1} \left[(\bar{Y}_{\tau_3,j+1} - \bar{Y}_{\tau_3,j} - \Delta_{\tau_3,n} b(\bar{Y}_{\tau_3,j-1}, \beta))^{\otimes 2} \right] \right),\end{aligned}$$

where $A_n^{\tau_3}(x, \alpha, \Lambda) := A(x, \alpha) + 3\Delta_n^{\frac{2-\tau_3}{\tau_3-1}} \Lambda$. As Kamatani and Uchida (2015) and Kamatani et al. (2016), let us denote

$$\begin{aligned}J_{1,n}(\alpha) &:= \frac{1}{k_{\tau_3,n}} \partial_\alpha^2 \mathbb{H}_{1,\tau_3,n}(\alpha|\hat{\Lambda}_n), & J_{1,n}(\beta) &:= \frac{1}{T_n} \partial_\beta^2 \mathbb{H}_{2,\tau_3,n}(\beta|\hat{\alpha}_{J_1,n}), \\ K_{1,n}(\alpha) &:= \{J_{1,n}(\alpha) \text{ is invertible.}\}, & K_{2,n}(\beta) &:= \{J_{2,n}(\beta) \text{ is invertible.}\}, \\ \bar{J}_{1,n}(\alpha) &:= J_{1,n}(\alpha) \mathbf{1}_{K_{1,n}(\alpha)} + I_{m_1} \mathbf{1}_{K_{1,n}^c(\alpha)}, & \bar{J}_{2,n}(\beta) &:= J_{2,n}(\beta) \mathbf{1}_{K_{2,n}(\beta)} + I_{m_2} \mathbf{1}_{K_{2,n}^c(\beta)},\end{aligned}$$

such that for all $k = 1, \dots, J_1$, $J_1 := \lfloor -\log_2(q_1(\eta_1 - \gamma/\tau_1)/(1 - \gamma'/\tau_3)) \rfloor$, and $\hat{\alpha}_{0,n} := \tilde{\alpha}_{q_1,\tau_1,n}^{(0)}$

$$\hat{\alpha}_{k,n} := \hat{\alpha}_{k-1,n} - \bar{J}_{1,n}^{-1}(\hat{\alpha}_{k-1,n}) \frac{1}{k_{\tau_3,n}} \partial_\alpha \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n}|\hat{\Lambda}_n),$$

and for all $k = 1, \dots, J_2$, $J_2 = \lfloor -\log_2(q_2(\eta_2 - \gamma)/(1 - \gamma')) \rfloor$, and $\hat{\beta}_{0,n} := \tilde{\beta}_{q_2,\tau_2,n}^{(0)}$,

$$\hat{\beta}_{k,n} := \hat{\beta}_{k-1,n} - \bar{J}_{2,n}^{-1}(\hat{\beta}_{k-1,n}) \frac{1}{T_n} \partial_\beta \mathbb{H}_{2,\tau_3,n}(\hat{\beta}_{k-1,n}|\hat{\alpha}_{J_1,n}).$$

3.2. PLDIs for the quasi likelihood functions. To examine the asymptotic behaviours of $\hat{\alpha}_{J_1}$ and $\hat{\beta}_{J_2,n}$, firstly we will see that the L^p -boundedness such that

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \underline{k}_{\eta_1,\tau_1,n}^{q_1} \left(\tilde{\alpha}_{q_1,\tau_1,n}^{(0)} - \alpha^* \right) \right|^M \right] + \sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \underline{T}_{\eta_2,n}^{q_2} \left(\tilde{\beta}_{q_2,\tau_2,n}^{(0)} - \beta^* \right) \right|^M \right] < \infty.$$

To show these boundedness, we define some random quantities: random fields such that

$$\begin{aligned}\mathbb{V}_{1,\tau_1,n}(\alpha; \vartheta^*) &:= \frac{1}{\underline{k}_{\eta_1,\tau_1,n}} \left(\mathbb{W}_{1,\tau_1,n}(\alpha|\hat{\Lambda}_n) - \mathbb{W}_{1,\tau_1,n}(\alpha^*|\hat{\Lambda}_n) \right) \\ &= -\frac{1}{2\underline{k}_{\eta_1,\tau_1,n}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} \left(\left\| \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \hat{\Lambda}_n) \right\|^2 - \left\| \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \hat{\Lambda}_n) \right\|^2 \right. \\ &\quad \left. - 2 \left(\Delta_{\tau_1,n}^{-1}(\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} \right) \left[\frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \hat{\Lambda}_n) \right] \right. \\ &\quad \left. + 2 \left(\Delta_{\tau_1,n}^{-1}(\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} \right) \left[\frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \hat{\Lambda}_n) \right] \right),\end{aligned}$$

$$\begin{aligned}\mathbb{V}_{2,\tau_2,n}(\beta; \vartheta^*) &:= \frac{1}{T_n} \left(\mathbb{W}_{2,\tau_2,n}(\beta) - \mathbb{W}_{2,\tau_2,n}(\beta^*) \right) \\ &= -\frac{1}{2\underline{k}_{\eta_2,\tau_2,n}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} \left(|b(\bar{Y}_{\tau_2,j-1}, \beta)|^2 - |b(\bar{Y}_{\tau_2,j-1}, \beta^*)|^2 \right. \\ &\quad \left. - 2\Delta_{\tau_2,n}^{-1}(\bar{Y}_{\tau_2,j+1} - \bar{Y}_{\tau_2,j}) [b(\bar{Y}_{\tau_2,j-1}, \beta)] \right)\end{aligned}$$

$$+2\Delta_{\tau_2,n}^{-1} (\bar{Y}_{\tau_2,j+1} - \bar{Y}_{\tau_2,j}) [b(\bar{Y}_{\tau_2,j-1}, \beta^*)];$$

score functions such that

$$S_{1,\tau_1,n}(\vartheta^*) := -\frac{2}{3\underline{k}_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{j-1}, \alpha^*)) \left[\Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \hat{\Lambda}_n) \right],$$

$$S_{2,\tau_2,n}(\vartheta^*) := -\frac{1}{\underline{T}_{\eta_1,n}^{1-q_2}} \sum_{j=1}^{\underline{k}_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [(\bar{Y}_{\tau_2,j+1} - \bar{Y}_{\tau_2,j}) - \Delta_{\tau_2,n} b(\bar{Y}_{\tau_2,j-1}, \beta^*)];$$

the observed information matrices such that

$$\begin{aligned} \Gamma_{1,\tau_1,n}(\alpha; \vartheta^*) [u_1^{\otimes 2}] &:= -\frac{2}{3\underline{k}_{\eta_1,\tau_1,n}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} (\partial_\alpha^2 A(\bar{Y}_{j-1}, \alpha)) \left[u_1^{\otimes 2}, \Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \hat{\Lambda}_n) \right] \\ &\quad + \frac{4}{9\underline{k}_{\eta_1,\tau_1,n}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha)) [u_1^{\otimes 2}, \partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha)], \\ \Gamma_{2,\tau_2,n}(\beta; \vartheta^*) [u_2^{\otimes 2}] &:= -\frac{1}{\underline{k}_{\eta_2,\tau_2,n}} \sum_{j=1}^{\underline{k}_{\eta_2,\tau_2,n}-2} (\partial_\beta^2 b(\bar{Y}_{j-1}, \beta)) [u_2^{\otimes 2}, \Delta_{\tau_2,n}^{-1} (\bar{Y}_{\tau_2,j+1} - \bar{Y}_{\tau_2,j}) - b(\bar{Y}_{\tau_2,j-1}, \beta)] \\ &\quad + \frac{1}{\underline{k}_{\eta_2,\tau_2,n}} \sum_{j=1}^{\underline{k}_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta)) [u_2^{\otimes 2}, \partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta)]; \end{aligned}$$

and the limiting information matrices such that

$$\begin{aligned} \Gamma_{1,\tau_1}(\vartheta^*) [u_1^{\otimes 2}] &:= \frac{4}{9} \int_{\mathbf{R}^d} (\partial_\alpha A(x, \alpha^*)) [u_1^{\otimes 2}, \partial_\alpha A(x, \alpha^*)] \nu_0(dx), \\ \Gamma_2(\vartheta^*) [u_2^{\otimes 2}] &:= \int_{\mathbf{R}^d} (\partial_\beta b(x, \beta^*)) [u_2^{\otimes 2}, \partial_\beta b(x, \beta^*)] \nu_0(dx). \end{aligned}$$

Lemma 1. Assume [A1]-[A2] and [A5]-[A6]. Moreover, assume $\underline{k}_{\eta_1,\tau_1,n}^{q_1} \Delta_{\tau_1,n} \rightarrow 0$.

(1) For every $p > 1$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} [|S_{1,\tau_1,n}(\vartheta^*)|^p] < \infty.$$

(2) Let $\epsilon_1 = \epsilon_0/2$. Then for every $p > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\sup_{\alpha \in \Theta_1} \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} |\mathbb{V}_{1,\tau_1,n}(\alpha; \vartheta^*) - \mathbb{V}_{1,\tau_1}(\alpha; \vartheta^*)| \right)^p \right] < \infty.$$

(3) For any $M_3 > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E}_{\theta^*} \left[\left(\underline{k}_{\eta_1,\tau_1,n}^{-1} \sup_{\vartheta \in \Xi} |\partial_\alpha^3 \mathbb{W}_{1,\tau_1,n}(\alpha; \Lambda)| \right)^{M_3} \right] < \infty.$$

(4) Let $\epsilon_1 = \epsilon_0/2$. Then for $M_4 > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E}_{\vartheta^*} \left[\left(\underline{k}_{\eta_1, \tau_1, n}^{\epsilon_1} |\Gamma_{1, \tau_1, n}(\alpha^*; \vartheta^*) - \Gamma_{1, \tau_1}(\vartheta^*)| \right)^{M_4} \right] < \infty.$$

Remark 4. Let us assume $h_n := n^{-7/10}$, $\tau_1 = 2$; then $T_n = n^{3/10}$, $p_{\tau_1, n} = h_n^{-1/2} = n^{7/20}$, $\Delta_{\tau_1, n} = p_{\tau_1, n} h_n = n^{-7/20}$ and $k_{\tau_1, n} = n^{13/20}$. If we set $\eta_1 = 47/60$, then $\underline{k}_{\eta_1, \tau_1, n} := n^{13/30}$, $\underline{k}_{\eta_1, \tau_1, n} \Delta_{\tau_1, n} = n^{1/12} \rightarrow \infty$ and for all $q_1 \in (0, \frac{1}{2}]$, $\underline{k}_{\eta_1, \tau_1, n}^{q_1} \Delta_{\tau_1, n} = n^{13q_1/30 - 7/20} \rightarrow 0$ as $n \rightarrow \infty$.

Now we give an example of $\underline{k}_{\eta_1, \tau_1, n}$ which directly affects the burden of derivation of $\tilde{\alpha}_{q_1, \tau_1, n}^{(0)}$. If we have $n = 10^8$, then $\underline{k}_{\eta_1, \tau_1, n} = 10^{52/15} \approx 2929$, while $k_{\tau_1, n} = 10^{26/5} \approx 158489$.

With respect to J_1 , as $\gamma = \gamma' = 7/10$, when we set $\tau_3 = 1.9$ and $q_1 = 1/4$, then $J_1 = \lfloor -\log_2(q_1(\eta_1 - \gamma/\tau_1)/(1 - \gamma'/\tau_3)) \rfloor = \lfloor -\log_2(0.25(47/60 - 7/20)/(1 - 7/19)) \rfloor = 2$.

Lemma 2. Assume [A1]-[A2] and [A5]-[A6]. Moreover, assume $\underline{k}_{\eta_2, \tau_2, n}^{q_2} \Delta_{\tau_2, n}^{1+q_2} \rightarrow 0$.

(1) For every $p > 1$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} [|S_{2, \tau_2, n}(\vartheta^*)|^p] < \infty.$$

(2) Let $\epsilon_1 = \epsilon_0/2$. Then for every $p > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\sup_{\beta \in \Theta_2} \underline{T}_{\eta_2, n}^{\epsilon_1} |\mathbb{V}_{2, \tau_2, n}(\beta; \vartheta^*) - \mathbb{V}_2(\beta; \vartheta^*)| \right)^p \right] < \infty.$$

(3) For every $M_3 > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\underline{T}_{\eta_2, n}^{-1} \sup_{\beta \in \Theta_2} |\partial_{\beta}^3 \mathbb{W}_{2, \tau_2, n}(\beta)| \right)^{M_3} \right] < \infty.$$

(4) Let $\epsilon_1 = \epsilon_0/2$. Then for every $M_4 > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\underline{T}_{\eta_2, n}^{\epsilon_1} |\Gamma_{2, \tau_2, n}(\beta^*; \vartheta^*) - \Gamma_2(\vartheta^*)| \right)^{M_4} \right] < \infty.$$

Remark 5. Let us assume $h_n := n^{-7/10}$, $\tau_2 = 1.2$; then $T_n = n^{3/10}$, $p_{\tau_2, n} = h_n^{-5/6} = n^{7/12}$, $\Delta_{\tau_2, n} = p_{\tau_2, n} h_n = n^{-7/60}$, $\eta_2 = 5/6$ and $\underline{k}_{\eta_2, \tau_2, n} = n^{1/4}$. It holds for all $q_2 \in (0, \frac{1}{2}]$, $\underline{k}_{\eta_2, \tau_2, n}^{q_2} \Delta_{\tau_2, n}^{1+q_2} = n^{q_2/4 - 7(1+q_2)/60} = n^{2q_2/15 - 7/60} \rightarrow 0$.

A concrete example of $\underline{k}_{\eta_2, \tau_2, n}$, which directly affects the burden of derivation of $\tilde{\beta}_{q_2, \tau_2, n}^{(0)}$, is given as follows. If we have $n = 10^8$, then $\underline{k}_{\eta_2, \tau_2, n} = 10^2$, while $k_{\tau_2, n} = 10^{26/5} \approx 158489$ for $\tau = 2$. With respect to J_2 , if $q_2 = 2^{-8}$, we have $J_2 = \lfloor -\log_2(q_2(\eta_2 - \gamma)/(1 - \gamma')) \rfloor = 9$.

In addition to these evaluations, we define the sets for all $r > 0$,

$$\begin{aligned} \mathbb{U}_{1, q_1, n}^{(0)}(\alpha^*) &:= \{u_1 \in \mathbf{R}^{m_1}; \alpha^* + \underline{k}_{\eta_1, \tau_1, n}^{-q_1} u_1 \in \Theta_1\}, \\ V_{1, q_1, n}^{(0)}(r) &:= \{u_1 \in \mathbb{U}_{1, q_1, n}^{(0)}(\alpha^*); r \leq |u_1|\}, \\ \mathbb{U}_{2, q_2, n}^{(0)}(\beta^*) &:= \{u_2 \in \mathbf{R}^{m_2}; \beta^* + \underline{T}_{\eta_2, n}^{-q_2} u_2 \in \Theta_2\}, \\ V_{2, q_2, n}^{(0)}(r) &:= \{u_2 \in \mathbb{U}_{2, q_2, n}^{(0)}(\beta^*); r \leq |u_2|\} \end{aligned}$$

and the statistical random fields:

$$\begin{aligned} \mathbb{Z}_{1, \tau_1, n}^{(0)}(u_1; \vartheta^*) &:= \exp \left(\mathbb{H}_{1, \tau_1, n}^{(0)} \left(\alpha^* + \underline{k}_{\eta_1, \tau_1, n}^{-q_1} u_1 | \hat{\Lambda}_n \right) - \mathbb{H}_{1, \tau_1, n}^{(0)} \left(\alpha^* | \hat{\Lambda}_n \right) \right), \\ \mathbb{Z}_{2, \tau_2, n}^{(0)}(u_2; \vartheta^*) &:= \exp \left(\mathbb{H}_{2, \tau_2, n}^{(0)} \left(\beta^* + \underline{T}_{\eta_2, n}^{-q_2} u_2 \right) - \mathbb{H}_{2, \tau_2, n}^{(0)} \left(\beta^* \right) \right). \end{aligned}$$

We have the following results as for the random fields and consequently the Bayes type estimators $\tilde{\alpha}_{q_1, \tau_1, n}^{(0)}$ and $\tilde{\beta}_{q_2, \tau_2, n}^{(0)}$ using Lemma 1 and Lemma 2.

Theorem 1. *Let $L > 0$ and $i = 1, 2$. Assume [A1]-[A3] and [A5]-[A6]. Then, there exists $C(L) > 0$ such that*

$$P \left[\sup_{u_i \in V_{i, q_i, n}(r)} \mathbb{Z}_{i, \tau_i, n}^{(0)}(u_i) \geq e^{-r} \right] \leq \frac{C(L)}{r^L}$$

for all $r > 0$ and $n \in \mathbf{N}$. Moreover, for all $M > 0$,

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \underline{k}_{\eta_1, \tau_1, n}^{q_1} \left(\tilde{\alpha}_{q_1, \tau_1, n}^{(0)} - \alpha^* \right) \right|^M \right] + \sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \underline{T}_{\eta_2, n}^{q_2} \left(\tilde{\beta}_{q_2, \tau_2, n}^{(0)} - \beta^* \right) \right|^M \right] < \infty.$$

Remark 6. *Note that*

$$\begin{aligned} \underline{k}_{\eta_1, \tau_1, n} &= n^{\eta_1} h_n^{1/\tau_1} \geq n^{\eta_1} (n^{-\gamma})^{1/\tau_1} = n^{\eta_1 - \gamma/\tau_1} = n^{(1-\gamma'/\tau_3)(1-\gamma'/\tau_3)^{-1}(\eta_1 - \gamma/\tau_1)} \\ &= \left(n n^{-\gamma'/\tau_3} \right)^{(\eta_1 - \gamma/\tau_1)/(1-\gamma'/\tau_3)} \geq \left(n h^{1/\tau_3} \right)^{(\eta_1 - \gamma/\tau_1)/(1-\gamma'/\tau_3)} = k_{\tau_3, n}^{(\eta_1 - \gamma/\tau_1)/(1-\gamma'/\tau_3)} \end{aligned}$$

and

$$\underline{T}_{\eta_2, n} = n^{\eta_2} h_n \geq n^{\eta_2} n^{-\gamma} = \left(n^{1-\gamma'} \right)^{(\eta_2 - \gamma)/(1-\gamma')} \geq T_n^{(\eta_2 - \gamma)/(1-\gamma')};$$

therefore,

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| k_{\tau, n}^{q_1(\eta_1 - \gamma/\tau_1)/(1-\gamma'/\tau_3)} \left(\tilde{\alpha}_{q_1, \tau_1, n}^{(0)} - \alpha^* \right) \right|^M \right] \leq \sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \underline{k}_{\eta_1, \tau_1, n}^{q_1} \left(\tilde{\alpha}_{q_1, \tau_1, n}^{(0)} - \alpha^* \right) \right|^M \right] < \infty,$$

and

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| T_n^{q_2(\eta_2 - \gamma)/(1-\gamma')} \left(\tilde{\beta}_{q_2, \tau_2, n}^{(0)} - \beta^* \right) \right|^M \right] \leq \sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \underline{T}_{\eta_2, n}^{q_2} \left(\tilde{\beta}_{q_2, \tau_2, n}^{(0)} - \beta^* \right) \right|^M \right] < \infty.$$

In order to state asymptotic properties of the hybrid multi-step estimators, we introduce the notation as follows. Let $V((l_1, l_2), (l_3, l_4))$ be the real-valued function as for $l_1, l_2, l_3, l_4 = 1, \dots, d$,

$$\begin{aligned} &V((l_1, l_2), (l_3, l_4)) \\ &:= \sum_{k=1}^d \left(\Lambda_{\star}^{1/2} \right)^{(l_1, k)} \left(\Lambda_{\star}^{1/2} \right)^{(l_2, k)} \left(\Lambda_{\star}^{1/2} \right)^{(l_3, k)} \left(\Lambda_{\star}^{1/2} \right)^{(l_4, k)} \left(\mathbf{E}_{\theta^*} \left[\left| \epsilon_0^{(k)} \right|^4 \right] - 3 \right) \\ &\quad + \frac{3}{2} \left(\Lambda_{\star}^{(l_1, l_3)} \Lambda_{\star}^{(l_2, l_4)} + \Lambda_{\star}^{(l_1, l_4)} \Lambda_{\star}^{(l_2, l_3)} \right), \end{aligned}$$

and the function σ is defined as for $i = 1, \dots, d$ and $j = i, \dots, d$,

$$\sigma(i, j) := \begin{cases} j & \text{if } i = 1, \\ \sum_{\ell=1}^{i-1} (d - \ell + 1) + j - i + 1 & \text{if } i > 1. \end{cases}$$

Furthermore, for $i_1, i_2 = 1, \dots, d(d+1)/2$,

$$W_1^{(i_1, i_2)} := V(\sigma^{-1}(i_1), \sigma^{-1}(i_2)).$$

Let $\{B_{\kappa}(x) \mid \kappa = 1, \dots, m_1\}$ and $\{f_{\lambda}(x) \mid \lambda = 1, \dots, m_2\}$ be sequences of $\mathbf{R}^d \otimes \mathbf{R}^d$ -valued functions and \mathbf{R}^d -valued ones respectively such that their components and their derivatives with respect to x are polynomial growth functions for all κ and λ . For $\bar{B}_{\kappa}(x) := \frac{1}{2} (B_{\kappa}(x) + B_{\kappa}(x)^T)$,

$$\begin{aligned} &\left(W_2^{(\tau)}(\{B_{\kappa}(x) : \kappa = 1, \dots, m_1\}) \right)^{(\kappa_1, \kappa_2)} \\ &:= \begin{cases} \int_{\mathbf{R}^d} \text{tr} \left\{ \left(\bar{B}_{\kappa_1} A \bar{B}_{\kappa_2} A \right)(x) \right\} \nu(dx) & \text{if } \tau \in (1, 2), \\ \int_{\mathbf{R}^d} \text{tr} \left\{ \left(\bar{B}_{\kappa_1} A \bar{B}_{\kappa_2} A + 4 \bar{B}_{\kappa_1} A \bar{B}_{\kappa_2} \Lambda_{\star} + 12 \bar{B}_{\kappa_1} \Lambda_{\star} \bar{B}_{\kappa_2} \Lambda_{\star} \right)(x) \right\} \nu(dx) & \text{if } \tau = 2. \end{cases} \end{aligned}$$

Set

$$\begin{aligned}\mathcal{I}^\tau(\vartheta^*) &:= \text{diag} \left\{ W_1, \mathcal{I}^{(2,2),\tau}, \mathcal{I}^{(3,3)} \right\}(\vartheta^*), \\ \mathcal{J}^\tau(\vartheta^*) &:= \text{diag} \left\{ I_{d(d+1)/2}, \mathcal{J}^{(2,2),\tau}, \mathcal{J}^{(3,3)} \right\}(\vartheta^*), \\ \mathcal{I}^{(2,2),\tau}(\vartheta^*) &:= W_2^{(\tau)} \left(\left\{ \frac{3}{4} (A^\tau)^{-1} (\partial_{\alpha^{(k_1)}} A) (A^\tau)^{-1} (\cdot, \vartheta^*) : k_1 = 1, \dots, m_1 \right\} \right), \\ \mathcal{J}^{(2,2),\tau}(\vartheta^*) &:= \left(\frac{1}{2} \int_{\mathbf{R}^d} \text{tr} \left\{ (A^\tau)^{-1} (\partial_{\alpha^{(i_1)}} A) (A^\tau)^{-1} (\partial_{\alpha^{(i_2)}} A) \right\} (x, \vartheta^*) \nu(dx) \right)_{i_1, i_2=1, \dots, m_1}, \\ \mathcal{I}^{(3,3)}(\vartheta^*) = \mathcal{J}^{(3,3)}(\vartheta^*) &:= \left(\int_{\mathbf{R}^d} (A)^{-1} \left[\partial_{\beta^{(j_1)}} b, \partial_{\beta^{(j_2)}} b \right] (x, \vartheta^*) \nu(dx) \right)_{j_1, j_2=1, \dots, m_2}.\end{aligned}$$

Let $\hat{\theta}_{\varepsilon, n} := \text{vech} \hat{\Lambda}_n$, $\theta_\varepsilon^* := \text{vech} \Lambda^*$ and

$$(\zeta_0, \zeta_1, \zeta_2) \sim N_{d(d+1)/2+m_1+m_2} \left(\mathbf{0}, (\mathcal{J}^\tau(\vartheta^*))^{-1} (\mathcal{I}^\tau(\vartheta^*)) (\mathcal{J}^\tau(\vartheta^*))^{-1} \right).$$

The asymptotic normality and convergence of moments of the hybrid multi-step estimators with the initial Bayes type estimators are as follows.

Theorem 2. *Assume [A1]-[A6]. Then, under $k_{\tau_3, n} \Delta_{\tau_3, n}^2 \rightarrow 0$,*

$$\left(\sqrt{n} \left(\hat{\theta}_{\varepsilon, n} - \theta_\varepsilon^* \right), \sqrt{k_{\tau_3, n}} (\hat{\alpha}_{J_1, n} - \alpha^*), \sqrt{T_n} \left(\hat{\beta}_{J_2, n} - \beta^* \right) \right) \rightarrow^{\mathcal{L}} (\zeta_0, \zeta_1, \zeta_2).$$

Moreover,

$$\mathbf{E} \left[f \left(\sqrt{n} \left(\hat{\theta}_{\varepsilon, n} - \theta_\varepsilon^* \right), \sqrt{k_{\tau_3, n}} (\hat{\alpha}_{J_1, n} - \alpha^*), \sqrt{T_n} \left(\hat{\beta}_{J_2, n} - \beta^* \right) \right) \right] \rightarrow \mathbf{E} [f(\zeta_0, \zeta_1, \zeta_2)]$$

for all continuous functions f of at most polynomial growth.

4. EXAMPLE AND SIMULATION RESULTS

We consider the three-dimensional diffusion process.

$$dX_t = b(X_t, \beta) dt + a(X_t, \alpha) dw_t, \quad t \geq 0, \quad X_0 = (1, 1, 1)^*,$$

where

$$\begin{aligned}b(X_t, \beta) &= \begin{pmatrix} 1 - \beta_1 X_{t,1} - 10 \sin(\beta_2 X_{t,2}^2) \\ 1 - \beta_3 X_{t,2} - 10 \sin(\beta_4 X_{t,3}^2) \\ 1 - \beta_5 X_{t,3} - 10 \sin(\beta_6 X_{t,1}^2) \end{pmatrix}, \\ a(X_t, \alpha) &= \begin{pmatrix} \sqrt{\alpha_1(2 + \cos(X_{t,3}^2))} & 0 & 0 \\ 0 & \sqrt{\alpha_2(2 + \cos(X_{t,1}^2))} & 0 \\ 0 & 0 & \sqrt{\alpha_3(2 + \cos(X_{t,2}^2))} \end{pmatrix}.\end{aligned}$$

Moreover, the true parameter values are

$$(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*, \beta_5^*, \beta_6^*) = (1, 2, 2, 3, 3, 4)$$

and $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (1, 2, 3)$. The parameter space is $\Theta = [0.01, 10]^9$.

The noisy data $\{Y_{ih_n}\}_{i=0, \dots, n}$ are defined as for all $i = 0, \dots, n$,

$$Y_{ih_n} := X_{ih_n} + \Lambda^{1/2} \varepsilon_{ih_n},$$

where $n = 5 \times 10^7$, $h_n = \frac{4}{10^6}$, $T = nh_n = 200$, $\Lambda = 10^{-3} I_3$, I_3 is the 3×3 -identity matrix, $\{\varepsilon_{ih_n}\}_{i=0, \dots, n}$ is the i.i.d. sequence of 3-dimensional normal random vectors with $\mathbf{E}[\varepsilon_0] = \mathbf{0}$ and $\text{Var}(\varepsilon_0) = I_3$.

For the true model, 100 independent sample paths are generated by the Euler-Maruyama scheme, and the mean and the standard deviation (s.d.) for the estimators in Theorems 1 and

2 are computed and shown in Tables 1-9 below. The personal computer with Intel i7-6950X (3.00GHz) was used for the simulations. In each table, the time means the computation time of estimators for one sample path.

Table 1 shows the simulation results of the estimator $\hat{\Lambda}_n = (\Lambda_{n,i,j})_{i,j=1,2,3}$ of $\Lambda = (\Lambda_{ij})_{i,j=1,2,3}$.

Tables 2 and 3 show the simulation results of the adaptive ML type estimator $(\hat{\alpha}_{A,n}, \hat{\beta}_{A,n})$ with the initial value being the true value, where

$$\begin{aligned}\hat{\alpha}_{A,n} &= \arg \sup_{\alpha \in \Theta_1} \mathbb{H}_{1,n}(\alpha | \hat{\Lambda}_n), \\ \hat{\beta}_{A,n} &= \arg \sup_{\beta \in \Theta_2} \mathbb{H}_{2,n}(\beta | \hat{\alpha}_{A,n}),\end{aligned}$$

the quasi log likelihood functions are that

$$\begin{aligned}\mathbb{H}_{1,n}(\alpha | \Lambda) &= -\frac{1}{2} \sum_{j=1}^{k_{\tau_3,n}-2} \left(\left(\frac{2}{3} \Delta_{\tau_3,n} A_n^{\tau_3}(\bar{Y}_{\tau_3,j-1}, \alpha, \Lambda) \right)^{-1} \left[(\bar{Y}_{\tau_3,j+1} - \bar{Y}_{\tau_3,j})^{\otimes 2} \right] \right. \\ &\quad \left. + \log \det A_n^{\tau_3}(\bar{Y}_{\tau_3,j-1}, \alpha, \Lambda) \right), \\ \mathbb{H}_{2,n}(\beta | \alpha) &= -\frac{1}{2} \sum_{j=1}^{k_{\tau_3,n}-2} \left((\Delta_{\tau_3,n} A(\bar{Y}_{\tau_3,j-1}, \alpha))^{-1} \left[(\bar{Y}_{\tau_3,j+1} - \bar{Y}_{\tau_3,j} - \Delta_{\tau_3,n} b(\bar{Y}_{\tau_3,j-1}, \beta))^{\otimes 2} \right] \right),\end{aligned}$$

the local mean $\{\bar{Y}_{\tau,j}\}_{j=0,\dots,k_{\tau,n}-1}$ is defined as

$$\bar{Y}_{\tau,j} = \frac{1}{p_{\tau,n}} \sum_{i=0}^{p_{\tau,n}-1} Y_{j\Delta_{\tau,n}+ih_n}.$$

Here $\tau_3 = 2.0$, $k_{\tau_3,n} = 10^5$, $p_{\tau_3,n} = 500$, $\Delta_{\tau_3,n} = 2 \times 10^{-3}$, $T = k_{\tau_3,n} \Delta_{\tau_3,n} = 200$, $A_n^{\tau_3}(x, \alpha, \Lambda) = A(x, \alpha) + 3\Delta_{\tau_3,n}^{\frac{2-\tau_3}{\tau_3-1}} \Lambda = A(x, \alpha) = aa^T(x, \alpha)$. The adaptive ML type estimator $(\hat{\alpha}_{A,n}, \hat{\beta}_{A,n})$ are obtained by means of **optim()** based on the "L-BFGS-B" method in the R Language.

TABLE 1. estimator of Λ

$\hat{\Lambda}_{11}(0.001)$	$\hat{\Lambda}_{12}(0)$	$\hat{\Lambda}_{13}(0)$	$\hat{\Lambda}_{22}(0.001)$	$\hat{\Lambda}_{23}(0)$	$\hat{\Lambda}_{33}(0.001)$	time(sec.)
0.001	0.000	0.000	0.001	0.000	0.001	0.47
(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	

TABLE 2. adaptive ML type estimator of α with the initial value being the true value

	$\hat{\alpha}_1(1)$	$\hat{\alpha}_2(2)$	$\hat{\alpha}_3(3)$	time(sec.)
true	1.048	2.057	3.053	17
	(0.005)	(0.01)	(0.014)	

TABLE 3. adaptive ML type estimator of β with the initial value being the true value

	$\hat{\beta}_1(1)$	$\hat{\beta}_2(2)$	$\hat{\beta}_3(2)$	$\hat{\beta}_4(3)$	$\hat{\beta}_5(3)$	$\hat{\beta}_6(4)$	time(sec.)
true	1.021	1.988	2.043	2.953	3.048	3.984	61
	(0.046)	(0.01)	(0.099)	(0.038)	(0.162)	(0.028)	

From Tables 1-3, we see that all estimators have good behaviour.

Tables 4 and 5 show the simulation results of the adaptive ML type estimator $(\hat{\alpha}_{A,n}, \hat{\beta}_{A,n})$ with the initial value being the uniform random number on Θ . Most of estimators of β have

TABLE 4. adaptive ML type estimator of α with the initial value being the uniform random number on Θ

	$\hat{\alpha}_1(1)$	$\hat{\alpha}_2(2)$	$\hat{\alpha}_3(3)$	time(sec.)
	1.048	2.057	3.053	
unif	(0.005)	(0.01)	(0.014)	33

TABLE 5. adaptive ML type estimator of β with the initial value being the uniform random number on Θ

	$\hat{\beta}_1(1)$	$\hat{\beta}_2(2)$	$\hat{\beta}_3(2)$	$\hat{\beta}_4(3)$	$\hat{\beta}_5(3)$	$\hat{\beta}_6(4)$	time(sec.)
	0.384	2.634	1.344	2.860	2.415	4.948	
true	(0.246)	(3.934)	(0.398)	(3.847)	(0.470)	(4.496)	53

considerable biases since the initial value may be far from the true value. As is well known, it is essential to select an appropriate initial value for optimisation.

Tables 6 and 7 show the simulation results of the initial Bayes type estimators with uniform priors defined as

$$\tilde{\alpha}_{q_1, \tau_1, n}^{(0)} = \frac{\int_{\Theta_1} \alpha \exp\left(\mathbb{H}_{1, \tau_1, n}^{(0)}(\alpha | \hat{\Lambda}_n)\right) d\alpha}{\int_{\Theta_1} \exp\left(\mathbb{H}_{1, \tau_1, n}^{(0)}(\alpha | \hat{\Lambda}_n)\right) d\alpha},$$

$$\tilde{\beta}_{q_2, \tau_2, n}^{(0)} = \frac{\int_{\Theta_2} \beta \exp\left(\mathbb{H}_{2, \tau_2, n}^{(0)}(\beta)\right) d\beta}{\int_{\Theta_2} \exp\left(\mathbb{H}_{2, \tau_2, n}^{(0)}(\beta)\right) d\beta},$$

where

$$\mathbb{H}_{1, \tau_1, n}^{(0)}(\alpha | \Lambda) = \frac{1}{\underline{k}_{\eta_1, \tau_1, n}^{1-2q_1}} \mathbb{W}_{1, \tau_1, n}(\alpha | \Lambda), \quad \mathbb{H}_{2, \tau_2, n}^{(0)}(\beta) = \frac{1}{\underline{T}_{\eta_2, n}^{1-2q_2}} \mathbb{W}_{2, \tau_2, n}(\beta),$$

$$\mathbb{W}_{1, \tau_1, n}(\alpha | \Lambda) := -\frac{1}{2} \sum_{j=1}^{\underline{k}_{\eta_1, \tau_1, n}-2} \left\| \Delta_{\tau_1, n}^{-1} (\bar{Y}_{\tau_1, j+1} - \bar{Y}_{\tau_1, j})^{\otimes 2} - \frac{2}{3} A_{\tau_1, n}(\bar{Y}_{\tau_1, j-1}, \alpha, \Lambda) \right\|^2,$$

$$\mathbb{W}_{2, \tau_2, n}(\beta) := -\frac{1}{2} \sum_{j=1}^{\underline{k}_{\eta_2, \tau_2, n}-2} \Delta_{\tau_2, n}^{-1} |\bar{Y}_{\tau_2, j+1} - \bar{Y}_{\tau_2, j} - \Delta_{\tau_2, n} b(\bar{Y}_{\tau_2, j-1}, \beta)|^2.$$

Here we set $\eta_1 = \eta_2 = 61/70$ and the initial Bayes estimator of α are obtained by the reduced data with $q_1 = 1/2$, $\tau_1 = 2.0$, $\underline{k}_{\eta_1, \tau_1, n} = 10^4$, $\Delta_{\tau_1, n} = 2 \times 10^{-3}$. Moreover, the initial Bayes estimator of β are derived from the reduced data with $q_2 = 1/2$, $\tau_2 = 2.0$, $\underline{k}_{\eta_2, \tau_2, n} = 10^4$, $\Delta_{\tau_2, n} = 2 \times 10^{-3}$, $\underline{T}_{\eta_2, n} = \underline{k}_{\eta_2, \tau_2, n} \Delta_{\tau_2, n} = 20$.

Furthermore, the initial Bayes type estimators of α and β are calculated with MpCN method proposed by Kamatani (2018) for 10^3 and 10^6 Markov chains and 10^2 and 10^5 burn-in iterations, respectively.

TABLE 6. initial Bayes type estimator of α with reduced data ($\underline{k}_{\eta_1, \tau_1, n} = 10^4$)

	$\tilde{\alpha}_1^{(0)}(1)$	$\tilde{\alpha}_2^{(0)}(2)$	$\tilde{\alpha}_3^{(0)}(3)$	time(sec.)
	1.029	2.015	3.019	
	(0.023)	(0.035)	(0.047)	22

TABLE 7. initial Bayes type estimator of β with reduced data ($\underline{k}_{\eta_2, \tau_2, n} = 10^4$, $\underline{T}_{\eta_2, n} = 20$)

$\hat{\beta}_1^{(0)}(1)$	$\hat{\beta}_2^{(0)}(2)$	$\hat{\beta}_3^{(0)}(2)$	$\hat{\beta}_4^{(0)}(3)$	$\hat{\beta}_5^{(0)}(3)$	$\hat{\beta}_6^{(0)}(4)$	time(min.)
1.077	1.990	2.169	2.933	3.111	3.989	46
(0.197)	(0.034)	(0.453)	(0.130)	(0.523)	(0.108)	

Since we set that $\eta_1 = \eta_2 = 61/70$, $\gamma = \gamma' = 7/10$, $\tau_1 = \tau_3 = 2.0$, $q_1 = q_2 = 1/2$, one has that

$$J_1 = \lfloor -\log_2(q_1(\eta_1 - \gamma/\tau_1)/(1 - \gamma'/\tau_3)) \rfloor = 1,$$

$$J_2 = \lfloor -\log_2(q_2(\eta_2 - \gamma)/(1 - \gamma')) \rfloor = 1.$$

Tables 8 and 9 show the simulation results of the hybrid multi-step estimators ($\hat{\alpha}_{J_1, n}, \hat{\beta}_{J_2, n}$) with the initial estimator ($\tilde{\alpha}_{q_1, \tau_1, n}^{(0)}, \tilde{\beta}_{q_2, \tau_2, n}^{(0)}$) in Tables 6 and 7.

TABLE 8. hybrid multi-step estimator of α with the initial Bayes type estimator ($\underline{k}_{\eta_2, \tau_2, n} = 10^4$)

$\hat{\alpha}_1(1)$	$\hat{\alpha}_2(2)$	$\hat{\alpha}_3(3)$	time(sec.)
1.048	2.057	3.053	13
(0.005)	(0.010)	(0.014)	

TABLE 9. hybrid multi-step estimator of β with the initial Bayes type estimator ($\underline{k}_{\eta_2, \tau_2, n} = 10^4$, $\underline{T}_{\eta_2, n} = 20$)

$\hat{\beta}_1(1)$	$\hat{\beta}_2(2)$	$\hat{\beta}_3(2)$	$\hat{\beta}_4(3)$	$\hat{\beta}_5(3)$	$\hat{\beta}_6(4)$	time(sec.)
1.021	1.988	2.044	2.953	3.048	3.983	70
(0.046)	(0.010)	(0.099)	(0.038)	(0.162)	(0.028)	

From Tables 8 and 9, we can see that the hybrid multi-step estimators with the initial Bayes estimators improve the initial Bayes estimators in Tables 6 and 7. It is worth mentioning that the performance of the hybrid multi-step estimator with the initial Bayes estimator is almost the same as that of the estimator in Tables 2 and 3.

5. CONCLUDING REMARKS

In this paper, we have provided the hybrid estimation for noisily ergodic diffusion processes based on ultra high frequency data from the viewpoint of computational cost. In order to get the adaptive ML type estimators, we need optimisation of the quasi likelihood function and selecting a suitable initial value is important, but it comes difficult when the dimension of the parameter space is large. While the computation of the Bayes type estimator is generally free from a choice of the initial value, there is a serious problem that it takes so much time to compute the Bayes type estimator when the sample size is large. Kamatani and Uchida (2015) proposed the multi-step ML type estimator based on the initial Bayes type estimator with the full data, and Kaino et al. (2017) and Kaino and Uchida (2018a,b) studied the adaptive ML type estimator with the initial Bayes type estimator derived from the reduced data by applying the result of Kutoyants (2017) to high frequency data analysis. In this paper, we have proposed the initial Bayes type estimator with the reduced data based on the local means obtained from the high frequency data with noise and hybrid multi-step estimator with the initial Bayes type estimator. The proposed hybrid multi-step estimators have asymptotic normality and convergence of moments and we see from the numerical examples in Section 4 that they have good performance.

6. PROOF

6.1. Some lemmas with respect to local means. We note some useful lemmas for moment evaluations. Most of them are proved in Nakakita and Uchida (2018b) and Nakakita and Uchida (2018c). We denote $\tau = \tau_i$ for $i = 1, 2, 3$, and define the following random variables:

$$\zeta_{\tau,j+1,n} := \frac{1}{p_{\tau,n}} \sum_{i=0}^{p_{\tau,n}-1} \int_{j\Delta_{\tau,n}+ih_n}^{(j+1)\Delta_{\tau,n}} dw_s, \quad \zeta'_{\tau,j+2,n} := \frac{1}{p_{\tau,n}} \sum_{i=0}^{p_{\tau,n}-1} \int_{(j+1)\Delta_{\tau,n}}^{(j+1)\Delta_{\tau,n}+ih_n} dw_s.$$

The next lemma is Lemma 11 in Nakakita and Uchida (2018b).

Lemma 3. $\zeta_{\tau,j+1,n}$ and $\zeta'_{\tau,j+1,n}$ are $\mathcal{G}_{j+1,n}^\tau$ -measurable, independent of $\mathcal{G}_{j,n}^\tau$ and Gaussian. These variables have the next decomposition:

$$\begin{aligned} \zeta_{\tau,j+1,n} &= \frac{1}{p_{\tau,n}} \sum_{k=0}^{p_{\tau,n}-1} (k+1) \int_{j\Delta_{\tau,n}+kh_n}^{j\Delta_{\tau,n}+(k+1)h_n} dw_s, \\ \zeta'_{\tau,j+1,n} &= \frac{1}{p_{\tau,n}} \sum_{k=0}^{p_{\tau,n}-1} (p_{\tau,n} - k - 1) \int_{j\Delta_{\tau,n}+kh_n}^{j\Delta_{\tau,n}+(k+1)h_n} dw_s. \end{aligned}$$

The evaluation of the following conditional expectations holds:

$$\begin{aligned} \mathbf{E} [\zeta_{\tau,j,n} | \mathcal{G}_{j,n}^\tau] &= \mathbf{E} [\zeta'_{\tau,j+1,n} | \mathcal{G}_{j,n}^\tau] = \mathbf{0}, \\ \mathbf{E} [\zeta_{\tau,j+1,n} (\zeta_{\tau,j+1,n})^T | \mathcal{G}_{j,n}^\tau] &= m_{\tau,n} \Delta_{\tau,n} I_r, \\ \mathbf{E} [\zeta'_{\tau,j+1,n} (\zeta'_{\tau,j+1,n})^T | \mathcal{G}_{j,n}^\tau] &= m'_{\tau,n} \Delta_{\tau,n} I_r, \\ \mathbf{E} [\zeta_{\tau,j+1,n} (\zeta'_{\tau,j+1,n})^T | \mathcal{G}_{j,n}^\tau] &= \chi_{\tau,n} \Delta_{\tau,n} I_r, \end{aligned}$$

where $m_{\tau,n} = \left(\frac{1}{3} + \frac{1}{2p_{\tau,n}} + \frac{1}{6p_{\tau,n}^2}\right)$, $m'_{\tau,n} = \left(\frac{1}{3} - \frac{1}{2p_{\tau,n}} + \frac{1}{6p_{\tau,n}^2}\right)$, and $\chi_{\tau,n} = \frac{1}{6} \left(1 - \frac{1}{p_{\tau,n}^2}\right)$.

The next lemma is from Nakakita and Uchida (2018c).

Lemma 4. Assume [A1]-[A2] and [A5]-[A6]. Moreover, assume the components of the functions $f, g \in C^2(\mathbf{R}^d; \mathbf{R})$, $\partial_x f$, $\partial_x g$, $\partial_x^2 f$, $\partial_x^2 g$ are polynomial growth functions. Then we have

$$\left| \mathbf{E} [f(\bar{Y}_{\tau,j}) g(X_{(j+1)\Delta_{\tau,n}}) - f(X_{j\Delta_{\tau,n}}) g(X_{j\Delta_{\tau,n}}) | \mathcal{H}_{j,n}^\tau] \right| \leq C \Delta_{\tau,n} (1 + |X_{j\Delta_{\tau,n}}|)^C.$$

The next lemma is from Nakakita and Uchida (2018b) and Nakakita and Uchida (2018c).

Lemma 5. Assume [A1]-[A2] and [A5]-[A6].

(1) The next expansion holds:

$$\begin{aligned} \bar{Y}_{\tau,j+1} - \bar{Y}_{\tau,j} &= \Delta_{\tau,n} b(X_{j\Delta_{\tau,n}}) + a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n}) \\ &\quad + e_{\tau,j,n} + (\Lambda_\star)^{1/2} (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j}), \end{aligned}$$

where $e_{\tau,j,n}$ is a $\mathcal{H}_{j+2,n}^\tau$ -measurable random variable such that $\left| \mathbf{E} [e_{\tau,j,n} | \mathcal{H}_{j,n}^\tau] \right| \leq C \Delta_{\tau,n}^2 (1 + |X_{j\Delta_{\tau,n}}|)^C$ and $\|e_{\tau,j,n}\|_p \leq C(p) \Delta_{\tau,n}$ for $j = 1, \dots, k_n - 2$, $n \in \mathbf{N}$ and $p \geq 1$.

(2) For any $p \geq 1$ and $\mathcal{H}_{j,n}^\tau$ -measurable $\mathbf{R}^d \otimes \mathbf{R}^r$ -valued random variable $\mathbb{B}_{j,n} \in \bigcap_{p>0} L^p(P_{\theta^\star})$, we have the next L^p -boundedness

$$\mathbf{E} \left[\left\| \sum_{j=1}^{k_{\tau,n}-2} \mathbb{B}_{j,n} [e_{\tau,j,n} (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n})^T] \right\|^p \right]^{1/p} \leq C(p) k_n \Delta_{\tau,n}^2.$$

(3) For any $p \geq 1$ and $\mathcal{H}_{j,n}^\tau$ -measurable $\mathbf{R}^d \otimes \mathbf{R}^d$ -valued random variable $\mathbb{C}_{j,n} \in \bigcap_{p>0} L^p(P_{\theta^\star})$, we have the next L^p -boundedness

$$\mathbf{E} \left[\left\| \sum_{j=1}^{k_{\tau,n}-2} \mathbb{C}_{j,n} [e_{\tau,j,n}] \right\|^p \right]^{1/p} \leq C(p) k_n \Delta_{\tau,n}^{3/2}.$$

We define for any $\tau \in (1, 2]$ and $j = 0, \dots, k_{\tau,n} - 2$,

$$\begin{aligned} & \widehat{A}_{\tau,j,n} \\ & := \Delta_{\tau,n}^{-1} \left[(m_{\tau,n} + m'_{\tau,n})^{-\frac{1}{2}} a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n}) + \sqrt{\frac{3}{2}} (\Lambda_\star)^{\frac{1}{2}} (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j}) \right]^{\otimes 2}. \end{aligned}$$

Lemma 6. Assume [A1]-[A2] and [A5]-[A6]. Moreover, assume $M : \mathbf{R}^d \times \Xi \rightarrow \mathbf{R}^d \otimes \mathbf{R}^d$ is a polynomial growth function uniformly in ϑ . Then, for $\underline{k} \leq k_{\tau,n}$,

$$\left\| \sum_{j=1}^{\underline{k}-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\bar{Y}_{\tau,j+1} - \bar{Y}_{\tau,j})^{\otimes 2} - \frac{2}{3} \widehat{A}_{\tau,j,n} \right] \right\|_p \leq C(p) \underline{k} \Delta_{\tau,n}.$$

Proof. We have

$$\begin{aligned} & \mathbf{E} \left[\left\| \Delta_{\tau,n} \widehat{A}_{\tau,j,n} - \frac{3}{2} \left[a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n}) + (\Lambda_\star)^{\frac{1}{2}} (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j}) \right]^{\otimes 2} \right\|^p \right]^{1/p} \\ & \leq \left| \frac{1}{m_{\tau,n} + m'_{\tau,n}} - \frac{3}{2} \right| \mathbf{E} \left[\left\| [a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n})]^{\otimes 2} \right\|^p \right]^{1/p} \\ & \quad + \sqrt{6} \left| \sqrt{\frac{1}{m_{\tau,n} + m'_{\tau,n}}} - \sqrt{\frac{3}{2}} \right| \\ & \quad \times \mathbf{E} \left[\left\| [a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n}) (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j})^T (\Lambda_\star)^{1/2}] \right\|^p \right]^{1/p} \\ & \leq \left| \frac{1}{2/3 + 1/(3p_{\tau,n}^2)} - \frac{1}{2/3} \right| C(p) \Delta_{\tau,n} + \left| \sqrt{\frac{1}{2/3 + 1/(3p_{\tau,n}^2)}} - \sqrt{\frac{1}{2/3}} \right| \frac{C(p) \Delta_{\tau,n}^{1/2}}{p_{\tau,n}^{1/2}} \\ & \leq \frac{C(p) \Delta_{\tau,n}}{p_{\tau,n}^2} \end{aligned}$$

with Taylor's expansion for $f_1(x) = 1/x$ and $f_2(x) = \sqrt{1/x}$ around $x = 2/3$. Using this evaluation, we obtain

$$\begin{aligned} & \mathbf{E} \left[\left\| \sum_{j=1}^{\underline{k}-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\bar{Y}_{\tau,j+1} - \bar{Y}_{\tau,j})^{\otimes 2} - \frac{2}{3} \widehat{A}_{\tau,j,n} \right] \right\|^p \right]^{1/p} \\ & \leq \sum_{j=1}^{\underline{k}-2} \|M(\bar{Y}_{\tau,j-1}, \alpha)\|_{2p} \\ & \quad \times \left\| \widehat{A}_{\tau,j,n} - \frac{3}{2\Delta_{\tau,n}} \left[a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n}) + (\Lambda_\star)^{\frac{1}{2}} (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j}) \right]^{\otimes 2} \right\|_{2p} \\ & \quad + \mathbf{E} \left[\left\| \sum_{j=1}^{\underline{k}-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\Delta_{\tau,n} b(X_{j\Delta_{\tau,n}}) + e_{\tau,j,n})^{\otimes 2} \right] \right\|^p \right]^{1/p} \end{aligned}$$

$$\begin{aligned}
& + \mathbf{E} \left[\left| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\Delta_{\tau,n} b(X_{j\Delta_{\tau,n}})) (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n})^T a(X_{j\Delta_{\tau,n}})^T \right] \right|^p \right]^{1/p} \\
& + \mathbf{E} \left[\left| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} e_{\tau,j,n} (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n})^T a(X_{j\Delta_{\tau,n}})^T \right] \right|^p \right]^{1/p} \\
& + \mathbf{E} \left[\left| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\Delta_{\tau,n} b(X_{j\Delta_{\tau,n}}) + e_{\tau,j,n}) (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j})^T (\Lambda_{\star})^{1/2} \right] \right|^p \right]^{1/p}.
\end{aligned}$$

Because of the evaluation above, it holds

$$\begin{aligned}
& \sum_{j=1}^{k-2} \|M(\bar{Y}_{\tau,j-1}, \alpha)\|_{2p} \\
& \quad \times \left\| \hat{A}_{\tau,j,n} - \frac{3}{2\Delta_{\tau,n}} \left[a(X_{j\Delta_{\tau,n}}) (\zeta_{\tau,j+1,n} + \zeta_{\tau,j+2,n}) + (\Lambda_{\star})^{1/2} (\bar{\varepsilon}_{\tau,j+1} - \bar{\varepsilon}_{\tau,j}) \right] \right\|_{2p}^{\otimes 2} \\
& \leq \frac{C(p)k}{p_{\tau,n}^2}
\end{aligned}$$

and note that $p_{\tau,n}^{-1} \leq \Delta_{\tau,n}$. With triangle inequality and Hölder's one,

$$\begin{aligned}
& \mathbf{E} \left[\left| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\Delta_{\tau,n} b(X_{j\Delta_{\tau,n}}) + e_{\tau,j,n})^{\otimes 2} \right] \right|^p \right]^{1/p} \\
& \leq \Delta_{\tau,n}^{-1} \sum_{j=1}^{k-2} \|M(\bar{Y}_{\tau,j-1}, \alpha)\|_{2p} \|\Delta_{\tau,n} b(X_{j\Delta_{\tau,n}}) + e_{\tau,j,n}\|_{4p}^2 \\
& \leq C(p)k\Delta_{\tau,n}.
\end{aligned}$$

In the next place, we can evaluate the following L^p -norm by the three norms, such that

$$\begin{aligned}
& \mathbf{E} \left[\left| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau,j-1}, \alpha) \left[\Delta_{\tau,n}^{-1} e_{\tau,j,n} (\zeta_{\tau,j+1,n} + \zeta'_{\tau,j+2,n})^T a(X_{j\Delta_{\tau,n}})^T \right] \right|^p \right]^{1/p} \\
& \leq \sum_{i=0}^2 \mathbf{E} \left[\left| \sum_{1 \leq 3j+i \leq k-2} M(\bar{Y}_{\tau,3j+i-1}, \alpha) \left[\Delta_{\tau,n}^{-1} (\Delta_{\tau,n} b(X_{(3j+1)\Delta_{\tau,n}})) \right. \right. \right. \\
& \quad \left. \left. \left. (\zeta_{\tau,3j+i+1,n} + \zeta'_{\tau,3j+i+2,n})^T a(X_{(3j+1)\Delta_{\tau,n}})^T \right] \right|^p \right]^{1/p};
\end{aligned}$$

and then Burkholder's inequality leads to

$$\begin{aligned}
& \mathbf{E} \left[\left| \sum_{1 \leq 3j \leq k-2} M(\bar{Y}_{\tau,3j-1}, \alpha) \left[b(X_{3j\Delta_{\tau,n}}) (\zeta_{\tau,3j+1,n} + \zeta'_{\tau,3j+2,n})^T a(X_{3j\Delta_{\tau,n}})^T \right] \right|^p \right]^{1/p} \\
& \leq \mathbf{E} \left[\left| \sum_{1 \leq 3j \leq k-2} \left| M(\bar{Y}_{\tau,3j-1}, \alpha) \left[b(X_{3j\Delta_{\tau,n}}) (\zeta_{\tau,3j+1,n} + \zeta'_{\tau,3j+2,n})^T a(X_{3j\Delta_{\tau,n}})^T \right] \right|^2 \right|^{p/2} \right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{1 \leq 3j \leq k-2} \left\| \left\| M(\bar{Y}_{\tau, 3j-1}, \alpha) \left[b(X_{3j\Delta_{\tau, n}}) (\zeta_{\tau, 3j+1, n} + \zeta'_{\tau, 3j+2, n})^T a(X_{3j\Delta_{\tau, n}})^T \right] \right\|_{p/2}^2 \right\|_{p/2} \right)^{1/2} \\
&\leq \left(\sum_{1 \leq 3j \leq k-2} C(p) \|\zeta_{\tau, 3j+1, n} + \zeta'_{\tau, 3j+2, n}\|_p^2 \right)^{1/2} \\
&\leq C(p) (k\Delta_{\tau, n})^{1/2}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\mathbf{E} \left[\left\| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau, j-1}, \alpha) \left[\Delta_{\tau, n}^{-1} e_{\tau, j, n} (\zeta_{\tau, j+1, n} + \zeta'_{\tau, j+2, n})^T a(X_{j\Delta_{\tau, n}})^T \right] \right\|_{p/2}^2 \right]^{1/p} \\
&\leq C(p) (k\Delta_{\tau, n})^{1/2}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbf{E} \left[\left\| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau, j-1}, \alpha) \left[\Delta_{\tau, n}^{-1} (\Delta_{\tau, n} b(X_{j\Delta_{\tau, n}}) + e_{\tau, j, n}) (\bar{\varepsilon}_{\tau, j+1} - \bar{\varepsilon}_{\tau, j})^T (\Lambda_{\star})^{1/2} \right] \right\|_{p/2}^2 \right]^{1/p} \\
&\leq C(p) \left(\frac{k}{p_{\tau, n}} \right)^{1/2}.
\end{aligned}$$

Finally because of Lemma 5

$$\mathbf{E} \left[\left\| \sum_{j=1}^{k-2} M(\bar{Y}_{\tau, j-1}, \alpha) \left[\Delta_{\tau, n}^{-1} e_{\tau, j, n} (\zeta_{\tau, j+1, n} + \zeta'_{\tau, j+2, n})^T a(X_{j\Delta_{\tau, n}})^T \right] \right\|_{p/2}^2 \right]^{1/p} \leq C(p) k\Delta_{\tau, n},$$

which completes the proof. \square

6.2. Derivation and evaluation for locally asymptotic quadratic form. We give the locally asymptotic quadratic form at $\vartheta^* \in \Xi$ for $u_1 \in \mathbf{R}^{m_1}$ and $u_2 \in \mathbf{R}^{m_2}$,

$$\begin{aligned}
\mathbb{Z}_{1, \tau_1, n}^{(0)}(u_1; \vartheta^*) &:= \exp \left(S_{1, \tau_1, n}(\vartheta^*) [u_1] - \frac{1}{2} \Gamma_{1, \tau_1}(\vartheta^*) [u_1^{\otimes 2}] + r_{1, \tau_1, n}(u_1; \vartheta^*) \right), \\
\mathbb{Z}_{2, \tau_2, n}^{(0)}(u_2; \vartheta^*) &:= \exp \left(S_{2, \tau_2, n}(\vartheta^*) [u_2] - \frac{1}{2} \Gamma_{2, \tau_2}(\vartheta^*) [u_2^{\otimes 2}] + r_{2, \tau_2, n}(u_2; \vartheta^*) \right),
\end{aligned}$$

where the residual terms are defined as

$$\begin{aligned}
r_{1, \tau_1, n}(u_1; \vartheta^*) &:= \int_0^1 (1-s) \left\{ \Gamma_{1, \tau_1}(\vartheta^*) [u_1^{\otimes 2}] - \Gamma_{1, \tau_1, n}(\alpha^* + s \underline{k}_{\eta_1, \tau_1, n}^{-q_1} u_1; \vartheta^*) [u_1^{\otimes 2}] \right\} ds, \\
r_{2, \tau_2, n}(u_2; \vartheta^*) &:= \int_0^1 (1-s) \left\{ \Gamma_{2, \tau_2}(\vartheta^*) [u_2^{\otimes 2}] - \Gamma_{2, \tau_2, n}(\beta^* + s \underline{T}_{\eta_2, n}^{-q_2} u_2; \vartheta^*) [u_2^{\otimes 2}] \right\} ds.
\end{aligned}$$

Proof of Lemma 1. We start with the proof of (1). Let us define

$$\begin{aligned}
\tilde{S}_{1, \tau_1, n}(\vartheta^*) [u_1] &:= -\frac{2}{3 \underline{k}_{\eta_1, \tau_1, n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1, \tau_1, n}-2} (\partial_{\alpha} A(\bar{Y}_{\tau_1, j-1}, \alpha^*)) \\
&\quad \left[u_1, \Delta_{\tau_1, n}^{-1} (\bar{Y}_{\tau_1, j+1} - \bar{Y}_{\tau_1, j})^{\otimes 2} - \frac{2}{3} A_{\tau_1, n}(\bar{Y}_{\tau_1, j-1}, \alpha^*, \Lambda_{\star}) \right].
\end{aligned}$$

Because of $\mathbf{E} \left[\left\| \sqrt{n} \left(\hat{\Lambda}_n - \Lambda_\star \right) \right\|^p \right] < \infty$ shown in Nakakita and Uchida (2018c), we have the evaluation such that

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| S_{1,\tau_1,n}(\vartheta^\star) - \tilde{S}_{1,\tau_1,n}(\vartheta^\star) \right|^p \right] \leq \sup_{n \in \mathbf{N}} C(p) \frac{k_{\eta_1,\tau_1,n}^{pq_1}}{n^{\frac{p}{2}}} < \infty.$$

Hence it is enough to prove that

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left| \tilde{S}_{1,\tau_1,n}(\vartheta^\star) \right|^p \right] < \infty.$$

By Lemma 5, we obtain the decomposition

$$\begin{aligned} & \tilde{S}_{1,\tau_1,n}(\vartheta^\star) [u_1] \\ &= -\frac{2}{3k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha^\star)) \\ & \quad \left[u_1, \Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^\star, \Lambda_\star) \right] \\ &= -\frac{2}{3k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha^\star)) \left[u_1, \frac{2}{3} \hat{A}_{\tau_1,j,n} - \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^\star, \Lambda_\star) \right] \\ & \quad - \frac{2}{3k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{j-1}, \alpha^\star)) \left[u_1, \Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - \frac{2}{3} \hat{A}_{\tau_1,j,n} \right] \\ &= M_{1,\tau_1,n} + \dot{R}_{1,\tau_1,n} + \ddot{R}_{1,\tau_1,n}, \end{aligned}$$

where

$$\begin{aligned} M_{1,\tau_1,n} &:= -\frac{4}{9k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha^\star)) \left[u_1, \hat{A}_{\tau_1,j,n} - A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}) \right], \\ \dot{R}_{1,\tau_1,n} &:= -\frac{2}{3k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha^\star)) \left[u_1, \Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - \frac{2}{3} \hat{A}_{\tau_1,j,n} \right], \\ \ddot{R}_{1,\tau_1,n} &:= -\frac{4}{9k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{j=1}^{k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,j-1}, \alpha^\star)) \left[u_1, A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}) - A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}) \right], \end{aligned}$$

$a(x) := a(x, \alpha^\star)$ and $A_{\tau_1,n}(x) := A_{\tau_1,n}(x, \alpha^\star, \Lambda_\star)$. $\sup_{n \in \mathbf{N}} \left\| \dot{R}_{1,\tau_1,n} \right\|_p < \infty$ can be derived directly from Lemma 6 and the assumption $k_{\eta_1,\tau_1,n}^q \Delta_{\tau_1,n} \rightarrow 0$. We set $\ddot{R}_{1,i,\tau_1,n}$ as

$$\begin{aligned} \ddot{R}_{1,i,\tau_1,n} &:= -\frac{4}{9k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{1 \leq 3j+i \leq k_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{\tau_1,3j+i-1}, \alpha^\star)) \\ & \quad \left[u_1, A_{\tau_1,n}(X_{(3j+i)\Delta_{\tau_1,n}}) - A_{\tau_1,n}(\bar{Y}_{3j+i-1\tau_1}) \right], \end{aligned}$$

and examine only the case $i = 0$ without loss of generality. We also define $\ddot{R}_{1,0,\tau_1,n}$ as

$$\begin{aligned} \ddot{R}_{1,0,\tau_1,n} &:= -\frac{4}{9k_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{1 \leq 3j \leq k_{\eta_1,\tau_1,n}-2} \mathbf{E} \left[(\partial_\alpha A(\bar{Y}_{\tau_1,3j-1}, \alpha^\star)) \right. \\ & \quad \left. \left[u_1, A_{\tau_1,n}(X_{3j\Delta_{\tau_1,n}}) - A_{\tau_1,n}(\bar{Y}_{\tau_1,3j-1}) \right] \middle| \mathcal{H}_{3j-1}^{\tau_1} \right]. \end{aligned}$$

Because of Burkholder's inequality, it holds

$$\mathbf{E} \left[\left| \ddot{R}_{1,0,\tau_1,n} - \ddot{R}_{1,0,\tau_1,n} \right|^{p\gamma} \right]^{1/p} \leq \frac{C(p)}{\underline{k}_{\eta_1,\tau_1,n}^{1-2q}} < \infty.$$

Hence it is sufficient to see $\|\ddot{R}_{1,0,\tau_1,n}\|_p < \infty$, and because of Lemma 4, we can have

$$\left\| \ddot{R}_{1,0,\tau_1,n} \right\|_p \leq C(p) \underline{k}_{\eta_1,\tau_1,n}^{q_1} \Delta_{\tau_1,n},$$

and then $\sup_{n \in \mathbf{N}} \left\| \ddot{R}_{1,\tau_1,n} \right\|_p < \infty$. We define $M_{1,i,\tau_1,n}$ for $i = 0, 1, 2$ as

$$\begin{aligned} M_{1,i,\tau_1,n} &= -\frac{4}{9\underline{k}_{\eta_1,\tau_1,n}^{1-q_1}} \sum_{1 \leq 3j+i \leq \underline{k}_{\eta_1,\tau_1,n}-2} (\partial_\alpha A(\bar{Y}_{3j+i-1\tau_1}, \alpha^*)) \left[u_1, \hat{A}_{\tau_1,3j+i,n} - A_{\tau_1,n}(X_{(3j+i)\Delta_{\tau_1,n}}) \right], \end{aligned}$$

and since the property of conditional expectation $\mathbf{E} \left[\hat{A}_{\tau_1,j,n} | \mathcal{H}_{j,n}^{\tau_1} \right] = A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}})$ holds, Burkholder's inequality verifies

$$\mathbf{E} [|M_{1,i,\tau_1,n}|^p] \leq \frac{C(p)}{\underline{k}_{\eta_1,\tau_1,n}^{2-2q_1}} < \infty,$$

for all i because of the integrability.

In the second place, we show (2) holds. Let us define

$$\begin{aligned} \mathbb{V}_{1,\tau_1,n}^{(\dagger)}(\alpha) &= -\frac{1}{2\underline{k}_{\eta_1,\tau_1,n}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left(\left\| \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star) \right\|^2 - \left\| \frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \Lambda_\star) \right\|^2 \right. \\ &\quad - 2 \left(\Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} \right) \left[\frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star) \right] \\ &\quad \left. + 2 \left(\Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} \right) \left[\frac{2}{3} A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \Lambda_\star) \right] \right), \end{aligned}$$

and then the evaluation $\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\sup_{\alpha \in \Theta_1} \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \left| \mathbb{V}_{1,\tau_1,n}(\alpha; \vartheta^\star) - \mathbb{V}_{1,\tau_1,n}^{(\dagger)}(\alpha; \vartheta^\star) \right| \right)^p \right] < \infty$ can be easily obtained due to $\mathbf{E} \left[\left\| \sqrt{n} (\hat{\Lambda}_n - \Lambda_\star) \right\|^p \right] < \infty$. We also define

$$\begin{aligned} \mathbb{V}_{1,\tau_1,n}^{(\ddagger)}(\alpha) &:= -\frac{2}{9\underline{k}_{\eta_1,\tau_1,n}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left(\left\| A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star) \right\|^2 - \left\| A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \Lambda_\star) \right\|^2 \right. \\ &\quad - 3 \left(A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^*, \Lambda_\star) \right) \left[A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star) \right] \\ &\quad \left. + 3 \left(A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^*, \Lambda_\star) \right) \left[A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \Lambda_\star) \right] \right); \end{aligned}$$

then

$$\begin{aligned} &\underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \left(\mathbb{V}_{1,\tau_1,n}^{(\dagger)}(\alpha) - \mathbb{V}_{1,\tau_1,n}^{(\ddagger)}(\alpha) \right) \\ &= \frac{2\underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1}}{3\underline{k}_{\eta_1,\tau_1,n}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left(\left(\Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^*, \Lambda_\star) \right) \right. \\ &\quad \left. \left[A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star) \right] \right. \\ &\quad \left. - \left(\Delta_{\tau_1,n}^{-1} (\bar{Y}_{\tau_1,j+1} - \bar{Y}_{\tau_1,j})^{\otimes 2} - A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^*, \Lambda_\star) \right) \right. \\ &\quad \left. \left[A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^*, \Lambda_\star) \right] \right). \end{aligned}$$

Using Lemma 6 as $\dot{R}_{1,\tau_1,n}$, it is easy to have

$$\left\| \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \left(\mathbb{V}_{1,\tau_1,n}^{(\dagger)}(\alpha) - \mathbb{V}_{1,\tau_1,n}^{(\ddagger)}(\alpha) \right) \right\|_p \leq C(p) \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \Delta_{\tau_1,n},$$

and similarly

$$\left\| \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \partial_\alpha \left(\mathbb{V}_{1,\tau_1,n}^{(\dagger)}(\alpha) - \mathbb{V}_{1,\tau_1,n}^{(\ddagger)}(\alpha) \right) \right\|_p \leq C(p) \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \Delta_{\tau_1,n};$$

then Sobolev's inequality verifies

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\sup_{\alpha \in \Theta_1} \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \left| \mathbb{V}_{1,\tau_1,n}^{(\dagger)}(\alpha; \vartheta^*) - \mathbb{V}_{1,\tau_1,n}^{(\ddagger)}(\alpha; \vartheta^*) \right| \right)^p \right] < \infty.$$

Hence it is sufficient to obtain the evaluation

$$\sup_{n \in \mathbf{N}} \mathbf{E} \left[\left(\sup_{\alpha \in \Theta_1} \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} \left| \mathbb{V}_{1,\tau_1,n}(\alpha; \vartheta^*) - \mathbb{V}_{1,\tau_1,n}^{(\ddagger)}(\alpha; \vartheta^*) \right| \right)^p \right] < \infty.$$

Let us define $M_{1,\tau_1,n}^{(\dagger)}$ and $R_{1,\tau_1,n}^{(\dagger)}$ as

$$\begin{aligned} M_{1,\tau_1,n}^{(\dagger)} &= -\frac{2}{9\underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left(\|A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha, \Lambda_\star)\|^2 \right. \\ &\quad - 3(A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)) [A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha, \Lambda_\star)] \\ &\quad - \|A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)\|^2 \\ &\quad \left. + 3(A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)) [A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)] \right), \\ R_{1,\tau_1,n}^{(\dagger)} &= -\frac{2}{9\underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left(\|A_{\tau_1,n}(X_{j\Delta_{\tau_1,n}}, \alpha, \Lambda_\star)\|^2 \right. \\ &\quad - 3(A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)) [A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha, \Lambda_\star)] \\ &\quad - \|A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)\|^2 \\ &\quad \left. + 3(A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)) [A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)] \right) \\ &\quad - \frac{2}{9\underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1}} \sum_{j=1}^{\underline{k}_{\eta_1,\tau_1,n}-2} \left(\|A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star)\|^2 \right. \\ &\quad - 3(A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)) [A_{\tau,n}(\bar{Y}_{\tau_1,j-1}, \alpha, \Lambda_\star)] \\ &\quad - \|A_{\tau_1,n}(\bar{Y}_{\tau_1,j-1}, \alpha^\star, \Lambda_\star)\|^2 \\ &\quad \left. + 3(A_{\tau,n}(X_{j\Delta_{\tau_1,n}}, \alpha^\star, \Lambda_\star)) [A_{\tau,n}(\bar{Y}_{\tau_1,j-1}, \alpha^\star, \Lambda_\star)] \right). \end{aligned}$$

$\sup_{n \in \mathbf{N}} \left\| \sup_{\alpha \in \Theta_1} \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} R_{1,\tau_1,n}^{(\dagger)}(\alpha) \right\|_p \leq C(p) \Delta_{\tau_1,n}^{\frac{1}{2}}$ and $\sup_{n \in \mathbf{N}} \left\| \sup_{\alpha \in \Theta_1} \underline{k}_{\eta_1,\tau_1,n}^{\epsilon_1} M_{1,\tau_1,n}^{(\dagger)}(\alpha) \right\|_p < \infty$ are easily obtained by the discussion parallel to Nakakita and Uchida (2018c) and Yoshida (2011) respectively.

(3) and (4) are shown in the way parallel to Nakakita and Uchida (2018c) and the discussion above respectively. \square

Proof of Lemma 2. We decompose

$$\begin{aligned}
S_{2,\tau_2,n}(\vartheta^*) &:= -\frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [\bar{Y}_{\tau_2,j+1} - \bar{Y}_{\tau_2,j} - \Delta_{\tau_2,n} b(\bar{Y}_{\tau_2,j-1}, \beta^*)] \\
&= -\frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [a(X_{j\Delta_{\tau_2,n}})(\zeta_{\tau_2,j+1,n} + \zeta'_{\tau_2,j+2,n})] \\
&\quad - \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [(\Lambda_\star)^{1/2}(\bar{\varepsilon}_{\tau_2,j+1} - \bar{\varepsilon}_{\tau_2,j})] \\
&\quad - \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [e_{\tau_2,j,n}] \\
&= M_{2,\tau_2,n} + R_{2,\tau_2,n},
\end{aligned}$$

where

$$\begin{aligned}
M_{2,\tau_2,n} &:= -\frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [a(X_{j\Delta_{\tau_2,n}})(\zeta_{\tau_2,j+1,n} + \zeta'_{\tau_2,j+2,n})] \\
&\quad - \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [(\Lambda_\star)^{1/2}(\bar{\varepsilon}_{\tau_2,j+1} - \bar{\varepsilon}_{\tau_2,j})], \\
R_{2,\tau_2,n} &= -\frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{j=1}^{k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,j-1}, \beta^*)) [e_{\tau_2,j,n}].
\end{aligned}$$

Since Burkholder's inequality verifies the evaluations such that

$$\begin{aligned}
&\mathbf{E} \left[\left| \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{1 \leq 3j \leq k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,3j-1}, \beta^*)) [a(X_{3j\Delta_{\tau_2,n}})(\zeta_{\tau_2,3j+1,n} + \zeta'_{\tau_2,3j+2,n})] \right|^p \right] \\
&\leq \frac{C(p)}{\underline{T}_{\eta_2,n}^{p-pq_2}} \mathbf{E} \left[\left| \sum_{1 \leq 3j \leq k_{\eta_2,\tau_2,n}-2} |\partial_\beta b(\bar{Y}_{\tau_2,3j-1}, \beta^*)|^2 |a(X_{3j\Delta_{\tau_2,n}})(\zeta_{\tau_2,3j+1,n} + \zeta'_{\tau_2,3j+2,n})|^2 \right|^{p/2} \right] \\
&\leq \frac{C(p) \underline{T}_{\eta_2,n}^{\frac{p}{2}}}{\underline{T}_{\eta_2,n}^{p(1-q_2)}},
\end{aligned}$$

and

$$\mathbf{E} \left[\left| \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{1 \leq 3j \leq k_{\eta_2,\tau_2,n}-2} (\partial_\beta b(\bar{Y}_{\tau_2,3j-1}, \beta^*)) [(\Lambda_\star)^{1/2}(\bar{\varepsilon}_{\tau_2,j+1} - \bar{\varepsilon}_{\tau_2,j})] \right|^p \right] \leq \frac{C(p) \underline{T}_{\eta_2,n}^{\frac{p}{2}}}{\underline{T}_{\eta_2,n}^{p(1-q_2)}},$$

we obtain $\sup_{n \in \mathbf{N}} \|M_{2,\tau_2,n}\|_p < \infty$. With respect to $R_{2,\tau_2,n}$, Burkholder's inequality also leads to

$$\begin{aligned} & \mathbf{E} \left[\left| \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{1 \leq 3j \leq k_{\eta_2,\tau_2,n}-2} (\partial_{\beta} b(\bar{Y}_{\tau_2,3j-1}, \beta^*)) \left[e_{\tau_2,3j,n} - \mathbf{E} \left[e_{\tau_2,3j,n} | \mathcal{H}_{3j,n}^{\tau_2} \right] \right] \right|^p \right] \\ & \leq C(p) \mathbf{E} \left[\left| \frac{1}{\underline{T}_{\eta_2,n}^{2-2q_2}} \sum_{1 \leq 3j \leq k_{\eta_2,\tau_2,n}-2} |\partial_{\beta} b(\bar{Y}_{\tau_2,3j-1}, \beta^*)|^2 \left| e_{\tau_2,3j,n} - \mathbf{E} \left[e_{\tau_2,3j,n} | \mathcal{H}_{3j,n}^{\tau_2} \right] \right|^2 \right|^{p/2} \right] \\ & \leq \frac{C(p) k_{\eta_2,\tau_2,n}^{\frac{p}{2}} \Delta_{\tau_2,n}^p}{\underline{T}_{\eta_2,n}^{p(1-q_2)}} \leq C(p) \left[k_{\eta_2,\tau_2,n}^{q_2 - \frac{1}{2}} \Delta_{\tau_2,n}^{q_2} \right]^p, \end{aligned}$$

and we also have

$$\begin{aligned} & \mathbf{E} \left[\left| \frac{1}{\underline{T}_{\eta_2,n}^{1-q_2}} \sum_{1 \leq 3j \leq k_{\eta_2,\tau_2,n}-2} (\partial_{\beta} b(\bar{Y}_{\tau_2,3j-1}, \beta^*)) \left[\mathbf{E} \left[e_{\tau_2,3j,n} | \mathcal{H}_{3j,n}^{\tau_2} \right] \right] \right|^p \right] \\ & \leq \frac{C(p) k_{\eta_2,\tau_2,n}^p \Delta_{\tau_2,n}^{2p}}{\underline{T}_{\eta_2,n}^{p(1-q_2)}} \leq \frac{C(p) k_{\eta_2,\tau_2,n}^p \Delta_{\tau_2,n}^{2p}}{(k_{\eta_2,\tau_2,n} \Delta_{\tau_2,n})^{p(1-q_2)}} \leq C(p) \left[k_{\eta_2,\tau_2,n}^{q_2} \Delta_{\tau_2,n}^{1+q_2} \right]^p, \end{aligned}$$

which is led by $\left| \mathbf{E} \left[e_{\tau_2,j,n} | \mathcal{H}_{j,n}^{\tau_2} \right] \right| \leq C \Delta_{\tau_2,n}^2 (1 + |X_{j\Delta_{\tau_2,n}}|)^C$ shown in Nakakita and Uchida (2018c). The derivation of (ii) is parallel to that of (ii) in Lemma 1.

(3) and (4) are shown in an analogous way to Nakakita and Uchida (2018c) and the discussion above respectively. \square

Proof of Theorem 1. Lemma 1, Lemma 2 and Theorem 3 in Yoshida (2011) lead to the PLDI as discussed in Nakakita and Uchida (2018c). The L^M -evaluation of estimators also can be obtained with a parallel discussion to Nakakita and Uchida (2018c). \square

Proof of Theorem 2. First of all, we show the L^p -boundedness

$$\mathbf{E} \left[\left| \sqrt{k_{\tau_3,n}} (\hat{\alpha}_{J_1,n} - \alpha^*) \right|^p \right] < \infty, \mathbf{E} \left[\left| \sqrt{T_n} (\hat{\beta}_{J_2,n} - \beta^*) \right|^p \right] < \infty.$$

As discussed in Kaino and Uchida (2018b), for all $k = 1, \dots, J_1$, on $K_n(\hat{\alpha}_{k-1,n})$,

$$\begin{aligned} & \partial_{\alpha} \mathbb{H}_{1,\tau_3,n}(\alpha^* | \hat{\Lambda}_n) \\ & = \partial_{\alpha} \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n) + \partial_{\alpha}^2 \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n) [\alpha^* - \hat{\alpha}_{k-1,n}] \\ & \quad + \int_0^1 (1-s) \partial_{\alpha}^3 \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} + s(\alpha^* - \hat{\alpha}_{k-1,n}) | \hat{\Lambda}_n) ds \left[(\alpha^* - \hat{\alpha}_{k-1,n})^{\otimes 2} \right], \end{aligned}$$

and

$$\begin{aligned} \hat{\alpha}_{k,n} & = \hat{\alpha}_{k-1,n} - \left(\frac{1}{k_{\tau_3,n}} \partial_{\alpha}^2 \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n) \right)^{-1} \frac{1}{k_{\tau_3,n}} \partial_{\alpha} \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n) \\ & = \hat{\alpha}_{k-1,n} - \left(\frac{1}{k_{\tau_3,n}} \partial_{\alpha}^2 \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n) \right)^{-1} \frac{1}{k_{\tau_3,n}} \partial_{\alpha} \mathbb{H}_{1,\tau_3,n}(\alpha^* | \hat{\Lambda}_n) \\ & \quad + (\alpha^* - \hat{\alpha}_{k-1,n}) \\ & \quad + \left(\frac{1}{k_{\tau_3,n}} \partial_{\alpha}^2 \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n) \right)^{-1} \\ & \quad \times \frac{1}{k_{\tau_3,n}} \int_0^1 (1-s) \partial_{\alpha}^3 \mathbb{H}_{1,\tau_3,n}(\hat{\alpha}_{k-1,n} + s(\alpha^* - \hat{\alpha}_{k-1,n}) | \hat{\Lambda}_n) ds \left[(\alpha^* - \hat{\alpha}_{k-1,n})^{\otimes 2} \right], \end{aligned}$$

that is,

$$\begin{aligned}
& \hat{\alpha}_{k,n} - \alpha^* \\
&= - \left(\frac{1}{k_{\tau_3,n}} \partial_\alpha^2 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n \right) \right)^{-1} \frac{1}{k_{\tau_3,n}} \partial_\alpha \mathbb{H}_{1,\tau_3,n} \left(\alpha^* | \hat{\Lambda}_n \right) \\
&+ \left(\frac{1}{k_{\tau_3,n}} \partial_\alpha^2 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n \right) \right)^{-1} \\
&\quad \times \frac{1}{k_{\tau_3,n}} \int_0^1 (1-s) \partial_\alpha^3 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} + s(\alpha^* - \hat{\alpha}_{k-1,n}) | \hat{\Lambda}_n \right) ds \left[(\alpha^* - \hat{\alpha}_{k-1,n})^{\otimes 2} \right].
\end{aligned}$$

The analogous argument of Kamatani and Uchida (2015) verifies

$$\mathbf{E} \left[\left\| \left(\frac{1}{k_{\tau_3,n}} \partial_\alpha^2 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n \right) \right)^{-1} \right\|^p \mathbf{1}_{K_n(\hat{\alpha}_{k-1,n})} \right] < \infty, \quad P(K_n^c(\hat{\alpha}_{k-1,n})) \leq \frac{C(L)}{k_{\tau_3,n}^L}.$$

Then we obtain the L^p -boundedness for any $p \geq 1$ and $k \in \mathbf{N}$ such that $2^k q_1' \leq 1/2$, where $q_1' = q_1(\eta_1 - \gamma/\tau_1) / (1 - \gamma'/\tau_3)$, that is to say, $k \leq -\log_2 q_1' - 1$, as the discussion in Kamatani and Uchida (2015),

$$\begin{aligned}
& \mathbf{E} \left[\left| k_{\tau_3,n}^{2^k q_1'} (\hat{\alpha}_{k,n} - \alpha^*) \right|^p \right] \\
&\leq C(p) \mathbf{E} \left[\left\| \left(\frac{1}{k_{\tau_3,n}} \partial_\alpha^2 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n \right) \right)^{-1} \right\|^{2p} \mathbf{1}_{K_n(\hat{\alpha}_{k-1,n})} \right]^{1/2} \\
&\quad \times \mathbf{E} \left[\left| \frac{k_{\tau_3,n}^{2^k q_1'}}{k_{\tau_3,n}} \partial_\alpha \mathbb{H}_{1,\tau_3,n} \left(\alpha^* | \hat{\Lambda}_n \right) \right|^{2p} \right]^{1/2} \\
&+ C(p) \mathbf{E} \left[\left\| \left(\frac{1}{k_{\tau_3,n}} \partial_\alpha^2 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} | \hat{\Lambda}_n \right) \right)^{-1} \right\|^{2p} \mathbf{1}_{K_n(\hat{\alpha}_{k-1,n})} \right]^{1/2} \\
&\quad \times \mathbf{E} \left[\left\| \frac{1}{k_{\tau_3,n}} \int_0^1 (1-s) \partial_\alpha^3 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} + s(\alpha^* - \hat{\alpha}_{k-1,n}) | \hat{\Lambda}_n \right) ds \right\|^{4p} \right]^{1/4} \\
&\quad \times \mathbf{E} \left[\left| k_{\tau_3,n}^{2^{k-1} q_1'} (\alpha^* - \hat{\alpha}_{k-1,n}) \right|^{8p} \right]^{1/4} \\
&+ C(p).
\end{aligned}$$

The result of Nakakita and Uchida (2018c) leads to the L^p -boundednesses for all $p \geq 1$ such as

$$\begin{aligned}
& \mathbf{E} \left[\left| \frac{k_{\tau_3,n}^{2^k q_1'}}{k_{\tau_3,n}} \partial_\alpha \mathbb{H}_{1,\tau_3,n} \left(\alpha^* | \hat{\Lambda}_n \right) \right|^p \right] < \infty, \\
& \mathbf{E} \left[\left\| \frac{1}{k_{\tau_3,n}} \int_0^1 (1-s) \partial_\alpha^3 \mathbb{H}_{1,\tau_3,n} \left(\hat{\alpha}_{k-1,n} + s(\alpha^* - \hat{\alpha}_{k-1,n}) | \hat{\Lambda}_n \right) ds \right\|^p \right] < \infty.
\end{aligned}$$

Hence we obtain

$$\mathbf{E} \left[\left| k_{\tau_3,n}^{2^{k-1} q_1'} (\hat{\alpha}_{k-1,n} - \alpha^*) \right|^p \right] < \infty \Rightarrow \mathbf{E} \left[\left| k_{\tau_3,n}^{2^k q_1'} (\hat{\alpha}_{k,n} - \alpha^*) \right|^p \right] < \infty,$$

and as shown in Theorem 1, $\mathbf{E} \left[\left| k_{\tau_3, n}^{q'_1} (\hat{\alpha}_{0, n} - \alpha^*) \right|^p \right] < \infty$, and therefore we have the L^p -evaluation $\mathbf{E} \left[\left| k_{\tau_3, n}^{2^{J_1-1} q'_1} (\hat{\alpha}_{J_1, n} - \alpha^*) \right|^p \right] < \infty$ as a result. Then

$$\begin{aligned}
& \mathbf{E} \left[\left| k_{\tau_3, n}^{1/2} (\hat{\alpha}_{J_1, n} - \alpha^*) \right|^p \right] \\
& \leq C(p) \mathbf{E} \left[\left\| \left(\frac{1}{k_{\tau_3, n}} \partial_{\alpha}^2 \mathbb{H}_{1, \tau_3, n} (\hat{\alpha}_{J_1-1, n} | \hat{\Lambda}_n) \right)^{-1} \right\|^{2p} \mathbf{1}_{K_n(\hat{\alpha}_{J_1-1, n})} \right]^{1/2} \\
& \quad \times \mathbf{E} \left[\left| \frac{k_{\tau_3, n}^{1/2}}{k_{\tau_3, n}} \partial_{\alpha} \mathbb{H}_{1, \tau_3, n} (\alpha^* | \hat{\Lambda}_n) \right|^{2p} \right]^{1/2} \\
& + C(p) \mathbf{E} \left[\left\| \left(\frac{1}{k_{\tau_3, n}} \partial_{\alpha}^2 \mathbb{H}_{1, \tau_3, n} (\hat{\alpha}_{J_1-1, n} | \hat{\Lambda}_n) \right)^{-1} \right\|^{2p} \mathbf{1}_{K_n(\hat{\alpha}_{J_1-1, n})} \right]^{1/2} \\
& \quad \times \mathbf{E} \left[\left\| \frac{1}{k_{\tau_3, n}} \int_0^1 (1-s) \partial_{\alpha}^3 \mathbb{H}_{1, \tau_3, n} (\hat{\alpha}_{J_1-1, n} + s(\alpha^* - \hat{\alpha}_{J_1-1, n}) | \hat{\Lambda}_n) ds \right\|^{4p} \right]^{1/4} \\
& \quad \times \mathbf{E} \left[\left| k_{\tau_3, n}^{1/4} (\alpha^* - \hat{\alpha}_{J_1-1, n}) \right|^{8p} \right]^{1/4} \\
& + C(p) \\
& < \infty
\end{aligned}$$

because the discussion above holds, $J_1 > -\log_2 q'_1 - 1$ and hence $2^{J_1-1} q'_1 > 1/4$. The same result for $\hat{\beta}_{J_2, n}$ can be derived from the parallel discussion.

In the second place, we will see the convergence in law. Actually it is sufficient to see

$$k_{\tau_3, n}^{1/4} (\hat{\alpha}_{J_1-1, n} - \alpha^*) \xrightarrow{P} \mathbf{0},$$

and it can be verified because

$$\mathbf{E} \left[\left| k_{\tau_3, n}^{1/4} (\alpha^* - \hat{\alpha}_{J_1-1, n}) \right|^p \right] = k_{\tau_3, n}^{1/4-2^{J_1-1} q'_1} \mathbf{E} \left[\left| k_{\tau_3, n}^{2^{J_1-1} q'_1} (\alpha^* - \hat{\alpha}_{J_1-1, n}) \right|^p \right] \rightarrow 0.$$

The same argument holds for $\hat{\beta}_{J_2, n}$ too. □

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