REGULARITY AND SOLVABILITY OF LINEAR DIFFERENTIAL OPERATORS IN GEVREY SPACES: OMITTED PROOFS

GABRIEL ARAÚJO

This is an addendum to [1] which aims to provide the proofs of some results in that paper (Theorem 7.5 and Proposition 9.15) which were removed from its final version. The reason for such omission is that these proofs follow quite closely others already present in the literature, with minor modifications. I make them publicly available for the sake of completeness.

1. Proof of Theorem 7.5

Our main reference here is Hörmander [4]. We will start by proving analogous versions of several auxiliary lemmas used in his book, which we did not find in the literature (especially the ones not covered by Björck [2]). Although the proofs of these lemmas are very much like their counterparts in [4], we chose to present them here for the sake of completeness. We will, however, make free use of the results already proven in [2].

For the first result in this section (which is an adaptation of [4, Theorem 10.1.5]) we recall that for each $k \in \mathscr{K}_\omega$ one defines, in accordance with [2] and [4],

$$M_k(\xi) \doteq \sup_\eta \frac{k(\xi+\eta)}{k(\eta)}$$

Also, for $\lambda > 0$ let $\mathscr{K}_\omega^\lambda$ stand for the set of functions $k \in \mathscr{K}_\omega$ such that

$$k(\xi+\eta) \leq e^{\lambda|\xi|^{\frac{1}{\sigma}}} k(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{1.1}$$

so $\mathscr{K}_\omega$ is exactly the union of all $\mathscr{K}_\omega^\lambda$.

Lemma 1.1. *For each $\lambda > 0$, each $k \in \mathscr{K}_\omega^\lambda$ and each $\delta > 0$ there exist $k_\delta \in \mathscr{K}_\omega^\lambda$ and $C_\delta > 0$ such that, for every $\xi \in \mathbb{R}^n$ one has*

1. $1 \leq k_\delta(\xi)/k(\xi) \leq C_\delta$ *and*
2. $1 \leq M_{k_\delta}(\xi) \leq e^{\delta|\xi|}$.

Proof. For $\delta > 0$ let

$$k_\delta(\xi) \doteq \sup_\eta e^{-\delta|\eta|} k(\xi-\eta), \quad \xi \in \mathbb{R}^n,$$

which defines and element of $\mathscr{K}_\omega^\lambda$. Indeed, for $\xi, \xi' \in \mathbb{R}^n$ we have

$$k_\delta(\xi+\xi') = \sup_\eta e^{-\delta|\eta|} k(\xi+\xi'-\eta) \leq \sup_\eta e^{-\delta|\eta|} e^{\lambda|\xi'|^{\frac{1}{\sigma}}} k(\xi-\eta) = e^{\lambda|\xi'|^{\frac{1}{\sigma}}} k_\delta(\xi).$$

Notice that

$$k(\xi) \leq k_\delta(\xi) = \sup_\eta e^{-\delta|\eta|} k(\xi-\eta) \leq \sup_\eta e^{-\delta|\eta|} e^{\lambda|\eta|^{\frac{1}{\sigma}}} k(\xi) = k(\xi) \sup_\eta e^{\lambda|\eta|^{\frac{1}{\sigma}} - \delta|\eta|}$$

where the constant on the far right (call it C_δ) is finite, proving the first statement. A change of variables allows us to write, for $\xi \in \mathbb{R}^n$,

$$k_\delta(\xi) = \sup_\eta e^{-\delta|\xi-\eta|} k(\eta)$$

and so we have

$$k_\delta(\xi+\xi') = \sup_\eta e^{-\delta|\xi+\xi'-\eta|} k(\eta) \leq e^{\delta|\xi'|} \sup_\eta e^{-\delta|\xi-\eta|} k(\eta) = e^{\delta|\xi'|} k_\delta(\xi)$$

Date: March 3, 2022.

thus implying that $M_{k_\delta}(\xi') \leq e^{\delta|\xi'|}$. □

Now we present a version of [4, Lemma 13.3.1].

Lemma 1.2. *Let $k \in \mathscr{K}_\omega$ and, for each $\delta > 0$, let $k_\delta \in \mathscr{K}_\omega$ be as in Lemma 1.1. Then for each $\phi \in \mathcal{D}_\omega(\mathbb{R}^n)$ there exists $\delta_0 > 0$ such that*

$$\|\phi u\|_{p,k_\delta} \leq 2\|\phi\|_{1,1}\|u\|_{p,k_\delta}$$

for every $0 < \delta < \delta_0$ and every $u \in \mathcal{B}_{p,k_\delta} = \mathcal{B}_{p,k}$.

Proof. From [2, Theorem 2.2.7] we have, for every $\delta > 0$,

$$\|\phi u\|_{p,k_\delta} \leq \|\phi\|_{1,M_{k_\delta}}\|u\|_{p,k_\delta}$$

so it is enough to prove the existence of a $\delta_0 > 0$ such that

$$\|\phi\|_{1,M_{k_\delta}} \leq 2\|\phi\|_{1,1}$$

for every $0 < \delta < \delta_0$. But from the definition of the norms we have

$$\|\phi\|_{1,M_{k_\delta}} = \frac{1}{(2\pi)^n}\int M_{k_\delta}(\xi)\ |\hat{\phi}(\xi)|\ \mathrm{d}\xi \to \frac{1}{(2\pi)^n}\int |\hat{\phi}(\xi)|\ \mathrm{d}\xi = \|\phi\|_{1,1}$$

because $M_{k_\delta} \to 1$ uniformly on compact set as $\delta \to 0^+$: this follows immediately from Lemma 1.1, which also implies that $\mathcal{B}_{p,k_\delta}$ and $\mathcal{B}_{p,k}$ are the same as topological vector spaces, since their norms are equivalent. □

Now we proceed with the proof of Theorem 7.5 from [1]. We shall not reproduce its statement here. Due to [4, Lemma 13.1.2] there exist operators with constant coefficients $P_1(D), \dots, P_r(D)$ and functions $c_0, c_1, \dots, c_r \in C^\infty(\Omega)$, that are uniquely determined by the following properties:

- $P_j \prec P_0$ for every $j \in \{1, \dots, r\}$;
- $c_j(x_0) = 0$ for every $j \in \{0, \dots, r\}$;
- and, in Ω,

$$P(x, D) = P_0(D) + \sum_{j=1}^{r} c_j(x)P_j(D).$$

Since we are also assuming that the coefficients of $P(x, D)$ belong to $G^{\sigma_0}(\Omega)$ one can actually show that $c_0, c_1, \dots, c_r \in G^{\sigma_0}(\Omega)$. For every $\epsilon > 0$, define

$$X_\epsilon \doteq \{x \in \mathbb{R}^n\ ;\ |x - x_0| < \epsilon\}$$

and select $\epsilon_0 > 0$ such that $X_{\epsilon_0} \subset \Omega$. Let $\chi \in G_c^{\sigma_0}(\mathbb{R}^n)$ be equal to 1 in a neighborhood of $\{x \in \mathbb{R}^n\ ;\ |x| \leq 2\epsilon_0\}$ and

$$E_0 \in B^{\mathrm{loc}}_{\infty,\tilde{P}_0}(\mathbb{R}^n)$$

be a fundamental solution of $P_0(D)$, and define

$$F_0 \doteq \chi E_0 \in B_{\infty,\tilde{P}_0}.$$

If $g \in \mathcal{E}'_\omega(\mathbb{R}^n)$ has its support in X_{ϵ_0} then $F_0 * g = E_0 * g$ in X_{ϵ_0}, hence

$$P_0(D)(F_0 * g) = F_0 * P_0(D)g = g \text{ in } X_{\epsilon_0}.$$

Now let $\psi \in G_c^{\sigma_0}(\mathbb{R}^n)$ be such that

$$\psi = 1 \text{ in } \{x \in \mathbb{R}^n\ ;\ |x| \leq 1\}$$
$$\psi = 0 \text{ in } \{x \in \mathbb{R}^n\ ;\ |x| > 2\}$$

and define $\psi_\epsilon(x) \doteq \psi((x - x_0)/\epsilon)$. We claim the existence of $0 < \epsilon_1 < \epsilon_0/2$ such that for each $0 < \epsilon < \epsilon_1$ and each $f \in \mathcal{E}'_\omega(\mathbb{R}^n)$ the equation

$$g + \sum_{j=0}^{r} \psi_\epsilon c_j P_j(D)(F_0 * g) = \psi_\epsilon f \tag{1.2}$$

has a unique solution $g \in \mathcal{E}'_\omega(\mathbb{R}^n)$. Proceeding as in [4], we provisionally assume this claim and define the operator E as

$$Ef \doteq F_0 * g$$

where $g \in \mathcal{E}'_\omega(\mathbb{R}^n)$ is the unique solution of (1.2), which yields a linear map $E : \mathcal{E}'_\omega(\mathbb{R}^n) \to \mathcal{E}'_\omega(\mathbb{R}^n)$: we will prove that if $\epsilon > 0$ is small enough then this operator has the properties described in the statement above.

First, since $\operatorname{supp}\psi_\epsilon \subset X_{\epsilon_0}$ equation (1.2) implies that $\operatorname{supp} g \subset X_{\epsilon_0}$, so in X_ϵ

$$\begin{aligned} P(x,D)Ef &= P(x,D)(F_0 * g) \\ &= P_0(D)(F_0 * g) + \sum_{j=0}^r c_j P_j(D)(F_0 * g) \\ &= g + \sum_{j=0}^r \psi_\epsilon c_j P_j(D)(F_0 * g) \\ &= \psi_\epsilon f \\ &= f \end{aligned}$$

thus proving the first property claimed. Second, let $u \in \mathcal{E}'_\omega(\mathbb{R}^n)$ be such that $\operatorname{supp} u \subset X_\epsilon$ and $f \doteq P(x,D)u$: putting $g \doteq P_0(D)u$ in the left-hand side of (1.2) we get

$$\begin{aligned} g + \sum_{j=0}^r \psi_\epsilon c_j P_j(D)(F_0 * g) &= P_0(D)u + \sum_{j=0}^r \psi_\epsilon c_j P_j(D)(F_0 * P_0(D)u) \\ &= P_0(D)u + \sum_{j=0}^r \psi_\epsilon c_j P_j(D)u \\ &= P(x,D)u \\ &= f \\ &= \psi_\epsilon f \end{aligned}$$

that is, g solves equation (1.2), and by uniqueness we have

$$Ef = F_0 * g = F_0 * P_0(D)u = u.$$

This proves the second property of E.

The last property of E – the estimate between norms – will follow from the proof of our claim about existence and uniqueness of solutions of equation (1.2), so now we proceed in that direction. For every $\epsilon > 0$ we define a linear map $A_\epsilon : \mathcal{D}'_\omega(\mathbb{R}^n) \to \mathcal{D}'_\omega(\mathbb{R}^n)$ by the expression

$$A_\epsilon g \doteq \sum_{j=0}^r \psi_\epsilon c_j P_j(D)(F_0 * g)$$

which is well-defined for every $g \in \mathcal{D}'_\omega(\mathbb{R}^n)$, for F_0 is compactly supported. Let $k \in \mathscr{K}_\omega$ and, for $\delta > 0$, let $k_\delta \in \mathscr{K}_\omega$ as in Lemma 1.2 (in which case $\mathcal{B}_{p,k_\delta} = \mathcal{B}_{p,k}$, with equivalent defining norms): according to it, there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ one has

$$\begin{aligned} \|A_\epsilon g\|_{p,k_\delta} &\leq \sum_{j=0}^r \|\psi_\epsilon c_j P_j(D)(F_0 * g)\|_{p,k_\delta} \\ &\leq 2 \sum_{j=0}^r \|\psi_\epsilon c_j\|_{1,1} \|P_j(D)(F_0 * g)\|_{p,k_\delta} \end{aligned}$$

as long as $P_j(D)(F_0 * g) \in \mathcal{B}_{p,k}$ (recall that $\psi_\epsilon c_j \in \mathcal{D}_\omega(\mathbb{R}^n)$ for every $j \in \{0,\dots,r\}$ according to [1, Lemma 7.4]. Now, since $P_j \prec P_0$ and $F_0 \in B_{\infty,\tilde{P}_0}$ there are constants $C_1, C_2 > 0$ such that

$$|P_j(\xi)||\hat{F}_0(\xi)| \leq |\tilde{P}_j(\xi)||\hat{F}_0(\xi)| \leq C_1 |\tilde{P}_0(\xi)||\hat{F}_0(\xi)| \leq C_1 C_2$$

for every $\xi \in \mathbb{R}^n$, so if we define $C \doteq C_1 C_2 > 0$ we have that

$$\|P_j(D)(F_0 * g)\|_{p,k_\delta} = \|k_\delta \ P_j \ \hat{F}_0 \ \hat{g}\|_{L^p} \leq C\|k_\delta \ \hat{g}\|_{L^p} = C\|g\|_{p,k_\delta}$$

for every $g \in \mathcal{B}_{p,k}$: therefore

$$\|A_\epsilon g\|_{p,k_\delta} \leq 2C \sum_{j=0}^{r} \|\psi_\epsilon c_j\|_{1,1} \|g\|_{p,k_\delta}$$

and thus $A_\epsilon : \mathcal{B}_{p,k} \to \mathcal{B}_{p,k}$ continuously. Now [4, Lemma 13.3.2] allows us to choose $0 < \epsilon_1 < \epsilon_0/2$ such that

$$\sum_{j=0}^{r} \|\psi_\epsilon c_j\|_{1,1} \leq \frac{1}{4C}$$

for every $0 < \epsilon < \epsilon_1$. We stress that such a choice is independent of k, and hence

$$\|A_\epsilon g\|_{p,k_\delta} \leq \frac{1}{2}\|g\|_{p,k_\delta} \tag{1.3}$$

for every $g \in \mathcal{B}_{p,k}$. We conclude that $I + A_\epsilon : \mathcal{B}_{p,k} \to \mathcal{B}_{p,k}$ is invertible, which means that equation (1.2) has a unique solution $g \in \mathcal{B}_{p,k}$ whenever $f \in \mathcal{B}_{p,k}$, which must have compact support for reasons already mentioned. We need one more lemma to finish this argument.

Lemma 1.3. *Let $1 \leq p \leq \infty$. Every $u \in \mathcal{E}'_\omega(\mathbb{R}^n)$ belongs to $\mathcal{B}_{p,k}$ for some $k \in \mathscr{K}_\omega$.*

Proof of Lemma 1.3. For $u \in \mathcal{E}'_\omega(\mathbb{R}^n)$, [2, Theorem 1.8.14] ensures, among other things, the existence of constants $\lambda \in \mathbb{R}$ and $C > 0$ such that

$$|\hat{u}(\xi)| \leq Ce^{\lambda|\xi|^{\frac{1}{\sigma}}}, \quad \forall \xi \in \mathbb{R}^n.$$

Of course we can assume $\lambda > 0$, so $k(\xi) \doteq e^{-2\lambda|\xi|^{\frac{1}{\sigma}}}$ defines an element of $\mathscr{K}_\omega$ and

$$k(\xi)|\hat{u}(\xi)| \leq Ce^{-\lambda|\xi|^{\frac{1}{\sigma}}}, \quad \forall \xi \in \mathbb{R}^n,$$

so $k\hat{u} \in L^p(\mathbb{R}^n)$ (i.e. $u \in \mathcal{B}_{p,k}$) no matter what p is. □

Now we turn back to the deduction of estimate (7.2) in the statement of the theorem (see [1]). Let $f \in \mathcal{E}'_\omega(\mathbb{R}^n) \cap \mathcal{B}_{p,k}$ and take $g \in \mathcal{E}'_\omega(\mathbb{R}^n) \cap \mathcal{B}_{p,k}$ the unique solution of (1.2): by (1.3) we have

$$\|g\|_{p,k_\delta} \leq 2\|\psi_\epsilon f\|_{p,k_\delta}$$

thus

$$\|Ef\|_{p,\tilde{P}_0 k_\delta} = \|F_0 * g\|_{p,\tilde{P}_0 k_\delta} \leq \|F_0\|_{\infty,\tilde{P}_0}\|g\|_{p,k_\delta} \leq 2\|F_0\|_{\infty,\tilde{P}_0}\|\psi_\epsilon f\|_{p,k_\delta} \leq 4\|F_0\|_{\infty,\tilde{P}_0}\|\psi_\epsilon\|_{1,1}\|f\|_{p,k_\delta}$$

where we used Lemma 1.2 again. On the other hand, Lemma 1.1 ensures that the norms $\|\cdot\|_{p,k_\delta}$ and $\|\cdot\|_{p,k}$ are equivalent: an explicit calculation actually shows that

$$\|u\|_{p,k} \leq \|u\|_{p,k_\delta} \leq C_\delta \|u\|_{p,k}, \quad \forall u \in \mathcal{B}_{p,k}.$$

In the same manner one obtains

$$\|u\|_{p,\tilde{P}_0 k} \leq \|u\|_{p,\tilde{P}_0 k_\delta} \leq C_\delta \|u\|_{p,\tilde{P}_0 k}, \quad \forall u \in \mathcal{B}_{p,k}$$

so now we have

$$\|Ef\|_{p,\tilde{P}_0 k} \leq \|Ef\|_{p,\tilde{P}_0 k_\delta} \leq 4\|F_0\|_{\infty,\tilde{P}_0}\|\psi_\epsilon\|_{1,1}\|f\|_{p,k_\delta} \leq 4C_\delta\|F_0\|_{\infty,\tilde{P}_0}\|\psi_\epsilon\|_{1,1}\|f\|_{p,k}.$$

□

2. Proof of Proposition 9.15

In this section we follow very closely the arguments in [3, pp. 53–56]; this is indeed the "Gevrey version" of them. Again, the reader is referred to our main article for the statement of Proposition 9.15, which we shall not recall here.

We assume that g and u are such that $\operatorname{supp} \mathrm{d}g \subset U_0^-$ and $\operatorname{supp} \mathrm{d}u \subset U_0^+ \cap V_0$: the other case (i.e. the opposite choice of signs) can be treated analogously. First of all, compactness of $\overline{U}$ ensures the existence of a constant $A > 0$ (which does not depend on x_0) such that

$$|\Phi(x,t) - \Phi(x_0,t)| \leq A|x - x_0|, \quad \forall (x,t) \in U.$$

Fix some $\phi \in G^\sigma(\mathbb{C})$ and define

$$\phi^\sharp \doteq Z^*\phi = \phi \circ Z \tag{2.1}$$

which belongs, for instance, to $G^\sigma(U)$ since Z is a real-analytic map. Denoting by

$$\begin{array}{ccccc} \pi & : & \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & (x,t) & \longmapsto & t \end{array}$$

the projection onto the t-variable, we have $U_0 = \pi(U)$ since U is cylindrical, and so

$$\pi^* g \in G^\sigma(U; \wedge^{q-1}\mathbb{C}T^*\mathbb{R}^{n+1}).$$

This observation allows us to define

$$F \doteq \phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^* g \tag{2.2}$$

which belongs to $G^\sigma(U; \wedge^q\mathbb{C}T^*\mathbb{R}^{n+1})$ and, recalling that over Ω we have an identification $\wedge^q\mathbb{C}T^*\mathbb{R}^{n+1} \cong \Lambda^{0,q} \oplus \mathrm{T}'^{1,q-1}$ we can define $f \in G^\sigma(U; \Lambda^{0,q})$ as the (unique) component of F in that direct sum. We claim that if the support of ϕ is conveniently chosen we can achieve $\mathrm{d}'f = 0$ i.e. $\mathrm{d}F$ will be a section of $\mathrm{T}'^{1,q}$. Indeed, without extra assumptions we have

$$\begin{aligned} \mathrm{d}F &= \mathrm{d}\left(\phi^\sharp \wedge \mathrm{d}\bar{Z}\right) \wedge \pi^* g - \phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \mathrm{d}\left(\pi^* g\right) \\ &= \mathrm{d}\phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^* g - \phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^*(\mathrm{d}g). \end{aligned}$$

However

$$\mathrm{d}\phi^\sharp = \mathrm{d}(Z^*\phi) = Z^*(\mathrm{d}\phi) = Z^*\left(\frac{\partial \phi}{\partial z} \wedge \mathrm{d}z + \frac{\partial \phi}{\partial \bar{z}} \wedge \mathrm{d}\bar{z}\right) = \left(\frac{\partial \phi}{\partial z} \circ Z\right) \wedge \mathrm{d}Z + \left(\frac{\partial \phi}{\partial \bar{z}} \circ Z\right) \wedge \mathrm{d}\bar{Z}$$

hence

$$\mathrm{d}\phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^* g = \left(\frac{\partial \phi}{\partial z} \circ Z\right) \wedge \mathrm{d}Z \wedge \mathrm{d}\bar{Z} \wedge \pi^* g$$

is a section of $\mathrm{T}'^{1,q}$ over U: if we can prove that $\phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^*(\mathrm{d}g)$ is also a section of $\mathrm{T}'^{1,q}$ then our claim will follow. This is where the choice of ϕ (or, rather, its support) kicks in: we can choose it so that this summand is actually zero.

Indeed, let $a > 0$ and $b \in \mathbb{R}$ and define the strip

$$E(a,b) \doteq \{x + iy \in \mathbb{C} \ ; \ |x - x_0| \leq a, \ y \geq b\}.$$

From the definition of U_0^- we have

$$\begin{aligned} \pi^{-1}(U_0^-) &= \{(x,t) \in U \ ; \ \Phi(x_0,t) < y_0\} \\ Z^{-1}(E(a,b)) &= \{(x,t) \in U \ ; \ |x - x_0| \leq a, \ \Phi(x,t) \geq b\} \end{aligned}$$

and if $(x,t) \in Z^{-1}(E(a,b)) \cap \pi^{-1}(U_0^-)$ then

$$b \leq \Phi(x,t) \leq |\Phi(x,t) - \Phi(x_0,t)| + \Phi(x_0,t) < A|x - x_0| + y_0 \leq Aa + y_0.$$

So if we choose a, b such that $y_0 + Aa \leq b$ then $Z^{-1}(E(a,b)) \cap \pi^{-1}(U_0^-) = \emptyset$. In particular, choosing $\operatorname{supp} \phi \subset E(a,b)$ yields

$$\operatorname{supp} \phi^\sharp = \operatorname{supp} Z^*\phi = Z^{-1}(\operatorname{supp} \phi) \subset Z^{-1}(E(a,b)).$$

Since we already had

$$\operatorname{supp}\pi^*(\mathrm{d}g) = \pi^{-1}(\operatorname{supp}\mathrm{d}g) \subset \pi^{-1}(U_0^-)$$

for $\operatorname{supp}\mathrm{d}g \subset U_0^-$ by hypothesis, we must have $\operatorname{supp}\phi^\sharp$ and $\operatorname{supp}\pi^*(\mathrm{d}g)$ disjoint, hence $\phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^*(\mathrm{d}g)$ vanishes in U. We conclude that $\mathrm{d}' f = 0$.

We introduce a new parameter $r > 0$ (to be specified later) and let $\chi \in G^\sigma_c(\mathbb{R})$ be such that $0 \leq \chi \leq 1$ and

$$\chi(x) = 1 \text{ if } |x - x_0| < r/2,$$
$$\chi(x) = 0 \text{ if } |x - x_0| > r.$$

Let also $\tilde{\chi} \in G^\sigma(\mathbb{R} \times \mathbb{R}^n)$ be defined as

$$\tilde{\chi}(x,t) \doteq \chi(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^n,$$

hence

$$v \doteq \tilde{\chi} \wedge \mathrm{d}Z \wedge \pi^* u$$

is a section of $\Lambda^{1,n-q}$ with G^σ coefficients. Since $\operatorname{supp} u \subset V_0$ we have that

$$\operatorname{supp} v \subset \operatorname{supp}\tilde{\chi} \cap \pi^{-1}(\operatorname{supp} u) \subset \{(x,t) \in V \ ; \ |x - x_0| \leq r, \ t \in \operatorname{supp} u\}$$

the latter being a compact subset of V if we choose $r > 0$ sufficiently small: in that case $v \in G^\sigma_c(V; \Lambda^{1,n-q})$. It follows from all the definitions that

$$f \wedge v = F \wedge v = \phi^\sharp \wedge \mathrm{d}\bar{Z} \wedge \pi^* g \wedge \tilde{\chi} \wedge \mathrm{d}Z \wedge \pi^* u = \pm \left(\tilde{\chi}\ \phi^\sharp\right) \wedge \mathrm{d}Z \wedge \mathrm{d}\bar{Z} \wedge \pi^*(g \wedge u).$$

We remark that the first identity follows from the fact that $f - F$ is a section of $\mathrm{T}'^{1,q-1}$ (so its wedge with v is zero) and that the correct sign in the last identity is irrelevant for our purposes: we are only interested in studying the vanishing of their integrals. Also, recalling that $\operatorname{supp}\phi^\sharp \subset Z^{-1}(E(a,b))$ and that $\tilde{\chi}(x,t) = 1$ if $|x - x_0| < r/2$, it is clear that if we further impose that $a < r/2$ then $\tilde{\chi} = 1$ on $\operatorname{supp}\phi^\sharp$, and hence

$$f \wedge v = \pm\ \phi^\sharp \wedge \mathrm{d}Z \wedge \mathrm{d}\bar{Z} \wedge \pi^*(g \wedge u).$$

Now notice that

$$\phi^\sharp \wedge \mathrm{d}Z \wedge \mathrm{d}\bar{Z} = (Z^*\phi) \wedge \mathrm{d}Z \wedge \mathrm{d}\bar{Z} = Z^*\left(\phi \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}\right) = 2i\ Z^*\left(\phi \wedge \mathrm{d}y \wedge \mathrm{d}x\right).$$

We will now assume that ϕ is non-negative, and define $\psi_0 \in G^\sigma(\mathbb{C}; \mathbb{R})$ as

$$\psi_0(x + iy) \doteq \int_{-\infty}^{y} \phi(x + is)\ \mathrm{d}s$$

which clearly satisfies

$$\frac{\partial \psi_0}{\partial y} = \phi.$$

A simple calculation also shows that since $E(a,b)$ contains $\operatorname{supp}\phi$ then it also contains $\operatorname{supp}\psi_0$. Letting $\psi \doteq \psi_0 \wedge \mathrm{d}x \in G^\sigma(\mathbb{C}; \wedge^1 T^*\mathbb{C})$ we conclude that

$$\mathrm{d}\psi = \mathrm{d}\psi_0 \wedge \mathrm{d}x = \frac{\partial \psi_0}{\partial y} \wedge \mathrm{d}y \wedge \mathrm{d}x = \phi \wedge \mathrm{d}y \wedge \mathrm{d}x$$

hence

$$\begin{aligned} f \wedge v &= \pm\ 2i\ Z^*\left(\phi \wedge \mathrm{d}y \wedge \mathrm{d}x\right) \wedge \pi^*(g \wedge u) \\ &= \pm\ 2i\ Z^*\left(\mathrm{d}\psi\right) \wedge \pi^*(g \wedge u) \\ &= \pm\ 2i\ \mathrm{d}(Z^*\psi) \wedge \pi^*(g \wedge u). \end{aligned}$$

We claim that, for the choices above, $Z^*\psi \wedge \pi^*(g \wedge u)$ is compactly supported in U. Indeed, since $\operatorname{supp}\psi = \operatorname{supp}\psi_0 \subset E(a,b)$ we have $\operatorname{supp} Z^*\psi \subset Z^{-1}(E(a,b))$ and thus

$$(\operatorname{supp} Z^*\psi) \cap (\operatorname{supp}\pi^* u) \subset \{(x,t) \in U \ ; \ |x - x_0| \leq a, \ t \in \operatorname{supp} u\}$$

the latter a compact subset of U, while the former clearly contains the support of $Z^*\psi \wedge \pi^*(g \wedge u)$, hence our claim. It then follows from Stokes's Theorem that

$$0 = \int \mathrm{d}\left(Z^*\psi \wedge \pi^*(g \wedge u)\right) = \int \mathrm{d}(Z^*\psi) \wedge \pi^*(g \wedge u) \pm \int Z^*\psi \wedge \mathrm{d}\pi^*(g \wedge u)$$

which in turn implies

$$\int f \wedge v = \pm\, 2i \int Z^*\psi \wedge \mathrm{d}\pi^*(g \wedge u).$$

But notice that

$$Z^*\psi \wedge \mathrm{d}\pi^*(g \wedge u) = Z^*\psi \wedge \pi^*(\mathrm{d}g \wedge u) \pm Z^*\psi \wedge \pi^*(g \wedge \mathrm{d}u)$$

where the first summand is zero since $\operatorname{supp}(Z^*\psi) \cap \operatorname{supp} \pi^*(\mathrm{d}g) = \emptyset$: this follows from the fact that $\operatorname{supp}\psi \subset E(a,b)$ and $\operatorname{supp} \mathrm{d}g \subset U_0^-$, and thus implies that

$$\int f \wedge v = \pm\, 2i \int Z^*\psi \wedge \pi^*(g \wedge \mathrm{d}u).$$

Now we are going to impose further restrictions on ϕ. Recall that $\operatorname{supp} \mathrm{d}u \subset U_0^+ \cap V_0$, meaning that $\Phi(x_0, t) > y_0$ for all $t \in \operatorname{supp} \mathrm{d}u$: by compactness, there exists $\rho > 0$ such that

$$\Phi(x_0, t) > y_0 + \rho, \quad \forall t \in \operatorname{supp} \mathrm{d}u.$$

Once again we shrink $r > 0$ so that $2Ar < \rho$, and thus $y_0 + Ar < -Ar + y_0 + \rho$, which allows us to choose $b, b', b'' \in \mathbb{R}$ such that

$$y_0 + Ar \leq b < b' < b'' < -Ar + y_0 + \rho.$$

If we further assume that

$$\operatorname{supp}\phi \subset \{x + iy \ ; \ |x - x_0| \leq a, \ b \leq y \leq b'\}$$

then it follows from the definition of ψ_0 that

$$y > b' \Rightarrow \psi_0(x + iy) = \psi_0(x + ib'), \quad \forall x \in \mathbb{R}.$$

For $|x - x_0| \leq a$ and $t \in \operatorname{supp} \mathrm{d}u$ we then have

$$\begin{aligned}
\Phi(x,t) &= (\Phi(x,t) - \Phi(x_0,t)) + \Phi(x_0,t) \\
&\geq -A|x - x_0| + \Phi(x_0,t) \\
&\geq -Aa + y_0 + \rho \\
&> -Ar + y_0 + \rho \\
&> b'
\end{aligned}$$

which implies that

$$\psi_0(Z(x,t)) = \psi_0(x + i\Phi(x,t)) = \psi_0(x + ib')$$

holds whenever $|x - x_0| \leq a$ and $t \in \operatorname{supp} \mathrm{d}u$.

Now recall that U is a cylindrical open set centered at the origin, hence there exists an open interval $I \subset \mathbb{R}$ centered at 0 such that $U = I \times U_0$. Hence

$$C(x) \doteq \psi_0(x + ib') = \int_{-\infty}^{b'} \phi(x + is)\mathrm{d}s$$

defines a function $C : I \to \mathbb{R}$ which allows us to write

$$Z^*\psi = Z^*(\psi_0 \wedge \mathrm{d}x) = (\psi_0 \circ Z) \wedge \mathrm{d}(x \circ Z) = C(x) \wedge \mathrm{d}x$$

for $(x,t) \in U$ such that $|x - x_0| \leq a$ and $t \in \operatorname{supp} \mathrm{d}u$. It is also clear that

$$\operatorname{supp}\left(Z^*\psi \wedge \pi^*(g \wedge \mathrm{d}u)\right) \subset \{(x,t) \in U \ ; \ |x - x_0| \leq a, \ t \in \operatorname{supp} \mathrm{d}u\}$$

and therefore

$$\int f \wedge v = \pm\, 2i \int Z^*\psi \wedge \pi^*(g \wedge \mathrm{d}u) = \pm\, 2i \int C(x) \wedge \mathrm{d}x \wedge \pi^*(g \wedge \mathrm{d}u) = \pm\, 2i \left(\int C(x)\ \mathrm{d}x\right) \int g \wedge \mathrm{d}u$$

where

$$\int C(x)\ \mathrm{d}x = \int \int_{-\infty}^{b'} \phi(x+is)\ \mathrm{d}s\ \mathrm{d}x = \int_{\mathbb{C}} \phi \neq 0$$

if we assume ϕ nonzero: equivalence (9.5) from [1] is proven.

We now turn to the second part of the statement: we will prove that if we shrink $a > 0$ as well as the difference $b' - b > 0$ (but keeping b fixed) then there exists $H \in \mathcal{O}(\mathbb{C})$ such that

$$\Re H \leq 0 \text{ in } Z(\operatorname{supp} f), \quad \Re H > 0 \text{ in } Z(\operatorname{supp} \mathrm{d}' v).$$

Recall that $\operatorname{supp} f \subset \operatorname{supp} F$, and from (2.2) and (2.1) we have

$$\operatorname{supp} F \subset \operatorname{supp} \phi^\sharp \cap \operatorname{supp} \pi^* g = Z^{-1}(\operatorname{supp} \phi) \cap \pi^{-1}(\operatorname{supp} g)$$

and thus

$$Z(\operatorname{supp} F) \subset \operatorname{supp} \phi \cap Z(\pi^{-1}(\operatorname{supp} g)) \subset \operatorname{supp} \phi \subset \{x + iy \in \mathbb{C}\ ;\ |x - x_0| \leq a,\ b \leq y \leq b'\}.$$

We denote by $\mathcal{R}$ the latter set above, and also define the quantities

$$\begin{aligned} M &\doteq \max \{\Phi(x,t)\ ;\ |x - x_0| \leq r,\ t \in \operatorname{supp} \mathrm{d}u\} \\ M_+ &\doteq \max \left\{\Phi(x,t)\ ;\ \frac{r}{2} \leq |x - x_0| \leq r,\ t \in \operatorname{supp} u\right\} \\ M_- &\doteq \min \left\{\Phi(x,t)\ ;\ \frac{r}{2} \leq |x - x_0| \leq r,\ t \in \operatorname{supp} u\right\} \end{aligned}$$

as well as the following subsets of the complex plane

$$\begin{aligned} \mathcal{A} &\doteq \{x + iy \in \mathbb{C}\ ;\ |x - x_0| \leq r,\ b'' \leq y \leq M\} \\ \mathcal{B} &\doteq \left\{x + iy \in \mathbb{C}\ ;\ \frac{r}{2} \leq |x - x_0| \leq r,\ M_- \leq y \leq M_+\right\}. \end{aligned}$$

We claim that $Z(\operatorname{supp} \mathrm{d}' v) \subset \mathcal{A} \cup \mathcal{B}$. In order to check this, notice first that since v is a section of $\Lambda^{1,n-q}$ we have

$$\mathrm{d}' v = \mathrm{d}v\mathrm{d}\left(\tilde{\chi} \wedge \mathrm{d}Z \wedge \pi^* u\right) = \mathrm{d}\tilde{\chi} \wedge \mathrm{d}Z \wedge \pi^* u - \tilde{\chi} \wedge \mathrm{d}Z \wedge \pi^*(\mathrm{d}u)$$

hence, clearly,

$$\operatorname{supp} \mathrm{d}' v \subset \left(\operatorname{supp} \mathrm{d}\tilde{\chi} \cap \operatorname{supp} \pi^* u\right) \cup \left(\operatorname{supp} \tilde{\chi} \cap \operatorname{supp} \pi^*(\mathrm{d}u)\right).$$

On the other hand

$$\begin{aligned} \operatorname{supp} \tilde{\chi} &\subset \{(x,t) \in \mathbb{R} \times \mathbb{R}^n\ ;\ |x - x_0| \leq r\} \\ \operatorname{supp} \mathrm{d}\tilde{\chi} &\subset \left\{(x,t) \in \mathbb{R} \times \mathbb{R}^n\ ;\ \frac{r}{2} \leq |x - x_0| \leq r\right\} \\ \operatorname{supp} \pi^* u &\subset \{(x,t) \in U\ ;\ t \in \operatorname{supp} u\} \\ \operatorname{supp} \pi^*(\mathrm{d}u) &\subset \{(x,t) \in U\ ;\ t \in \operatorname{supp} \mathrm{d}u\} \end{aligned}$$

which, together, ensure that $\operatorname{supp} \mathrm{d}' v$ is contained in the union of the sets below:

$$\begin{aligned} S_1 &\doteq \left\{(x,t) \in U\ ;\ \frac{r}{2} \leq |x - x_0| \leq r,\ t \in \operatorname{supp} u\right\} \\ S_2 &\doteq \{(x,t) \in U\ ;\ |x - x_0| \leq r,\ t \in \operatorname{supp} \mathrm{d}u\}. \end{aligned}$$

Clearly, Z maps S_1 into $\mathcal{B}$. Also, if $(x,t) \in S_2$ we have

$$\begin{aligned} \Phi(x,t) &= (\Phi(x,t) - \Phi(x_0,t)) + \Phi(x_0,t) \\ &\geq -A|x - x_0| + \Phi(x_0,t) \\ &> -Ar + y_0 + \rho \\ &> b'' \end{aligned}$$

and from the definitions of Z, M, S_2 and $\mathcal{A}$ we have $Z(x,t) \in \mathcal{A}$, proving our claim.

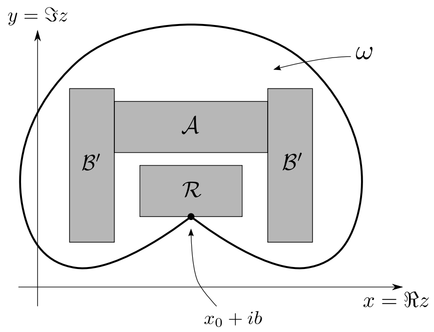

FIGURE 1. The compact sets $\mathcal{H} = \mathcal{A} \cup \mathcal{B}'$ and $\mathcal{R}$, which are disjoint; and the open set ω, which contains both of them.

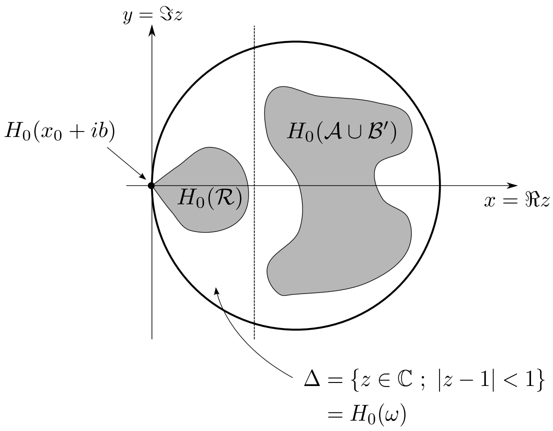

FIGURE 2. The scheme presented in Figure 1, now deformed by the homeomorphism H_0.

For a better visualization of the argument, we define the sets

$$\mathcal{B}' \doteq \Big\{x + iy \in \mathbb{C}\ ;\ \frac{r}{2} \leq |x - x_0| \leq r,\ \min\{M_-, b\} \leq y \leq \max\{M_+, M\}\Big\}$$

(which contains $\mathcal{B}$) and $\mathcal{H} \doteq \mathcal{A} \cup \mathcal{B}'$ which, on the one hand, contains $\mathcal{A} \cup \mathcal{B}$, and, on the other hand, does not intercept $\mathcal{R}$ (see Figure 1). It is clear that there exists a bounded open set $\omega \subset \mathbb{C}$, which is connected and simply connected, such that:

(1) it contains $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{R}$, except for the point $x_0 + ib \in \partial\mathcal{R}$;
(2) its boundary is a Jordan curve that contains the point $x_0 + ib$; and
(3) $\mathbb{C} \setminus \overline{\omega}$ is connected.

Let $\Delta \subset \mathbb{C}$ stand for the unit open disc centered at 1: a result due to Carathéodory ensures the existence of a homeomorphism $H_0 : \overline{\omega} \to \overline{\Delta}$ which is a biholomorphism between interiors, and we can assume without loss of generality that $H_0(x_0 + ib) = 0$ (see Figure 2). In particular, $\Re H_0(z) > 0$ for every $z \in \overline{\omega}$ except for $z = x_0 + ib$. Since $\mathcal{H} \subset \omega$ is a compact set, there exists $c > 0$ such that

$$\Re H_0 > 2c \text{ in } \mathcal{H}.$$

Also, if we further shrink a and choose b' sufficiently close to b (so that $\mathcal{R}$ is "thin" in the y-direction) then

$$\Re H_0 < \frac{c}{4} \text{ in } \mathcal{R}.$$

Finally, Mergelyan's Theorem allows us to approximate H_0 by an entire function H_1 such that

$$\Re H_1 > \frac{3c}{2} \text{ in } \mathcal{H}$$
$$\Re H_1 < \frac{c}{2} \text{ in } \mathcal{R}$$

thus setting $H \doteq H_1 - c$ finishes the proof. □

References

[1] G. Araújo. Regularity and solvability of linear differential operators in Gevrey spaces. *Math. Nachr.*, 291:729–758, 2018.

[2] G. Björck. Linear partial differential operators and generalized distributions. *Ark. Mat.*, 6:351–407 (1966), 1966.

[3] P. D. Cordaro and F. Treves. Homology and cohomology in hypo-analytic structures of the hypersurface type. *J. Geom. Anal.*, 1(1):39–70, 1991.

[4] L. Hörmander. *The analysis of linear partial differential operators. II*, volume 257 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Differential operators with constant coefficients.