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FINITENESS OF HOMOTOPY GROUPS RELATED TO THE NON-ABELIAN TENSOR PRODUCT

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ABSTRACT. By using finiteness related result of non-abelian tensor product we prove finiteness conditions for the homotopy groups $\pi_n(X)$ in terms of the number of tensors. In particular, we establish a quantitative version of the classical Blakers-Massey triad connectivity theorem. Moreover, we study others finiteness conditions and equivalence properties that arise from the non-abelian tensor square. In the end, we give applications to homotopy pushout, especially in the case of Eilenberg-MacLane spaces.

1. INTRODUCTION

In [8] Brown and Loday presented a topological significance for the non-abelian tensor product of groups. The non-abelian tensor product is used to describe the third relative homotopy group of a triad as a *non-abelian tensor product* of the second homotopy groups of appropriate subspaces. More specifically, in [8, Corollary 3.2], the third triad homotopy group is

$$\pi_3(X, A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C),$$

where X is a pointed space and $\{A, B\}$ is an open cover of X such that A, B and $C = A \cap B$ are connected and $(A, C), (B, C)$ are 1-connected.

In [12], Ellis gave a finiteness criterion for the triad homotopy group in terms of the finiteness of the involved groups (see also [2, 3]). More generally, finiteness conditions to $\pi_n(X)$ when excision theorem holds is given by the finiteness of $\pi_n(A)$, $\pi_n(B)$ and $\pi_{n-1}(C)$. However, when the excision property does not hold, the failure is mesured by the triad homotopy groups $\pi_n(X, A, B)$ with $n \geq 3$. Therefore, finiteness of $\pi_n(X)$ also depends on the triad homotopy groups $\pi_n(X, A, B)$. Thus, in order to give a bound for $\pi_n(X)$, it is needed to study finiteness conditions on $\pi_n(X, A, B)$. In [9], another application of the non-abelian

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tensor product is done by Brown and Loday, where they extended the classical Blakers-Massey triad connectivity theorem, which states that if A, B, and $A \cap B$ are connected, $\{A, B\}$ is an open cover of X, $(A, A \cap B)$ is p-connected, and $(B, A \cap B)$ is q-connected, then $\pi_{p+q+1}(X, A, B)$ is isomorphic to the non-abelian tensor product

$$\pi_{p+1}(A, A \cap B) \otimes \pi_{q+1}(B, A \cap B).$$

The hypothesis $p, q \geq 2$ is broadened to $p, q \geq 1$, and the hypothesis $\pi_1(A \cap B) = 0$ is removed.

For the convenience of the reader we repeat the relevant definitions (cf. [2, 3, 16]). Let G and H be groups each of which acts upon the other (on the right),

$$G \times H \to G, \ (g, h) \mapsto g^h; \ \ H \times G \to H, \ (h, g) \mapsto h^g$$

and on itself by conjugation, in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{(h^{g_1})} = \left(\left(g^{g_1^{-1}} \right)^h \right)^{g_1} \ \text{ and } \ h^{(g^{h_1})} = \left(\left(h^{h_1^{-1}} \right)^g \right)^{h_1}.$$

In this situation we say that G and H act *compatibly* on each other. Let H^φ be an extra copy of H, isomorphic via $\varphi : H \to H^\varphi$, $h \mapsto h^\varphi$, for all $h \in H$. Consider the group $\eta(G, H)$ defined in [16] as

$$\eta(G, H) = \langle G \cup H^\varphi \ | \ [g, h^\varphi]^{g_1} = [g^{g_1}, (h^{g_1})^\varphi], \ [g, h^\varphi]^{h_1^\varphi} = [g^{h_1}, (h^{h_1})^\varphi],$$
$$\forall g, g_1 \in G, \ h, h_1 \in H \rangle.$$

It is a well known fact (see [16, Proposition 2.2]) that the subgroup $[G, H^\varphi]$ of $\eta(G, H)$ is canonically isomorphic with the *non-abelian tensor product* $G \otimes H$, as defined by Brown and Loday in their seminal paper [8], the isomorphism being induced by $g \otimes h \mapsto [g, h^\varphi]$ (see also Ellis and Leonard [13]). It is clear that the subgroup $[G, H^\varphi]$ is normal in $\eta(G, H)$ and one has the decomposition

$$\eta(G, H) = ([G, H^\varphi] \cdot G) \cdot H^\varphi,$$

where the dots mean (internal) semidirect products. We observe that when $G = H$ and all actions are conjugations, $\eta(G, H)$ becomes the group $\nu(G)$ introduced in [20]. Recall that an element $\alpha \in \eta(G, H)$ is called a *tensor* if $\alpha = [a, b^\varphi]$ for suitable $a \in G$ and $b \in H$. We write $T_\otimes(G, H)$ to denote the set of all tensors (in $\eta(G, H)$). When $G = H$ and all actions are by conjugation, we simply write $T_\otimes(G)$ instead of $T_\otimes(G, G)$. A number of structural results for the non-abelian tensor product of groups (and related constructions) in terms of the set of tensors where presented in [2, 3, 4, 5, 21]).

Our contribution is to give finiteness conditions and bounds to the triad homotopy groups $\pi_n(X, A, B)$, furthermore finiteness condition and bound is given to $\pi_n(X)$. We establish the following related results.

Theorem A. *Let X be a union of open subspaces A, B such that A, B and $C = A \cap B$ are path-connected, and the pairs (A, C) and (B, C) are respectively p-connected and q-connected. Suppose that $\pi_n(A)$, $\pi_n(B)$, $\pi_{n-1}(C)$ and the set of tensors $\mathrm{T}_\otimes(\pi_{p+1}(A, C), \pi_{q+1}(B, C))$ are finite, where $n = p + q + 1$. Then $\pi_n(X)$ is a finite group with $\{a, b, c, m\}$-bounded order, where $|\pi_n(A)| = a$, $|\pi_n(B)| = b$, $|\pi_{n-1}(C)| = c$ and $m = |\mathrm{T}_\otimes(\pi_{p+1}(A, C), \pi_{q+1}(B, C))|$.*

An application of Theorem A is that $\pi_3(K(C_{r^\infty}, 2) \vee K(C_{s^\infty}, 2))$ is trivial, where "$\vee$" is the wedge sum, r and s are primes, and $K(G, n)$ is an *Eilenberg-MacLane space* (i.e., a topological space having just one nontrivial homotopy group $\pi_n(K(G, n)) \cong G$). See also Remark 2.1 and Corollary 2.2 below.

In [8], Brown and Loday shown that the third homotopy group of the suspension of an Eilenberg-MacLane space $K(G, 1)$ satisfies

$$\pi_3(SK(G, 1)) \cong J_2(G),$$

where $J_2(G)$ denotes the kernel of the derived map $\kappa : [G, G^\varphi] \to G'$, given by $[g, h^\varphi] \mapsto [g, h]$ (cf. [21, Chapter 2 and 3]). Many authors had studied bounds to the order of $\pi_3(SK(G, 1))$ (cf. [1, 6, 7, 17]). Now, we can deduce a finiteness criterion for $\pi_3(SX)$ in terms of $\pi_2(X)$ and the number of tensors $\mathrm{T}_\otimes(G)$, where $\pi_1(X) \cong G$ and SX is the suspension of a space X (see Remark 2.4, below).

Theorem B. *Let X be a connected space and $\pi_1(X) = G$. Suppose that the set of tensors $\mathrm{T}_\otimes(G)$ has exactly m tensors in $\nu(G)$ and $\pi_2(X)$ is finite with $|\pi_2(X)| = a$. Then $\pi_3(SX)$ is a finite group with $\{a, m\}$-bounded order.*

It is well known that the finiteness of the non-abelian tensor square $[G, G^\varphi]$ does not imply the finiteness of the group G (see Remark 2.7(b), below). In [18], Parvizi and Niroomand prove that if G is a finitely generated subgroup and the non-abelian tensor square $[G, G^\varphi]$ is finite, then G is finite. We obtain equivalence conditions (see the following theorem) and a topological related result (see Corollary 3.3).

Theorem C. *Let G be a finitely generated group. The following properties are equivalents.*

   (a) *The group G is finite;*
   (b) *The set of tensors $\mathrm{T}_\otimes(G)$ is finite;*

(c) *The non-abelian tensor $[G, G^\varphi]$ is finite;*
(d) *The derived subgroup G' is locally finite and the kernel $J_2(G) \cong \pi_3(SK(G,1))$ is periodic;*
(e) *The derived subgroup G' is locally finite and the Diagonal subgroup $\Delta(G)$ is periodic;*
(f) *The derived subgroup G' is locally finite and the subgroup $\tilde{\Delta}(G) = \langle [g, h^\varphi][h, g^\varphi] \mid g, h \in G \rangle$ is periodic;*
(g) *The non-abelian tensor square $[G, G^\varphi]$ is locally finite.*

In algebraic topology, non-abelian tensor product arises from a homotopy pushout, see [8, 10]. The homotopy pushout (or homotopy amalgameted sum) is well known for the application in the classical Seifert-van Kampen theorem as well as Higher Homotopy Seifert-van Kampen theorem in the case of a covering by two open sets. Our contribution is to apply the idea of Theorem A and related construction on tensor product in order to obtain finiteness results related to homotopy pushout. For instance, see Proposition 4.2 for an application to homotopy pushout of Eilenberg-Maclane spaces.

The paper is organized as follows. In the next section we describe finiteness criteria for the group $\pi_n(X)$ in terms of the number of tensors. In particular, we establish a quantitative version of the classical Blakers-Massey triad connectivity theorem. In the third section we examine some finite necessary conditions for the group G in terms of certain torsion elements of the non-abelian tensor square $[G, G^\varphi]$. In the final section, as an application we obtain finiteness criteria for the homotopy pushout that depends on the number of tensor of non-abelian tensor product of groups.

2. Finiteness Conditions

Let (X, A, B) be a triad, that is, A and B are subspaces of X, containing the base-point in $C = A \cap B$, such that the triad homotopy group $\pi_n(X, A, B)$ for $n \geq 3$ fit into a long exact sequence

$$\cdots \to \pi_n(B, C) \to \pi_n(X, A) \to \pi_n(X, A, B) \to \pi_{n-1}(B, C) \to \cdots .$$

Let X be a pointed space and $\{A, B\}$ an open cover of X such that A, B and $C = A \cap B$ are connected and $(A, C), (B, C)$ are 1-connected, see [22].

By using the relative homotopy long exact sequences and the third triad homotopy group, Ellis and McDermott obtain interesting bound to the order of $\pi_3(X)$ (cf. [14, Proposition 5]). We have obtained (as a consequence of Theorem A) a more general version. In contrast to

their bound, we do not require that $\pi_2(A, C)$ and $\pi_2(B, C)$ be finite groups and also there is no need to estimate $\pi_2(X, C)$.

Following the same setting, we apply the extended Blakers-Massey triad connectivity [9, Theorem 4.2]

$$\pi_{p+q+1}(X, A, B) \cong \pi_{p+1}(A, A \cap B) \otimes \pi_{q+1}(B, A \cap B),$$

in order to present finiteness condition and bound to $\pi_n(X)$ in terms of set of tensors $T_\otimes(\pi_{p+1}(A, C), \pi_{q+1}(B, C))$, where $n = p + q + 1$.

Proof of Theorem A. Consider the relative homotopy long exact sequences, as in [22],

$$\pi_n(B) \to \pi_n(B, C) \to \pi_{n-1}(C)$$
$$\pi_n(A) \to \pi_n(X) \to \pi_n(X, A)$$
$$\pi_n(B, C) \to \pi_n(X, A) \to \pi_n(X, A, B).$$

By the first exact sequence, we deduce that $\pi_n(B, C)$ is a finite group with $\{b, c\}$-bounded order. According to Brown-Loday's result [9, Theorem 4.2], the group $\pi_n(X, A, B)$ is isomorphic to the non-abelian tensor product $M \otimes N$, where $M = \pi_{p+1}(A, C)$ and $N = \pi_{q+1}(B, C)$. As $|T_\otimes(\pi_{p+1}(A, C), \pi_{q+1}(B, C))| = m$, we have $\pi_n(X, A, B)$ is finite with m-bounded order ([3, Theorem B]). From this we conclude that the group $\pi_n(X, A)$ is finite with $\{b, m\}$-bounded order. In the same manner we can see that $\pi_n(X)$ is finite with $\{a, b, c, m\}$-bounded order. The proof is complete. $\qquad\square$

REMARK 2.1. *A direct application of Theorem A is that* $\pi_3(K(G, 2) \vee K(H, 2))$ *is a finite group with m-bounded order, when* $|T_\otimes(G, H)| = m$. *In particular,* $\pi_3(K(C_{r^\infty}, 2) \vee K(C_{s^\infty}, 2))$ *is trivial, where s and r are primes and* $K(G, n)$ *is Eilenberg-MacLane space (a topological space having just one nontrivial homotopy group* $\pi_n(K(G, n)) \cong G$).

By using the same idea of the previous remark we have the following corollary.

COROLLARY 2.2. *Let A and B be p-connected and q-connected locally contractible spaces, respectively. Suppose that* $|\pi_n(A)| = a$, $|\pi_n(B)| = b$ *and* $|T_\otimes(\pi_{p+1}(A), \pi_{q+1}(B))| = m$, *where* $n = p+q+1$. *Then* $\pi_n(A \vee B)$ *is finite with* $\{a, b, m\}$-*bounded order.*

The previous corollary is an example that we can use the non-abelian tensor product to overcome the failure of the excision property, which reflects in $\pi_n(A \vee B)$ being different of $\pi_n(A) \oplus \pi_n(B)$ in general, where $n \geq 2$.

The following result provide a finiteness criterion to the group G in terms of the number of tensors in the non-abelian tensor square $[G, G^\varphi]$.

COROLLARY 2.3. *Let X be connected space and $\pi_1(X) = G$. Suppose that the first homology group of X, $H_1(X, \mathbb{Z})$, is finitely generated and the set of tensors $\mathrm{T}_\otimes(G) \subseteq \nu(G)$ has exactly m tensors. Then $H_1(X, \mathbb{Z})$ and $\pi_1(X) = G$ are finite with m-bounded orders.*

Proof. Recall that X is a connected space and $\pi_1(X) = G$. Suppose that the first homology of X, $H_1(X, \mathbb{Z})$, is finitely generated and the set of tensors $\mathrm{T}_\otimes(G) \subseteq \nu(G)$ has exactly m tensors. We need to prove that $H_1(X, \mathbb{Z})$ and $\pi_1(X) = G$ are finite with m-bounded order.

By Theorem A, the non-abelian tensor square $[G, G^\varphi]$ is finite with m-bounded order. Consequently, the derived subgroup G' is finite with m-bounded order. Since G^{ab} is finitely generated, it follows that the abelianization G^{ab} is isomorphic to an subgroup of the non-abelian tensor square $[G, G^\varphi]$ ([2, Theorem C (a)]). Therefore the abelianization G^{ab} is finite with m-bounded order. Consequently, G is finite with m-bounded order. The proof is complete. $\square$

In the case of the suspension triad $(SX; C_+X, C_-X)$, see [8, 22], we can obtained finiteness criteria and bounds to the order of $\pi_3(SX)$, where C_-X and C_+X are the two cones of X in SX.

Proof of Theorem B. Consider the long exact sequence

$$\cdots \to \pi_2(X) \to \pi_3(SX) \to \pi_2(\Omega SX, X) \to \cdots$$

where ΩSX is the space of loops in SX, maps from the circle S^1 to SX, equipped with the compact–open topology. The result follows by applying Theorem A on $[G, G^\varphi] \cong \pi_2(\Omega SX, X)$. $\square$

REMARK 2.4. *In the above result, it is worth noting that $\pi_2(X)$ does not need to be trivial, therefore the result is more general compared to the bounds for $\pi_3(SK(G, 1))$ when $X = K(G, 1)$. However, in [8], proved that if $\pi_1(X) = G$ and $\pi_2(X)$ is trivial, then $\pi_3(SX) \cong J_2(G) = \ker(\kappa)$, where $\kappa : [G, G^\varphi] \to G'$, given by $[g, h^\varphi] \mapsto [g, h]$.*

Combining the above bounds to the order of the non-abelian tensor square and [8, Proposition 4.10] we can obtained, under appropriate conditions in the set of tensors $\mathrm{T}_\otimes(G)$, some finiteness criteria and bounds to the orders of $\pi_3(SK(G, 1))$ and $\pi_2^S(K(G, 1))$ the second stable homotopy of Eilenberg-MacLane space.

COROLLARY 2.5. *Let G be a group. Suppose that the set $\mathrm{T}_\otimes(G)$ has exactly m tensors in $\nu(G)$. Then*

(a) *The second stable homotopy group $\pi_2^S(K(G, 1))$ is finite with m-bounded order;*
(b) *$\pi_3(SK(G, 1))$ is finite with m-bounded order.*

A particular case of the previous corollary is when G is a Prüfer group C_{p^∞} so $\pi_2^S(K(G,1))$ and $\pi_3(SK(G,1))$ are trivial groups.

Next we give a sufficient and necessary conditions to the suspension of Eilenberg-Maclane space $K(G,1)$ be a finite group.

PROPOSITION 2.6. *Let G be a BFC-group such that G^{ab} is finitely generated. Then, $\pi_3(SK(G,1))$ is finite if and only if $\pi_1(K(G,1))$ is finite.*

Proof. Assume that $J_2(G) \cong \pi_3(SK(G,1))$ is finite. We have the following short exact sequence

$$0 \to J_2(G) \to [G, G^\varphi] \to G' \to 1,$$

since G' is finite by Neumann's Theorem [19, 14.5.11], we have that the non-abelian tensor square $[G, G^\varphi]$ is finite. Since G^{ab} is finitely generated, it follows that the abelianization G^{ab} is isomorphic to a subgroup of the non-abelian tensor square $[G, G^\varphi]$ ([2, Theorem C (a)]). Moreover, we can deduce that the abelianization G^{ab} is a subgroup of the diagonal subgroup $\Delta(G)$. From this we deduce that the abelianization G^{ab} is finite and so, G is finite.

Conversely, suppose that $\pi_1(K(G,1))$ is finite. Consequently, the non-abelian tensor square $[G, G^\varphi]$ is finite and so, $\pi_3(SK(G,1))$ is finite. The proof is complete. $\square$

REMARK 2.7. *(a) Assume that G is a BFC-group and the abelianization G^{ab} is finitely generated. Consider the following short exact sequence (cf. [21, Section 2]),*

$$1 \to \Delta(G) \to J_2(G) \to H_2(G) \to 1,$$

where $J_2(G)$ is isomorphic to $\pi_3(SK(G,1))$ and $H_2(G)$ is the second homology group. Recall that the Diagonal subgroup $\Delta(G)$ is given by $\Delta(G) = \langle [g, g^\varphi] \mid g \in G \rangle$. The subgroup $\Delta(G)$ is finite if and only if $\pi_1(K(G,1))$ is finite (if and only if the group $\pi_3(SK(G,1))$ is finite). See [8, Section 2] for more details.

(b) Note that the hypothesis of the finitely generated abelianization is really needed. For instance, the Prufer group $G = C_{p^\infty}$ is an infinite group such that the non-abelian tensor square $[G, G^\varphi]$ is trivial and so, finite. In particular, $J_2(G) \cong \pi_3(SK(G,1))$ is also trivial.

A direct application of Corollary 2.3 and Corollary 2.5 for the suspension of Eilenberg-Maclane space $K(G,1)$ and the second stable homotopy of $K(G,1)$ is the following finiteness condition.

COROLLARY 2.8. *Let G be a group. Then, $\pi_3(SK(G,1))$ and G' are finite if and only if $T_\otimes(G)$ is finite. Moreover, suppose that G is perfect then, $\pi_2^S(K(G,1))$ and G are finite if and only if $T_\otimes(G)$ is finite.*

3. Torsion elements in the non-abelian tensor square

This section is will be devoted to obtain some finiteness conditions for the group G in terms of the torsion elements in the non-abelian tensor square. Specifically, our proofs involve looking at the description of the diagonal subgroup $\Delta(G) \leqslant [G, G^\varphi]$. Such a description has previously been used by the authors [2, 21].

It is well known that the finiteness of the non-abelian tensor square $[G, G^\varphi]$ does not imply the finiteness of the group G (see Remark 2.7, above). In [18], Parvizi and Niroomand prove that if G is a finitely generated subgroup and the non-abelian tensor square $[G, G^\varphi]$ is finite, then G is finite. Later, in [2], the authors prove that if G is a finitely generated locally graded group and the exponent of the non-abelian tensor square $\exp([G, G^\varphi])$ is finite, then G is finite. The next result can be viewed as a generalization of the above result.

Theorem C. *Let G be a finitely generated group. The following properties are equivalents.*
  (a) *The group G is finite;*
  (b) *The set of tensors $\mathrm{T}_\otimes(G)$ is finite;*
  (c) *The non-abelian tensor $[G, G^\varphi]$ is finite;*
  (d) *The derived subgroup G' is locally finite and the kernel $J_2(G) \cong \pi_3(SK(G, 1))$ is periodic;*
  (e) *The derived subgroup G' is locally finite and the Diagonal subgroup $\Delta(G)$ is periodic;*
  (f) *The derived subgroup G' is locally finite and the subgroup $\tilde{\Delta}(G)$ is periodic;*
  (g) *The non-abelian tensor square $[G, G^\varphi]$ is locally finite.*

Proof. $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ are directly.

$(b) \Rightarrow (c)$. Suppose that $\mathrm{T}_\otimes(G)$. By [2, Theorem A], the non-abelian tensor square $[G, G^\varphi]$ is finite.

$(f) \Rightarrow (g)$. By Schmidt's Theorem [19, 14.3.1], it suffices to prove that the abelianization G^{ab} is finite.
Since G^{ab} is finitely generated, we deduce that

$$G^{ab} = T \times F,$$

where T is the torsion part and F the free part of G^{ab} (cf. [19, 4.2.10]). If the abelianization G^{ab} is not periodic, then there exists an element of infinite order $x \in G$ such that $xG^{ab} \in F$. In particular, $[x, x^\varphi][x, x^\varphi] = [x, x^\varphi]^2$ is an infinite element in $\tilde{\Delta}(G)$. Consequently,

F is trivial and $G^{ab} = T$ is finite.

$(g) \Rightarrow (a)$. First we prove that the abelianization G^{ab} is finite. Arguing as in the above paragraph, we deduce that $G^{ab} = T \times F$, where T is the torsion part and F the free part of G^{ab}. From [21, Remark 5] we conclude that $\Delta(G^{ab})$ is isomorphic to

$$\Delta(T) \times \Delta(F) \times (T \otimes_{\mathbb{Z}} F),$$

where $T \otimes_{\mathbb{Z}} F$ is the usual tensor product of $\mathbb{Z}$-modules. In particular, the free part of $\Delta(G^{ab})$ is precisely $\Delta(F)$. Now, the canonical projection $G \twoheadrightarrow G^{ab}$ induces an epimorphism $q : \Delta(G) \twoheadrightarrow \Delta(G^{ab})$. Since $\Delta(G)$ is locally finite, it follows that $\Delta(G^{ab})$ is also locally finite. Consequently, F is trivial and thus G^{ab} is periodic and, consequently, finite.

It remains to prove that the derived subgroup G' is finite. Since G is finitely generated and G^{ab} is finite, it follows that the derived subgroup G' is finitely generated ([19, 1.6.11]). As G' is an homomorphic image of the non-abelian tensor square $[G, G^{\varphi}]$, we have G' is finite. From this we deduce that G is finite. The proof is complete. $\square$

PROPOSITION 3.1. *Let G be a polycyclic-by-finite group. Suppose that the non-abelian tensor $[G, G^{\varphi}]$ is periodic. Then G is finite.*

Proof. Since the derived subgroup G' is an epimorphic image of the non-abelian tensor square $[G, G^{\varphi}]$, it follows that G' is also periodic. In particular, we can deduce that G' is finite. Now, arguing as in the proof of Theorem 3, we deduce that G^{ab} is finite. The proof is complete. $\square$

It is well known that if G is a group with exponent $\exp(G) \in \{2, 3, 4, 6\}$, then G is locally finite (Levi-van der Waerden, Sanov, see [19, Section 14.2] for more details). We also examine the finiteness of the group G, when the non-abelian tensor square $[G, G^{\varphi}]$ has small exponent.

COROLLARY 3.2. *Let $n \in \{2, 3, 4, 6\}$ and G a finitely generated group. Assume that the exponent of the non-abelian tensor square $\exp([G, G^{\varphi}])$ is exactly n. Then G is finite.*

Proof. It suffices to see that the non-abelian tensor square $[G, G^{\varphi}]$ is locally finite (see [19, Section 14.2] for more details). By Theorem 3, the group G is finite. The proof is complete. $\square$

The following corollary is a topological version of Theorem C.

COROLLARY 3.3. *Let G be a finitely generated group and X be a topological space such that $\pi_1(X) = G$ and $\pi_2(X)$ is trivial. Then the following properties are equivalents.*

(a) *The group $\pi_1(X) = G$ is finite;*
(b) *The derived subgroup G' is locally finite and $\pi_3(SX)$ is periodic;*
(c) *The derived subgroup G' is locally finite and $F(\pi_3(SX)) \cong F(H_2(G))$, where $F(H)$ is the free part of a group H;*
(d) *The derived subgroup G' is locally finite and $F(\pi_3(SX)) \cong F(\pi_4(S^2X))$, where $F(H)$ is the free part of a group H.*

Proof. Since $\pi_2(X)$ is trivial, it follows that $J_2(G) \cong \pi_3(SX)$ (cf. [8, Proposition 3.3]). Consider the short exact sequences, as in [8],

$$1 \to J_2(G) \to [G, G^\varphi] \to G' \to 1,$$

$$1 \to \tilde{\Delta}(G) \to J_2(G) \to \pi_4(S^2X) \to 1.$$

Therefore, the result follows by applying Theorem C. ∎

4. APPLICATION TO HOMOTOPY PUSHOUT

We end this paper by proving some finiteness criteria for the homotopy pushout. Consider the following commutative square of spaces

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ \downarrow g & & \downarrow a \\ B & \xrightarrow{b} & X \end{array}$$

and denote $F(f), F(g)$ and $F(a)$ the homotopy fibre of f, g and a, respectively, and let $F(X)$ be the homotopy fibre of $F(g) \to F(a)$. The previous square is called a *homotopy pushout* when the canonical map of squares from the double mapping cylinder $M(f,g)$ to X is a weak equivalence of spaces at the four corners, for more details see [8]. For the homotopy pushout, we have an analogous of Theorem A.

PROPOSITION 4.1. *Let the following square of spaces be a homotopy pushout*

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ \downarrow g & & \downarrow a \\ B & \xrightarrow{b} & X \end{array}$$

Suppose that $\pi_3(A)$, $\pi_2(F(g))$ and the set of tensors $T_\otimes(G,H)$ are finite, where $G = \pi_1(F(f))$ and $H = \pi_1(F(g))$. Then $\pi_3(X)$ is a finite group with $\{n_a, n_b, m\}$-bounded order, where $|\pi_3(A)| = n_a$, $|\pi_2F(g)| = n_b$ and $|T_\otimes(G,H)| = m$.

Proof. By [8, Theorem 3.1] we have that $\pi_1(F(X)) \simeq \pi_1(F(f)) \otimes \pi_1(F(g))$. Since $|T_\otimes(\pi_1(F(f)), \pi_1(F(g)))| = m$ then $\pi_1(F(X))$ is finite with m-bounded order ([3, Theorem B]). By using $|\pi_3(A)| = n_a$, $|\pi_2 F(g)| = n_b$ and the long exact sequences of the fibrations

$$F(X) \to F(g) \to F(a)$$

$$F(a) \to A \to X,$$

it follows that $\pi_3(X)$ is a finite group with $\{n_a, n_b, m\}$-bounded.   □

Many authors had studied some finiteness conditions for the non-abelian tensor product of groups (cf. [2, 4, 11, 12, 15, 18]). For instance, in [15], Moravec proved that if G, H are locally finite groups acting compatibly on each other, then so is $G \otimes H$. In [11], Donadze, Ladra and Thomas proved interesting finiteness criteria for non-abelian tensor product in terms of the involved groups. In [2, 3], the authors prove a finiteness criterion for the non-abelian tensor product of groups in terms of the number of tensors. In the case of homotopy pushout for Eilenberg-Maclane spaces is possible to obtain results direct from the study of the non-abelian tensor product of groups. We obtain the following related result.

PROPOSITION 4.2. *Let M, N be normal subgroups of a group G, and form the homotopy pushout*

$$
\begin{array}{ccc}
K(G,1) & \longrightarrow & K(G/N,1) \\
\downarrow & & \downarrow \\
K(G/M,1) & \longrightarrow & X
\end{array}
$$

(a) *Suppose that the subgroups M and N are locally finite. Then $\pi_2(X)$ and $\pi_3(X)$ are locally finite.*
(b) *Suppose that M is a non-abelian free group of finite rank and N is a finite group. Then $\pi_2(X)$ and $\pi_3(X)$ are finite.*
(c) *Suppose that the set of tensors $T_\otimes(M, N) \subseteq \eta(M, N)$ is finite. Then $\pi_3(X)$ is a finite group with m-bounded order, where $|T_\otimes(M, N)| = m$.*

Proof. According to Brown and Loday's result [8, Corollary 3.4], the group $\pi_3(X)$ is isomorphic to a subgroup of the non-abelian tensor product $[M, N^\varphi]$ and $\pi_2(X)$ is isomorphic to $(M \cap N)/[M, N]$. In particular, the group $\pi_2(X)$ is an homomorphic image of $M \cap N$.

(a). As $M \cap N$ is locally finite we have $\pi_2(X)$ is locally finite. Now, since M and N are locally finite, it follows that the non-abelian tensor

product $[M, N^\varphi]$ is locally finite (Moravec, [15]). Consequently, the group $\pi_3(X)$ is locally finite.

(b). Since M is finite and N is a non-abelian free group, we deduce that $\pi_2(X)$ is finite. According to Donadze, Ladra and Thomas' result [11, Corollary 4.7], we conclude that the non-abelian tensor product $[M, N^\varphi]$ is finite and so, $\pi_3(X)$ is finite.

(c). According to Theorem A, the non-abelian tensor product $[M, N^\varphi]$ is finite with m-bounded order. In particular, the group $\pi_3(X)$ is finite with m-bounded order. The proof is complete. $\qquad\square$

REMARK 4.3. *Note that Proposition 4.2 (c) in a certain sense cannot be improved. For instance, if $M = N = C_{p^\infty}$, then the group $\pi_2(X) \cong C_{p^\infty}$ is infinite and $\pi_3(X)$ is trivial.*

As an interesting consequence of Proposition 4.2, we obtain that: if $G = MN$ such that $M \cap N$ and $[M, N^\varphi]$ are trivial, then X is 3-connected, i.e., $\pi_n(X)$ is trivial for $n = 1, 2$ and 3, see the following example.

EXAMPLE 4.4. *When $G = C_{r^\infty} \times C_{s^\infty}$ in Proposition 4.2 with r and s primes (not necessarily distinct), we have that X is 3-connected. In fact, $\pi_1(X)$ is trivial by Van Kampen theorem, as well $\pi_2(X) = 0$ as a consequence of the Van Kampen theorem for maps, and finally the triviality of $\pi_3(X)$ follows from the proposition above.*

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