

COMPARISON OF THE BERGMAN KERNEL AND THE CARATHÉODORY–EISENMAN VOLUME

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ABSTRACT. It is proved that for any domain in \mathbb{C}^n the Carathéodory–Eisenman volume is comparable with the volume of the indicatrix of the Carathéodory metric up to small/large constants depending only on n . Then the “multidimensional Suita conjecture” theorem of Blocki and Zwonek implies a comparable relationship between these volumes and the Bergman kernel.

In recent years, the interest in holomorphically invariant objects has grown from quantities stemming from maps to or from the one-dimensional disc to quantities related to the n -dimensional ball. The main focus of interest has been the squeezing function, which measures how big the one-to-one image of a domain can be while remaining inside the unit ball (and sending a base point to the origin of the ball).

The Carathéodory–Eisenman “volume” is a variant on that idea, at the infinitesimal level. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Given D be a domain in \mathbb{C}^n , and $z \in D$,

$$CE_D(z) = \sup\{|\det F'(z)|^2 : F \in \mathcal{O}(D, \mathbb{D}^n)\}.$$

We are using the polydisc \mathbb{D}^n for technical reasons. Replacing it by the unit ball in \mathbb{C}^n , we get the same function up to small/large constants independent of D .

Unfortunately, the lack of a higher-dimensional analogue to the Koebe quarter theorem prevents us from relating our results to the squeezing function, but the behaviour of $CE_D(z)$ can be related to some basic geometric objects associated to the domain. We need more definitions.

Definition 1. Let D be a domain in \mathbb{C}^n , $z, w \in D$, and $X \in \mathbb{C}^n$. The pluricomplex Green function g_D , the Azukawa metric A_D and the Carathéodory metric C_D are defined in the following way:

$$g_D(z, w) = \sup\{u(w) : u \in PSH(D), u < 0, u(\zeta) < \log \|\zeta - z\| + C\},$$

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$$A_D(z; X) = \limsup_{\lambda \rightarrow 0} \frac{\exp(g_D(z, z + \lambda X))}{|\lambda|},$$

$$C_D(z; X) = \sup\{|f'(z)X| : f \in \mathcal{O}(D, \mathbb{D})\}.$$

Let $L_h^2(D)$ be the Bergman space of D , i.e. the Hilbert space of all square-integrable holomorphic functions f on D . Let K_D be the restriction to the diagonal of the Bergman kernel of D . Recall that

$$K_D(z) = \sup\{|f(z)|^2 : f \in L_h^2(D), \|f\|_{L^2(D)} \leq 1\}.$$

Denote by $\delta_D(z; X)$ the distance from z to ∂D along the vector X :

$$\delta_D(z; X) := \sup\{r > 0 : z + \lambda X \in D \text{ if } |\lambda| < r\}.$$

Observe that $\delta_D^{-1}(z; X)$ is 1-homogeneous in X , that is,

$$\delta_D^{-1}(z; \mu X) = |\mu| \delta_D^{-1}(z; X), \quad \mu \in \mathbb{C},$$

so it is this quantity that we want to compare to the various infinitesimal metrics that occur in complex analysis.

Let $I_D(z)$ be the indicatrix of $\delta_D^{-1}(z; \cdot)$, that is,

$$I_D(z) = \{X \in \mathbb{C}^n : \delta_D^{-1}(z; X) < 1\}.$$

Then $z + I_D(z)$ is the maximal balanced subdomain of D centered at z .

Set IA_D and IC_D to be the indicatrices of A_D and C_D . Note that $IC_D(z)$ is a convex set.

Definition 2. Denote V_D , VA_D and VC_D the Euclidean volumes of I_D , IA_D and IC_D , respectively.

Since $C_D \leq A_D \leq \delta_D^{-1}$, then $I_D \subset IA_D \subset IC_D$ and hence

$$(1) \quad V_D \leq VA_D \leq VC_D.$$

On the other hand, if G is a balanced domain, then $(1, f)_{L_2(G)} = 0$ for any $f \in L_2(G)$ with $f(0) = 0$. Therefore $K_G(0) = (\text{Vol } G)^{-1}$. In particular, applying this to $G = z + I_D(z) \subset D$,

$$K_D(z) \leq (V_D(z))^{-1}.$$

The following opposite inequality is called the multidimensional Suita conjecture (see [1, Theorem 7.5] and [2, Theorem 2]).

Theorem 3. If D is a pseudoconvex domain in \mathbb{C}^n , then

$$K_D \geq (VA_D)^{-1}.$$

Corollary 4. If D is a domain in \mathbb{C}^n and $A_D \geq c\delta_D^{-1}$ for some $c > 0$, then

$$c^{-2n}(VA_D)^{-1} \geq (V_D)^{-1} \geq K_D \geq (VA_D)^{-1}.$$

Note that if D is \mathbb{C} -convex, resp. convex, then $A_D \geq C_D \geq c\delta_D^{-1}$ with $c = 1/4$, resp. $c = 1/2$ (see e.g. [5, Proposition 1], resp. the remark after this proposition). Thus, Corollary 4 applies to those cases and we reobtain $K_D \leq c^{-2n}(VA_D)^{-1}$ as in [2, Theorem 5].

The aim of this note is to prove a version of Theorem 3 for CE_D , comparing it to the volume of the Carathéodory indicatrix (see Definition 2).

Theorem 5. *Let D be a domain in \mathbb{C}^n . There are constants $C_n > c_n > 0$ depending only on n such that*

$$C_n(VC_D)^{-1} \geq CE_D \geq c_n(VC_D)^{-1}.$$

In particular, if D is pseudoconvex, then by Theorem 3 and (1),

$$C_n K_D \geq CE_D.$$

Corollary 6. *If D is domain in \mathbb{C}^n and $C_D \geq c\delta_D^{-1}$ for some $c > 0$, then*

$$C_n K_D \geq CE_D \geq c^{2n} c_n K_D.$$

Proof of Corollaries 4 and 6. We only have to show that under the assumption $A_D \geq c\delta_D^{-1}$ (resp. $C_D \geq c\delta_D^{-1}$), D is pseudoconvex. Suppose it is not. By [3, Theorem 4.1.25], after an affine change of coordinates, we may suppose that $0 \in \partial D$ and

$$D \supset D_{c,r} := \{z \in r\mathbb{D}^2 : \operatorname{Re}z_1 + (\operatorname{Re}z_2)^2 < c(\operatorname{Im}z_2)^2\}, \quad c > 1, r > 0.$$

Recall that the Kobayashi-Royden metric of a domain G in \mathbb{C}^n is given by

$$\kappa_G(z; X) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, G) : \varphi(0) = z, \alpha\varphi'(0) = X\}.$$

It follows by the proof of [4, Theorem 1.1] that

$$\limsup_{\delta \rightarrow 0^+} \delta^{3/4} \kappa_{D_{c,r}}((-\delta, 0); e_1) < \infty.$$

Then the inequalities $C_D \leq A_D \leq \kappa_D \leq \kappa_{D_{c,r}}$ lead to a contradiction.

Proof of Theorem 5. Let $a \in D$. Note that $D_a := IC_D(a)$ is a convex balanced domain centered at a , and hence

$$(2) \quad C_D(a; X) = C_{D_a}(a; X) = \delta_{D_a}^{-1}(a; X).$$

The proof of [5, Proposition 14] rests on the construction, in a \mathbb{C} -convex domain D , of an orthonormal basis e_1, \dots, e_n of \mathbb{C}^n such that for $1 \leq j \leq n$,

$$\delta_D(a; e_j) = \operatorname{dist}(a, a + \operatorname{Span}(e_k, k \geq j) \setminus D).$$

Since D_a is convex, using (2), we deduce from [5, (4)] that one may find a constant $k_n > 0$ depending only on n such that

$$(3) \quad k_n \sum_{j=1}^n |X_j| r_j(a) \leq C_D(a; X) \leq \sum_{j=1}^n |X_j| r_j(a),$$

where the X_j are the coordinates of X in the basis e_1, \dots, e_n and $r_j(a) = C_D(a; e_j)$.

Let $\Pi_D(a) = \prod_{j=1}^n r_j(a)$.

Lemma 7. *There exists a map $F = (f_1, \dots, f_n) \in \mathcal{O}(D, \mathbb{D})$ such that*

$$|\det F'(a)| \geq (k_n)^n \Pi_D(a).$$

Proof. Let $f_1 \in \mathcal{O}(D, \mathbb{D})$ be extremal for the Carathéodory metric in the e_1 direction, thus $\left| \frac{\partial f_1}{\partial z_1}(a) \right| = |f_1'(a)e_1| = r_1(a)$. We proceed recursively: suppose we already have chosen $f_i \in \mathcal{O}(D, \mathbb{D})$, $1 \leq i \leq m$, such that

$$\left| \det \left(\frac{\partial f_i}{\partial z_j}(a) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \right| \geq (k_n)^m \prod_{j=1}^m r_j(a).$$

Then define $V \in \mathbb{C}^{m+1}$ to be the vector of cofactors

$$V_j := (-1)^{m+1+j} \det \left(\frac{\partial f_i}{\partial z_l}(a) \right)_{\substack{1 \leq i \leq m \\ 1 \leq l \leq m+1, l \neq j}}, \quad 1 \leq j \leq m+1.$$

Choose $f_{m+1} \in \mathcal{O}(D, \mathbb{D})$ to be extremal for the Carathéodory metric in the V direction, so that $|f_{m+1}'(a)V| = C_D(a; V)$. By our choice of V ,

$$\det \left(\frac{\partial f_i}{\partial z_j}(a) \right)_{\substack{1 \leq i \leq m+1 \\ 1 \leq j \leq m+1}} = \sum_{j=1}^{m+1} V_j \frac{\partial f_{m+1}}{\partial z_j}(a) = f_{m+1}'(a)V.$$

By (3) and the recursion assumption,

$$\begin{aligned} C_D(a; V) &\geq k_n \sum_{j=1}^{m+1} |V_j| r_j(a) \geq k_n |V_{m+1}| r_{m+1}(a) \\ &= k_n r_{m+1}(a) \left| \det \left(\frac{\partial f_i}{\partial z_j}(a) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \right| \geq (k_n)^{m+1} \prod_{j=1}^{m+1} r_j(a). \end{aligned}$$

□

On the other hand, from the definition of the Carathéodory metric, for any map $F \in \mathcal{O}(D, \mathbb{D}^n)$ one has that

$$|\det F'(a)| \leq n! \Pi_D(a).$$

It follows that

$$(4) \quad (k_n)^{2n} \leq C E_D(a) \cdot (\Pi_D(a))^{-2} \leq (n!)^2.$$

Now we compare $\Pi_D(a)^{-2}$ with $V C_D(a)$. Define the diamond domain

$$E_a := \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n r_j(a) |z_j - a_j| < 1 \right\},$$

the inequalities (3) imply that $k_n D_a \subset E_a \subset D_a$, and so

$$(5) \quad k_n^{2n} VC_D(a) \leq \text{Vol}(E_a) = \frac{(2\pi)^n}{(2n)!(\Pi_D(a))^2} \leq VC_D(a).$$

Combining (4) and (5), we get that

$$(k_n)^{2n} \leq \frac{(2n)!}{(2\pi)^n} CE_D(a) VC_D(a) \leq \frac{(n!)^2}{(k_n)^{2n}}.$$

□

Remark. Let $P_D(a) = 1/\min \Pi_D(a)$, where the minimum is taken over all orthonormal bases of \mathbb{C}^n . Denote by $V_D^i(a)$ and $V_D^e(a)$ the maximal volume of a polydisc Δ_a centered at a such that $\Delta_a \subset D_a$, respectively the minimal volume of a polydisc Δ_a centered at a such that $\Delta_a \supset D_a$. It follows from the proof above that the functions V_D^i, V_D^e, VE_D, P_D and $(CE_D)^{-1}$ are equal up to small/large constants depending only on n .

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