AN ELEMENTARY APPROACH TO THE DIMENSION OF MEASURES SATISFYING A FIRST-ORDER LINEAR PDE CONSTRAINT

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ABSTRACT. We give a simple criterion on the set of probability tangent measures $\mathrm{Tan}(\mu, x)$ of a positive Radon measure μ, which yields lower bounds on the Hausdorff dimension of μ. As an application, we give an *elementary* and purely algebraic proof of the sharp Hausdorff dimension lower bounds for first-order linear PDE-constrained measures; bounds for closed (measure) differential forms and normal currents are further discussed. A *weak* structure theorem in the spirit of [Ann. Math. 184(3) (2016), pp. 1017–1039] is also discussed for such measures.

1. INTRODUCTION

The question of determining the dimension of a vector-valued Radon measure satisfying a PDE-constraint is a longstanding one. A good starting point are curl-free measure fields. The seminal work of DE GIORGI [7] on the structure of *sets of finite perimeter* and the *co-area formula* [11] from FLEMING & RISHEL yield the estimate $|Du| \ll \mathcal{H}^{d-1}$ for all distributional gradients Du represented by a Radon measure. Later on, FEDERER extended (see [10, Sec. 4.1.21]) this result to the estimate $\|T\| \ll \mathcal{I}^m \ll \mathcal{H}^m$ for m-dimensional normal currents $T \in \mathbf{N}_m(\mathbb{R}^d)$.[1] Recently, these results have been further extended to deal with more general differential constraints (in the context of $\mathcal{A}$-free measures). Namely, in [6] it is shown that $|\mu| \ll \mathcal{I}^{\ell_{\mathbb{P}^k}} \ll \mathcal{H}^{\ell_{\mathbb{P}^k}}$ for measures μ satisfying a generic constraint of the form $P(D)\mu = 0$, where $P(D)$ is a kth-order linear partial differential operator with constant coefficients and $\ell_{\mathbb{P}^k}$ is a positive integer depending only on the principal symbol $\mathbb{P}^k$ of $P(D)$. This $\ell_{\mathbb{P}^k}$ dimensional estimate turns out to be sharp for first-order operators; for higher-order operators it is an open question whether it remains an optimal bound (see [6, Conjecture 1.6]).

The compendium of results mentioned above are of stronger structural character than the ones presented on this note, since only bounds on the *Hausdorff dimension* of such measures will be discussed here. However, they also require a significantly stronger machinery. Our main interest is to give a self-contained and "elementary" proof of the Hausdorff dimension (sharp)

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[1]Here, $\mathcal{I}^m$ is the m-integral-geometric measure on $\mathbb{R}^d$.

bounds for measures $\mu \in \mathcal{M}(\Omega, E)$ solving, in the sense of the distributions, an equation of the form

$$P(D)\mu := \sum_{i=1}^{d} P_i[\partial_i \mu] + P_0\mu = 0\,, \quad P_0, P_i \in F \otimes E\,, \tag{1}$$

where E, F are finite dimensional euclidean spaces.

The angular stone of our proof rests on a rather simple *invariance criterion* affecting all normalized blow-ups of a given positive Radon measure σ, which effortlessly yields a lower bound on the Hausdorff dimension $\dim_{\mathcal{H}}(\sigma)$, where as usual

$$\dim_{\mathcal{H}}(\sigma) := \sup\big\{\, 0 \le \kappa \le d : \sigma \ll \mathcal{H}^{\kappa} \,\big\}\,.$$

This criterion (contained in Lemma 9) links the *vector-space dimension*, of those directions with respect to which a blow-up of σ may be an invariant measure, to a lower bound of the Hausdorff dimension. In particular, this re-directs the study of dimensional estimates for measures satsifying (1), to the study of the structural rigidity of their sets $\mathrm{Tan}(|\mu|, x)$ of *probability tangent measures* (described in Sec. 3). (A similar method for establishing dimensional estimates has been considered in [4] by Ambrosio & Soner; see also [13] for the slightly more restrictive context of *tangent spaces* $T_\sigma(x) \subset \mathbb{R}^d$ introduced by Bouchitté, Buttazzo and Seppecher.)

The advantage of this viewpoint lies in the fact that the principal symbol

$$\xi \mapsto \mathbb{P}(\xi) := \sum_{i=1}^{d} \xi_i P_i, \quad \xi \in \mathbb{R}^d,$$

being linear as function of ξ, precisely characterizes those directions where tangent measures are invariant measures. Thus, allowing one to define a dimension associated to the principal part of the operator:

$$\ell_{\mathbb{P}} := \min_{e \in E \setminus \{0\}} \dim\big(\, \{\mathbb{P}[e] \equiv 0\}^{\perp} \,\big)\,. \tag{2}$$

Here, we have used the short-hand notation $\{\mathbb{P}[e] \equiv 0\} := \{\, \xi : \mathbb{P}(\xi)[e] = 0 \,\}$. Note that this definition of dimension agrees with the definition given in [6, eq. (1.6)]. It may be worth to mention that, in the context of cocancelling operators (introduced by Van Schaftingen [18] and further extended in [6]; see also [16, 17]), $P(D)$ is an $(\ell_{\mathbb{P}} - 1)$-cocancelling operator.

Our main result is contained in the following theorem:

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $P(D)$ be a first-order differential operator as in (1), and let $\mu \in \mathcal{M}(\Omega; E)$ be a solution of the equation*

$$P(D)\mu = 0 \quad \textit{in the sense of distributions on } \Omega.$$

Then,

$$\dim_{\mathcal{H}}(|\mu|) \ge \ell_{\mathbb{P}}.$$

Moreover, this estimate is sharp since the measure

$$\mu = e\,\mathcal{H}^{\ell_{\mathbb{P}}} \llcorner \{\mathbb{P}[e] \equiv 0\}^{\perp}$$

is a solution of (1) on $\mathbb{R}^d$, whenever $e \in E$ is any vector at which the minimum in (2) is attained.

Remark 2. The proof of Theorem 1 does not require, in any way, the *structure theorem for PDE-constrained measures* [9, Theorem 1.1].

At all points x where $[P(D) \circ \frac{d\mu}{d|\mu|}(x)]$ is elliptic, that is, precisely when the polar $\frac{d\mu}{d|\mu|}(x)$ does not belong to the *wave cone* set

$$\Lambda_{\mathbb{P}} := \bigcup_{\xi \in \mathbb{R}^d} \ker \mathbb{P}(\xi) \subset E,$$

the sets $\mathrm{Tan}(|\mu|, x)$ turn out to be trivial (containing only fully-invariant measures). The invariance criterion then allows us to give the following *soft* version of [9, Theorem 1.1] (see also [1] in the case of gradients):

Corollary 3 (weak structure theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $P(D)$ be a first-order differential operator as in (1), and let $\mu \in \mathcal{M}(\Omega; E)$ be a solution of the equation*

$$P(D)\mu = 0 \quad \textit{in the sense of distributions on } \Omega.$$

Then,

$$|\mu| \llcorner \left\{ x \in \Omega : \tfrac{d\mu}{d|\mu|}(x) \notin \Lambda_{\mathbb{P}} \right\} \ll \mathcal{H}^{\kappa} \quad \textit{for all } 0 \leq \kappa < d.$$

Remark 4. The results contained in Theorem 1 and Corollary 3 apply to solutions of the inhomogeneous equation

$$P(D)\mu = \tau \in \mathcal{M}(\Omega; F).$$

To see this, let $\tilde{\mu} = (\mu, \tau)$, $\tilde{E} = E \times F$, and consider the operator $\tilde{P}(D)\tilde{\mu} = P(D)\mu - \tau$.

Further comments. Both Theorem 1 and Lemma 9 *do not* lead to rectifiability, nor estimates of the form $|\mu| \ll \mathcal{I}^{\ell_{\mathbb{P}}}$, or even $|\mu| \ll \mathcal{H}^{\ell_{\mathbb{P}}}$ by the methods presented on this note. This assertion is in line with the following observation. The shortcoming of Corollary 3 —with respect to the (strong) structure theorem— lies in the requirement of κ being strictly smaller than d. As it has been remarked by De Lellis (see [8, Proposition 3.3]), Preiss' example [15, Example 5.8(1)] of a purely singular measure with only trivial *tangent measures* hinders the hope for a traditional *blow-up strategy* leading to the estimate in the critical case $\kappa = d$.[2]

In a forthcoming paper [5], it will be shown that all functions $u : \Omega \to \mathbb{R}^d$ of *bounded deformation* satisfy the following rigidity property: every probability tangent measure $\tau \in \mathrm{Tan}(Eu, x)$ can be split as a sum of 1-directional measures (here, $Eu = \frac{1}{2}(Du + Du^{t}) \in \mathcal{M}(\Omega; \mathrm{sym}(\mathbb{R}^d \otimes \mathbb{R}^d))$ is the distributional symmetric gradient of u). Hence, by Lemma 9, one may recover the dimensional estimate $\dim_{\mathcal{H}}(|Eu|) \geq d - 1$ from [2] through a completely different method. Note however that symmetric gradients satisfy the St. Venant compatibility conditions (see [12, Example 3.10(e)]) which is a 2nd-order differential constraint.

[2] The definition of *tangent measure* introduced by Preiss in [15] is slightly different than our definition of probability tangent measure. However, the same triviality in the cited example can be inferred for our notion of tangent measure (see [14, Remark 14.4(1)]).

Organization. Applications of our results for several relevant first-order operators are discussed in Section 2; dimension bounds for closed differential forms and normal currents are discussed in Corollaries 6-8. A brief list of definitions (required for the proofs) and the invariance criterion (contained in Lemma 9) are given in Section 3. Section 4 is devoted to the proofs. Lastly, an appendix on multilinear algebra operations has been included, this may be of use for the applications on differential forms and normal currents discussed below.

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2. Applications

In this section we discuss explicit dimensional bounds for several relevant first-order differential operators.

Here and in what follows $\Omega \subset \mathbb{R}^d$ is an open set.

2.1. **Gradients.** The space $\mathrm{BV}(\Omega;\mathbb{R}^m)$ of functions of bounded variation consists of functions $u : \Omega \to \mathbb{R}^m$ whose distributional derivative Du can be represented by a Radon measure μ in $\mathcal{M}(\Omega;\mathbb{R}^m \otimes \mathbb{R}^d)$. We recall (see [12]) that the gradient $\mu = Du$ is (locally) a curl-free field in the sense that

$$\operatorname{curl}(\mu) := (\partial_i \mu_{kj} - \partial_j \mu_{ki})_{kij} = 0, \qquad 1 \le i,j \le d,\ 1 \le k \le m.$$

In the case $P(D) = \operatorname{curl}$ we have

$$\ker \mathbb{P}_{\mathrm{curl}}(\xi) = \left\{ a \otimes \xi : a \in \mathbb{R}^m \right\}, \quad \xi \in \mathbb{R}^d,$$

and therefore $\ell_{\mathrm{curl}} = d-1$. Theorem 1 then recovers the well-known (see [2]) dimensional bound for gradients

$$u \in \mathrm{BV}(\Omega;\mathbb{R}^m) \implies \dim_{\mathcal{H}}(|Du|) \ge d-1.$$

2.2. **Fields of bounded divergence.** Consider the divergence operator defined on matrix-fields $\boldsymbol{\mu} \in \mathcal{M}(\Omega;\mathbb{R}^k \otimes \mathbb{R}^d)$ defined as

$$\operatorname{div} \boldsymbol{\mu} = \left(\sum_{i=1}^d \partial_i \mu_{ij} \right)_j, \quad 1 \le j \le k. \tag{3}$$

In this case we get $\mathbb{P}_{\mathrm{div}}(\xi)[M] = M \cdot \xi$ over the space of tensors $M \in \mathbb{R}^k \otimes \mathbb{R}^d$, and $\{\mathbb{P}_{\mathrm{div}}[M] \equiv 0\}^\perp = (\ker M)^\perp \cong \operatorname{ran} M$. It follows from Riesz' representation theorem ($\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}|\boldsymbol{\mu}|}(x) \neq 0$ for $|\boldsymbol{\mu}|$-a.e. x) and Theorem 1 that

$$\operatorname{div} \boldsymbol{\mu} \in \mathcal{M}(\Omega;\mathbb{R}^k) \quad \implies \quad \dim_{\mathcal{H}}(|\boldsymbol{\mu}|) \ge 1.$$

In a further refinement, we get the following corollary:

Corollary 5. *Let $\boldsymbol{\mu} \in \mathcal{M}(\Omega;\mathbb{R}^k \otimes \mathbb{R}^d)$ satisfy the non-homogeneous equation $\operatorname{div} \boldsymbol{\mu} = \tau$ for some $\tau \in \mathcal{M}(\Omega;\mathbb{R}^k)$. Further, assume the set*

$$\left\{ x \in \Omega : \operatorname{rank} \frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}|\boldsymbol{\mu}^s|}(x) \ge \ell \right\}$$

has full $|\boldsymbol{\mu}^s|$-measure on Ω. Then, $\dim_{\mathcal{H}}(|\boldsymbol{\mu}|) \ge \ell$.

Proof. In this case $\dim(\{\mathbb{P}_{\mathrm{div}}[M] \equiv 0\}^\perp) = \operatorname{rank} M \geq \ell$. Then, by (4) and Lemma 9, one gets the desired bound $\dim_{\mathcal{H}}(|\boldsymbol{\mu}|) \geq \ell$. □

2.3. **Measure differential forms.** Let $m \in \{0, \dots, d-1\}$ and let $\omega \in \mathcal{M}(\Omega; \bigwedge^m \mathbb{R}^d)$ be a *measure m-form*. The *exterior derivative* of ω is the $(m+1)$-form distribution

$$\mathrm{d}\omega := \sum_{\substack{i=1,\dots,n \\ 1 \leq i_1 < \cdots < i_m \leq n}} \partial_i \omega_{i_1 \cdots i_m} [\, \mathrm{d}x_i \wedge \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \, \mathrm{d}x_{i_m}],$$

where the $\omega_{i_1 \cdots i_m} = \langle \omega, \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \, \mathrm{d}x_{i_m} \rangle \in \mathcal{M}(\Omega)$ are the coefficients of ω. The exterior derivative defines a first-order operator of the form (1) with $V = \bigwedge^m \mathbb{R}^d$ and $F = \bigwedge^{m+1} \mathbb{R}^d$, and a principal symbol $\mathbb{d}(\xi) : \bigwedge^m \mathbb{R}^d \to \bigwedge^{m+1} \mathbb{R}^d$ acting on m-co-vectors as

$$\mathbb{d}(\xi)[v^*] = \xi^* \wedge v^*.$$

Here, $w^* \in \bigwedge^m \mathbb{R}^d$ is the image of $w \in \bigwedge_m \mathbb{R}^d$ under the canonical isomorphism. By Lemma 11 in the Appendix, we get $\{\mathbb{d}[v^*] \equiv 0\} = \operatorname{Ann}^1(v^*)$ (see (5) in the Appendix) and therefore $\ell_{\mathbb{d}} = d - m$.

Corollary 6. *Let $\omega \in \mathcal{M}(\Omega; \bigwedge^m)$ be a measure m-form satisfying $\mathrm{d}\omega = \eta$ for some $\eta \in \mathcal{M}(\Omega; \bigwedge^{m+1} \mathbb{R}^d)$. Then, ω satisfies the dimensional estimate*

$$\dim_{\mathcal{H}}(|\omega|) \geq d - m.$$

2.4. **Normal currents.** Let $1 \leq m \leq d$ be an integer. The space of *m-currents* consists of all distributions $T \in \mathcal{D}'(\Omega; \bigwedge_m \mathbb{R}^d)$. In duality with the space of smooth differential forms and the exterior derivative, one defines the *boundary* of a current T as the $(m-1)$-current acting on $\mathrm{C}_c^\infty(\Omega; \bigwedge^{m-1} \mathbb{R}^d)$ as $\partial T[\omega] = T(\mathrm{d}\omega)$. The space $\mathbf{N}_m(\Omega)$ of *m-dimensional normal currents* is defined as the space of m-currents T, such that both T and ∂T can be represented by a measure, that is,

$$\mathbf{N}_m(\Omega) \cong \big\{ T \in \mathcal{M}(\Omega; \textstyle\bigwedge_m \mathbb{R}^d)) : \partial T \in \mathcal{M}(\Omega; \bigwedge_{m-1} \mathbb{R}^d) \big\}.$$

The total variation of a normal current T is denoted by $\|T\|$; and we write $T = \vec{T}\,\|T\|$ to denote its polar decomposition. The boundary operator on $\mathbf{N}_m(\Omega)$ defines a first-order operator of the form (1), with a principal symbol $\mathbb{d}^*(\xi) : \bigwedge_m \mathbb{R}^d \to \bigwedge_{m-1} \mathbb{R}^d$ acting on m-vectors as the interior multiplication

$$\mathbb{d}^*(\xi)[v] = v \llcorner \xi^* \quad \text{where } \langle v \llcorner \xi^*, z^* \rangle = \langle v, \xi^* \wedge z^* \rangle.$$

Using the notation contained in the appendix, we readily check that $\{\mathbb{d}^*[v] \equiv 0\} = \operatorname{Ann}_1(v)$. By means of Lemma 12 and definition (2), we conclude $\ell_{\mathbb{d}^*} = m$. Theorem 1 gives an alternative proof of the known dimensional estimates for normal currents:

Corollary 7. *Let $T = \vec{T}\|T\| \in \mathbf{N}_m(\Omega)$ be an m-dimensional normal current on Ω. Then, $\|T\|$ satisfies the dimensional estimate*

$$\dim_{\mathcal{H}}(\|T\|) \geq m.$$

Moreover, by the natural association between fields with bounded divergence and one-dimensional normal currents, Corollary 3 and Proposition 5 yield a simple proof of the following soft version of [9, Corollary 1.12]:

Corollary 8. *Let* $T_1 = \vec{T}_1\|T_1\|, \ldots, T_d = \vec{T}_d\|T_d\| \in \mathbf{N}_1(\Omega)$ *be one-dimensional normal currents and assume there exists a positive Radon measure* $\sigma \in \mathcal{M}(\Omega)$ *satisfying the following properties:*

(i) $\sigma \ll \|T_i\|$ *for all* $i = 1, \ldots, d$,
(ii) $\operatorname{span}\{\vec{T}_1(x), \ldots, \vec{T}_d(x)\} = \mathbb{R}^d$ *for* σ*-almost every* $x \in \mathbb{R}^d$.

Then, $\sigma \ll \mathcal{H}^\kappa$ *for all* $0 \le \kappa < d$.

3. Preliminaries

Let E be a finite dimensional euclidean space. We denote by $\mathcal{M}(\Omega; E) \cong \mathrm{C}_c(\Omega; E)^*$ the space of E-valued Radon measures over Ω. For a vector-valued measure $\mu \in \mathcal{M}(\Omega; E)$, we write the Radon–Nykodým–Lebesgue decomposition of μ as

$$\mu = \mu^{ac}\mathscr{L}^d \llcorner \Omega + g_\mu|\mu^s|, \qquad |g_\mu|_E = 1,$$

where $\mu^{ac} \in \mathrm{L}^1(\Omega; E)$, $|\mu^s| \perp \mathscr{L}^d \llcorner \Omega$, and $g_\mu \in \mathrm{L}^1(\Omega, |\mu^s|; E)$.

The map $T^{r,x}(y) = (y - x)/r$, which maps the open ball $B_r(x) \subset \mathbb{R}^d$ into the open unit ball $B_1 \subset \mathbb{R}^d$, induces a (isometry) push-forward $T^{r,x}_\# : \mathcal{M}(\mathbb{R}^d; E) \to \mathcal{M}(\mathbb{R}^d; E)$. A (normalized) sequence of the form

$$\gamma_j = \frac{1}{|\mu|(B_{r_j}(x))}\,(T^{r_j,x_0}_\# \mu)\llcorner B_1, \qquad r_j \downarrow 0, \quad j \in \mathbb{N},$$

is called a *bounded blow-up* sequence of μ at x_0. If $\tau = \text{w*-}\lim \gamma_j$ on $\mathcal{M}(\overline{B_1})$, we say that σ is a *probability tangent measure* of μ at x_0, symbolically we denote this by

$$\tau \in \mathrm{Tan}(\mu, x_0).$$

Observe that $|\tau|(\overline{B_1}) = 1$ and, at a $|\mu|$-Lebesgue point $x_0 \in \Omega$, it holds

$$\tau \in \mathrm{Tan}(\mu, x_0) \quad \iff \quad \tau = \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x_0)|\tau| \quad \text{and} \quad |\sigma| \in \mathrm{Tan}(|\mu|, x_0).$$

For this an other facts about $\mathrm{Tan}(\mu, x)$, we refer the interested reader to the monograph [3, Sec. 2.7].

For a finite dimensional euclidean vector space W, we write $\mathrm{Gr}(W)$ to denote the *Grassmanian* of all linear subspaces of W, and $\mathrm{Gr}(\ell, W)$ to denote the set of ℓ-dimensional subspaces of W; when $W = \mathbb{R}^d$ we shall simply write $\mathrm{Gr}(d)$ and $\mathrm{Gr}(\ell, d)$ respectively. For given $V \in \mathrm{Gr}(d)$, a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is called V*-invariant* if

$$\tau_\#\mu = \mu \quad \text{for all translations } \tau : \mathbb{R}^d \to \mathbb{R}^d \text{ satisfying } \tau(V) = V.$$

The subspace of V-invariant measures is denoted $\mathcal{M}^V(\mathbb{R}^d)$. Note that this space is sequentially weak-$*$ closed in $\mathcal{M}(\mathbb{R}^d)$.

The dimension criterion is contained in the next result:

Lemma 9 (invariance criterion)**.** *Let* $0 \le \ell \le d$ *be a positive integer and let* $\sigma \in \mathcal{M}(\Omega)$ *be a positive measure. Assume that at,* σ^s*-almost every* $x \in \Omega$*, every bounded tangent measure* $\tau \in \mathrm{Tan}(\sigma^s, x)$ *can be split on* B_1 *as a finite sum*

$$\text{(C)} \qquad \tau = (\tau_1 + \cdots + \tau_k)\llcorner B_1, \qquad k = k(\sigma) \in \mathbb{N},$$

where, for each $1 \le h \le k$, τ_h *is a* V_h*-invariant measure for some* $V_h \in \mathrm{Gr}(\ell_h, d)$ *with* $\ell_h \ge \ell$*. Then,* σ *satisfies the dimensional estimate*

$$\dim_{\mathcal{H}}(\sigma) \ge \ell.$$

4. Proofs

We begin by proving an estimate on the upper Hausdorff densities.

Lemma 10. *Let* $0 \le \ell \le d$ *be a positive integer and let* $\sigma \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ *be a positive measure. Let* $x \in \Omega$ *be a* $|\mu^s|$*-Lebesgue point and assume that every bounded tangent measure* $\tau \in \mathrm{Tan}(\mu^s, x)$ *can be split on* B_1 *as a finite sum*

$$\tau = \tau_1 + \cdots + \tau_k, \qquad k = k(\sigma) \in \mathbb{N},$$

where each τ_h *is a* V_h*-invariant measure for some* $V_h \in \mathrm{Gr}(\ell_h, d)$ *with* $\ell \le \ell_h$*.*

Then, the upper κ*-density of* μ *at* x *is equal to zero for all* $\kappa \in [0, \ell)$*, that is,*

$$\theta^{*\kappa}(\mu^s, x) := \limsup_{r \downarrow 0} \frac{\mu(Q_r(x))}{r^\kappa} = 0 \quad \forall\, \kappa \in [0, \ell).$$

Proof. It suffices to show that $\theta^{*\kappa}(\mu^s, x)$ is finite for all $\kappa \in [0, \ell)$. The fact that $\theta^{*\kappa}(\mu^s, x)$ is equally zero will then follow from the next simple observation: if $\theta^{*\kappa_1}(\mu^s, x) > 0$, then $\theta^{*\kappa}(\mu^s, x) = \infty$ for all $\kappa \in (\kappa_1, \ell)$.

We argue by contradiction. Assume that $\theta^{*\kappa}(\mu^s, x) = \infty$ for some $\kappa \in [0, \ell)$ and let $t \in (0, 4^{-\frac{d}{\ell - \kappa}})$. Then, by [3, Proposition 2.42], there exists a bounded tangent measure $\tau \in \mathrm{Tan}(\mu^s, x)$ with $t^\kappa \le \tau(\overline{B_t}) \le \tau(B_1) \le 1$. On the other hand, by assumption, we may find a positive integer $k = k(\tau)$ such that

$$\tau = \tau_1 + \cdots + \tau_k,$$

where each τ_h is a positive V_h-directional measure, for all $1 \le h \le k$. Let us denote by $\mathbf{p}_h : \mathbb{R}^d \to V_h^\perp$ the canonical projection so that

$$\tau_h(F) \le \mathscr{L}^{\ell_h}((\mathbf{1} - \mathbf{p}_h)F) \cdot \tilde{\tau}_h(\mathbf{p}_h F) \quad F \subset B_1,$$

where up to a linear isometry transformation we have $\tau_h = \mathscr{L}^{\ell_h} \otimes \tilde{\tau}_h$.

Next, we use that $4t < 1$ and that $\ell \le \ell_h$ (for all $1 \le h \le k$) to obtain the estimate

$$\begin{aligned} t^\kappa \le \tau(\overline{B_t}) &= \tau_1(\overline{B_t}) + \cdots + \tau_k(\overline{B_t}) \\ &\le (2t)^{\ell_1} \tilde{\tau}_1(\mathbf{p}_1 B_{\frac{1}{4}}) + \cdots + (2t)^{\ell_k} \tilde{\tau}_k(\mathbf{p}_k B_{\frac{1}{4}}) \\ &\le (2t)^{\ell_1} 2^{-(d-\ell_1)} \tau_1(B_1) + \cdots + (2t)^{\ell_k} 2^{-(d-\ell_k)} \tau_k(B_1) \\ &\le 2^d t^\ell \tau(B_1) \le 2^d t^\ell. \end{aligned}$$

This chain of inequalities implies $2^{-\frac{d}{\ell - \kappa}} \le t$, which directly contradicts our choice of t. This shows $\theta^{*\kappa}(\mu, x) < \infty$, as desired. □

Proof of Lemma 9. Fix an arbitrary $\kappa \in [0, \ell)$. By the previous lemma and the assumption we know that the set $\Theta_0^\kappa := \{\, x \in \Omega : \theta^{*\kappa}(\sigma, x) = 0 \,\}$ has full $|\sigma^s|$-measure on Ω. Hence, $\sigma^s \llcorner \Theta_0^\kappa = \sigma^s$. Moreover, for every $\varepsilon > 0$, it holds $\theta^{*\kappa}(\sigma^s, x) \le \varepsilon$ for all $x \in \Theta_0^\kappa$. Then, the upper-density criterion contained in [3, Theorem 2.56] holds and therefore

$$\sigma^s \llcorner \Theta_0^\kappa \le 2^\kappa \varepsilon\, \mathcal{H}^\kappa \llcorner \Theta_0^\kappa \quad \text{for all } \varepsilon > 0.$$

Letting $\varepsilon \downarrow 0$ we deduce that $\sigma^s(F) = 0$ whenever $\mathcal{H}^\kappa(F \cap \Theta_0^\kappa) < \infty$ for a Borel set $F \subset \Omega$. By the definition of Hausdorff dimension, this implies $\dim_{\mathcal{H}}(\sigma^s) \geq \kappa$. Since $\kappa \in [0, \ell)$ was chosen arbitrarily and $\dim_{\mathcal{H}}(\sigma) = \dim(\sigma^s)$, we conclude that $\dim_{\mathcal{H}}(\sigma) \geq \ell$. □

Proof of Theorem 1. Let $x \in \Omega$ be a $|\mu|^s$-Lebesgue point so that every probability tangent measure $\sigma \in \mathrm{Tan}(\mu^s, x)$ can be written as $\sigma = e|\sigma|$ with $e = \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in E$ and $|\sigma| \in \mathrm{Tan}(|\mu|^s, x)$.

Fix $\sigma \in \mathrm{Tan}(\mu^s, x)$. Note that $P^1(D)\sigma = 0$ in the sense of distributions on B_1, where $P^1(D)$ is the principal part of $P(D)$. This follows from the scaling rule

$$P^1(D)[T_\#^{r_j,x}\mu] = -r_j \cdot P_0[T_\#^{r_j,x}\mu],$$

where the term in the right-hand side converges strongly to zero (in the sense of distributions) as $j \to \infty$. We now use the fact that B_1 is a star-shaped domain to define smooth approximations of σ on B_1 as follows. Fix $\delta > 0$ to be a small parameter and define $\sigma_\delta := (T_\#^{1-\delta,0}\sigma) \star \rho_\delta \in \mathrm{C}^\infty(B_1; E)$, where ρ_δ is a standard mollifier at scale δ. In this way $\sigma_\delta \mathscr{L}^d \llcorner B_1 \overset{*}{\rightharpoonup} \sigma$ and $|\sigma_\delta| \mathscr{L}^d \llcorner B_1 \overset{*}{\rightharpoonup} |\sigma|$ as $\delta \downarrow 0$ on B_1. Observe that, for each $\delta > 0$, the measure σ_δ (which satisfies $\sigma_\delta = e|\sigma_\delta|$) solves (in the classical sense) the homogeneous equation

$$P^1(D)\sigma_\delta = \sum_{i=1}^n P_i[e]\,(\partial_i|\sigma_\delta|) = 0 \text{ on } B_1.$$

In symbolic language this reads $\mathbb{P}(\nabla|\sigma_\delta|)[e] = 0$, or equivalently, in terms of the differential inclusion,

$$\nabla(|\sigma_\delta|) \in \{\mathbb{P}[e] \equiv 0\} \text{ on } B_1.$$

We deduce that $\nabla(|\sigma_\delta|)(x)[\xi] = 0$ for all $\xi \in \{\mathbb{P}[e] \equiv 0\}^\perp$ and all $x \in B_1$. In particular, for every $\delta > 0$, the measure $|\sigma_\delta|\mathscr{L}^d \llcorner B_1$ is $\{\mathbb{P}[e] \equiv 0\}^\perp$-invariant. Since the space of $\{\mathbb{P}[e] \equiv 0\}^\perp$-invariant measures is sequentially weak-$*$ closed, we infer that

$$|\sigma| \in \mathrm{Tan}(|\mu|^s, x) \text{ is a } \{\mathbb{P}[e] \equiv 0\}^\perp\text{-invariant measure on } B_1. \tag{4}$$

Finally, since x was chosen to be an arbitrary $|\mu|^s$-Lebesgue point, $|\mu|$ satisfies (C) with $\ell = \ell_{\mathbb{P}}$. We conclude, by Lemma 9, that $\dim_{\mathcal{H}}(|\mu|) \geq \ell_{\mathbb{P}}$. □

Proof of Corollary 3. By the very definition of $\Lambda_{\mathbb{P}}$, it follows that $\{\mathbb{P}[e] \equiv 0\} = \{0\}$ for all $e \notin \Lambda_{\mathbb{P}}$. Let us write $S_{\mathbb{P},\mu} := \left\{ x \in \Omega : \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \notin \Lambda_{\mathbb{P}} \right\}$. From (4), it follows that $|\mu| \llcorner S_{\mathbb{P},\mu}$ satisfies the assumptions of Lemma 9 with $\ell = d$. Therefore $\dim_{\mathcal{H}}(|\mu| \llcorner S_{\mathbb{P},\mu}) = d$. The sought estimate is then an immediate consequence of the definition of Hausdorff dimension. □

Appendix A. Multilinear algebra

Let V be a finite dimensional euclidean space. The exterior algebra $\bigwedge^* V$ is a graded algebra with the "$\wedge$" product. Specifically $\wedge : \bigwedge^p V \times \bigwedge^q V \to \bigwedge^{p+q} V : (\xi^*, \omega^*) \mapsto \xi^* \wedge \omega^*$.

In the particular situation when $p = 1$, this is the multiplication by 1-covectors. As such, we can define the annihilator of this map on a fixed m-vector by setting

$$\mathrm{Ann}^1(v^*) := \{\, \xi \in V : \xi^* \wedge v^* = 0 \,\}. \tag{5}$$

Lemma 11. *Let V be an euclidean space of dimension d, let $m \in \{0, \dots, d\}$ be a positive integer, and let $v^* \in \bigwedge^m V$ be a non-zero m-covector. Then*

$$\mathrm{Ann}^1(v^*) \in \mathrm{Gr}(\ell, V) \quad \textit{for some } 0 \le \ell \le m.$$

Moreover, if v^* is a simple m-covector, then $\ell = m$.

Proof. The assertion that $\mathrm{Ann}_1(v)$ is in fact a linear space follows immediately from the bi-linearity of the wedge product. Notice also that, on simple vectors $v^* = v_1^* \wedge \cdots \wedge v_m^*$, the result is immediate since then $\mathrm{Ann}^1(v^*) = \mathrm{span}\{v_1, \dots, v_m\}$ (so that $\ell = m$ in this case). Any automorphism φ of V lifts to an automorphism Φ on $\bigwedge^* V$ satisfying

$$\Phi(v_1^* \wedge \cdots \wedge v_m^*) = \varphi(v_1^*) \wedge \cdots \wedge \varphi(v_m^*).$$

Hence, once $v^* \in \bigwedge_m V$ is fixed, we may assume without loss of generality that $\mathrm{Ann}^1(v^*) = \mathrm{span}\{e_1, \dots, e_\ell\}$ for some $0 \le \ell \le d$, where $\{e_1, \dots, e_d\}$ is an orthonormal basis of V. Indeed, let $\{\xi_1, \dots, \xi_\ell\}$ be a normal basis of $\mathrm{Ann}^1(v)$ and let φ be the automorphism of V satisfying $\varphi(\xi_i^*) = e_i^*$ for all $1 \le i \le \ell$ and $\varphi(w^*) = w^*$ for all $w^* \in \mathrm{Ann}^1(v^*)^\perp$. Then,

$$\Phi(\xi_i^* \wedge v^*) = e_i^* \wedge \varphi(v^*).$$

Let us fix $i_0 \in \{1, \dots, \ell\}$ and observe that

$$e_{i_0}^* \wedge v^* = \sum_{\substack{1 \le i_1 < \cdots < i_m \le d \\ i_1, \dots, i_m \ne i_0}} v_{i_1 \cdots i_m} (e_{i_0}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_m}^*),$$

where $v_{i_1 \cdots i_m} = \langle v^*, e_{i_1} \wedge \cdots \wedge e_{i_m} \rangle$. On the other hand, the set

$$\{\, e_{i_0}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_m}^* : 1 \le i_1 < \cdots < i_m \le d, i_1, \dots i_m \ne i_0 \,\}$$

conforms a set of linearly independent m-covectors in $\bigwedge^{m+1} V$. Therefore, $e_i^* \wedge v^* = 0$ if and only if $e_{i_0}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_m}^* = 0$ for all $1 \le i_1 < \cdots < i_m \le d$ such that $v_{i_1 \cdots i_m} \ne 0$. Since $1 \le i_0 \le \ell$ was chosen arbitrarily, this yields the set contention

$$\mathrm{Ann}^1(v^*) \subset \bigcap_{\substack{i_1, \dots, i_m \ne i_0 \\ v_{i_1 \cdots i_m} \ne 0}} \mathrm{Ann}^1(e_{i_1}^* \wedge \cdots \wedge e_{i_m}^*).$$

By the first observation, on the dimension of annihilators of simple vectors, we conclude that $\dim[\mathrm{Ann}^1(v^*)] = \ell \le m$. □

By duality, the exterior product induces tan interior multiplication on the algebra of vectors $\bigwedge_* V$. This is a bilinear map $\llcorner : \bigwedge_q V \times \bigwedge^p V \mapsto \bigwedge_{q-p} V$: $(v, w^*) \mapsto v \llcorner w^*$, where $v \llcorner w$ acts on $(q-p)$-co-vectors z^* as

$$\langle v \llcorner w^*, z^* \rangle = \langle v, w^* \wedge z^* \rangle.$$

Similarly as before, when $p = 1$, we may consider its corresponding annihilator

$$\mathrm{Ann}_1(v) := \{\, \xi \in \mathbb{R}^d : v \llcorner \xi^* = 0 \,\}.$$

A similar (dual) proof to the one of Lemma 11 yields the following result:

Lemma 12. *Let V be an euclidean space of dimension d, let $m \in \{0, \dots, d\}$, and let $v \in \bigwedge_m V$ be a non-zero m-vector. Then*

$$\mathrm{Ann}_1(v) \in \mathrm{Gr}(d-\ell, V) \quad \text{for some } 0 \le \ell \le m.$$

Furthermore, if v is a simple m-vector, then $\ell = d - m$.

References

[1] G. Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 2, 239–274. MR1215412

[2] L. Ambrosio, A. Coscia, and G. Dal Maso, *Fine properties of functions with bounded deformation*, Arch. Rational Mech. Anal. **139** (1997), no. 3, 201–238. MR1480240

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000. MR1857292

[4] L. Ambrosio and H. M. Soner, *A measure theoretic approach to higher codimension mean curvature flows*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 4, 25** (1997), no. 1-2, 27–49 (en). MR1655508

[5] A. Arroyo-Rabasa, *Rigidity of tangent functions of bounded variation and its applications in the calculus of variations*. In preparation.

[6] A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and F. Rindler, *Dimensional estimates and rectifiability for measures satisfying linear pde constraints*, ArXiv e-prints: 1811.01847 (2018), available at 1811.01847.

[7] E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61, Editrice Tecnico Scientifica, Pisa, 1961. MR0179651

[8] C. De Lellis, *A note on Alberti's rank-one theorem*, Transport equations and multi-D hyperbolic conservation laws, 2008, pp. 61–74. MR2504174

[9] G. De Philippis and F. Rindler, *On the structure of $\mathcal{A}$-free measures and applications*, Ann. Math. **184** (2016), no. 3, 1017–1039. MR3549629

[10] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325

[11] W. H. Fleming and R. Rishel, *An integral formula for total gradient variation*, Arch. Math. (Basel) **11** (1960), 218–222. MR0114892

[12] I. Fonseca and S. Müller, *$\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young measures*, SIAM J. Math. Anal. **30** (1999), no. 6, 1355–1390. MR1718306

[13] I. Fragalà and C. Mantegazza, *On some notions of tangent space to a measure*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 2, 331–342. MR1686704

[14] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995. MR1333890

[15] D. Preiss, *Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability, and densities*, Ann. of Math. (2) **125** (1987), no. 3, 537–643. MR890162

[16] B. Raita, *L^1-estimates and $\mathbb{A}$-weakly differentiable functions*, University of Oxford, Technical Report OxPDE-18/01, 2018.

[17] D. Spector and J. V. Schaftingen, *Optimal embeddings into lorentz spaces for some vector differential operators via gagliardo's lemma*, ArXiv e-prints: 1811.02691 (2018), available at 1811.02691.

[18] J. Van Schaftingen, *Limiting Sobolev inequalities for vector fields and canceling linear differential operators*, J. Eur. Math. Soc. **15** (2013), no. 3, 877–921. MR3085095

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