Faulhaber and Bernoulli[*]

Ryan Zielinski

ryan_zielinski@fastmail.com

7 December 2018

Abstract

In this note we will use Faulhaber's Formula to explain why the odd Bernoulli numbers are equal to zero.

1 Introduction

For odd numbers greater than or equal to seven, why are the Bernoulli numbers equal to zero? Because Faulhaber's Formula tells us that $\sum_{k=1}^{n} k^{2m+1}$ is a polynomial in $\left(\sum_{k=1}^{n} k\right)^2$, and $\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2+2n^3+n^4}{4}$.

2 Faulhaber's Formula

We might know already

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

Through inductive reasoning like that in [6] we might discover further

$$2^1 \cdot \left(\frac{n(n+1)}{2}\right)^2 = \binom{2}{1} \cdot \sum k^3$$

$$2^2 \cdot \left(\frac{n(n+1)}{2}\right)^3 = \binom{3}{0} \cdot \sum k^3 + \binom{3}{2} \cdot \sum k^5$$

$$2^3 \cdot \left(\frac{n(n+1)}{2}\right)^4 = \binom{4}{1} \cdot \sum k^5 + \binom{4}{3} \cdot \sum k^7 \tag{1}$$

$$2^4 \cdot \left(\frac{n(n+1)}{2}\right)^5 = \binom{5}{0} \cdot \sum k^5 + \binom{5}{2} \cdot \sum k^7 + \binom{5}{4} \cdot \sum k^9$$

$$2^5 \cdot \left(\frac{n(n+1)}{2}\right)^6 = \binom{6}{1} \cdot \sum k^7 + \binom{6}{3} \cdot \sum k^9 + \binom{6}{5} \cdot \sum k^{11}.$$

(We abbreviate $\sum_{k=1}^n k^m$ by $\sum k^m$.) The general case is

$$2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^m = \binom{m}{0} \cdot \sum k^m + \binom{m}{2} \cdot \sum k^{m+2} \tag{2}$$
$$+ \cdots + \binom{m}{m-3} \cdot \sum k^{2m-3} + \binom{m}{m-1} \cdot \sum k^{2m-1}$$

or

$$2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^m = \binom{m}{1} \cdot \sum k^{m+1} + \binom{m}{3} \cdot \sum k^{m+3} \tag{3}$$
$$+ \cdots + \binom{m}{m-3} \cdot \sum k^{2m-3} + \binom{m}{m-1} \cdot \sum k^{2m-1}.$$

If we wish to prove such expressions, by [1, 2, 3] we may proceed using Pascal's observation of telescoping sums. Consider the special case of

$$\left(\frac{3 \cdot 4}{2}\right)^m = \left(\frac{1 \cdot 2}{2}\right)^m - 0 + \left(\frac{2 \cdot 3}{2}\right)^m - \left(\frac{1 \cdot 2}{2}\right)^m + \left(\frac{3 \cdot 4}{2}\right)^m - \left(\frac{2 \cdot 3}{2}\right)^m$$
$$= \sum_{k=1}^3 \left[\left(\frac{k(k+1)}{2}\right)^m - \left(\frac{(k-1)k}{2}\right)^m\right].$$

For the general case of

$$\left(\frac{n(n+1)}{2}\right)^m = \sum_{k=1}^n \left[\left(\frac{k(k+1)}{2}\right)^m - \left(\frac{(k-1)k}{2}\right)^m\right], \tag{4}$$

the expression in brackets we may rewrite as

$$\left(\frac{k}{2}\right)^m \left((1+k)^m - (-1+k)^m\right)$$

2

and then use the binomial theorem to arrive at (2) or (3).

What if we want to rewrite such expressions in terms of a particular $\sum k^{2m+1}$? For example, by looking at (1) we may write

$$\sum k^5 = \frac{1}{\binom{3}{2}} \cdot \left[2^2 \cdot \left(\frac{n(n+1)}{2} \right)^3 - \binom{3}{0} \cdot \sum k^3 \right]$$

$$= \frac{1}{3} \cdot \left[4 \cdot \frac{n(n+1)}{2} - 1 \right] \cdot \left(\sum k \right)^2,$$

which then implies

$$\sum k^7 = \frac{1}{\binom{4}{3}} \cdot \left[2^3 \cdot \left(\frac{n(n+1)}{2} \right)^4 - \binom{4}{1} \cdot \sum k^5 \right]$$

$$= \frac{1}{4} \cdot \left[2^3 \cdot \left(\frac{n(n+1)}{2} \right)^2 - 4 \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1 \right) \right] \cdot \left(\sum k \right)^2.$$

By a proof by mathematical induction on the m of (2) or (3) we may arrive at the general result of

$$\sum k^{2m+1} = \frac{1}{m+1} \cdot \left[2^m \cdot \left(\frac{n(n+1)}{2} \right)^{m-1} - a_2 \cdot \left(\frac{n(n+1)}{2} \right)^{m-2} + a_3 \cdot \left(\frac{n(n+1)}{2} \right)^{m-3} \right.$$

$$\left. \mp \cdots \mp a_{m-2} \cdot \left(\frac{n(n+1)}{2} \right)^2 \mp a_{m-1} \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1 \right) \right] \cdot \left(\sum k \right)^2, \quad (5)$$

where the a_i are rational numbers and $m \geqq 3$. This relationship we will call Faulhaber's Formula. (For some of the history of the problem, see [1, 2, 3, 4].)

3   Bernoulli Numbers

By [1, 2] or Chapter 1 of [5] we may define the Bernoulli numbers B_n by

$$B_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} \cdot B_k = 0,$$

where $n \geqq 1$. For example, to find B_1 we write

$$\sum_{k=0}^{1} \binom{1+1}{k} \cdot B_k = \binom{2}{0} \cdot B_0 + \binom{2}{1} \cdot B_1 = 0,$$

which implies $B_1 = -\frac{1}{2}$. It turns out $B_3 = B_5 = 0$. Following the section "Back to Faulhaber's form" of [4] we may write

$$\sum k^{2m+1} = \frac{1}{2m+2} \cdot \left[\binom{2m+2}{0} \cdot B_0 \cdot n^{2m+2} - \binom{2m+2}{1} \cdot B_1 \cdot n^{2m+1} \right.$$

$$\left. \pm \cdots + \binom{2m+2}{2m} \cdot B_{2m} \cdot n^2 - \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n \right], \quad (6)$$

3

for which we will assume $m \geqq 3$.

4 Conclusion

With regard to the claim at the start, suppose we set (5) and (6) equal to one another. If we multiply out (5), we see it does not contain the term n. That means the last term of (6),

$$-\frac{1}{2m+2} \cdot \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n,$$

must be equal to zero. In other words, for all $m \geqq 3$, $B_{2m+1} = 0$.

Acknowledgements

References

[1] A. F. Beardon, Sums of powers of integers, *Amer. Math. Monthly* 103 (1996) 201-213.

[2] A. W. F. Edwards, Sums of powers of integers: a little of the history, *Math. Gazette* 66 (1982) 22-28.

[3] A. W. F. Edwards, A quick route to sums of powers, *Amer. Math. Monthly* 93 (1986) 451-455.

[4] D. E. Knuth, Johann Faulhaber and sums of powers, *Mathematics of Computation* 61 (1993) 277–294.

[5] H. Rademacher, *Topics in Analytic Number Theory.* Springer-Verlag, New York, 1973.

[6] R. Zielinski, Induction and analogy in a problem of finite sums, https://arxiv.org/abs/1608.04006.