ON THE QUASICONFORMAL EQUIVALENCE OF DYNAMICAL CANTOR SETS

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ABSTRACT. The complement of a Cantor set in the complex plane is itself regarded as a Riemann surface of infinite type. The problem of this paper is the quasiconformal equivalence of such Riemann surfaces. Particularly, we are interested in Riemann surfaces given by Cantor sets which are created through dynamical methods. We discuss the quasiconformal equivalence for the complements of Cantor Julia sets of rational functions and random Cantor sets. We also consider the Teichmüller distance between random Cantor sets.

1. INTRODUCTION

Let E be a Cantor set in the Riemann sphere $\widehat{\mathbb{C}}$, that is, a totally disconnected perfect set in $\widehat{\mathbb{C}}$. The standard middle one-third Cantor set $\mathcal{C}$ is a typical example. We consider the complement $X_E := \widehat{\mathbb{C}} \setminus E$ of the Cantor set E. It is an open Riemann surface with uncountable boundary components. We are interested in the quasiconformal equivalence of such Riemann surfaces. In the previous paper [11], we show that the complement of the limit set of a Schottky group is quasiconformally equivalent to $X_{\mathcal{C}}$, the complement of $\mathcal{C}$ ([11] Theorem 6.2). In this paper, we discuss the quasiconformal equivalence for the complements of Cantor Julia sets of hyperbolic rational functions and random Cantor sets (see §2 for the terminologies). We establish the following theorems.

Theorem I. *Let f be a rational function of degree $d > 1$ and $\mathcal{J}$ be the Julia set of f. Suppose that f is hyperbolic and $\mathcal{J}$ is a Cantor set. Then, the complement $X_{\mathcal{J}}$ of $\mathcal{J}$ is quasiconformally equivalent to $X_{\mathcal{C}}$.*

As for random Cantor sets, we obtain the followings.

Theorem II. *Let $\omega = (q_n)_{n=1}^{\infty}$ and $\widetilde{\omega} = (\tilde{q}_n)_{n=1}^{\infty}$ be sequences with δ-lower bound. We put*

$$d(\omega, \widetilde{\omega}) = \sup_{n \in \mathbb{N}} \max \left\{ \left| \log \frac{1 - \tilde{q}_n}{1 - q_n} \right|, |\tilde{q}_n - q_n| \right\}. \tag{1.1}$$

Date: August 30, 2019.

2010 *Mathematics Subject Classification.* Primary 30C62, Secondary 30F25.

Key words and phrases. Cantor sets, Quasiconformal maps.

The author was partially supported by the Ministry of Education, Science, Sports and Culture, Japan; Grant-in-Aid for Scientific Research (B), 16H03933, 2016–2020.

(1) *If* $d(\omega, \widetilde{\omega}) < \infty$*, then there exists an* $\exp(C(\delta)d(\omega, \widetilde{\omega}))$*-quasiconformal mapping* φ *on* $\widehat{\mathbb{C}}$ *such that* $\varphi(E(\omega)) = E(\widetilde{\omega})$*, where* $C(\delta) > 0$ *is a constant depending only on* δ*;*

(2) *if* $\lim_{n\to\infty} \log \frac{1-\tilde{q}_n}{1-q_n} = 0$*, then* $E(\widetilde{\omega})$ *is asymptotically conformal to* $E(\omega)$*, that is, there exists a quasiconformal mapping* φ *on* $\widehat{\mathbb{C}}$ *with* $\varphi(E(\omega)) = E(\widetilde{\omega})$ *such that for any* $\varepsilon > 0$*,* $\varphi|_{U_\varepsilon}$ *is* $(1+\varepsilon)$*-quasiconformal on a neighborhood* U_ε *of* $E(\omega)$*.*

From above results and a result [11] Theorem 6.2, immediately we obtain;

Corollary 1.1. *Let* E *be a Cantor set which is a Julia set of a rational function satisfying the conditions in Theorem I. Then, the complement of the limit set of a Schottky group* G *is quasiconformally equivalent to* X_E*.*

As consequences of Theorem II (1), we obtain;

Corollary 1.2. *Let* $E(\omega)$ *be a random Cantor set for* $\omega = (q_n)_{n=1}^{\infty}$*. Suppose that* ω *has lower and upper bounds. Then,* $X_{E(\omega)}$ *is quasiconformally equivalent to* $X_{\mathcal{C}}$*.*

We have also the following.

Corollary 1.3. *Let* E *be a Cantor set as in Corollaries 1.1 or 1.2. Then, the Cantor set* E *is quasiconformally removable, that is, every quasiconformal mapping on the complement of* E *is extended to a quasiconformal mapping on the Riemann sphere.*

It is known ([7] V. 11F. Theorem) that the random Cantor set $E(\omega)$ for ω is of capacity zero if and only if

$$\prod_{n=1}^{\infty}(1-q_n)^{2^{-n}} = 0. \tag{1.2}$$

Hence if $\{q_n\}_{n=1}^{\infty}$ rapidly converges to one as it satisfies (1.2), then $X_{E(\omega)}$ is not quasiconformally equivalent to $X_{\mathcal{C}}$ because the positivity of the capacity of closed sets in the plane is preserved by quasiconformal mappings (cf. [7] III. Theorem 8 H). In fact, we can say more:

Theorem III. *If* ω *does not have an upper bound, then* $X_{E(\omega)}$ *is not quasiconformally equivalent to* $X_{\mathcal{C}}$*.*

The proof of Theorem II gives the following.

Corollary 1.4. *Let* ω *and* $\widetilde{\omega}$ *be sequences satisfying the same conditions as in Theorem II (2). Then, the Hausdorff dimension of* $E(\widetilde{\omega})$ *is the same as that of* $E(\omega)$*.*

Acknowledgement. The author thanks Prof. K. Matsuzaki for his valuable comments. This research was partly done during the author's stay in the Erwin Schrödinger institute at Vienna. He also thanks the institute for its brilliant support to his research.

2. Preliminaries

2.1. **Complex dynamics.** We begin with a small and brief introduction of complex dynamics. We may refer textbooks on the topic, e. g. [5] for a general theory of complex dynamics.

Let f be a rational function of degree $d > 1$ on $\mathbb{C}$. We denote by f^n the n times iterations of f. The Fatou set $\mathcal{F}$ of f is the set of points in $\widehat{\mathbb{C}}$ which have neighborhoods where $\{f^n\}_{n=1}^{\infty}$ is a normal family. The complement of $\mathcal{F}$, which is denoted by $\mathcal{J}$, is called the Julia set of f.

A rational function f is called *hyperbolic* if it is expanding near $\mathcal{J}$. More precisely, if $\mathcal{J} \not\ni \infty$, then f is hyperbolic if there exist a constant $A > 1$ and a smooth metric $\sigma(z)|dz|$ in a neighborhood U of $\mathcal{J}$ such that

$$\sigma(f(z))|f'(z)| \geq A\sigma(z), \quad z \in \mathcal{J}$$

(see [5] V. 2). If $\infty \in \mathcal{J}$, the hyperbolicity of f is defined by conjugation of Möbius transformations as usual.

The hyperbolicity is also characterized in terms of the Euclidean metric and the dynamical behavior of rational functions as well.

Proposition 2.1 ([5] V. 2. Lemma 2.1 and Theorem 2.2)**.** *A rational function f is hyperbolic if and only if every critical point belongs to $\mathcal{F}$ and is attracted to an attracting cycle. If $\infty \notin \mathcal{J}$, then the hyperbolicity of f is equivalent to the existence of $m \geq 1$ such that $|(f^m)'| > 1$ on $\mathcal{J}$.*

2.2. **Random Cantor sets.** (cf. [7] I. 6). Let $\omega = (q_n)_{n=1}^{\infty} = (q_1, q_2, \dots)$ be a sequence of real numbers with $0 < q_n < 1$ for each $n \in \mathbb{N}$. We construct a Cantor set $E(\omega)$ for ω inductively as follows.

First, we remove an open interval J_1 of length q_1 from $E_0 := I = [0, 1]$ so that $I \setminus J_1$ consists of two closed intervals I_1^1, I_1^2 of the same length. We put $E_1 = \cup_{i=1}^{2} I_1^i$. We remove an open interval of length $|I_1^i|q_2$ from each I_1^i so that the remainder E_2 consists of four closed intervals of the same length, where $|J|$ is the length of an interval J. Inductively, we define E_{k+1} from $E_k = \cup_{i=1}^{2^k} I_k^i$ by removing an open interval of length $|I_k^i|q_{k+1}$ from each closed interval I_k^i of E_k so that E_{k+1} consists of 2^{k+1} closed intervals of the same length. The random Cantor set $E(\omega)$ for ω is defined by

$$E(\omega) = \cap_{k=1}^{\infty} E_k.$$

It is a generalization of the standard middle one-third Cantor set $\mathcal{C}$. In fact, $\mathcal{C} = E(\omega_0)$ for $\omega_0 = (\frac{1}{3})_{n=1}^{\infty} = (\frac{1}{3}, \frac{1}{3}, \dots)$.

We say that a sequence $\omega = (q_n)_{n=1}^{\infty}$ as above is of (δ-)lower bound if there exists a $\delta > 0$ such that $q_n \geq \delta$ for any $n \in \mathbb{N}$. We also say that a sequence ω has a (δ-)upper bound if $q_n \leq 1 - \delta$ for any $n \in \mathbb{N}$.

2.3. **Hausdorff dimension.** Let E be a subset of $\mathbb{C}$ and $\alpha > 0$. We consider a countable open covering $\{U_i\}_{i \in \mathbb{N}}$ of E with $\operatorname{diam}(U_i) < r$ for a given

$r > 0$. Then, we set

$$\Lambda_\alpha^r(E) := \inf\left\{\sum_{i\in\mathbb{N}} (\mathrm{diam}(U_i))^\alpha\right\},$$

where the infimum is taken over all countable open covering $\{U_i\}_{i\in\mathbb{N}}$ with $\mathrm{diam}(U_i) < r$. We put

$$\Lambda_\alpha(E) = \lim_{r\to 0} \Lambda_\alpha^r(E)$$

and the Hausdorff dimension $\dim_H(E)$ of E by

$$\dim_H(E) = \inf\{\alpha \mid \Lambda_\alpha(E) = 0\}.$$

2.4. **The quasiconformal equivalence of open Riemann surfaces.** We say that two Riemann surfaces R_1, R_2 are quasiconformally equivalent if there exists a quasiconformal homeomorphism between them. We also say that they are quasiconformally equivalent near the ideal boundary if there exist compact subset K_j of R_j ($j = 1, 2$) and quasiconformal homeomorphism φ from $R_1 \setminus K_1$ onto $R_2 \setminus K_2$.

It is obvious that if R_1, R_2 are quasiconformally equivalent, then they are quasiconformally equivalent near the ideal boundary. On the other hand, we have shown that the converse is not true in general. In fact, we have constructed two Riemann surfaces which are not quasiconformally equivalent while they are homeomorphic to each other and quasiconformally equivalent near the ideal boundary ([11] Example 3.1). We also give a sufficient condition for Riemann surfaces to be quasiconformally equivalent ([11] Theorem 5.1).

Proposition 2.2. *Let R_1, R_2 be open Riemann surfaces which are homeomorphic to each other and quasiconformally equivalent near the ideal boundary. If the genus of R_1 is finite, then R_1 and R_2 are quasiconformally equivalent.*

3. Proof of Theorem I

Let f be a hyperbolic rational function with the Cantor Julia set $\mathcal{J}$. We show that $X_{\mathcal{J}}$ is quasiconformally equivalent to $X_{\mathcal{C}}$. By Proposition 2.2, it suffices to show that there exists a compact subset K of $\mathcal{F}$ such that $\mathcal{F}\setminus K$ is quasiconformally equivalent to the complement of a compact subset of $X_{\mathcal{C}}$.

Considering the conjugation by Möbius transformations, we may assume that $\mathcal{J}$ does not contain ∞. Since $\mathcal{J}$ is a Cantor set, the Fatou set $\mathcal{F}$ is connected. Furthermore, it follows from Proposition 2.1 that $\mathcal{F}$ contains an attracting fixed point z_0 of f. Then, we may find a simply connected neighborhood Ω_0 of z_0 such that $f(\overline{\Omega}_0) \subset \Omega_0$. We may take Ω_0 so that the boundary $\partial\Omega_0$ is a smooth Jordan curve and it does not contain the forward orbits of critical points of f.

For each $k \in \mathbb{N}$, let Ω_k be a connected component of $f^{-k}(\Omega_0)$ containing z_0. We may assume that Ω_1 is bounded by at least two Jordan curves. Then, each Ω_k is bounded by mutually disjoint finitely many smooth Jordan curves and we have

$$z_0 \in \Omega_0 \subset \overline{\Omega}_0 \subset \Omega_1 \subset \overline{\Omega_1} \cdots \subset \overline{\Omega}_k \subset \Omega_{k+1} \subset \dots$$

and

$$\mathcal{F} = \cup_{k=0}^{\infty} \Omega_k.$$

Since f is hyperbolic, the Julia set $\mathcal{J}$ does not contain critical points. Moreover, there exists a simply connected neighborhood V_z of each $z \in \mathcal{J}$ such that $f|_{V_z}$ is injective on V_z (Proposition 2.1). Hence, from compactness of $\mathcal{J}$ there exist disks $V_1, \dots, V_n$ for some $n \in \mathbb{N}$ such that $\mathcal{J} \subset \cup_{j=1}^{n} V_j$ and $f|_{V_j}$ is injective ($1 \leq j \leq n$). Then, we show;

Lemma 3.1. *There exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, each connected component of $\Omega_{k+1} \setminus \overline{\Omega}_k$ is contained in some V_j $(1 \leq j \leq n)$.*

Proof. Since $f(\Omega_{k+1} \setminus \overline{\Omega_k}) = \Omega_k \setminus \overline{\Omega_{k-1}}$ and $\Omega_{k+1} \supset \Omega_k$, we see that if every connected component of $\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$ is contained in some V_j, then so is for $k \geq k_0$. Hence, we may find such a k_0 to show the statement of the lemma.

Suppose that for any $k \in \mathbb{N}$, there exists a connected component Δ_k of $\Omega_{k+1} \setminus \Omega_k$ such that Δ_k is not contained in any V_j $(j = 1, 2, \dots, n)$. Thus, for sufficiently large k, Δ_k is contained in $\cup_{j=1}^{n} V_j$ but it is not contained in any V_j. By taking a subsequence if necessary, we may assume that $\Delta_k \cap V_j \neq \emptyset$ and $\Delta_k \cap V_{j'} \neq \emptyset$ for some $j, j' \in \{1, 2, \dots, n\}$. Let x be an accumulation point of $\{\Delta_k\}_{k=1}^{\infty}$. Then, x has to be in $\mathcal{J}$ because $\mathcal{F} = \cup_{k=1}^{\infty} \Omega_k$.

The Julia set $\mathcal{J}$ is totally disconnected. Hence, if a sequence $\{x_{k_m}\}_{m=1}^{\infty}$ $(x_{k_m} \in \Delta_{k_m})$ converges to x, then $\{\Delta_{k_m}\}_{m=1}^{\infty}$ also converges to $\{x\}$. In other words, for any neighborhood U of x, there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, Δ_{k_m} is contained in U. However, $x \in \mathcal{J}$ is in some V_{j_0} because $\mathcal{J} \subset \cup_{j=1}^{n} V_j$. Therefore, Δ_{k_m} is contained in V_{j_0} if m is sufficiently large. Thus, we have a contradiction. □

We take $k_0 \in \mathbb{N}$ in the above lemma. Let $\omega_1, \omega_2, \dots, \omega_\ell$ be the set of connected components of $\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$. Each ω_j is bounded by a finite number, say $L(j)+1$, of mutually disjoint simple closed curves. We may assume that $L(j) > 1$ $(j = 1, 2, \dots \ell)$. Note that the number of boundary components of $\partial\Omega_{k_0} \cap \partial\Omega_{k_0+1}$ is equal to ℓ. It is because $\partial(\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}})$ consists of mutually disjoint simple closed curves in $\widehat{\mathbb{C}}$, and $\overline{\Omega_{k_0}}$ is compact.

For any $k > k_0$ and for a connected component Δ of $\Omega_{k+1} \setminus \overline{\Omega_k}$, we have $f^{k-k_0}(\Delta) \subset \Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$ and f^{k-k_0} is conformal in Δ since Δ is contained in some V_j. Hence, Δ is conformally equivalent to ω_J for some $J \in \{1, 2, \dots, \ell\}$. Therefore, if $k > k_0$, then $\Omega_{k+1} \setminus \overline{\Omega_k}$ contains at most ℓ conformally different connected components.

Now, we consider the middle one-third Cantor set $\mathcal{C}$ and $X_{\mathcal{C}} := \widehat{\mathbb{C}} \setminus \mathcal{C}$. It is not hard to see that $X_{\mathcal{C}}$ admits a pants decomposition $\{P_{i,j}\}_{i \in \mathbb{Z} \setminus \{0\}, j \in \{1, \dots, 2^{|i|-1}\}}$ as in Figure 1. In fact, we may take all $P_{i,j}$ so that they are conformally equivalent to each other. Let P_N $(N \in \mathbb{N})$ be a subdomain of $X_{\mathcal{C}}$ consisting

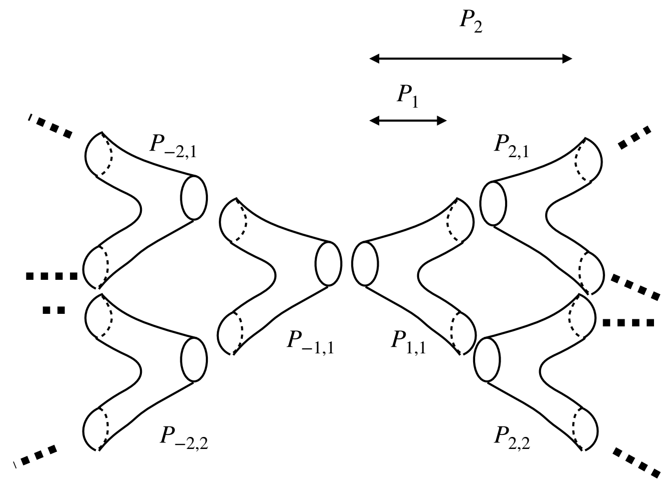

FIGURE 1. The middle one-third Cantor set

of $P_{i,j}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, 2^{i-1}$. We see that P_N is bounded by $2^N + 1$ mutually disjoint simple closed curves.

Let $N_0 \in \mathbb{N}$ be the largest number with $2^{N_0} + 1 \leq \ell$. We put

$$K := \overline{P_{N_0} \cup_{j=1}^{\ell_0} P_{N_0+1,j}},$$

where $\ell_0 = \ell - 2^{N_0} - 1$. Then, K is a compact subset of $X_{\mathcal{C}}$ bounded by ℓ simple closed curves. We denote by $C_1, \ldots, C_\ell$, where $C_1 \subset \partial P_{1,1}$. We may take a subdomain G_1 of $X_{\mathcal{C}}$ so that $G_1 \setminus K$ is quasiconformally equivalent to $\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$ as follows.

We take the largest number L_1 with $2^{L_1} \leq L(1)$. Then,

$$\overline{G_{1,1}} := \overline{\left(\cup_{i=1}^{L_1} \cup_{j=1,\ldots,2^{|i|}} P_{-i,j}\right) \cup \left(\cup_{j=1,\ldots,L(1)-2^{L_1}} P_{-L_1-1,j}\right)}$$

is a closed subdomain of $X_{\mathcal{C}}$ with $L(1) + 1$ boundary curves. Hence, $G_{1,1}$ is quasiconformally equivalent to ω_1 since both of them are planar domains bounded by the same number of closed curves.

Similarly, we may construct subdomains $G_{1,2}, \ldots, G_{1,\ell}$ such that $\partial G_{1,j} \cap \partial K = C_j$ and each $G_{1,j}$ is quasiconformally equivalent to ω_j $(j = 1, 2, \ldots, \ell)$. Combining K with $G_{1,1}, \ldots, G_{1,\ell}$, we obtain a desired subdomain G_1.

By using the same argument as above, we have a subdomain G_2 of $X_{\mathcal{C}}$ such that $G_1 \subset G_2$ and $G_2 \setminus \overline{G_1}$ is quasiconformally equivalent to $\Omega_{k_0+2} \setminus \overline{\Omega_{k_0+1}}$. Moreover, we may use this argument inductively and we obtain a exhaustion $\{G_i\}_{i=1}^{\infty}$ of $X_{\mathcal{C}}$ such that

$$K \subset G_1 \subset G_2 \subset \cdots \subset G_i \subset G_{i+1} \subset \dots, \quad X_{\mathcal{C}} = \cup_{i=1}^{\infty} G_i,$$

and $G_{i+1} \setminus \overline{G_i}$ are quasiconformally equivalent to $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$.

Now, we note the following.

Proposition 3.1. *Let R_1, R_2 be Riemann surfaces. We consider simple closed curves α_i in R_i with $R_i \setminus \alpha_i = S_1^{(i)} \cup S_2^{(i)}$, where $S_1^{(i)}$ and $S_2^{(i)}$ are mutually disjoint subsurface of R_i $(i = 1, 2)$. Suppose that there exist quasiconformal mappings $f_j : S_j^{(1)} \to S_j^{(2)}$ $(j = 1, 2)$ such that $f_1(\alpha_1) = f_2(\alpha_1) = \alpha_2$. Then, there exists a quasiconformal mapping $f : R_1 \to R_2$. Moreover, the maximal dilatation of f depends only on those of f_1, f_2 and the local behavior of those mappings near α_1.*

We may apply this proposition to domains $G_{i+1} \setminus \overline{G_i}$ and $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$ $(i = 1, 2, \dots)$. Noting that there only finitely many conformal equivalence classes in those domains, we verify that $X_{\mathcal{C}} \setminus K = \cup_{i \in \mathbb{N}} (G_{i+1} \setminus \overline{G_i})$ and $\mathcal{F} \setminus \overline{\Omega_{k_0+1}} = \cup_{i \in \mathbb{N}} (\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}})$ are quasiconformally equivalent. □

4. Proof of Theorem II

Proof of (1). We divide the proof into several steps.

Step 1 : Analyzing random Cantor sets. Let $\omega = (q_n)_{n=1}^{\infty}$ and $\widetilde{\omega} = (\tilde{q}_n)_{n=1}^{\infty}$ be sequences with δ-lower bound. We take $E_k = \cup_{i=1}^{2^k} I_k^i$ and $\widetilde{E}_k = \cup_{i=1}^{2^k} \widetilde{I}_k^i$ as in §2.2 for ω and $\widetilde{\omega}$, respectively. In fact, I_k^i (resp. $\widetilde{I}_k^i$) is located at the left of I_k^{i+1} (resp. $\widetilde{I}_k^{i+1}$) for $i = 1, 2, \dots, 2^k - 1$. The set $[0, 1] \setminus E_k$ (resp. $[0, 1] \setminus \widetilde{E}_k$) consists of $2^k - 1$ open intervals $J_k^1, \dots, J_k^{2^k-1}$ (resp. $\widetilde{J}_k^1, \dots, \widetilde{J}_k^{2^k-1}$). Each J_k^i (resp. $\widetilde{J}_k^i$) is located between I_k^i and I_k^{i+1} (resp. $\widetilde{I}_k^i$ and $\widetilde{I}_k^{i+1}$).

Because of the construction, we have

$$|I_{k+1}^i| = \frac{1}{2}(1 - q_k)|I_k^i|.$$

Therefore, we have

$$|I_k^i| = 2^{-k} \prod_{j=1}^{k} (1 - q_j). \tag{4.1}$$

Next, we estimate the length of J_k^i.

In construction E_{k+1} from E_k, we obtain open intervals I_{k+1}^{2i-1}, I_{k+1}^{2i} and the closed interval J_{k+1}^{2i-1} such that $I_k^i = I_{k+1}^{2i-1} \cup J_{k+1}^{2i-1} \cup I_{k+1}^{2i}$ for each i, k (Figure 2).

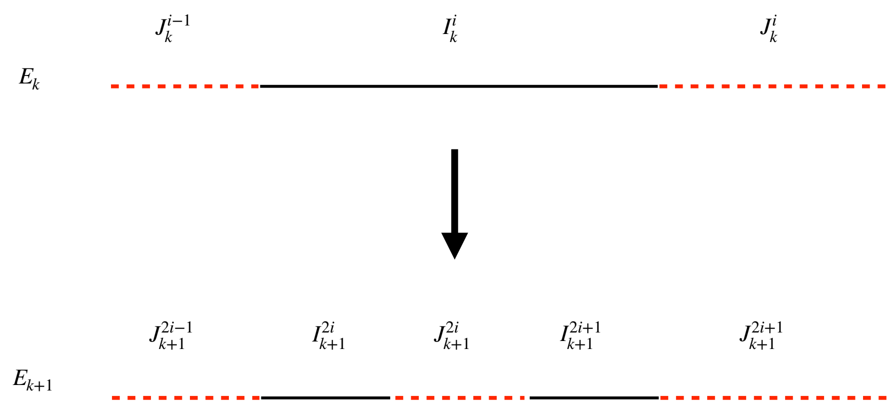

FIGURE 2.

If i is odd, we have

$$|J_{k+1}^i| = |I_k^i| q_{k+1} = \frac{2q_{k+1}}{1-q_{k+1}} |I_{k+1}^1| \geq 2\delta |I_{k+1}^1|, \tag{4.2}$$

as $q_{k+1} \geq \delta$.

If i is even, then $i = 2^\ell m$ for an integer ℓ with $1 \leq \ell \leq k$ and an odd number m. Since J_{k+1}^i is located between I_{k+1}^i and I_{k+1}^{i+1}, we see that $J_{k+1}^i = J_k^{i/2} = J_k^{2^{\ell-1}m}$. Repeating this argument, we have $J_{k+1}^i = J_{k-\ell+1}^m$. Since m is odd, we conclude from (4.2) that

$$\begin{aligned} |J_{k+1}^i| &= |J_{k-\ell+1}^m| = 2^{-k+\ell} q_{k-\ell+1} \prod_{j=1}^{k-\ell} (1-q_j) \\ &\geq 2^{-k+1} \delta \prod_{j=1}^{k} (1-q_j) \geq 4\delta |I_{k+1}^1| \end{aligned} \tag{4.3}$$

as $q_{k-\ell+1} \geq \delta$.

Thus, we obtain the following from (4.2) and (4.3).

Lemma 4.1. *Let* I_k^i *and* J_{k+1}^i *be the same ones as above for a sequence* $\omega = (q_n)_{n=1}^{\infty}$ *with* δ*-lower bound. Then,*

$$|J_{k+1}^i| \geq 2\delta |I_{k+1}^1| \tag{4.4}$$

hold for $i = 1, 2, \ldots, 2^{k+1} - 1$.

Step 2: Constructing a pants decomposition. We draw a circle C_k^i centered at the midpoint of I_k^i with radius $\frac{1}{2}(1+\delta)|I_k^1|$ for each $k \in \mathbb{N}$ and $1 \leq i \leq 2^k$. From (4.4), we see that $C_k^i \cap C_k^j = \emptyset$ if $i \neq j$. Since

$$\frac{1}{2} \cdot \delta |I_{k+1}^1| < \frac{1}{2} \cdot \delta |I_k^1|,$$

we also see that $C_{k+1}^i \cap C_k^j = \emptyset$. Therefore, $\cup_{k=1}^{\infty} \cup_{i=1}^{2^k} C_k^i$ gives a pants decomposition for $X_{E(\omega)}$.

We draw circles $\widetilde{C}_k^i$ for $\widetilde{\omega}$ by the same way. Then, we also see that $\cup_{k=1}^{\infty} \cup_{i=1}^{2^k} \widetilde{C}_k^i$ gives a pants decomposition for $X_{E(\widetilde{\omega})}$.

Step 3: Analyzing a pair of pants. We denote by P_k^i a pair of pants bounded by C_k^i, C_{k+1}^{2i-1} and C_{k+1}^{2i}. We consider the complex structure of P_k^i so that we may assume that the center of C_k^i is the origin with radius $\frac{1}{2}(1+\delta)|I_k^1|$. Then, the centers of C_{k+1}^{2i-1} and C_{k+1}^{2i} are

$$-\frac{1}{2} q_{k+1} |I_k^1| - \frac{1}{4}\left(1+\delta\right)\left(1-q_{k+1}\right)|I_k^1|$$

and

$$\frac{1}{2} q_{k+1} |I_k^1| + \frac{1}{4}\left(1+\delta\right)\left(1-q_{k+1}\right)|I_k^1|,$$

respectively.

By applying an affine map $z \mapsto \alpha z + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}$ to P_k^i so that the circle C_k^i is sent a circle centered at the origin with radius $1+\delta$. We denote the circle by $C_{k,1}$. Then, the circle C_{k+1}^{2i-1} is sent a circle $C_{k,2}$ centered at

$$-x_k := -q_{k+1} - \frac{1}{2}\left(1+\delta\right)\left(1-q_{k+1}\right) = -\frac{1}{2}\left\{\left(1+\delta\right) + \left(1-\delta\right) q_{k+1}\right\}$$

with radius

$$r_k := \frac{1}{2}\left(1+\delta\right)\left(1-q_{k+1}\right)$$

and C_{k+1}^{2i} is sent a circle $C_{k,3}$ centered at x_k with radius r_k. We may conformally identify P_k^i with a pair of pants $\mathcal{P}_k$ bounded by $C_{k,1}, C_{k,2}$ and $C_{k,3}$.

Similarly, we consider a pair of pants $\widetilde{P}_k^i$ bounded by $\widetilde{C}_k^i, \widetilde{C}_{k+1}^{2i-1}$ and $\widetilde{C}_{k+1}^{2i}$, and apply an affine map to the pair of pants $\widetilde{P}_k^i$ so that the circle $\widetilde{C}_k^i$ is mapped a circle centered at the origin with radius $1+\delta$, which is the same circle as the image of C_k^i above. We denote by $\widetilde{C}_{k,i}$ the image of $\widetilde{C}_k^i$ ($i = 1, 2, 3$). We may conformally identify $\widetilde{P}_k^i$ with a pair of pants $\widetilde{\mathcal{P}}_k$ bounded by $\widetilde{C}_{k,1}, \widetilde{C}_{k,2}$ and $\widetilde{C}_{k,3}$, where $\widetilde{C}_{k,1}$ is the same circle as $C_{k,1}$, $\widetilde{C}_{k,2}$ is centered at

$$-\widetilde{x}_k := -\frac{1}{2}\left\{(1+\delta) + (1-\delta)\,\widetilde{q}_{k+1}\right\}$$

wirh radius

$$\widetilde{r}_k := \frac{1}{2}(1+\delta)(1-\widetilde{q}_{k+1})$$

and $\widetilde{C}_{k,3}$ is centered at $\widetilde{x}_k$ with radius $\widetilde{r}_k$.

Step 4 : Constructing intermediate pairs of pants. By applying $z \mapsto (x_k/\widetilde{x}_k)z$ to $\widetilde{\mathcal{P}}_k$, we obtain a pair of pants $\widehat{P}_k$. The pair of pants $\widehat{P}_k$ is bounded by $\widehat{C}_{k,1}, \widehat{C}_{k,2}$ and $\widehat{C}_{k,3}$. Each $\widehat{C}_{k,i}$ is corresponding to $\widetilde{C}_{k,i}$ $(i = 1, 2, 3)$. Note that for each i, the center of $\widehat{C}_{k,i}$ is x_k, the same as that of $C_{k,i}$, and $\widehat{P}_k$ is conformally equivalent to $\widetilde{\mathcal{P}}_k$. The radius of $\widehat{C}_{k,1}$ is

$$(1+\delta)\cdot\frac{x_k}{\widetilde{x}_k} = (1+\delta)\,\frac{(1+\delta) + (1-\delta)\,q_{k+1}}{(1+\delta) + (1-\delta)\,\widetilde{q}_{k+1}},$$

and the radius of $\widehat{C}_{k,2}, \widehat{C}_{k,3}$ is

$$\widehat{r}_k := \frac{1}{2}(1+\delta)(1-\widetilde{q}_{k+1})(1+\delta)\,\frac{(1+\delta) + (1-\delta)\,q_{k+1}}{(1+\delta) + (1-\delta)\,\widetilde{q}_{k+1}}.$$

Now, we take an intermediate pair of pants P'_k bounded by $\widehat{C}_{k,1}, C_{k,2}$ and $C_{k,3}$.

Step 5 : Making quasiconformal mappings, I. In the following the argument, we use a notation $d(\varphi)$ for a quasiconformal mapping φ as

$$d(\varphi) = \log K(\varphi),$$

where $K(\varphi)$ is the maximal dilatation of φ.

We suppose that $q_{k+1} \geq \widetilde{q}_{k+1}$. Then, we have

$$\widehat{r}_k \geq r_k = \frac{1}{2}(1+\delta)(1-q_{k+1}).$$

In other words, the radius of $\widehat{C}_{k,2}$, $\widehat{C}_{k,3}$ is not smaller than that of $C_{k,2}$, $C_{k,3}$.

Let $C_{k,+}$ be a circle centered at x_k with radius

$$\widetilde{R}_k := (1+\delta)\,\frac{x_k}{\widetilde{x}_k} - x_k,$$

so that $C_{k,+}$ is tangent with $\widehat{C}_{k,1}$.

We consider two circular annuli $A_{k,+}$ bounded by $C_{k,+}$ and $\widehat{C}_{k,3}$, $A'_{k,+}$ bounded by $C_{k,+}$ and $C_{k,3}$. Here, we use the following well-known fact.

Lemma 4.2. *For annuli $A_i = \{0 < r_i < |z| < R_i < \infty\}$ $(i = 1, 2)$, there exists a quasiconformal mapping $\varphi : A_1 \to A_2$ such that*

$$\varphi(r_1 e^{i\theta}) = r_2 e^{i\theta}$$
$$\varphi(R_1 E^{i\theta}) = R_2 e^{i\theta}$$

and

$$K(\varphi) = e^{d(\varphi)},$$

where

$$d(\varphi) = \left| \log \frac{\log R_1 - \log r_1}{\log R_2 - \log r_2} \right| .$$

It follows from Lemma 4.2 that there exists a quasiconformal mapping $\varphi_{k,+} : A_{k,+} \to A'_{k,+}$ such that

$$d(\varphi_{k,+}) = \log \frac{\log \widetilde{R}_k - \log r_k}{\log \widetilde{R}_k - \log \widehat{r}_k},$$

$$\varphi_{k,+}(z) = z, \tag{4.5}$$

for any $z \in C_{k,+}$ and

$$\arg(\varphi_{k,+}(z) - x_k) = \arg(z - x_k) \tag{4.6}$$

for $z \in \widehat{C}_{k,3}$.

Since

$$\log \frac{c-a}{c-b} = \log \left(1 + \frac{b-a}{c-b}\right) \leq \frac{b-a}{c-b}$$

for $0 < a \leq b < c$, we obtain

$$d(\varphi_{k,+}) \leq \frac{\log \widehat{r}_k - \log r_k}{\log \widetilde{R}_k - \log \widehat{r}_k}. \tag{4.7}$$

Moreover, we have

$$\begin{aligned} \log \widetilde{R}_k - \log \widehat{r}_k &= \log \frac{(1+\delta) - (1-\delta)\widetilde{q}_{k+1}}{(1+\delta) - (1+\delta)\widetilde{q}_{k+1}} \\ &\geq \log \frac{(1+\delta) - (1-\delta)\delta}{(1+\delta) - (1+\delta)\delta} > 0, \end{aligned} \tag{4.8}$$

and

$$\log \widehat{r}_k - \log r_k = \log \frac{1 - \widetilde{q}_{k+1}}{1 - q_{k+1}} + \log \frac{(1+\delta) + (1-\delta)q_{k+1}}{(1+\delta) + (1-\delta)\widetilde{q}_{k+1}}. \tag{4.9}$$

We also see that

$$\begin{aligned} &\log \tfrac{(1+\delta)+(1-\delta)q_{k+1}}{(1+\delta)+(1-\delta)\widetilde{q}_{k+1}} \\ &= \log \left\{ 1 + \tfrac{(1-\delta)(q_{k+1} - \widetilde{q}_{k+1})}{(1+\delta)+(1-\delta)\widetilde{q}_{k+1}} \right\} \leq \tfrac{(1-\delta)(q_{k+1} - \widetilde{q}_{k+1})}{(1+\delta)+(1-\delta)\widetilde{q}_{k+1}} \\ &\leq q_{k+1} - \widetilde{q}_{k+1}, \end{aligned} \tag{4.10}$$

because

$$(1+\delta) + (1-\delta)\,\widetilde{q}_{k+1} > 1 - \delta > 0.$$

From (4.7)–(4.10), we obtain

$$\begin{aligned} d(\varphi_{k,+}) &\leq \left(\log \tfrac{(1+\delta)-(1-\delta)\delta}{(1+\delta)-(1+\delta)\delta} \right)^{-1} \\ &\times \left\{ \log \tfrac{1-\widetilde{q}_{k+1}}{1-q_{k+1}} + (q_{k+1} - \widetilde{q}_{k+1}) \right\} \leq C(\delta) d(\omega, \widetilde{\omega}) \end{aligned} \tag{4.11}$$

for some constant $C(\delta) > 0$ depending only on δ.

We may do the same operation, symmetrically; we take a circle $C_{k,-}$ centered at $-x_k$ of radius $\widetilde{R}_k$ and consider two annuli $A_{k,-}$ and $A'_{k,-}$. The annulus $A_{k,-}$ is bounded by $C_{k,-}$ and $\widehat{C}_{k,2}$, and $A'_{k,-}$ is bounded by $C_{k,-}$ and $C_{k,2}$. Then, we obtain a quasiconformal mapping $\varphi_{k,-} : A_{k,-} \to A'_{k,-}$ such that

$$\varphi_{k,-}(z) = z \tag{4.12}$$

for $z \in C_{k,-}$ and

$$\arg(\varphi_{k,-}(z) + x_k) = \arg(z + x_k). \tag{4.13}$$

for $z \in \widehat{C}_{k,2}$. Moreover, the mapping satisfies an inequality,

$$d(\varphi_{k,-}) \leq C(\delta) d(\omega, \widetilde{\omega}). \tag{4.14}$$

We define a homeomorphism $\varphi_k : \widehat{P}_k \to P'_k$ by

$$\varphi_k(z) = \begin{cases} \varphi_{k,+}(z), & z \in A_{k,+} \\ \varphi_{k,-}(z), & z \in A_{k,-} \\ z, & \text{otherwise.} \end{cases}$$

The homeomorhpism φ_k is quasiconformal except circles $C_{k,+}, C_{k,-}$. Hence, it has to be quasiconformal on $\widehat{P}_k$ with

$$d(\varphi_k) \leq C(\delta) d(\omega, \widetilde{\omega}). \tag{4.15}$$

Step 6 : Making quasiconformal mappings, II. In this step, we make a quasiconformal mapping from P'_k to $\mathcal{P}_k$. Recall that P'_k is a pair of pants bounded by $\widehat{C}_{k,1}$, $C_{k,2}$ and $C_{k,3}$, and $\mathcal{P}_k$ is bounded by $C_{k,1}$, $C_{k,2}$ and $C_{k,3}$.

Let $C_{k,0}$ be a circle centered at the origin of radius $x_k + r_k$, so that $C_{k,0}$ is tangent with $C_{k,2}$, $C_{k,3}$. We consider circular annuli B'_k bounded by $C_{k,0}$ and $\widehat{C}_{k,1}$, and B_k bounded by $C_{k,0}$ and $C_{k,1}$. It follows from Lemma 4.2 that there exists a quasiconformal mapping $\psi_{k,0} : B'_k \to B_k$ such that

$$d(\psi_{k,0}) = \log \frac{\log(1+\delta)\frac{x_k}{\widetilde{x}_k} - \log(x_k + r_k)}{\log(1+\delta) - \log(x_k + r_k)}$$

and $\psi_{k,0}|_{C_0}$ is the identity.

As in Step 5, we have

$$d(\psi_{k,0}) \leq \frac{\log x_k - \log \widetilde{x}_k}{\log(1+\delta) - \log(x_k + r_k)}.$$

Now, we see that

$$\begin{aligned} \log(1+\delta) - \log(x_k + r_k) &= \log \frac{1+\delta}{1+\delta(1-q_{k+1})} \\ &\geq \log \frac{1+\delta}{1+\delta(1-\delta)} > 0, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\text{(4.17)}\quad \log x_k - \log \widetilde{x}_k &= \log\left(1+(1-\delta)\frac{q_{k+1}-\widetilde{q}_{k+1}}{(1+\delta)+(1-\delta)\widetilde{q}_{k+1}}\right)\\ &\leq (1-\delta)\,\frac{q_{k+1}-\widetilde{q}_{k+1}}{(1+\delta)+(1-\delta)\widetilde{q}_{k+1}}\\ &\leq q_{k+1}-\widetilde{q}_{k+1}.\end{aligned}$$

From (4.16) and (4.17), we have

$$d(\psi_{k,0}) \leq \left(\log\frac{1+\delta}{1+\delta(1-\delta)}\right)^{-1}(q_{k+1}-\widetilde{q}_{k+1}). \tag{4.18}$$

We define a homeomorphism $\psi_k : P'_k \to \mathcal{P}_k$ by

$$\psi_k(z) = \begin{cases}\psi_{k,0}(z), & z \in B'_k\\ z, & \text{otherwise.}\end{cases}$$

Then, as in Step 5, we see that ψ_k is quasiconformal on P'_k with

$$d(\psi_k) \leq C(\delta)d(\omega,\widetilde{\omega}). \tag{4.19}$$

In the case where $q_{k+1} \leq \widetilde{q}_{k+1}$, the same argument still works in Steps 5 and 6; we obtain the same results.

Step 7 : Making a global quasiconformal mapping. In Steps 5 and 6, we have made quasiconformal mappings $\varphi_k : \widehat{P}_k \to P'_k$ and $\psi_k : P'_k \to \mathcal{P}_k$. Thus, $\Phi_k := \psi_k \circ \varphi_k : \widehat{P}_k \to \mathcal{P}_k$ gives a quasiconformal mapping with

$$d(\Phi_k) \leq C(\delta)d(\omega,\widetilde{\omega})$$

for each $k \in \mathbb{N}$.

Because of the boundary behaviors (4.5), (4.6), (4.12) and (4.13), we see that those mappings give a quasiconformal mapping Φ from $X_{E(\omega)}$ onto $X_{E(\widetilde{\omega})}$ with

$$d(\Phi) \leq C(\delta)d(\omega,\widetilde{\omega}).$$

Furthermore, from our construction of the mapping, we see that $\Phi(\mathbb{H}) = \mathbb{H}$. Therefore, Φ is extended to a quasiconformal self-mapping of $\widehat{\mathbb{C}}$ as desired. □

Proof of (2). Take any $\varepsilon > 0$. Since, $\log\frac{1-\widetilde{q}_n}{1-q_n} \to 0$ as $n \to \infty$, we also see that $q_n \to \widetilde{q}_n \to 0$. Viewing (4.11) and (4.18), we verify that there exists an $N \in \mathbb{N}$ such that

$$d(\varphi_k) < \frac{1}{2}\log(1+\varepsilon) \quad \text{and} \quad d(\psi_k) < \frac{1}{2}\log(1+\varepsilon),$$

if $k > N$. Hence, if $k > N$, then

$$d(\Phi_k) = d(\psi_k \circ \phi_k) \leq d(\psi_k) + d(\varphi_k) < \log(1+\varepsilon). \tag{4.20}$$

Since the pants decompositions in Step 2 of the proof (1) give exhaustions $X_{E(\omega)}$ and $X_{E(\widetilde{\omega})}$, (4.20) implies the maximal dilatation $K(\Phi) = e^{d(\Phi)}$ is less than $(1+\varepsilon)$ on the outside of a sufficiently large compact subset of $X_{E(\omega)}$. Therefore, $\Phi : X_{E(\omega)} \to X_{E(\widetilde{\omega})}$ is asymptotically conformal. □

5. Proof of Theorem III

Suppose that there exists a K-quasiconformal map from $X_{\mathcal{C}}$ to $X_{E(\omega)}$. Let $d > 0$ be the smallest hyperbolic length in all simple closed curves in $X_{\mathcal{C}}$. By Wolpert's formula (cf. [10], [12]), the hyperbolic length of any simple closed curve in $X_{E(\omega)}$ is not less than $K^{-1}d$.

Let $\varepsilon > 0$ be an arbitrary small constant. Since $\sup\{q_n \mid n \in \mathbb{N}\} = 1$, there exist a sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ and $N_0 \in \mathbb{N}$ such that

$$1 - \varepsilon < q_{n_k} < 1,$$

if $k > N_0$.

Now we look at $I^1_{q_k-1}$ of E_{q_k-1} for $k > N_0$. Then, $I^1_{q_k} \subset E_{q_k}$ is an interval of length $\frac{1}{2}(1-q_k)|I^1_{q_k-1}| < \frac{1}{2}\varepsilon|I^1_{q_k-1}|$. Therefore, we may take an annulus A_k in $X_{E(\omega)}$ bounded by two concentrated circles C^1_k, C^2_k such that the radius of C^1_k is $\frac{1}{4}\varepsilon|I^1_{q_k-1}|$ and that of C^2_k is $(\frac{1}{2} - \frac{1}{4}\varepsilon)|I^1_{q_k-1}|$. If we take $\varepsilon > 0$ sufficiently small, then the length of the core curve of A_k with respect to the hyperbolic metric on A_k becomes smaller than $K^{-1}d$. Since $A_k \subset X_{E(\omega)}$, the length of the core curve of A_k with respect to the hyperbolic metric of $X_{E(\omega)}$ is not greater than the length with respect to the hyperbolic metric of A_k. Thus, we find a closed curve in $X_{E(\omega)}$ whose length is less that $K^{-1}d$. It is a contradiction and we complete the proof of the theorem.

6. Proofs of Corollaries

Proof of Corollary 1.1. Let Λ be the limit set of the Schottky group G. We have shown ([11] Theorem 6.2) that X_Λ is quasiconformally equivalent to $X_{\mathcal{C}}$. Hence, it follows from Theorem I that X_E is quasiconformally equivalent to X_Λ as desired. □

Proof of Corollary 1.2. Since $\mathcal{C} = E(\omega_0)$ for $\omega_0 = (\frac{1}{3})_{n=1}^{\infty}$, the statement follows immediately from Theorem II (1). □

Proof of Corollary 1.3. Let $\varphi : X_\Lambda \to X_E$ be a quasiconformal map given by Corollary 1.1. Take any quasiconformal map ψ on X_E to $\widehat{\mathbb{C}}$. Then, $\Phi := \psi \circ \varphi$ be a quasiconformal map on X_Λ. It is known that any quasiconformal map on X_Λ is extended to a quasiconformal map on $\widehat{\mathbb{C}}$ (cf. [9]). Hence, both φ and Φ are extended to $\widehat{\mathbb{C}}$ and so is $\psi = \Phi \circ \varphi^{-1}$. □

Proof of Corollary 1.4. Let $\Psi : \mathbb{C} \to \mathbb{C}$ be the quasiconformal mapping given in §4. We put $D = \dim_H(E(\omega))$ and $\widetilde{D} = \dim_H(E(\widetilde{\omega}))$. We use the argument in the proof of Theorem II (2).

For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$K(\Phi_k) < 1 + \varepsilon$$

if $k > N$, where Φ_k is the quasiconformal mapping given in §4. Therefore, $\Phi|_{U_N}$ is a $(1+\varepsilon)$-quasiconformal mapping on $U_N := E(\omega) \cup \bigcup_{k>N} \bigcup_{i=1}^{2^k} P_k^i$. Here, we use the following result by Astala [3].

Proposition 6.1. *Let Ω, Ω' be planar domains and $f : \Omega \to \Omega'$ K-quasiconformal mapping. Suppose that $E \subset \Omega$ is a compact subset of Ω. Then,*

$$dim_H(f(E)) \leq \frac{2K\, dim_H(E)}{2 + (K-1)\, dim_H(E)}. \tag{6.1}$$

It follows from (6.1) that

$$\dim_H(E(\widetilde{\omega})) \leq \frac{2(1-\varepsilon)\dim_H(E(\omega))}{2 + \varepsilon \dim_H(E(\omega))}.$$

Since $\varepsilon > 0$ could be an arbitrary small, we obtain

$$\dim_H(E(\widetilde{\omega})) \leq \dim_H(E(\omega)).$$

By considering Φ^{-1}, we get the reverse inequality for $\dim_H(E(\omega))$ and $\dim_H(E(\widetilde{\omega}))$. Thus, we conclude that $\dim_H(E(\omega)) = \dim_H(E(\widetilde{\omega}))$ as desired. □

7. Examples

Example 7.1. Let $f_c(z) = z^2 + c$. Suppose that c is not in the Mandelbrot set. Then, it is well known that f_c is hyperbolic and the Julia set $\mathcal{J}_{f_c}$ is a Cantor set. Thus, f_c satisfies the condition of Theorem I.

Example 7.2. Let $B_0(z)$ be a Blaschke product of degree $d > 1$. Suppose that B_0 has an attracting fixed point on the unit circle $T := \{|z| = 1\}$. Since the Julia set $\mathcal{J}_{B_0}$ of B_0 is included in T, it has to be a Cantor set. It is also easy to see that B_0 is hyperbolic. Thus, B_0 satisfies the condition on Theorem I.

In Theorem II, we have estimated the maximal dilatations for sequences with lower bound. In next example, we may estimate the maximal dilatation for sequences without lower bound.

Example 7.3. For $0 < a < 1$ and a fixed $L \in \mathbb{N}$, we put $q_n = a^n$ and $\widetilde{q}_n = a^{n+L}$ and we consider $E(\omega)$, $E(\widetilde{\omega})$ for $\omega = (q_n)_{n=1}^{\infty}$, $\widetilde{\omega} = (\widetilde{q}_n)_{n=1}^{\infty}$. By using the same idea as in the proof of Theorem II, we claim that there exists an $\exp(Ca^{-L})$-quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi(E(\omega)) = E(\widetilde{\omega})$, where $C > 0$ is a constant independent of ω and $\widetilde{\omega}$.

Proof of the claim. We use the same notations for $E(\omega)$ and $E(\widetilde{\omega})$ as those in the proof of Theorem II. Then,

$$E_k = \cup_{i=1}^{2^k} I_k^i, \quad [0,1] = E_k \cup \bigcup_{i=1}^{2^k-1} J_k^i$$

and for $i = 1, 2, \ldots, 2^k$,

$$|I_k^i| = \left(\frac{1}{2}\right)^k \prod_{j=1}^{k} (1 - a^j).$$

If i is odd, then

$$|J_{k+1}^i| = a^{k+1}|I_k^1| \geq 2a^{k+1}|I_{k+1}^1|.$$

If $i = 2^\ell m$ $(1 \leq \ell \leq k; m$ is odd$)$, then we have

$$|J_{k+1}^i| = |J_{k-\ell+1}^m| \geq 4a^{k+1}|I_{k+1}^1|.$$

Thus, we conclude that

$$|J_{k+1}^i| \geq 2a^{k+1}|I_{k+1}^1|, \tag{7.1}$$

for $i = 1, 2, \ldots 2^{k+1} - 1$.

We draw a circle C_k^i centered at the midpoint of I_k^i with radius $\frac{1}{2}(1+a^k)|I_k^1|$ for each $k \in \mathbb{N}$ and $1 \leq i \leq 2^k$. From (7.1), we see that $C_k^i \cap C_k^j = \emptyset$ if $i \neq j$. Therefore, $\cup_{k=1}^{\infty} \cup_{i=1}^{2^k} C_k^i$ gives a pants decomposition of $X_{E(\omega)}$. We also draw circles $\widetilde{C}_k^i$ for $\widetilde{\omega}$ by the same way. Then, $\cup_{k=1}^{\infty} \cup_{i=1}^{2^k} \widetilde{C}_k^i$ gives a pants decomposition of $X_{E(\widetilde{\omega})}$.

We denote by P_k^i a pair of pants bounded by C_k^i, C_{k+1}^{2i-1} and C_{k+1}^{2i}. As in Step 3 of the proof of Theorem II, we may identify P_k^i with a pair of pants $\mathcal{P}_k$ bounded by $C_{k,1}, C_{k,2}$ and $C_{k,3}$, where $C_{k,1}$ is a circle centered at the origin with radius $1 + a^k$, $C_{k,2}$ is centered at

$$-x_k := -a^{k+1} - \frac{1}{2}(1 + a^{k+1})(1 - a^{k+1})$$

with radius

$$r_k := \frac{1}{2}(1 + a^{k+1})(1 - a^{k+1})$$

and $C_{k,3}$ is centered at x_k with radius r_k.

Similarly, we take a pair of pants $\widetilde{P}_k^i$ bounded by $\widetilde{C}_k^i, \widetilde{C}_{k+1}^{2i-1}$ and $\widetilde{C}_{k+1}^{2i}$, which is conformally equivalent to a pair of pants $\widetilde{\mathcal{P}}_k$ bounded by $\widetilde{C}_{k,1}, \widetilde{C}_{k,2}$ and $\widetilde{C}_{k,3}$, where $\widetilde{C}_{k,1}$ is the same circle as $C_{k,1}$, $\widetilde{C}_{k,2}$ is centered at

$$-\widetilde{x}_k := - - a^{k+L+1} - \frac{1}{2}(1 + a^{k+L+1})(1 - a^{k+L+1})$$

wirh radius

$$\widetilde{r}_k := \frac{1}{2}(1 + a^{k+L+1})(1 - a^{k+L+1})$$

and $\widetilde{C}_{k,3}$ is centered at $\widetilde{x}_k$ with radius $\widetilde{r}_k$.

We also take an intermediate pair of pants, $\widehat{P}_k$ similar to that of the proof of Theorem II. Then, by using exactly the same method, we may construct a $\exp(Ca^{-L})$-quasiconformal mapping from P_k^i onto $\widetilde{P}_k^i$, where $C > 0$ is a constant independent of k and i. Since the calculation is a bit long but the same as in §4, we may leave it to the reader.

By gluing those quasiconformal mappings together, we get an $\exp(Ca^{-L})$-quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi(E(\omega)) = E(\widetilde{\omega})$ as desired. □

Cantor Julia sets of Blaschke products with parabolic fixed points. We showed ([11] Example 3.2) that a Cantor set which is the limit set of an extended Schottky group is not quasiconformally equivalent to the limit set of a Schottky group. We discuss the same thing for Cantor sets defined by non-hyperbolic rational functions.

Let $B_1(z)$ be a Blaschke product with a parabolic fixed point on the unit circle T. Suppose that there exists only one attracting petal at the parabolic fixed point. Then, we see that the Julia set $\mathcal{J}_{B_1}$ is a Cantor set on T (see [5] IV. 2. Example). However, B_1 is not hyperbolic since it has a parabolic fixed point.

It follows from Theorem I that two Riemann surfaces $X_{\mathcal{J}_{f_c}}$ for Example 7.1 and $X_{\mathcal{J}_{B_0}}$ for Example 7.2 are quasiconformally equivalent. While the Julia set $\mathcal{J}_{B_1}$ of B_1 is also a Cantor set, it is not hyperbolic. Therefore, we cannot apply Theorem I for B_1.

Now, we consider the Martin compactification of the complement. For a general theory of the Martin compactification, we may refer to [6]. Here, we note the following.

Proposition 7.1. *Let B be a hyperbolic Blaschke product of degree $d > 1$. Suppose that the Julia set $\mathcal{J}_B$ is a Cantor set in T. Then, the Martin compactification of $X_{\mathcal{J}_B}$ is homeomorphic to $\widehat{\mathbb{C}}$.*

Hence, the same statements as in Proposition 7.1 hold for $X_{\mathcal{J}_0} := \widehat{\mathbb{C}} \setminus \mathcal{J}_0$ and the quasiconformal map φ on $X_{\mathcal{J}_0}$ is extended to a homeomorphism of the Martin compactification of $X_{\mathcal{J}_0}$.

Next, we consider the Martin compactification of $X_{\mathcal{J}_1}$, especially the set of the Martin boundary over the parabolic fixed point of B_1. If the set contains at least two points, then it follows from Proposition 7.1 that there exists no quasiconformal map from $X_{\mathcal{J}_0}$ to $X_{\mathcal{J}_1}$.

Indeed, in [9] we observe the Martin compactification of the complement of the limit set of an extended Schottky group and show that the set of the Martin boundary over a parabolic fixed point consists of more than two points. It is a key fact to show that the limit set of the extended Schottky group is not quasiconformally equivalent to that of a Schottky group ([11]). However, by using an argument of Benedicks ([4]) (see also Segawa [8]) on the Martin compactification, we may show the following.

Lemma 7.1. *In the Martin compactification of $X_{\mathcal{J}_1}$, there is exactly one minimal point over the parabolic fixed point of B_1.*

Remark 7.1. In the Martin compactification of a Riemann surface, the set corresponding to a topological boundary component of the Riemann surface is connected and the minimal points in the set are regarded as extreme points of a convex set. Thus, if the set over a boundary component on the

Martin compactification contains only one minimal point, then it consists of only one point, that is, the minimal point.

Proof. To prove the lemma, we use a result by Benedicks.

We denote by $Q(t,r)$ $(t\in\mathbb{R}, r>0)$, the square

$$\left\{x+iy \mid |x-t|<\frac{r}{2}, |y|<\frac{r}{2}\right\}.$$

For a fixed α with $0<\alpha<1$ and every $x\in\mathbb{R}\setminus\{0\}$, we consider the solution of the Dirichlet problem on $Q(x,\alpha|x|)\setminus E$ with boundary values one on $\partial Q(x,\alpha|x|)$ and zero on $E\cap Q(x,\alpha|x|)$. We denote the solution by β_x^E. Then, Benedicks showed the following.

Proposition 7.2. *On the Martin compactification of $\widehat{\mathbb{C}}\setminus E$, there exist more than two points over ∞ if and only if*

$$\int_{|x|\geq 1}\frac{\beta_x^E(x)}{|x|}dx<\infty. \tag{7.2}$$

Let $a\in T$ be the parabolic fixed point B_1. We take a Möbius transformation γ so that $\gamma(T)=\mathbb{R}\cup\{\infty\}$ and $\gamma(a)=\infty$. For $\widehat{B}_1:=\gamma B_1\gamma^{-1}$, we see that ∞ is a parabolic fixed point with a unique attracting petal of $\widehat{B}_1$, and $\mathcal{J}_1:=\gamma(\mathcal{J}_{B_1})$ is contained in $\mathbb{R}\cup\{\infty\}$.

Since $z=\infty$ is a parabolic fixed point of $\widehat{B}_1$ with only one attracting pegtal, we may assume that there exists a sufficiently large $M>0$ such that $\mathcal{J}_1\cap\{\mathrm{Re}\ z<-M\}$ is empty while $\mathcal{J}_1\cap\{\mathrm{Re}\ z>M\}$ is not empty. Hence, $\mathcal{J}_1\cap Q(x,\alpha|x|)=\emptyset$ if $x<0$ and $|x|$ is sufficiently large. Therefore, $\beta_x^{\mathcal{J}_1}(x)=1$ for such x. Thus, we have

$$\int_{|x|\geq 1}\frac{\beta_x^{\mathcal{J}_1}(x)}{|x|}dx=\infty$$

and conclude that there exists exactly one point over ∞ from Proposition 7.2. □

Lemma 7.1 implies that we cannot use the argument used for extended Schottky groups. We exhibit the following conjecture at the end of this article.

Conjecture. $X_{\mathcal{J}_1}$ is not quasiconformally equivalent to $X_{\mathcal{C}}$.

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