arXiv:1812.07837v1 [math.AG] 19 Dec 2018

GLOBAL PRYM-TORELLI THEOREM FOR DOUBLE COVERINGS OF ELLIPTIC CURVES

ATSUSHI IKEDA

Abstract. The Prym variety for a branched double covering of a nonsingular projective curve is defined as a polarized abelian variety. We prove that any double covering of an elliptic curve which has more than 4 branch points is recovered from its Prym variety.

1. Introduction

Let C and C' be nonsingular projective curves, and let $\phi : C \to C'$ be a double covering branched at $2n$ points. In [14] the Prym variety $P(\phi)$ for the double covering ϕ is defined as a polarized abelian variety of dimension $d = g' - 1 + n$, where g' is the genus of C'. Let $\mathcal{R} = \mathcal{R}_{g',2n}$ be the moduli space of such coverings, and let $\mathcal{A} = \mathcal{A}_d$ be the moduli space of polarized abelian varieties of dimension d. Then the construction of the Prym variety defines the Prym map $P : \mathcal{R} \to \mathcal{A}$, and the Prym-Torelli problem asks whether the Prym map is injective. If $g' = 0$, then it is injective by the classical Torelli theorem for hyperelliptic curves. We consider the case $g' > 0$ and $\dim \mathcal{R} \leq \dim \mathcal{A}$, where we note that $\dim \mathcal{R} = 3g' - 3 + 2n$ and $\dim \mathcal{A} = \frac{(g'-1+n)(g'+n)}{2}$. The generically injectivity for the Prym map has been proved in most cases.

Theorem 1.1. *The Prym map is generically injective in the following cases;*

1. (Friedman and Smith [7], Kanev [9]) $n = 0$ *and* $\dim \mathcal{R} < \dim \mathcal{A}$*,*
2. (Marcucci and Pirola [11]) $g' > 1$*,* $n > 0$ *and* $\dim \mathcal{R} < \dim \mathcal{A} - 1$*,*
3. (Naranjo and Ortega [17]) $g' > 1$*,* $n > 0$ *and* $\dim \mathcal{R} = \dim \mathcal{A} - 1$*,*
4. (Marcucci and Naranjo [10]) $g' = 1$*,* $n > 0$ *and* $\dim \mathcal{R} \leq \dim \mathcal{A}$*.*

The Prym varieties for unramified coverings have been intensively studied because they are principally polarized abelian varieties. For ramified coverings, Nagaraj and Ramanan [15] proved the above Theorem 1.1 (2) for $n = 2$, and then Marcucci and Pirola [11] proved it for any $n > 0$. When $g' > 1$ and $\dim \mathcal{R} = \dim \mathcal{A}$, there are only two cases $(g', n) = (6, 0), (3, 2)$. If $(g', n) = (6, 0)$ then the Prym map is generically finite of degree 27 ([6]), and if $(g', n) = (3, 2)$ then it is generically finite of degree 3 ([15], [4]).

Although the Prym map is not injective for many cases in Theorem 1.1 ([5], [16], [15], [18]), we prove the injectivity when $g' = 1$. The following is the main result of this paper, which improves Theorem 1.1 (4).

2010 *Mathematics Subject Classification.* 14H40, 14C34.

Theorem 1.2 (Theorem 3.1). *If $g' = 1$, $n > 0$ and $\dim \mathcal{R} \leq \dim \mathcal{A}$, then the Prym map is injective.*

To prove this theorem we use the Gauss map for the polarization divisor, which is a standard approach to Torelli problems. Let $\mathcal{L}$ be an ample invertible sheaf which represents the polarization of the Prym variety $P = P(\phi)$. For a member $D \in |\mathcal{L}|$, we consider the Gauss map

$$\Psi_D : D \setminus D_{\text{sing}} \longrightarrow \mathbf{P}^{d-1} = \text{Grass}\,(d-1, H^0(P, \Omega_P^1)^\vee).$$

It is not difficult to show that there exists a member $D_0 \in |\mathcal{L}|$ such that the branch divisor of Ψ_{D_0} recovers the original covering $\phi : C \to C'$ in a similar way as Andreotti's proof [1] of Torelli theorem for hyperelliptic curves. The essential part of our proof is to distinguish the special member $D_0 \in |\mathcal{L}|$. We study the restriction $\Psi_D|_{\text{Bs}\,|\mathcal{L}|} : \text{Bs}\,|\mathcal{L}| \setminus D_{\text{sing}} \to \mathbf{P}^{d-1}$ of the Gauss map to the base locus of the linear system $|\mathcal{L}|$. Although Ψ_D is difficult to compute, the restriction $\Psi_D|_{\text{Bs}\,|\mathcal{L}|}$ is rather simple for any member $D \in |\mathcal{L}|$. By using the image of $\Psi_D|_{\text{Bs}\,|\mathcal{L}|}$ and the branch divisor of $\Psi_D|_{\text{Bs}\,|\mathcal{L}|}$, we can specify the member $D_0 \in |\mathcal{L}|$ which has the desired property.

In Section 2, we summarize some basic properties of bielliptic curves and their Prym varieties. In Section 3, we explain the strategy of the proof of Theorem 1.2 by using the key Propositions in Section 6. In Section 4, we explicitly describe the base locus of the linear system of polarization divisors. In Section 5, we show that the restricted Gauss map $\Psi_D|_{\text{Bs}\,|\mathcal{L}|}$ is the same map as the restriction of the Gauss map for the theta divisor on Jacobian variety of C. By giving a simple description for $\Psi_D|_{\text{Bs}\,|\mathcal{L}|}$, we prove some properties on the branch divisor of $\Psi_D|_{\text{Bs}\,|\mathcal{L}|}$. In Section 6, we present key Propositions, which are consequences of the results in Section 5.

In this paper, we work over an algebraically closed field k of characteristic $\neq 2$.

2. Properties of bielliptic curves and Prym varieties

Let C be a nonsingular projective curve of genus g over k, and let σ be an involution on C. In this paper, we call the pair (C, σ) a bielliptic curve of genus g, if $g > 1$ and the quotient $E = C/\sigma$ is a nonsingular curve of genus 1. We denote by $\phi : C \to E$ the quotient morphism. First we note the following.

Lemma 2.1 ([16] (3.3)). *Let (C, σ) be a bielliptic curve of genus g. If $g > 3$, then C is not a hyperelliptic curve.*

Let $N : J(C) \to J(E)$ be the norm map of ϕ, which is a homomorphism on their Jacobian varieties.

Lemma 2.2 ([14]). *Let $\phi : C \to E$ be the covering defined from a bielliptic curve (C, σ).*

1. $\phi^* : \text{Pic}^0(E) \to \text{Pic}^0(C)$ *is injective.*
2. *The kernel P of the norm map $N : J(C) \to J(E)$ is reduced and connected.*

By Lemma 2.2, the kernel P of the norm map N is an abelian variety of dimension $n = g - 1$. Let $P^\vee$ be the dual abelian variety of P, and let $\lambda_P : P \to P^\vee$ be the polarization isogeny which is defined as the restriction of the principal polarization on the Jacobian variety $J(C)$. Then the polarized abelian variety (P, λ_P) is called the Prym variety for the covering $\phi : C \to E$. We denote by $K(P) \subset P$ the kernel of the polarization $\lambda_P : P \to P^\vee$. An ample invertible sheaf $\mathcal{L}$ on P represents the the polarization λ_P, if the polarization isogeny λ_P is given by

$$\lambda_P : P(k) \longrightarrow P^\vee(k) = \mathrm{Pic}^0(P);\, x \longmapsto t_x^*\mathcal{L} \otimes \mathcal{L}^\vee,$$

where $t_x : P \to P$ denotes the translation by $x \in P(k)$.

Lemma 2.3 ([14])**.** *Let (P, λ_P) be the Prym variety defined from a bielliptic curve (C, σ), and let $\mathcal{L}$ be an ample invertible sheaf which represents λ_P.*

(1) $K(P) = \phi^* J(E)_2 \subset J(C)$*, where $J(E)_2$ denotes the set of points of order 2 on $J(E)$.*
(2) $\deg \lambda_P = 4$ *and* $h^0(P, \mathcal{L}) = 2$.

3. Proof of Main theorem

The main result of this paper is the following.

Theorem 3.1. *If $g > 3$, then the isomorphism class of a bielliptic curve of genus g is determined by the isomorphism class of its Prym variety.*

Let (P, λ_P) be the Prym variety of dimension $n \geq 3$ defined from a bielliptic curve (C, σ) of genus $g = n + 1$. We will recover the data $(E, e_1 + \cdots + e_{2n}, \eta)$ from the polarized abelian variety (P, λ_P), where $E = C/\sigma$ is the quotient curve, $e_1 + \cdots + e_{2n}$ is the branch divisor of the covering $\phi : C \to E$, and $\eta \in \mathrm{Pic}(E)$ is the invertible sheaf with $\phi^*\eta \cong \Omega_C^1$. We remark that $\eta^{\otimes 2} \cong \mathcal{O}_E(e_1 + \cdots + e_{2n})$, and η is the invertible sheaf which determines the double covering with the branch divisor $e_1 + \cdots + e_{2n}$.

Proof of Theorem 3.1. Let $\mathcal{L}$ be an ample invertible sheaf on P which represents the the polarization λ_P. We denote by $K(P)$ the Kernel of $\lambda_P : P \to P^\vee$. By Lemma 2.3, we have $\sharp K(P) = 4$ and $h^0(P, \mathcal{L}) = 2$. We define the subset $\Pi_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi_{\mathcal{L}} = \{D \in |\mathcal{L}| \mid t_x(D) = D \subset P \text{ for some } x \in K(P) \setminus \{0\}\},$$

where t_x is the translation by $x \in P(k)$. By Lemma 4.8, $\Pi_{\mathcal{L}}$ is a set of 6 members for any representative $\mathcal{L}$ of the polarization λ_P. For a member $D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, we consider the Gauss map

$$\Psi_D : D \setminus D_{\mathrm{sing}} \longrightarrow \mathbf{P}^{n-1} = \mathrm{Grass}\,(n-1, H^0(P, \Omega_P^1)^\vee),$$

where $\Psi_D(x)$ is defined by the inclusion $T_x(D) \subset T_x(P) \cong H^0(P, \Omega_P^1)^\vee$ of the tangent spaces at the point $x \in D \setminus D_{\mathrm{sing}} \subset P$. We set $U_D = \mathrm{Bs}\,|\mathcal{L}| \setminus D_{\mathrm{sing}}$, where $\mathrm{Bs}\,|\mathcal{L}| \subset P$ denotes the set of base points of the pencil $|\mathcal{L}|$. Let $X'_D = \overline{\Psi_D(U_D)} \subset \mathbf{P}^{n-1}$ be the Zariski closure of $\Psi_D(U_D) \subset \mathbf{P}^{n-1}$, and let $\nu_D : X_D \to$

X'_D be the normalization. By Lemma 6.1, U_D is a nonsingular variety, hence there is a unique morphism $\psi_D : U_D \to X_D$ such that $\Psi_D|_{U_D} = \nu_D \circ \psi_D$. We consider the closed subset $Z_D = \overline{\psi_D(\mathrm{Ram}\,(\psi_D))} \subset X_D$, where $\mathrm{Ram}\,(\psi_D) \subset U_D$ denotes the ramification divisor of ψ_D. By Proposition 6.2, Z_D has a canonical decomposition $Z_D = \bigcup_{i=1}^{2n} Z_{D,i}$, and there is a unique hyperplane $H_{D,i} \subset \mathbf{P}^{n-1}$ such that $\nu_D(Z_{D,i}) \subset H_{D,i}$ for any $1 \leq i \leq 2n$. Then the effective divisor $\nu_D^* H_{D,i} - Z_{D,i}$ on X_D has 2 irreducible components for general $D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, and these components coincide for special $D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}}$. We define the subset $\Pi'_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi'_{\mathcal{L}} = \{D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}} \mid \nu_D^* H_{D,i} - Z_{D,i} \text{ is irreducible for } 1 \leq i \leq 2n\}.$$

By Lemma 6.3, $\Pi'_{\mathcal{L}}$ is a set of 4 members for any representative $\mathcal{L}$ of the polarization λ_P. For a member $D \in \Pi'_{\mathcal{L}}$, we consider the dual variety $(X'_D)^\vee \subset (\mathbf{P}^{n-1})^\vee$ of $X'_D \subset \mathbf{P}^{n-1}$ and the dual variety $H_{D,i}^\vee \subset (\mathbf{P}^{n-1})^\vee$ of $H_{D,i} \subset \mathbf{P}^{n-1}$. By Proposition 6.4, $H_{D,i}^\vee$ is a point on $(X'_D)^\vee$, and we have an isomorphism

$$(E, e_1 + \cdots + e_{2n}, \eta) \cong ((X'_D)^\vee, H_{D,1}^\vee + \cdots + H_{D,2n}^\vee, \mathcal{O}_{(\mathbf{P}^{n-1})^\vee}(1)|_{(X'_D)^\vee}).$$

□

4. Pencil of polarization divisors

Let (C, σ) be a bielliptic curve of genus $g = n + 1 > 3$. For $\delta \in \mathrm{Pic}^n\,(C)$, we set the divisor $W_\delta \subset J(C)$ by

$$W_\delta(k) = \{L \in \mathrm{Pic}^0\,(C) = J(C)(k) \mid h^0(C, L \otimes \delta) > 0\}.$$

We remark that the singular locus of W_δ is given by

$$W_{\delta,\mathrm{sing}}(k) = \{L \in \mathrm{Pic}^0\,(C) \mid h^0(C, L \otimes \delta) > 1\},$$

and $\dim W_{\delta,\mathrm{sing}} = n - 3$ ([2, Proposition 8]), because C is not a hyperelliptic curve by Lemma 2.1. Let $\lambda_C : J(C) \to J(C)^\vee$ be the homomorphism defined by

$$\lambda_C :\ J(C)(k) \to J(C)^\vee(k) = \mathrm{Pic}^0\,(J(C));\ x \longmapsto [t_x^* \mathcal{O}_C(W_\delta) \otimes \mathcal{O}_C(-W_\delta)],$$

which does not depend on the choice of $\delta \in \mathrm{Pic}^n\,(C)$. Let $\iota_q : C \to J(C)$ be the morphism defined by

$$\iota_q :\ C(k) \longrightarrow \mathrm{Pic}^0\,(C) = J(C)(k);\ q' \longmapsto [\mathcal{O}_C(q' - q)].$$

for $q \in C(k)$.

Lemma 4.1.

$$x = \iota_q^*[\mathcal{O}_{J(C)}(W_\delta - W_{\delta+x})] \in \mathrm{Pic}^0\,(C)$$

for any $q \in C(k)$ and $x \in \mathrm{Pic}^0\,(C)$.

Proof. The statement means that $(-1) \circ \lambda_C$ is the inverse of the homomorphism $\iota_q^* : J(C)^\vee \to J(C)$ defined by the pull-buck $\iota_q^* : \mathrm{Pic}^0\,(J(C)) \to \mathrm{Pic}^0\,(C)$ of invertible sheaves. It is well-known ([13, Lemma 6.9]). □

Let P be the kernel of the Norm map $N : J(C) \to J(E)$, and let $D_\delta \subset P$ the fiber of the restriction of the norm map $N|_{W_\delta} : W_\delta \to J(E)$ at $0 \in J(E)$. We denote by $\mathcal{L}_\delta = \mathcal{O}_P(D_\delta) = \mathcal{O}_{J(C)}(W_\delta)|_P$ the restriction of $\mathcal{O}_{J(C)}(W_\delta)$ to P. Since W_δ is the theta divisor of $J(C)$, the ample invertible sheaf $\mathcal{L}_\delta$ represents the the polarization λ_P.

Lemma 4.2. *$D_{\delta+\phi^* s} \subset P$ is a member of the linear system $|\mathcal{L}_\delta|$ for any $s \in \mathrm{Pic}^0(E)$.*

Proof. By Lemma 4.1,

$$\phi^* s = \iota_q^*[\mathcal{O}_{J(C)}(W_\delta - W_{\delta+\phi^* s})] \in \mathrm{Pic}^0(C)$$

for $s \in \mathrm{Pic}^0(E)$ and $q \in C(k)$. We set $s' \in \mathrm{Pic}^0(J(E))$ by $s = \iota_{\phi(q)}^* s'$, where $\iota_{\phi(q)} : E \xrightarrow{\sim} J(E)$ is the isomorphism determined by $\iota_{\phi(q)}(\phi(q)) = 0$. Then we have

$$N^* s' = [\mathcal{O}_{J(C)}(W_\delta - W_{\delta+\phi^* s})] \in \mathrm{Pic}^0(J(C)),$$

because $\phi^* s = \iota_q^* N^* s'$ and $\iota_q^* : \mathrm{Pic}^0(J(C)) \to \mathrm{Pic}^0(C)$ is an isomorphism. Since $(N^* s')|_P = 0 \in \mathrm{Pic}^0(P)$, we have

$$\mathcal{O}_P(D_{\delta+\phi^* s}) \cong \mathcal{O}_{J(C)}(W_{\delta+\phi^* s})|_P \cong \mathcal{O}_{J(C)}(W_\delta)|_P = \mathcal{L}_\delta.$$

□

We denote by $C^{(i)}$ the i-th symmetric products of C. For $\delta \in \mathrm{Pic}^n(C)$, we define the morphism $\beta_\delta^i : C^{(n-2i)} \times E^{(i)} \to J(C)$ by

$$\beta_\delta^i : C^{(n-2i)}(k) \times E^{(i)}(k) \longrightarrow J(C)(k) = \mathrm{Pic}^0(C);$$

$$(q_1 + \cdots + q_{n-2i}, p_1 + \cdots + p_i) \longmapsto \mathcal{O}_C\Big(\sum_{j=1}^{n-2i} q_j\Big) \otimes \phi^* \mathcal{O}_E\Big(\sum_{j=1}^{i} p_j\Big) \otimes \delta^\vee.$$

We remark that $W_\delta = \mathrm{Image}(\beta_\delta^0)$, and we set

$$B_\delta^i = \begin{cases} \mathrm{Image}(\beta_\delta^i) & (1 \le 2i \le n), \\ \emptyset & (2i > n). \end{cases}$$

Lemma 4.3. *$B_\delta^1 \setminus W_{\delta,\mathrm{sing}} \neq \emptyset$ and $B_\delta^2 \subset W_{\delta,\mathrm{sing}}$.*

Proof. Let B^1 be the image of the morphism $\beta^1 : C^{(n-2)} \times E \to C^{(n)}$ defined by

$$\beta^1 : C^{(n-2)}(k) \times E(k) \to C^{(n)}(k);\ (q_1 + \cdots + q_{n-2}, \phi(q)) \longmapsto q_1 + \cdots + q_{n-2} + q + \sigma(q).$$

Since C is not a hyperelliptic curve, we have $\dim (\beta_\delta^0)^{-1}(W_{\delta,\mathrm{sing}}) = n-2 < n-1 = \dim B^1$, hence $B_\delta^1 = \beta_\delta^0(B^1) \nsubseteq W_{\delta,\mathrm{sing}}$.

To prove the second statement, we assume that $n \ge 4$, because $B_\delta^2 = \emptyset$ for $n = 3$. Let $F \subset C^{(n-4)} \times E^{(2)}$ be the fiber of the composition

$$C^{(n-4)} \times E^{(2)} \xrightarrow{\beta_\delta^2} J(C) \xrightarrow{N} J(E)$$

at $\eta - N(\delta) \in J(E)(k)$, where $\eta \in \mathrm{Pic}^n(E)$ denotes the unique invertible sheaf on E with $\phi^* \eta \cong \Omega_C^1$. We set $U = (C^{(n-4)} \times E^{(2)}) \setminus (F \cup (C^{(n-4)} \times \Delta_E))$, where $\Delta_E \subset E^{(2)}$

denotes the image of the diagonal in $E\times E$. For $y=(q_1+\cdots+q_{n-4},\phi(q)+\phi(r))\in U(k)$, there are points $q',r'\in C(k)$ such that

$$\begin{cases}\mathcal{O}_E(\phi(q'))\cong\eta\otimes\mathcal{O}_E(-\phi(q_1)-\cdots-\phi(q_{n-4})-\phi(q)-2\phi(r)),\\ \mathcal{O}_E(\phi(r'))\cong\eta\otimes\mathcal{O}_E(-\phi(q_1)-\cdots-\phi(q_{n-4})-2\phi(q)-\phi(r)).\end{cases}$$

Then we have

$$\begin{aligned}&\Omega^1_C(-q_1-\cdots-q_{n-4}-q-\sigma(q)-r-\sigma(r))\\ \cong&\phi^*\eta\otimes\mathcal{O}_C(-q_1-\cdots-q_{n-4}-q-\sigma(q)-r-\sigma(r))\\ \cong&\mathcal{O}_C(\sigma(q_1)+\cdots+\sigma(q_{n-4})+q'+\sigma(q')+r+\sigma(r))\\ \cong&\mathcal{O}_C(\sigma(q_1)+\cdots+\sigma(q_{n-4})+q+\sigma(q)+r'+\sigma(r')).\end{aligned}$$

We remark that $\phi(q)\neq\phi(q')$ and $\phi(q)\neq\phi(r)$, because $y\notin F$ and $y\notin C^{(n-4)}\times\Delta_E$. Hence we have $h^0(C,\Omega^1_C(-q_1-\cdots-q_{n-4}-q-\sigma(q)-r-\sigma(r)))>1$ and $\beta^2_\delta(y)\in W_{\delta,\mathrm{sing}}$. Since $U\subset(\beta^2_\delta)^{-1}(W_{\delta,\mathrm{sing}})$ is a dense subset of $C^{(n-4)}\times E^{(2)}$, we have $C^{(n-4)}\times E^{(2)}=(\beta^2_\delta)^{-1}(W_{\delta,\mathrm{sing}})$. □

Lemma 4.4. *For $s\in\mathrm{Pic}^0(E)$, $D_\delta=D_{\delta+\phi^*s}\subset P$ if and only if $s=0$ or $s=\eta-N(\delta)$.*

Proof. For $L\in P(k)\subset\mathrm{Pic}^0(C)$, we have $0=\phi^*N(L)=L+\sigma^*L\in\mathrm{Pic}^0(C)$. Hence we have $D_\delta=D_{\delta+\phi^*(\eta-N(\delta))}$, because

$$\begin{aligned}L\in D_\delta(k)\Longleftrightarrow h^0(C,L\otimes\delta)>0\Longleftrightarrow h^0(C,\sigma^*L\otimes\sigma^*\delta)>0\Longleftrightarrow h^0(C,L^\vee\otimes\sigma^*\delta)>0\\ \Longleftrightarrow h^0(C,\Omega^1_C\otimes L\otimes\sigma^*\delta^\vee)>0\Longleftrightarrow L\in D_{[\Omega^1_C]-\sigma^*\delta}(k)=D_{\delta+\phi^*(\eta-N(\delta))}(k).\end{aligned}$$

We assume that $D_\delta=D_{\delta+\phi^*s}$ for $s\neq0\in\mathrm{Pic}^0(E)$. Let $\alpha_{\delta+\phi^*s}:C^{(n-2)}\times C\to J(C)$ be the morphism defined by

$$\begin{aligned}\alpha_{\delta+\phi^*s}:\ &C^{(n-2)}(k)\times C(k)\longrightarrow\mathrm{Pic}^0(C)=J(C)(k);\\ &(q_1+\cdots+q_{n-2},q)\longmapsto\mathcal{O}_C(q_1+\cdots+q_{n-2}+2q)\otimes\delta^\vee\otimes\phi^*s^\vee.\end{aligned}$$

Then the set $D_\delta\setminus(W_{\delta,\mathrm{sing}}\cup B^1_\delta\cup\mathrm{Image}(\alpha_{\delta+\phi^*s}))$ is not empty, because

$$\dim D_\delta\cap(W_{\delta,\mathrm{sing}}\cup B^1_\delta\cup\mathrm{Image}(\alpha_{\delta+\phi^*s}))<n-1=\dim D_\delta.$$

For $L\in D_\delta(k)\setminus(W_{\delta,\mathrm{sing}}\cup B^1_\delta\cup\mathrm{Image}(\alpha_{\delta+\phi^*s}))$, there is $r_1+\cdots+r_n\in C^{(n)}(k)$ such that $L\otimes\delta\otimes\phi^*s\cong\mathcal{O}_C(r_1+\cdots+r_n)$, because $L\in D_\delta(k)=D_{\delta+\phi^*s}(k)\subset W_{\delta+\phi^*s}(k)$. Since $L\in W_\delta(k)\setminus W_{\delta,\mathrm{sing}}(k)$, we have $h^0(C,\Omega^1_C\otimes L^\vee\otimes\delta^\vee)=h^0(C,L\otimes\delta)=1$. Let $q_1+\cdots+q_n\in C^{(n)}(k)$ and $q'_1+\cdots+q'_n\in C^{(n)}(k)$ be the effective divisors defined by

$$L\otimes\delta\cong\mathcal{O}_C(q_1+\cdots+q_n),\quad\Omega^1_C\otimes L^\vee\otimes\delta^\vee\cong\mathcal{O}_C(q'_1+\cdots+q'_n).$$

Let $\phi(u_i)\in E(k)$ be the point determined by $s=[\mathcal{O}_E(\phi(r_i)-\phi(u_i))]$. Then

$$L\otimes\delta\otimes\mathcal{O}_C(\sigma(r_i))\cong\mathcal{O}_C(r_1+\cdots+r_n-r_i+u_i+\sigma(u_i)).$$

If $\sigma(r_i)\notin\{q'_1,\ldots,q'_n\}$, then

$$h^0(C,L\otimes\delta\otimes\mathcal{O}_C(\sigma(r_i)))=h^0(C,\Omega^1_C\otimes L^\vee\otimes\delta^\vee\otimes\mathcal{O}_C(-\sigma(r_i)))+1=1,$$

hence

$$q_1 + \cdots + q_n + \sigma(r_i) = r_1 + \cdots + r_n - r_i + u_i + \sigma(u_i).$$

Since $s \neq 0$, we have $\sigma(r_i) = r_j$ for some $j \neq i$, and $L \in B^1_\delta(k)$. It is a contradiction to $L \notin B^1_\delta(k)$, hence $\sigma(r_i) \in \{q'_1, \ldots, q'_n\}$ for any $1 \leq i \leq n$. Here the condition $L \notin \operatorname{Image}(\alpha_{\delta+\phi^*s})$ implies that $\sharp\{r_1, \ldots, r_n\} = n$ and

$$L^\vee \otimes \sigma^*\delta \otimes \phi^*s \cong \mathcal{O}_C(\sigma(r_1) + \cdots + \sigma(r_n)) = \mathcal{O}_C(q'_1 + \cdots + q'_n) \cong \Omega^1_C \otimes L^\vee \otimes \delta^\vee.$$

Hence we have $\phi^*s = [\Omega^1_C] - \delta - \sigma^*\delta = \phi^*(\eta - N(\delta))$, and $s = \eta - N(\delta)$ by Lemma 2.2. □

Let $B_\delta \subset J(C)$ be the subset

$$B_\delta = \bigcap_{s \in \operatorname{Pic}^0(E)} W_{\delta+\phi^*s}.$$

Lemma 4.5. $B_\delta \setminus W_{\delta,\mathrm{sing}} = B^1_\delta \setminus W_{\delta,\mathrm{sing}}$.

Proof. If $L \in B^1_\delta(k)$, then $L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + q + \sigma(q))$ for some $q_1, \ldots, q_{n-2}, q \in C(k)$. For $s \in \operatorname{Pic}^0(E)$, there is a point $q' \in C(k)$ such that $s = [\mathcal{O}_E(\phi(q') - \phi(q))]$. Since $L \otimes \delta \otimes \phi^*s \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + q' + \sigma(q'))$, we have $h^0(C, L \otimes \delta \otimes \phi^*s) > 0$ and $L \in W_{\delta+\phi^*s}(k)$. Hence the inclusion $B^1_\delta \subset B_\delta$ holds.

For $L \in B_\delta(k) \setminus W_{\delta,\mathrm{sing}}(k)$, there is a unique $r_1 + \cdots + r_n \in C^{(n)}(k)$ such that $\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(r_1 + \cdots + r_n)$, because $h^0(C, \Omega^1_C \otimes L^\vee \otimes \delta^\vee) = h^0(C, L \otimes \delta) = 1$. Let $\Sigma \subset \operatorname{Pic}^0(E)$ be the finite subset defined by

$$\Sigma = \{s \in \operatorname{Pic}^0(E) \mid L \otimes \phi^*s \in \beta^0_\delta(\Delta^{(n)})\},$$

where $\Delta = \{\sigma(r_1), \ldots, \sigma(r_n)\} \subset C$ and $\Delta^{(n)} \subset C^{(n)}$. For $s \in \operatorname{Pic}^0(E) \setminus (\Sigma \cup \{0\})$, there is a divisor $q_1 + \cdots + q_n \in C^{(n)}(k)$ such that $L \otimes \delta \otimes \phi^*s \cong \mathcal{O}_C(q_1 + \cdots + q_n)$, because $L \in B_\delta(k) \subset W_{\delta+\phi^*s}(k)$. Since $s \notin \Sigma$, we may assume that $q_n \notin \Delta$. The condition $\sigma(q_n) \notin \{r_1, \ldots, r_n\}$ implies that $h^0(C, \Omega^1_C \otimes L^\vee \otimes \delta^\vee \otimes \mathcal{O}_C(-\sigma(q_n))) = 0$ and $h^0(C, L \otimes \delta \otimes \mathcal{O}_C(\sigma(q_n))) = 1$. Let $\phi(q') \in E(k)$ be the point determined by $s = [\mathcal{O}_E(\phi(q_n) - \phi(q'))]$. Then

$$L \otimes \delta \otimes \mathcal{O}_C(\sigma(q_n)) \cong \mathcal{O}_C(q_1 + \cdots + q_{n-1} + q' + \sigma(q')).$$

Since $s \neq 0$, we have $\sigma(q_n) \in \{q_1, \ldots, q_{n-1}\}$ and $L \in B^1_\delta(k)$. □

Lemma 4.6. *The map*

$$\operatorname{Pic}^0(E) \longrightarrow |\mathcal{L}_\delta|;\ s \longmapsto D_{\delta+\phi^*s}$$

is a double covering, and the base locus $\operatorname{Bs}|\mathcal{L}_\delta|$ *of the linear system* $|\mathcal{L}_\delta|$ *is* $B_\delta \cap P$, *which is of dimension* $n-2$.

Proof. The map is well-defined by Lemma 4.2. Since $\dim|\mathcal{L}_\delta| = 1$, it is a double covering by Lemma 4.4. Hence we have

$$\operatorname{Bs}|\mathcal{L}_\delta| = \bigcap_{s \in \operatorname{Pic}^0(E)} D_{\delta+\phi^*s} = B_\delta \cap P.$$

By Lemma 4.3, B^1_δ is irreducible of dimension $n-1$. Since the restriction of the Norm map $N|_{B^1_\delta} : B^1_\delta \to J(E)$ is surjective, we have $\dim B^1_\delta \cap P = n-2$, hence $\dim B_\delta \cap P = n-2$ by Lemma 4.5. □

Lemma 4.7. *Let $\mathcal{L}$ be an ample invertible sheaf which represents the polarization λ_P on P, then there is $\delta \in \operatorname{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$.*

Proof. For any $\delta' \in \operatorname{Pic}^n(C)$, we have $\mathcal{L} \otimes \mathcal{L}^\vee_{\delta'} \in \operatorname{Pic}^0(P)$, because $\mathcal{L}_{\delta'}$ gives the same polarization as λ_P. Then $\mathcal{L} \cong t^*_x \mathcal{L}_{\delta'} \cong \mathcal{L}_{\delta'+x}$ for some $x \in P(k)$. Let $s \in \operatorname{Pic}^0(E)$ be a point with $2s = \eta - N(\delta' + x)$. For $\delta = \delta' + x + \phi^* s$, we have $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$ □

For an ample invertible sheaf $\mathcal{L}$ which represents the polarization λ_P, we set a subset in the linear system $|\mathcal{L}|$ by

$$\Pi_{\mathcal{L}} = \{D \in |\mathcal{L}| \mid t_x(D) = D \text{ for some } x \in K(P) \setminus \{0\}\},$$

where $K(P)$ is the kernel of the polarization λ_P.

Lemma 4.8. $\sharp \Pi_{\mathcal{L}} = 6$.

Proof. By Lemma 4.7, there is $\delta \in \operatorname{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$. For any $D \in |\mathcal{L}_\delta|$, by Lemma 4.6, there is $s \in \operatorname{Pic}^0(E)$ such that $D = D_{\delta+\phi^* s}$. If $D_{\delta+\phi^* s} \in \Pi_{\mathcal{L}_\delta}$, then by Lemma 2.3, there is $t \in J(E)_2 \setminus \{0\}$ such that $t_{\phi^* t}(D_{\delta+\phi^* s}) = D_{\delta+\phi^* s}$. Since $t_{\phi^* t}(D_{\delta+\phi^* s}) = D_{\delta+\phi^*(s-t)}$ and $t \neq 0$, by Lemma 4.4, we have

$$\delta + \phi^*(s-t) = \delta + \phi^* s + \phi^*(\eta - N(\delta + \phi^* s)) = \delta - \phi^* s,$$

hence $t = 2s$ by Lemma 2.2. It means that

$$\Pi_{\mathcal{L}_\delta} = \{D_{\delta+\phi^* s} \in |\mathcal{L}_\delta| \mid s \in J(E)_4 \setminus J(E)_2\}.$$

Since $\sharp(J(E)_4 \setminus J(E)_2) = 12$ and $D_{\delta+\phi^* s} = D_{\delta-\phi^* s}$, we have $\sharp \Pi_{\mathcal{L}} = 6$. □

5. Gauss maps

5.1. Gauss map for Jacobian and Gauss map for Prym.

Let

$$\Psi_{J(C),\delta} : W_\delta \setminus W_{\delta,\mathrm{sing}} \longrightarrow \mathbf{P}(H^0(C, \Omega^1_C)^\vee) = \operatorname{Grass}(n, H^0(C, \Omega^1_C)^\vee)$$

be the Gauss map for the subvariety $W_\delta \subset J(C)$. For $L \in W_\delta(k) \setminus W_{\delta,\mathrm{sing}}(k)$, the tangent space $T_L(W_\delta)$ of W_δ at L defines the image $\Psi_{J(C),\delta}(L)$ by the natural identifications

$$T_L(W_\delta) \subset T_L(J(C)) \cong (\Omega^1_{J(C)}(L))^\vee \cong H^0(J(C), \Omega^1_{J(C)})^\vee \cong H^0(C, \Omega^1_C)^\vee.$$

Lemma 5.1. *For $L \in W_\delta(k) \setminus W_{\delta,\mathrm{sing}}(k)$, the image $\Psi_{J(C),\delta}(L)$ of the Gauss map is identified with the canonical divisor*

$$q_1 + \cdots + q_n + q'_1 + \cdots + q'_n \in |\Omega^1_C| = \operatorname{Grass}(1, H^0(C, \Omega^1_C)) \cong \mathbf{P}(H^0(C, \Omega^1_C)^\vee),$$

where the effective divisors $q_1 + \cdots + q_n$ and $q'_1 + \cdots + q'_n$ are uniquely determined by $L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_n)$ and $\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(q'_1 + \cdots + q'_n)$.

Proof. It is a special case of Proposition (4.2) in [3, Chapter IV]. □

Lemma 5.2. *Let $K \in |\Omega^1_C|$ be an effective canonical divisor. If $q_1 + \sigma(q_1) \leq K$ for some $q_1 \in C(k)$, then $K = \sum_{i=1}^n (q_i + \sigma(q_i))$ for some $q_2, \dots, q_n \in C(k)$.*

Proof. When

$$K = \sum_{i=1}^{m} (q_i + \sigma(q_i)) + q + \sum_{j=1}^{2n-2m-1} r_j$$

for $1 \leq m \leq n-1$, we show that $\sigma(q) \in \{r_1, \dots, r_{2n-2m-1}\}$. First we assume that $1 \leq m \leq n-2$. Since C is not a hyperelliptic curve, by Clifford's theorem, we have

$$m+2 > h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i+\sigma(q_i))+q+\sigma(q))) \geq h^0(E, \mathcal{O}_E(\sum_{i=1}^{m} \phi(q_i)+\phi(q))) = m+1$$

and

$$m+1 > h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i)))) \geq h^0(E, \mathcal{O}_E(\sum_{i=1}^{m} \phi(q_i))) = m,$$

hence $h^0(C, \mathcal{O}_C(\sum_{i=1}^m (q_i + \sigma(q_i)) + q + \sigma(q))) = m+1$ and $h^0(C, \mathcal{O}_C(\sum_{i=1}^m (q_i + \sigma(q_i)))) = m$. Since $\sigma(q)$ is not a base point of $|\mathcal{O}_C(\sum_{i=1}^m (q_i + \sigma(q_i)) + q + \sigma(q))| = \phi^*|\mathcal{O}_E(\sum_{i=1}^m \phi(q_i) + \phi(q))|$, we have

$$m = h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i+\sigma(q_i))+q)) < h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i+\sigma(q_i))+q+\sigma(q))) = m+1,$$

hence

$$h^0(C, \mathcal{O}_C(\sum_{j=1}^{2n-2m-1} r_j - \sigma(q))) = h^0(C, \mathcal{O}_C(\sum_{j=1}^{2n-2m-1} r_j)) = n-m-1.$$

It implies that $\sigma(q) \leq \sum_{j=1}^{2n-2m-1} r_j$. We consider the case $m = n-1$. Let $\eta \in \operatorname{Pic}^n(E)$ be the invertible sheaf with $\phi^*\eta \cong \Omega^1_C$. There is a point $q' \in C(k)$ such that $\sum_{j=1}^{n-1} \phi(q_i) + \phi(q') \in |\eta|$. Then $\mathcal{O}_C(q + r_1) \cong \mathcal{O}_C(q' + \sigma(q'))$. Since C is not a hyperelliptic curve, we have $q + r_1 = q' + \sigma(q')$ and $\sigma(q) = r_1$. □

By the injective homomorphism

$$H^0(E, \eta) \longrightarrow H^0(C, \phi^*\eta) \cong H^0(C, \Omega^1_C),$$

we have the closed immersion

$$\iota : \mathbf{P}(H^0(E, \eta)^\vee) \longrightarrow \mathbf{P}(H^0(C, \Omega^1_C)^\vee).$$

Lemma 5.3. *For $L \in B_\delta(k) \setminus W_{\delta,\mathrm{sing}}(k)$,*

$$\Psi_{J(C),\delta}(L) \in \iota(\mathbf{P}(H^0(E, \eta)^\vee)) \subset \mathbf{P}(H^0(C, \Omega^1_C)^\vee).$$

Proof. For $L \in B_\delta(k) \setminus W_{\delta,\mathrm{sing}}(k)$, by Lemma 5.1, the image $\Psi_{J(C),\delta}(L)$ of the Gauss map is given by

$$q_1 + \cdots + q_n + q'_1 + \cdots + q'_n \in |\Omega^1_C| \cong \mathbf{P}(H^0(C, \Omega^1_C)^\vee),$$

where the effective divisors $q_1 + \cdots + q_n$ and $q'_1 + \cdots + q'_n$ are uniquely determined by $L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_n)$ and $\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(q'_1 + \cdots + q'_n)$. Since $L \in B_\delta(k)$, by Lemma 4.5, $\sigma(q_i) = q_j$ for some $i \neq j$. By Lemma 5.2, we have $\Psi_{J(C),\delta}(L) \in \iota(\mathbf{P}(H^0(E, \eta)^\vee))$. □

Let

$$\Psi_{P,\delta} : D_\delta \setminus D_{\delta,\text{sing}} \to \mathbf{P}(H^0(P, \Omega^1_P)^\vee) = \text{Grass}\,(n-1, H^0(P, \Omega^1_P)^\vee)$$

be the Gauss map for the subvariety $D_\delta \subset P$. For $L \in D_\delta(k) \setminus D_{\delta,\text{sing}}(k)$, the tangent space $T_L(D_\delta)$ of D_δ at L defines the image $\Psi_{P,\delta}(L)$ by the natural identifications

$$T_L(D_\delta) \subset T_L(P) \cong (\Omega^1_P(L))^\vee \cong H^0(P, \Omega^1_P)^\vee.$$

For $L \in P(k)$, the tangent space $T_L(P)$ of P at L is naturally identified with the orthogonal subspace

$$V_P = (\phi^* H^0(E, \Omega^1_E))^\perp \subset H^0(C, \Omega^1_C)^\vee \cong T_L(J(C))$$

to $\phi^* H^0(E, \Omega^1_E) \subset H^0(C, \Omega^1_C)$, and it corresponds to the ramification divisor $\text{Ram}\,(\phi) \in |\Omega^1_C|$ of the covering $\phi : C \to E$. We define the finite set Σ_δ by

$$\Sigma_\delta = \{\beta^0_\delta(r_1 + \cdots + r_n) \in J(C) \mid r_1 + \cdots + r_n \leq \text{Ram}\,(\phi)\}.$$

Lemma 5.4. $D_{\delta,\text{sing}} = (W_{\delta,\text{sing}} \cup \Sigma_\delta) \cap D_\delta$.

Proof. If $L \in D_\delta(k) \cap W_{\delta,\text{sing}}(k)$, then $L \in D_{\delta,\text{sing}}(k)$. If $L \in D_\delta(k) \setminus W_{\delta,\text{sing}}(k)$, then by Lemma 5.1,

$$L \in D_{\delta,\text{sing}}(k) \iff T_L(W_\delta) = T_L(P) \subset T_L(J(C)) \iff L \in \Sigma_\delta.$$

□

Lemma 5.5. $(B_\delta \cap P) \setminus W_{\delta,\text{sing}} = (B_\delta \cap P) \setminus D_{\delta,\text{sing}}$.

Proof. By Lemma 4.5, $(B_\delta \setminus W_{\delta,\text{sing}}) \cap \Sigma_\delta = (B^1_\delta \setminus W_{\delta,\text{sing}}) \cap \Sigma_\delta$, and it is empty because $\text{Ram}\,(\phi)$ is reduced. Hence by Lemma 5.4,

$$W_{\delta,\text{sing}} \cap B_\delta \cap P = W_{\delta,\text{sing}} \cap B_\delta \cap D_\delta = D_{\delta,\text{sing}} \cap B_\delta = D_{\delta,\text{sing}} \cap B_\delta \cap P.$$

□

We denote by

$$\begin{aligned} \pi :\ & \mathbf{P}(H^0(C, \Omega^1_C)^\vee) \setminus \{V_P\} \longrightarrow \mathbf{P}(H^0(P, \Omega^1_P)^\vee); \\ & [V \subset H^0(C, \Omega^1_C)^\vee] \longmapsto [V \cap V_P \subset V_P \cong H^0(P, \Omega^1_P)^\vee] \end{aligned}$$

the projection, where $V_P = (\phi^* H^0(E, \Omega^1_E))^\perp \subset H^0(C, \Omega^1_C)^\vee$ is the image of the dual of the restriction

$$H^0(C, \Omega^1_C) \cong H^0(J(C), \Omega^1_{J(C)}) \twoheadrightarrow H^0(P, \Omega^1_P).$$

Lemma 5.6. $\Psi_{P,\delta}(L) = \pi \circ \Psi_{J(C),\delta}(L)$ *for* $L \in D_\delta(k) \setminus D_{\delta,\text{sing}}(k)$.

Proof. For $L \in D_\delta(k) \setminus D_{\delta,\mathrm{sing}}(k)$, the tangent spaces at L satisfies

$$T_L(P) \cap T_L(W_\delta) = T_L(D_\delta) \subset T_L(J(C)),$$

because $P \cap W_\delta = D_\delta \subset J(C)$. Since $T_L(P) \subset T_L(J(C))$ is identified with $V_P \subset H^0(C, \Omega^1_C)^\vee$ by $T_L(J(C)) \cong H^0(C, \Omega^1_C)^\vee$, we have $\Psi_{P,\delta}(L) = \pi \circ \Psi_{J(C),\delta}(L)$. □

By Lemma 5.3, we have the morphism

$$\Psi^B_{J(C),\delta} : B_\delta \setminus W_{\delta,\mathrm{sing}} \longrightarrow \mathbf{P}(H^0(E, \eta)^\vee)$$

satisfying $\iota \circ \Psi^B_{J(C),\delta} = \Psi_{J(C),\delta}$.

Lemma 5.7. *The restriction $\Psi_{P,\delta}|_{(B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}}$ of the Gauss map $\Psi_{P,\delta}$ is identified with the restriction $\Psi^B_{J(C),\delta}|_{(B_\delta \cap P)\setminus D_{\delta,\mathrm{sing}}}$ of $\Psi^B_{J(C),\delta}$ by the isomorphism*

$$\pi \circ \iota : \mathbf{P}(H^0(E, \eta)^\vee) \xrightarrow{\sim} \mathbf{P}(H^0(P, \Omega^1_P)^\vee).$$

Proof. Since the composition

$$H^0(E, \eta) \hookrightarrow H^0(C, \Omega^1_C) \cong H^0(J(C), \Omega^1_{J(C)}) \twoheadrightarrow H^0(P, \Omega^1_P),$$

is an isomorphism, it is a consequence of Lemma 5.3 and Lemma 5.6. □

5.2. **Description for the restricted Gauss maps.** Let $\gamma_\delta : E^{(n-2)} \times E \to J(E)$ be the morphism defined by

$$\begin{aligned} \gamma_\delta : E^{(n-2)}(k) \times E(k) &\longrightarrow \mathrm{Pic}^0(E); \\ (p_1 + \cdots + p_{n-2}, p) &\longmapsto \mathcal{O}_E(p_1 + \cdots + p_{n-2} + 2p) \otimes (N(\delta))^\vee. \end{aligned}$$

Let $X_\delta \subset E^{(n-2)} \times E$ be the fiber of γ_δ at $0 \in J(E)$, and let $Y_\delta \subset C^{(n-2)} \times E$ be the fiber of the composition

$$C^{(n-2)} \times E \xrightarrow{\phi^{(n-2)} \times \mathrm{id}_E} E^{(n-2)} \times E \xrightarrow{\gamma_\delta} J(E)$$

at $0 \in J(E)$. We denote by $\psi_\delta : Y_\delta \to X_\delta$ the induced morphism by $\phi^{(n-2)} \times \mathrm{id}_E$. Let $\nu_\delta : X_\delta \to |\eta| \cong \mathbf{P}(H^0(E, \eta)^\vee)$ be the morphism defined by

$$\nu_\delta : X_\delta(k) \to |\eta|;\ (p_1 + \cdots + p_{n-2}, p) \longmapsto p_1 + \cdots + p_{n-2} + p + t_{\eta - N(\delta)}(p),$$

where $t_{\eta-N(\delta)}(p) \in E(k)$ is the point determined by

$$[\mathcal{O}_E(t_{\eta-N(\delta)}(p))] = [\mathcal{O}_E(p)] + \eta - N(\delta) \in \mathrm{Pic}(E).$$

We remark that $\beta^1_\delta(Y_\delta) = B^1_\delta \cap P \subset J(C)$, and we set

$$Y^\circ_\delta = (\beta^1_\delta)^{-1}((B^1_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}) = (\beta^1_\delta)^{-1}((B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}).$$

Lemma 5.8. *The diagram*

$$\begin{array}{ccc} Y^\circ_\delta & \xrightarrow{\beta^1_\delta} & (B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}} \\ \psi_\delta \downarrow & & \downarrow \Psi^B_{J(C),\delta} \\ X_\delta & \xrightarrow[\nu_\delta]{} & \mathbf{P}(H^0(E, \eta)^\vee) \end{array}$$

is commutative.

Proof. Let $L \in \mathrm{Pic}^0(C)$ be the invertible sheaf which represents the point $\beta^1_\delta(y) \in J(C)$ for $y = (q_1+\cdots+q_{n-2}, \phi(q)) \in Y^\circ_\delta(k)$. Then $q_1+\cdots+q_{n-2}+q+\sigma(q) \in C^{(n)}(k)$ is the unique effective divisor with $L \cong \mathcal{O}_C(q_1+\cdots+q_{n-2}+q+\sigma(q)) \otimes \delta^\vee$. Since

$$\nu_\delta \circ \psi_\delta(y) = \phi(q_1)+\cdots+\phi(q_{n-2})+\phi(q)+t_{\eta-N(\delta)}(\phi(q)) \in |\eta|,$$

we have

$$q_1+\sigma(q_1)+\cdots+q_{n-2}+\sigma(q_{n-2})+q+\sigma(q)+r+\sigma(r) \in |\Omega^1_C|,$$

where $r \in C(k)$ is given by $\phi(r) = t_{\eta-N(\delta)}(\phi(q))$. Then $\sigma(q_1)+\cdots+\sigma(q_{n-2})+r+\sigma(r) \in C^{(n)}(k)$ is the unique effective divisor with $\mathcal{O}_C(\sigma(q_1)+\cdots+\sigma(q_{n-2})+r+\sigma(r)) \cong \Omega^1_C \otimes L^\vee \otimes \delta^\vee$, and by Lemma 5.1, $\Phi^B_{J(C),\delta}(L)$ is equal to $\nu_\delta \circ \psi_\delta(y)$, □

Lemma 5.9. *X_δ and Y_δ are nonsingular projective varieties.*

Proof. We fix a point $p_0 \in E(k)$. Let $\gamma_{\delta,p_0} : E^{(n-2)} \to J(E)$ be the morphism defined by

$$\begin{aligned} \gamma_{\delta,p_0} :\ & E^{(n-2)}(k) \longrightarrow \mathrm{Pic}^0(E) = J(E)(k); \\ & p_1+\cdots+p_{n-2} \longmapsto \mathcal{O}_E(p_1+\cdots+p_{n-2}+2p_0) \otimes (N(\delta))^\vee. \end{aligned}$$

Then $\psi_\delta : Y_\delta \to X_\delta$ is the base change of $\phi^{(n-2)} : C^{(n-2)} \to E^{(n-2)}$ by the étale covering of degree 4;

$$\begin{array}{ccccc} Y_\delta & \xrightarrow{\psi_\delta} & X_\delta & \xrightarrow{\mathrm{pr}_2} & E \\ \mathrm{pr}_1 \downarrow & \Box & \downarrow \mathrm{pr}_1 & \Box & \downarrow (-2)_{J(E)} \circ \iota_{p_0} \\ C^{(n-2)} & \xrightarrow[\phi^{(n-2)}]{} & E^{(n-2)} & \xrightarrow[\gamma_{\delta,p_0}]{} & J(E), \end{array}$$

where $(-2)_{J(E)} \circ \iota_{p_0} : E \to J(E)$ is given by

$$(-2)_{J(E)} \circ \iota_{p_0} :\ E(k) \longrightarrow \mathrm{Pic}^0(E) = J(E)(k);\ p \longmapsto \mathcal{O}_E(2p_0 - 2p).$$

□

Lemma 5.10. *$\beta^1_\delta|_{Y^\circ_\delta} : Y^\circ_\delta \to (B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}$ is an isomorphism. In particular, $(B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}$ is a nonsingular variety.*

Proof. By Lemma 5.5, the image $\beta^1_\delta(Y^\circ_\delta) = (B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}$ is a closed subset in $W_\delta \setminus W_{\delta,\mathrm{sing}}$. We show that $\beta^1_\delta|_{Y^\circ_\delta} : Y^\circ_\delta \to W_\delta \setminus W_{\delta,\mathrm{sing}}$ is a closed immersion. Let $\beta^1 : C^{(n-2)} \times E \to C^{(n)}$ be the morphism given in the proof of Lemma 4.3. Since $\beta^1_\delta = \beta^0_\delta \circ \beta^1$ and $\beta^0_\delta : C^{(n)} \to J(C)$ induces the isomorphism

$$\beta^0_\delta :\ C^{(n)} \setminus (\beta^0_\delta)^{-1}(W_{\delta,\mathrm{sing}}) \xrightarrow{\cong} W_\delta \setminus W_{\delta,\mathrm{sing}},$$

it is enough to show that the finite morphism

$$\beta^1 :\ (C^{(n-2)} \times E) \setminus (\beta^1_\delta)^{-1}(W_{\delta,\mathrm{sing}}) \longrightarrow C^{(n)} \setminus (\beta^0_\delta)^{-1}(W_{\delta,\mathrm{sing}})$$

is a closed immersion. We remark that it is injective by Lemma 4.3. For $y = (q_1+\cdots+q_{n-2}, \phi(q_0)) \in (C^{(n-2)} \times E) \setminus (\beta^1_\delta)^{-1}(W_{\delta,\mathrm{sing}})$, we prove that the homomorphism

$$T_y(C^{(n-2)} \times E) \longrightarrow T_{\beta^1(y)}(C^{(n)})$$

on the tangent spaces is injective. If $\phi(q_0) \notin \{\phi(q_1), \dots, \phi(q_{n-2})\}$, then the point $y' = (q_1 + \cdots + q_{n-2}, q_0 + \sigma(q_0)) \in C^{(n-2)}(k) \times C^{(2)}(k)$ is not contained in the ramification divisor of the natural covering $C^{(n-2)} \times C^{(2)} \to C^{(n)}$. Since the morphism

$$E(k) \to C^{(2)}(k);\ \phi(q) \longmapsto q + \sigma(q)$$

is a closed immersion, the homomorphism

$$T_y(C^{(n-2)} \times E) \hookrightarrow T_{y'}(C^{(n-2)} \times C^{(2)}) \cong T_{\beta^1(y)}(C^{(n)})$$

is injective. We consider the case when $y = (q_1 + \cdots + q_{n-2-i} + iq_0, \phi(q_0))$ and $q_0 \notin \{q_1, \dots, q_{n-2-i}\}$ for some $i \geq 1$. First we assume that $\sigma(q_0) = q_0$. Then by Lemma 4.3, we have $i = 1$. The point $\tilde{y} = (q_1 + \cdots + q_{n-3}, q_0, \phi(q_0)) \in C^{(n-3)} \times C \times E$ is not contained in the ramification divisor of the natural covering $C^{(n-3)} \times C \times E \to C^{(n-2)} \times E$, and the point $\tilde{y}' = (q_1 + \cdots + q_{n-3}, 3q_0) \in C^{(n-3)} \times C^{(3)}$ is not contained in the ramification divisor of the natural covering $C^{(n-3)} \times C^{(3)} \to C^{(n)}$. Since the morphism

$$C(k) \times E(k) \to C^{(3)}(k);\ (q', \phi(q)) \longmapsto q' + q + \sigma(q)$$

is a closed immersion, the homomorphism

$$T_y(C^{(n-2)} \times E) \cong T_{\tilde{y}}(C^{(n-3)} \times C \times E) \hookrightarrow T_{\tilde{y}'}(C^{(n-3)} \times C^{(3)}) \cong T_{\beta^1(y)}(C^{(n)})$$

is injective. We assume that $\sigma(q_0) \neq q_0$. Then by Lemma 4.3, we have $\sigma(q) \notin \{q_1, \dots, q_{n-2-i}\}$. The point $\tilde{y} = (q_1 + \cdots + q_{n-2-i} + iq_0, q_0) \in C^{(n-2)} \times C$ is not contained in the ramification divisor of the covering $\mathrm{id}_{C^{(n-2)}} \times \phi : C^{(n-2)} \times C \to C^{(n-2)} \times E$, and the point $\tilde{y}' = (q_1 + \cdots + q_{n-2-i} + (i+1)q_0, \sigma(q_0)) \in C^{(n-1)} \times C$ is not contained in the ramification divisor of the natural covering $C^{(n-1)} \times C \to C^{(n)}$. Since the morphism

$$C^{(n-2)}(k) \times C(k) \longrightarrow C^{(n-1)}(k) \times C(k);$$
$$(q'_1 + \cdots + q'_{n-2}, q) \longmapsto (q'_1 + \cdots + q'_{n-2} + q, \sigma(q))$$

is a closed immersion, the homomorphism

$$T_y(C^{(n-2)} \times E) \cong T_{\tilde{y}}(C^{(n-2)} \times C) \hookrightarrow T_{\tilde{y}'}(C^{(n-1)} \times C) \cong T_{\beta^1(y)}(C^{(n)})$$

is injective. □

Let $X'_\delta = \overline{\Psi^B_{J(C),\delta}((B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}})}$ be the Zariski closure of the image of the restricted Gauss map $\Psi^B_{J(C),\delta}|_{(B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}}$ in $\mathbf{P}(H^0(E, \eta)^\vee)$.

Lemma 5.11. $\nu_\delta(X_\delta) = X'_\delta$.

Proof. By Lemma 5.8, we have

$$\Psi^B_{J(C),\delta}((B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}) = \nu_\delta(\psi_\delta(Y^\circ_\delta)) \subset \nu_\delta(X_\delta),$$

hence $X'_\delta = \overline{\nu_\delta(\psi_\delta(Y^\circ_\delta))} \subset \nu_\delta(X_\delta)$ and $Y^\circ_\delta \subset (\nu_\delta \circ \psi_\delta)^{-1}(X'_\delta)$. Since Y°_δ is dense in Y_δ, we have $Y_\delta \subset (\nu_\delta \circ \psi_\delta)^{-1}(X'_\delta)$ and $\nu_\delta(X_\delta) = (\nu_\delta \circ \psi_\delta)(Y_\delta) \subset X'_\delta$. □

Lemma 5.12. *If $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$, then $\nu_\delta : X_\delta \to X'_\delta$ is the normalization of X'_δ.*

Proof. We set morphisms $\alpha_\delta^\pm : E^{(n-3)} \times E \to E^{(n)}$, $\mu_\delta : E^{(n-3)} \times E \to E^{(n)}$ and $\nu_\delta^2 : E^{(n-4)} \times E^{(2)} \to E^{(n)}$ by

$$\begin{aligned} \alpha_\delta^+ :\ & E^{(n-3)}(k) \times E(k) \longrightarrow E^{(n)}(k); \\ & (p_1 + \cdots + p_{n-3}, p) \longmapsto p_1 + \cdots + p_{n-3} + 2p + t_{\eta - N(\delta)}(p), \\ \alpha_\delta^- :\ & E^{(n-3)}(k) \times E(k) \longrightarrow E^{(n)}(k); \\ & (p_1 + \cdots + p_{n-3}, p) \longmapsto p_1 + \cdots + p_{n-3} + 2p + t_{N(\delta) - \eta}(p), \\ \mu_\delta :\ & E^{(n-3)}(k) \times E(k) \longrightarrow E^{(n)}(k); \\ & (p_1 + \cdots + p_{n-3}, p) \longmapsto p_1 + \cdots + p_{n-3} + p + t_{\eta - N(\delta)}(p) + t_{N(\delta) - \eta}(p) \end{aligned}$$

and

$$\begin{aligned} \nu_\delta^2 :\ & E^{(n-4)}(k) \times E^{(2)}(k) \longrightarrow E^{(n)}(k); \\ & (p_1 + \cdots + p_{n-3}, p + p') \longmapsto p_1 + \cdots + p_{n-4} + p + t_{\eta - N(\delta)}(p) + p' + t_{\eta - N(\delta)}(p'). \end{aligned}$$

By the natural inclusion $X'_\delta \subset |\eta| \subset E^{(n)}$, the subset

$$U = X'_\delta \setminus (\mathrm{Image}\,(\alpha_\delta^+) \cup \mathrm{Image}\,(\alpha_\delta^-) \cup \mathrm{Image}\,(\mu_\delta) \cup \mathrm{Image}\,(\nu_\delta^2)),$$

is open dense in X'_δ, where we consider as $\mathrm{Image}\,(\nu_\delta^2) = \emptyset$ if $n = 3$. We show that the morphism

$$\nu_\delta :\ \nu_\delta^{-1}(U) \longrightarrow E^{(n)} \setminus (\mathrm{Image}\,(\alpha_\delta^+) \cup \mathrm{Image}\,(\alpha_\delta^-) \cup \mathrm{Image}\,(\mu_\delta) \cup \mathrm{Image}\,(\nu_\delta^2))$$

is a closed immersion. For $u = p_1 + \cdots + p_n \in U(k)$, we assume that

$$p + t_{\eta - N(\delta)}(p) \leq p_1 + \cdots + p_n \quad \text{and} \quad p' + t_{\eta - N(\delta)}(p') \leq p_1 + \cdots + p_n$$

for some $p \neq p' \in E(k)$. Since $u \notin \mathrm{Image}\,(\nu_\delta^2)$, we have

$$t_{\eta - N(\delta)}(p) = p' \quad \text{or} \quad p = t_{\eta - N(\delta)}(p'),$$

and furthermore $u \notin \mathrm{Image}\,(\mu_\delta)$ implies that

$$t_{\eta - N(\delta)}(p) = p' \quad \text{and} \quad p = t_{\eta - N(\delta)}(p'),$$

hence $N(\delta) - \eta \in J(E)_2 \setminus \{0\}$. This means that $\nu_\delta : \nu_\delta^{-1}(U) \to U$ is bijective if $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$. In the following, we prove that the homomorphism

$$T_x(E^{(n-2)} \times E) \longrightarrow T_{\nu_\delta(x)}(E^{(n)})$$

on the tangent spaces is injective for $x \in \nu_\delta^{-1}(U)$. Let $\tilde{\nu}_\delta : E^{(n-2)} \times E \to E^{(n-2)} \times E^{(2)}$ be the morphism defined by

$$\begin{aligned} \tilde{\nu}_\delta :\ & E^{(n-2)}(k) \times E(k) \longrightarrow E^{(n-2)}(k) \times E^{(2)}(k); \\ & (p_1 + \cdots + p_{n-2}, p) \longmapsto (p_1 + \cdots + p_{n-2}, p + t_{\eta - N(\delta)}(p)). \end{aligned}$$

If $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$, then the morphism $\tilde{\nu}_\delta$ is a closed immersion. For $x \in \nu_\delta^{-1}(U)$, the image $\tilde{\nu}_\delta(x)$ is not contained in the ramification divisor of the natural covering $E^{(n-2)} \times E^{(2)} \to E^{(n)}$, because $\nu_\delta(x) \notin \mathrm{Image}\,(\alpha_\delta^+) \cup \mathrm{Image}\,(\alpha_\delta^-)$. Hence the homomorphism

$$T_x(E^{(n-2)} \times E) \hookrightarrow T_{\tilde{\nu}_\delta(x)}(E^{(n)} \times E^{(2)}) \cong T_{\nu_\delta(x)}(E^{(n)})$$

is invective. By Lemma 5.9, the finite birational morphism $\nu_\delta : X_\delta \to X'_\delta$ gives the normalization of X'_δ. □

Remark 5.13. If $N(\delta) - \eta \in J(E)_2 \setminus \{0\}$, then $\nu_\delta : X_\delta \to X'_\delta$ is a covering of degree 2.

5.3. **The branch locus of the restricted Gauss maps.** Let $R_\delta \subset Y_\delta$ be the divisor defined by

$$R_\delta(k) = \{(q_1 + \cdots + q_{n-2}, p) \in Y_\delta(k) \mid p_i = \sigma(p_j) \text{ for some } i \neq j\}.$$

Lemma 5.14. $\beta^1_\delta(R_\delta) \subset W_{\delta,\mathrm{sing}}$.

Proof. It is a consequence of Lemma 4.3, because $\beta^1_\delta(R_\delta) \subset B^2_\delta$. □

Let $S_{\delta,r} \subset Y_\delta$ be the divisor defined by

$$S_{\delta,r}(k) = \{(q_1 + \cdots + q_{n-2}, p) \in Y_\delta(k) \mid q_1 + \cdots + q_{n-2} \geq r\}$$

for $r \in \mathrm{Ram}\,(\phi)$. Then the ramification divisor of $\psi_\delta : Y_\delta \to X_\delta$ is

$$\mathrm{Ram}\,(\psi_\delta) = R_\delta \cup \bigcup_{r \in \mathrm{Ram}\,(\phi)} S_{\delta,r}.$$

Lemma 5.15. $\beta^1_\delta(S_{\delta,r}) \not\subseteq W_{\delta,\mathrm{sing}}$*, and moreover* $\beta^1_\delta(S_{\delta,r}) \cap W_{\delta,\mathrm{sing}} = \emptyset$ *if* $n = 3$.

Proof. Let $W^1_{\delta,r} \subset J(C)$ be the subvariety defined by

$$W^1_{\delta,r}(k) = \{L \in \mathrm{Pic}^0\,(C) \mid h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-r) \otimes \delta) > 1\},$$

and let $T_{\delta,r} \subset J(C)$ be the image of the morphism

$$C^{(n-3)}(k) \times E(k) \longrightarrow \mathrm{Pic}^0\,(C) = J(C)(k);$$
$$(q_1 + \cdots + q_{n-3}, \phi(q)) \longmapsto \mathcal{O}_C(q_1 + \cdots + q_{n-3} + r + q + \sigma(q)) \otimes \delta^\vee.$$

Since C is not a hyperelliptic curve, by Martens' theorem [12, Theorem 1], we have $\dim W^1_{\delta,r} \leq n - 4$ and $T_{\delta,r} \not\subseteq W^1_{\delta,r}$, hence $\dim T_{\delta,r} = n-2$. We remark that $W^1_{\delta,r} = \emptyset$ in the case when $n = 3$. Since $\beta^1_\delta(S_{\delta,r}) = T_{\delta,r} \cap P$ is the fiber of the composition

$$T_{\delta,r} \subset J(C) \xrightarrow{N} J(E)$$

at $0 \in J(E)$, we have $\dim \beta^1_\delta(S_{\delta,r}) = n - 3$. Let $T_{\delta,2r} \subset T_{\delta,r}$ be the image of the morphism

$$C^{(n-4)}(k) \times E(k) \longrightarrow \mathrm{Pic}^0\,(C) = J(C)(k);$$
$$(q_1 + \cdots + q_{n-4}, \phi(q)) \longmapsto \mathcal{O}_C(q_1 + \cdots + q_{n-4} + 2r + q + \sigma(q)) \otimes \delta^\vee.$$

Since $2r = r + \sigma(r)$, we have $T_{\delta,2r} \subset B^2_\delta \subset W_{\delta,\mathrm{sing}}$ by Lemma 4.3. For $L \in (T_{\delta,r}(k) \cap W_{\delta,\mathrm{sing}}(k)) \setminus T_{\delta,2r}(k)$, there is $(q_1 + \cdots + q_{n-3}, \phi(q)) \in C^{(n-3)}(k) \times E(k)$ such that $L = \mathcal{O}_C(q_1 + \cdots + q_{n-3} + r + q + \sigma(q)) \otimes \delta^\vee$ and $r \notin \{q_1, \ldots, q_{n-3}\}$. If

$$q'_1 + \cdots + q'_{n-i} + ir \in |\Omega^1_C(-q_1 - \cdots - q_{n-3} - r - q - \sigma(q))| = |\Omega^1_C \otimes (L \otimes \delta)^\vee|$$

and $r \notin \{q'_1, \ldots, q'_{n-i}\}$, then by Lemma 5.2, the number i is odd. By the same way, any member in the linear system $|\Omega^1_C(-q'_1 - \cdots - q'_{n-i} - ir)| = |L \otimes \delta|$ has an odd multiplicity at r. It implies that $h^0(C, L\otimes \mathcal{O}_C(-r)\otimes\delta) = h^0(C, L\otimes\delta) > 1$. Hence we have

$$T_{\delta,r}\cap W_{\delta,\mathrm{sing}} = T_{\delta,2r}\cup (T_{\delta,r}\cap W^1_{\delta,r}).$$

When $n = 3$, it implies that $T_{\delta,r}\cap W_{\delta,\mathrm{sing}} = \emptyset$. When $n \geq 4$,

$$\beta^1_\delta(S_{\delta,r})\cap W_{\delta,\mathrm{sing}} = (T_{\delta,2r}\cap P)\cup(\beta^1_\delta(S_{\delta,r})\cap W^1_{\delta,r})$$

is a proper closed subset of $\beta^1_\delta(S_{\delta,r}) = T_{\delta,r}\cap P$, because $\dim(T_{\delta,2r}\cap P)\leq n-4$ and $\dim W^1_{\delta,r}\leq n-4$. □

Let $Z_{\delta,r} = \psi_\delta(S_{\delta,r})$ be the image of $S_{\delta,r}$ by $\psi_\delta : Y_\delta\to X_\delta$. Then

$$Z_{\delta,r}(k) = \{(p_1+\cdots+p_{n-2}, p)\in X_\delta \mid \phi(r)\leq p_1+\cdots+p_{n-2}\}.$$

Lemma 5.16. *If $n\geq 4$, then $Z_{\delta,r}$ is irreducible. If $n = 3$, then $X_\delta\cong E$, and $Z_{\delta,r}\subset X_\delta$ is a $J(X_\delta)_2$-orbit by the natural action of $J(X_\delta)$ on the curve X_δ of genus* 1.

Proof. Let $\gamma_{\delta,p_0} : E^{(n-2)}\to J(E)$ be the morphism given in the proof of Lemma 5.9 for fixed $p_0\in E(k)$, and let $i_r : E^{(n-3)}\to E^{(n-2)}$ be the morphism defined by

$$i_r:\ E^{(n-3)}(k)\longrightarrow E^{(n-2)}(k);\ p_1+\cdots+p_{n-3}\longmapsto p_1+\cdots+p_{n-3}+\phi(r).$$

If $n\geq 4$, then $Z_{\delta,r}$ is a $\mathbf{P}^{n-4}$-bundle over E by the base change

$$\begin{array}{ccccc} Z_{\delta,r} & \longrightarrow & X_\delta & \xrightarrow{\mathrm{pr}_2} & E \\ \downarrow & \square & \downarrow \mathrm{pr}_1 & \square & \downarrow (-2)_{J(E)}\circ\iota_{p_0} \\ E^{(n-3)} & \underset{i_r}{\longrightarrow} & E^{(n-2)} & \underset{\gamma_{\delta,p_0}}{\longrightarrow} & J(E) \end{array}$$

of the $\mathbf{P}^{n-4}$-bundle $\gamma_{\delta,p_0}\circ i_r : E^{(n-3)}\to J(E)$, hence $Z_{\delta,r}$ is irreducible. If $n = 3$, then $\mathrm{pr}_2 : X_\delta \to E$ is an isomorphism, and

$$Z_{\delta,r}\cong\{p\in E(k)\mid \mathcal{O}_E(\phi(r)+2p)\cong N(\delta)\}$$

is an orbit of $J(E)_2$-action. □

We denote by $\mathrm{Ram}\,(\psi^\circ_\delta)\subset Y^\circ_\delta$ the ramification divisor of $\psi^\circ_\delta = \psi_\delta|_{Y^\circ_\delta} : Y^\circ_\delta\to X_\delta$.

Lemma 5.17.

$$\overline{\psi^\circ_\delta(\mathrm{Ram}\,(\psi^\circ_\delta))} = \bigcup_{r\in\mathrm{Ram}\,(\phi)} Z_{\delta,r}.$$

Proof. Since $\mathrm{Ram}\,(\psi_\delta) = R_\delta\cup\bigcup_{r\in\mathrm{Ram}\,(\phi)} S_{\delta,r}$, by Lemma 5.14, we have $\mathrm{Ram}\,(\psi^\circ_\delta) = \bigcup_{r\in\mathrm{Ram}\,(\phi)} S_{\delta,r}\cap Y^\circ_\delta$. By Lemma 5.15, $S_{\delta,r}\cap Y^\circ_\delta\neq\emptyset$ for $n\geq 3$, and $S_{\delta,r}\cap Y^\circ_\delta = S_{\delta,r}$ for $n=3$. Since ψ_δ is a finite morphism, $\psi^\circ_\delta(S_{\delta,r}\cap Y^\circ_\delta)$ is of dimension $n-3$. By Lemma 5.16, we have $\overline{\psi^\circ_\delta(S_{\delta,r}\cap Y^\circ_\delta)} = \psi_\delta(S_{\delta,r}) = Z_{\delta,r}$. □

Let $H_r\subset\mathbf{P}(H^0(E,\eta)^\vee)$ be the hyperplane corresponding to the subspace

$$H^0(E,\eta\otimes\mathcal{O}_E(-\phi(r)))\subset H^0(E,\eta).$$

Lemma 5.18. *H_r is the unique hyperplane with the property $\nu_\delta(Z_{\delta,r})\subset H_r$.*

Proof. The inclusion $\nu_\delta(Z_{\delta,r}) \subset H_r$ is obvious. We prove the uniqueness of the hyperplane H_r. Let $z = (p_1 + \cdots + p_{n-3} + \phi(r), p) \in Z_{\delta,r}(k)$ be satisfying $p_i \neq t_{\eta-N(\delta)}(p_j)$ for $i \neq j$. We take a point $p' \in E(k) \setminus \{p\}$ such that $\mathcal{O}_E(2p') \cong \mathcal{O}_E(2p)$ and $\mathcal{O}_E(p'-p) \not\cong \eta \otimes N(\delta)^\vee$. Then $z' = (p_1 + \cdots + p_{n-3} + \phi(r), p')$ is contained in $Z_{\delta,r}(k)$, and $\nu_\delta(z) \neq \nu_\delta(z')$. It implies the uniqueness in the case when $n = 3$. When $n \geq 4$, we show that $\nu_\delta(Z_{\delta,r}) \subset H_r$ is a non-linear hypersurface in H_r. Let $l \subset H_r$ be the line containing the two points $\nu_\delta(z), \nu_\delta(z') \in \nu_\delta(Z_{\delta,r})$. Then the line $l \subset \mathbf{P}(H^0(E,\eta)^\vee)$ corresponds to the linear pencil

$$|\eta(-p_1 - \cdots - p_{n-3} - \phi(r))| \subset |\eta| \cong \mathbf{P}(H^0(E,\eta)^\vee).$$

For a point $p_0 \in E(k)$, there is a unique point $p_0' \in E(k)$ such that $p_0 + p_0' \in |\eta(-p_1 - \cdots - p_{n-3} - \phi(r))|$. If $\mathcal{O}_E(2p_0) \not\cong \mathcal{O}_E(2p)$, $\mathcal{O}_E(2p_0') \not\cong \mathcal{O}_E(2p)$ and

$$p_0, p_0' \notin \{t_{\eta-N(\delta)}(p_1), \ldots, t_{\eta-N(\delta)}(p_{n-3}), t_{N(\delta)-\eta}(p_1), \ldots, t_{N(\delta)-\eta}(p_{n-3})\},$$

then the point $p_1 + \cdots + p_{n-3} + \phi(r) + p_0 + p_0' \in |\eta|$ on the line l is not contained in $\nu_\delta(Z_{\delta,r})$. □

Lemma 5.19. *The pull-back of the divisor H_r by $\nu_\delta : X_\delta \to \mathbf{P}(H^0(E,\eta)^\vee)$ is*

$$\nu_\delta^* H_r = Z_{\delta,r} + M_{\delta,r} + M'_{\delta,r},$$

where $M_{\delta,y}$ and $M'_{\delta,y}$ are irreducible divisors on X_δ defined by

$$M_{\delta,r}(k) = \{(p_1 + \cdots + p_{n-2}, p) \in X_\delta(k) \mid p = \phi(r)\},$$
$$M'_{\delta,r}(k) = \{(p_1 + \cdots + p_{n-2}, p) \in X_\delta(k) \mid p = t_{N(\delta)-\eta}(\phi(r))\}.$$

Proof. Let I_r be an irreducible divisor on $E^{(n)}$ defined by

$$I_r(k) = \{p_1 + \cdots + p_n \in E^{(n)}(k) \mid p_1 + \cdots + p_n \geq \phi(r)\},$$

and let Z_r, M_r, M'_r be irreducible divisors on $E^{(n-2)} \times E$ defined by

$$Z_r(k) = \{(p_1 + \cdots + p_{n-2}, p) \in E^{(n-2)}(k) \times E(k) \mid p_1 + \cdots + p_{n-2} \geq \phi(r)\},$$
$$M_r(k) = \{(p_1 + \cdots + p_{n-2}, p) \in E^{(n-2)}(k) \times E(k) \mid p = \phi(r)\},$$
$$M'_r(k) = \{(p_1 + \cdots + p_{n-2}, p) \in E^{(n-2)}(k) \times E(k) \mid p = t_{N(\delta)-\eta}(\phi(r))\}.$$

Then the pull-back of the divisor I_r by the morphism

$$E^{(n-2)}(k) \times E(k) \longrightarrow E^{(n)}(k);$$
$$(p_1 + \cdots + p_{n-2}, p) \longmapsto p_1 + \cdots + p_{n-2} + p + t_{\eta-N(\delta)}(p)$$

is the divisor $Z_r + M_r + M'_r$ on $E^{(n-2)} \times E$. Since the restriction of I_r to $|\eta| \subset E^{(n)}$ is the divisor H_r on $\mathbf{P}(H^0(E,\eta)^\vee) \cong |\eta|$, the pull-back $\nu_\delta^* H_r$ is the restriction of $Z_r + M_r + M'_r$ to X_δ. □

Corollary 5.20. *$\nu_\delta^* H_r - Z_{\delta,r}$ is an irreducible divisor on X_δ if and only if $N(\delta) = \eta$.*

We consider the dual variety $(\Phi_{|\eta|}(E))^\vee \subset \mathbf{P}(H^0(E,\eta)^\vee)$ of the image of the closed immersion $\Phi_{|\eta|} : E \to \mathbf{P}(H^0(E,\eta))$.

Lemma 5.21. *The projective curve $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E,\eta))$ is reflexive. In particular, $\Phi_{|\eta|}(E) = ((\Phi_{|\eta|}(E))^\vee)^\vee \subset \mathbf{P}(H^0(E,\eta))$.*

Proof. If $1 \le i < n$, then $h^0(E, \eta \otimes \mathcal{O}_E(-ip)) = n - i$ for any $p \in E(k)$. If $n = 3$, then $h^0(E, \eta \otimes \mathcal{O}_E(-3p)) = 0$ for general $p \in E(k)$. Hence $h^0(E, \eta \otimes \mathcal{O}_E(-2p)) > h^0(E, \eta \otimes \mathcal{O}_E(-3p))$ for general $p \in E(k)$. Then there is a hyperplane $H \subset \mathbf{P}(H^0(E,\eta))$ which intersects $\Phi_{|\eta|}(E)$ at $\Phi_{|\eta|}(p)$ with the multiplicity 2. By [8, (3.5)], $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E,\eta))$ is reflexive, because the characteristic of the base field k is not equal to 2. ☐

Lemma 5.22. *If $N(\delta) = \eta$, then the dual variety of $X'_\delta \subset \mathbf{P}(H^0(E,\eta)^\vee)$ is $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E,\eta))$.*

Proof. By Lemma 5.21, we show that the dual variety of $\Phi_{|\eta|}(E)$ is X'_δ. For $L \in (B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}} \subset \mathrm{Pic}^0(C)$, there is a unique effective divisor $q_1 + \cdots + q_{n-2} + q + \sigma(q) \in C^{(n)}(k)$ such that

$$L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + q + \sigma(q)).$$

Since $L \in P(k)$, we have

$$\eta = N(\delta) = [\mathcal{O}_E(\phi(q_1) + \cdots + \phi(q_{n-2}) + 2\phi(q))],$$

hence

$$\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \phi^*\eta \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(\sigma(q_1) + \cdots + \sigma(q_{n-2}) + \sigma(q) + q)$$

and $\Psi^B_{J(C),\delta}(L) \in \mathbf{P}(H^0(E,\eta)^\vee)$ is defined by the effective divisor

$$\phi(q_1) + \cdots + \phi(q_{n-2}) + 2\phi(q) \in |\eta| \cong \mathbf{P}(H^0(E,\eta)^\vee).$$

It means that the hyperplane in $\mathbf{P}(H^0(E,\eta))$ corresponding $\Psi^B_{J(C),\delta}(L)$ is tangent to the image $\Phi_{|\eta|}(E)$. Hence we have

$$\Psi^B_{J(C),\delta}((B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}}) \subset (\Phi_{|\eta|}(E))^\vee.$$

Since $(\Phi_{|\eta|}(E))^\vee$ and $\Psi^B_{J(C),\delta}((B_\delta \cap P) \setminus D_{\delta,\mathrm{sing}})$ are irreducible hypersurfaces in $\mathbf{P}(H^0(E,\eta)^\vee)$, we have $X'_\delta = (\Phi_{|\eta|}(E))^\vee$. ☐

6. Key Propositions

Let $\mathcal{L}$ be an ample invertible sheaf on P which represents the the polarization λ_P.

Lemma 6.1. $U_D = \mathrm{Bs}\,|\mathcal{L}| \setminus D_{\mathrm{sing}}$ *is nonsingular for any* $D \in |\mathcal{L}|$.

Proof. Since $D = D_\delta$ for some $\delta \in \mathrm{Pic}^n(C)$, it is a consequence of Lemma 4.6 and Lemma 5.10. ☐

Let

$$\Psi_D : D \setminus D_{\mathrm{sing}} \longrightarrow \mathbf{P}^{n-1} = \mathrm{Grass}\,(n-1, H^0(P, \Omega^1_P)^\vee)$$

be the Gauss map for $D \in |\mathcal{L}|$, and let $\nu_D : X_D \to X'_D$ be the normalization of $X'_D = \overline{\Psi_D(U_D)} \subset \mathbf{P}^{n-1}$. Then by Lemma 6.1, there is a unique morphism

$\psi_D : U_D \to X_D$ such that $\Psi_D|_{U_D} = \nu_D \circ \psi_D$. Let $Z_D = \overline{\psi_D(\mathrm{Ram}\,(\psi_D))} \subset X_D$ be the Zariski closure of the image of the ramification divisor of ψ_D.

Proposition 6.2. *Let $D \subset P$ be a member in $|\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, where $\Pi_{\mathcal{L}} \subset |\mathcal{L}|$ is the subset in Lemma 4.8.*

1. *If $n = 3$, then X_D is a nonsingular projective curve of genus 1, and Z_D is a disjoint union of 6 orbits $Z_{D,1}, \ldots, Z_{D,6}$ by the $J(X_D)_2$-action.*
2. *If $n \geq 4$, then Z_D has $2n$ irreducible components $Z_{D,1}, \ldots, Z_{D,2n}$.*
3. *For any subset $Z_{D,i} \subset Z_D$ in (1) and (2), there is a unique hyperplane $H_{D,i} \subset \mathbf{P}^{n-1}$ such that $\nu_D(Z_{D,i}) \subset H_{D,i}$.*

Proof. By Lemma 4.7, there is $\delta \in \mathrm{Pic}^n\,(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$. By Lemma 4.6, there is $s \in \mathrm{Pic}^0\,(E)$ such that $D = D_{\delta+\phi^*s}$. By the proof of Lemma 4.8, $D \notin \Pi_{\mathcal{L}}$ implies $s \notin J(E)_4 \setminus J(E)_2$. By Lemma 5.7, the Gauss map $\Psi_D|_{U_D} : U_D \to \mathbf{P}^{n-1}$ is identified with $\Psi^B_{J(C),\delta+\phi^*s}|_{U_D} : U_D \to \mathbf{P}(H^0(E,\eta)^\vee)$. Since $N(\delta + \phi^*s) - \eta = 2s \notin J(E)_2 \setminus \{0\}$, by Lemma 5.11 and Lemma 5.12, the normalization of $X'_{\delta+\phi^*s} = X'_D$ is given by $\nu_{\delta+\phi^*s} : X_{\delta+\phi^*s} \to X'_{\delta+\phi^*s}$, and by Lemma 5.8 and Lemma 5.10, $\psi_D : U_D \to X_D$ is identified with $\psi^\circ_{\delta+\phi^*s} : Y^\circ_{\delta+\phi^*s} \to X_{\delta+\phi^*s}$. Hence the statements (1), (2) and (3) are consequence of Lemma 5.16, Lemma 5.17 and Lemma 5.18. □

We define the subset $\Pi'_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi'_{\mathcal{L}} = \{D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}} \mid \ \nu_D^* H_{D,i} - Z_{D,i} \text{ is irreducible for } 1 \leq i \leq 2n\}.$$

Lemma 6.3. $\sharp \Pi'_{\mathcal{L}} = 4$.

Proof. We use the same identification for Gauss maps as in the proof of Proposition 6.2. Then by Corollary 5.20,

$$D = D_{\delta+\phi^*s} \in \Pi'_{\mathcal{L}} \iff N(\delta + \phi^*s) = \eta \iff s \in J(E)_2,$$

and by Lemma 4.4, we have $\sharp \Pi'_{\mathcal{L}} = \sharp J(C)_2 = 4$. □

Let $e_1 + \cdots + e_{2n}$ be the branch divisor of the original covering $\phi : C \to E$, and let $\eta \in \mathrm{Pic}\,(E)$ be the invertible sheaf with $\phi^*\eta \cong \Omega^1_C$.

Proposition 6.4. *For any member $D \in \Pi'_{\mathcal{L}}$, there is an isomorphism*

$$(E, e_1 + \cdots + e_{2n}, \eta) \cong ((X'_D)^\vee, H^\vee_{D,1} + \cdots + H^\vee_{D,2n}, \mathcal{O}_{(\mathbf{P}^{n-1})^\vee}(1)|_{(X'_D)^\vee}),$$

where $H^\vee_{D,i} \in (\mathbf{P}^{n-1})^\vee$ is the point corresponding to the hyperplane $H_{D,i}$, and $(X'_D)^\vee \subset (\mathbf{P}^{n-1})^\vee$ is the dual variety of $X'_D \subset \mathbf{P}^{n-1}$.

Proof. We use the same identification for Gauss maps as in the proof of Proposition 6.2. When $D \in \Pi'_{\mathcal{L}}$, we may assume that $D = D_\delta$ and $N(\delta) = \eta$ by Corollary 5.20. Then the point $H^\vee_{D,i}$ is identified with the point $H^\vee_r = \Phi_{|\eta|}(\phi(r))$ for $r \in \mathrm{Ram}\,(\phi)$, and $(X'_D)^\vee$ is identified with $(X'_\delta)^\vee$, which coincides with $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E,\eta))$ by Lemma 5.22. □

Remark 6.5. For a member $D \in \Pi'_{\mathcal{L}}$ the Gauss map $\Psi_D : D \setminus D_{\text{sing}} \to \mathbf{P}^{n-1}$ is of degree 2^n, and $X'_D + \sum_{i=1}^{2n} H_{D,i}$ is the branch divisor of Ψ_D. But for $D \notin \Pi'_{\mathcal{L}}$ the Gauss map Ψ_D is not easy to compute.

Acknowledgments. The author would like to thank Juan Carlos Naranjo for the information on the recent paper [17].

References

[1] A. Andreotti, *On a theorem of Torelli,* Amer. J. Math. **80**, 801–828 (1958).
[2] A. Andreotti and A. L. Mayer, *On period relations for abelian integrals on algebraic curves,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) **21**, 189–238 (1967).
[3] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, Geometry of algebraic curves, Vol. I. Grundlehren der Mathematischen Wissenschaften, **267**, Springer-Verlag, New York, (1985).
[4] F. Bardelli, C. Ciliberto and A. Verra, *Curves of minimal genus on a general abelian variety,* Compos. Math. **96**, 115–147 (1995).
[5] R. Donagi, *The tetragonal construction,* Bull. Amer. Math. Soc. (N.S.) **4**, 181–185 (1981).
[6] R. Donagi, R. C. Smith, *The structure of the Prym map,* Acta Math. **146**, 25–102 (1981). Springer-Verlag, Berlin, (2004).
[7] R. Friedman and R. Smith, *The generic Torelli theorem for the Prym map,* Invent. Math. **67**, 473–490 (1982).
[8] A. Hefez and S. L. Kleiman, *Notes on the duality of projective varieties,* Geometry today, Progr. Math. **60**, Birkhäuser, 143–183 (1985).
[9] V. I, Kanev, *The global Torelli theorem for Prym varieties at a generic point,* Math. USSR Izv. **20**, 235–257 (1983).
[10] V. O. Marcucci and J. C. Naranjo, *Prym varieties of double coverings of elliptic curves,* Int. Math. Res. Not. **6**, 1689–1698 (2014).
[11] V. O. Marcucci and G. P. Pirola, *Generic Torelli theorem for Prym varieties of ramified coverings,* Compos. Math. **148**, 1147–1170 (2012).
[12] H. H. Martens, *On the varieties of special divisors on a curve,* J. Reine Angew. Math. **227**, 111–120 (1967).
[13] J. S. Milne, Jacobian varieties, Arithmetic geometry, edited by Gary Cornell and Joseph H. Silverman, 167–212, Springer, New York, (1986).
[14] D. Mumford, *Prym varieties. I,* Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 325–350 (1974).
[15] D. S. Nagaraj and S. Ramanan, *Polarisations of type* $(1, 2, \ldots, 2)$ *on abelian varieties,* Duke Math. J. **80**, 157–194 (1995).
[16] J. C. Naranjo, *Prym varieties of bi-elliptic curves,* J. Reine Angew. Math. **424**, 47–106 (1992).
[17] J. C. Naranjo and A. Ortega, (with an appendix by A. Verra,) *Generic injectivity of the Prym map for double ramified coverings,* Trans. Amer. Math. Soc. (to appear).
[18] A. Verra, *The Prym map has degree two on plane sextics,* The Fano Conference, Univ. Torino, 735–759 (2004).

Department of Mathematics, School of Engineering, Tokyo Denki University, Adachi-ku, Tokyo 120-8551, Japan
E-mail address: atsushi@mail.dendai.ac.jp