

On Generalized (m, n) -Jordan Derivations and Centralizers of Semiprime Rings

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Abstract. In this paper we give an affirmative answer to two conjectures on generalized (m, n) -Jordan derivations and generalized (m, n) -Jordan centralizers raised in [S. Ali and A. Fošner, *On Generalized (m, n) -Derivations and Generalized (m, n) -Jordan Derivations in Rings*, Algebra Colloq. **21** (2014), 411–420] and [A. Fošner, *A note on generalized (m, n) -Jordan centralizers*, Demonstratio Math. **46** (2013), 254–262]. Precisely, when R is a semiprime ring, we prove, under some suitable torsion restrictions, that every nonzero generalized (m, n) -Jordan derivation (resp., a generalized (m, n) -Jordan centralizer) is a derivation (resp., a two-sided centralizer).

2010 Mathematics Subject Classification. 16N60, 16W25

Key Words. semiprime ring, generalized (m, n) -derivation, generalized (m, n) -Jordan derivation, (m, n) -Jordan centralizer, generalized (m, n) -Jordan centralizer

1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. We denote by $\text{char}(R)$ the characteristic of a prime ring R . Let $n \geq 2$ be an integer.

A ring R is said to be n -torsion free if, for all $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if, for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A ring R is called semiprime if, for any $a \in R$, $aRa = \{0\}$ implies $a = 0$.

An additive mapping $d : R \rightarrow R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$, and it is called a Jordan derivation, if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. An additive mapping $T : R \rightarrow R$ is called a left (resp., right) centralizer if $T(xy) = T(x)y$ (resp., $T(xy) = xT(y)$) is fulfilled for all $x, y \in R$, and it is called a left (resp., right) Jordan centralizer if $T(x^2) = T(x)x$ (resp., $T(x^2) = xT(x)$) is fulfilled for all $x \in R$. We call an additive mapping $T : R \rightarrow R$ a two-sided centralizer (resp., a two-sided Jordan centralizer) if T is both a left as well as a right centralizer (resp., a left and a right Jordan centralizer).

An additive mapping $F : R \rightarrow R$ is called a generalized derivation if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where $d : R \rightarrow R$ is a derivation. The concept of generalized derivations was introduced by Brešar in [3] and covers both the concepts of derivations and left centralizers. It is easy to see that generalized derivations are exactly those additive mappings F which can be written in the form $F = d + T$, where d is a derivation and T is a left centralizer.

The Jordan counterpart of the notion of generalized derivation was introduced by Jing and Lu in [10] as follows: An additive mapping $F : R \rightarrow R$ is called a generalized Jordan derivation if $F(x^2) = F(x)x + xd(x)$ is fulfilled for all $x \in R$, where $d : R \rightarrow R$ is a Jordan derivation.

The study of relations between various sorts of derivations goes back to Herstein's classical result [9] which shows that any Jordan derivation on a 2-torsion free prime ring is a derivation (see also [5] for a brief proof of Herstein's result). In [7], Cusack generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). Motivated by these classical results, Vukman [17] proved that any generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation.

In the last few years several authors have introduced and studied various sorts of parameterized derivations. In [1], Ali and Fošner defined the notion of (m, n) -derivations as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $d : R \rightarrow R$ is called an (m, n) -derivation if

$$(m + n)d(xy) = 2md(x)y + 2nxd(y) \quad (1.1)$$

holds for all $x, y \in R$.

Obviously, a $(1, 1)$ -derivation on a 2-torsion free ring is a derivation.

In the same paper [1], a generalized (m, n) -derivation was defined as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $D :$

$R \longrightarrow R$ is called a generalized (m, n) -derivation if there exists an (m, n) -derivation $d : R \longrightarrow R$ such that

$$(m + n)D(xy) = 2mD(x)y + 2nxd(y) \quad (1.2)$$

holds for all $x, y \in R$.

Obviously, every generalized $(1, 1)$ -derivation on a 2-torsion free ring is a generalized derivation.

In [18], Vukman defined an (m, n) -Jordan derivation as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $d : R \longrightarrow R$ is called an (m, n) -Jordan derivation if

$$(m + n)d(x^2) = 2md(x)x + 2nxd(x) \quad (1.3)$$

holds for all $x, y \in R$.

Clearly, every $(1, 1)$ -Jordan derivation on a 2-torsion free ring is a Jordan derivation.

Recently, in [11], Kosi-Ulbl and Vukman proved the following result.

Theorem 1.1 ([11], **Theorem 1.5**) *Let $m, n \geq 1$ be distinct integers, R a $mn(m+n)|m-n|$ -torsion free semiprime ring and $d : R \longrightarrow R$ an (m, n) -Jordan derivation. Then d is a derivation which maps R into $Z(R)$.*

The (m, n) -generalized counterpart of the notion of an (m, n) -Jordan derivation is introduced by Ali and Fošner in [1] as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $F : R \longrightarrow R$ is called a generalized (m, n) -Jordan derivation if there exists an (m, n) -Jordan derivation $d : R \longrightarrow R$ such that

$$(m + n)F(x^2) = 2mF(x)x + 2nxd(x) \quad (1.4)$$

holds for all $x, y \in R$.

Based on some observations and inspired by the classical results, Ali and Fošner in [1] made the following conjecture.

Conjecture 1.2 ([1], **Conjecture 1**) *Let $m, n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $F : R \longrightarrow R$ be a nonzero generalized (m, n) -Jordan derivation. Then F is a derivation which maps R into $Z(R)$.*

The first aim of this paper is to give an affirmative answer to this conjecture. Namely, our first main result is the following theorem.

Theorem 1.3 *Let $m, n \geq 1$ be distinct integers, let R be a k -torsion free semiprime ring, where $k = 6mn(m+n)|m-n|$, and let $F : R \rightarrow R$ be a nonzero generalized (m, n) -Jordan derivation. Then F is a derivation which maps R into $Z(R)$.*

On the other hand and in parallel, there are similar works which study relations between various sorts of Jordan centralizers and centralizers. Namely, in [20], Zalar proved that any left (resp., right) Jordan centralizer on a 2-torsion free semiprime ring is a left (resp., right) centralizer. In [15], Vukman proved that, for a 2-torsion free semiprime ring R , every additive mapping $T : R \rightarrow R$ satisfying the relation “ $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$ ” is a two-sided centralizer. Motivated by these results and inspired by his work [15], Vukman in [19] introduced the notion of an (m, n) -Jordan centralizer as follows: Let $m, n \geq 0$ be two fixed integers with $m+n \neq 0$. An additive mapping $T : R \rightarrow R$ is called an (m, n) -Jordan centralizer if

$$(m+n)T(x^2) = mT(x)x + nxT(x) \quad (1.5)$$

holds for all $x, y \in R$.

Obviously, a $(1, 0)$ -Jordan centralizer (resp., $(0, 1)$ -Jordan centralizer) is a left (resp., a right) Jordan centralizer. When $n = m = 1$, we recover the maps studied in [15].

Based on some observations and results, Vukman conjectured that, on semiprime rings with suitable torsion restrictions, every (m, n) -Jordan centralizer is a two-sided centralizer (see [19]). Recently, this conjecture was solved affirmatively by Kosi-Ulbl and Vukman in [12]. Namely, they proved the following result.

Theorem 1.4 ([12], Theorem 1.5) *Let $m, n \geq 1$ be distinct integers, let R be an $mn(m+n)$ -torsion free semiprime ring, and let $T : R \rightarrow R$ be an (m, n) -Jordan centralizer. Then T is a two-sided centralizer.*

Inspired by the work of Vukman [15, 19], Fošner [8] introduced more generalized version of (m, n) -Jordan centralizers as follows: Let $m, n \geq 0$ be two fixed integers with $m+n \neq 0$. An additive mapping $T : R \rightarrow R$ is called a generalized (m, n) -Jordan centralizer if there exists an (m, n) -Jordan centralizer $T_0 : R \rightarrow R$ such that

$$(m+n)T(x^2) = mT(x)x + nxT_0(x) \quad (1.6)$$

holds for all $x \in R$.

Thus, a generalized $(1, 0)$ -Jordan centralizer is a left Jordan centralizer.

In [8], Fošner showed that, on a prime ring with a specific torsion condition, every generalized (m, n) -Jordan centralizer is a two-sided centralizer. This led Fošner to make the following conjecture.

Conjecture 1.5 ([8], **Conjecture 1**) *Let $m, n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \rightarrow R$ be a generalized (m, n) -Jordan centralizer. Then T is a two-sided centralizer.*

The second aim of this paper is to give an affirmative answer to Fošner's conjecture. Namely, our second main result is the following theorem.

Theorem 1.6 *Let $m, n \geq 1$ be two fixed integers, let R be an $6mn(m+n)(2n+m)$ -torsion free semiprime ring, and let $T : R \rightarrow R$ be a nonzero generalized (m, n) -Jordan centralizer. Then T is a two-sided centralizer.*

2 Proof of the main theorems

In the proof of our main results, Theorems 1.3 and 1.6, we shall use the following results.

Lemma 2.1 ([1], **Lemma 1**) *Let $m, n \geq 0$ be distinct integers with $m + n \neq 0$, let R be a 2-torsion free ring, and let $F : R \rightarrow R$ be a nonzero generalized (m, n) -Jordan derivation with an associated (m, n) -Jordan derivation d . Then, $(m + n)^2 F(xyx) = m(n - m)F(x)xy + m(3m + n)F(x)yx + m(m - n)F(y)x^2 + 4mnd(y)x + n(n - m)x^2d(y) + n(m + 3n)xyd(x) + n(m - n)yxd(x)$ for all $x, y \in R$.*

Lemma 2.2 ([6], **Theorem 3.3**) *Let $n \geq 2$ be a fixed integer and let R be a prime ring with $\text{char}(R) = 0$ or $\text{char}(R) \geq n$. If $T : R \rightarrow R$ is an additive mapping satisfying the relation $T(x^n) = T(x)x^{n-1}$ for all $x \in R$, then $T(xy) = T(x)y$ for all $x, y \in R$.*

Lemma 2.3 ([8], **Lemma 1**) *Let $m, n \geq 0$ be distinct integers with $m + n \neq 0$, let R be a ring, and let $T : R \rightarrow R$ be a nonzero generalized (m, n) -Jordan centralizer with an associated (m, n) -Jordan centralizer T_0 . Then, $2(m + n)^2 T(xyx) = mnT(x)xy + m(2m + n)T(x)yx - mnT(y)x^2 + 2mnxT_0(y)x - mnx^2T_0(y) + n(m + 2n)xyT_0(x) + mnyxT_0(x)$ for all $x, y \in R$.*

Lemma 2.4 ([16], **Lemma 3**) *Let R be a semiprime ring and let $T : R \rightarrow R$ be an additive mapping. If either $T(x)x = 0$ or $xT(x) = 0$ holds for all $x \in R$, then $T = 0$.*

We shall use the relation between semiprime rings and prime ideals. Namely, it is well-known that a ring R is semiprime if and only if the intersection of all prime ideals of R is zero if and only if R has no nonzero nilpotent (left, right) ideals (see for instance Lam's book [13] or the recent book of Brešar [4]). Due to

the classical Levitzki's paper [14], several authors prefer to refer to a such result by Levitzki's lemma.

Let I be an ideal of R . For an element $x \in R$, we use \bar{x} to denote the equivalence class of x modulo I .

Lemma 2.5 *Let R be both a 2-torsion free and a 3-torsion free semiprime ring and let $T : R \rightarrow R$ be an additive map such that $T(x)x^3 = 0$ and $T(x^4) = 0$ for all $x \in R$. Then $T(xy) = T(x)y$ for all $x, y \in R$.*

Proof. Let $x, y \in R$. We prove that $T(xy) = T(x)y$. We may assume that x and y are not 0. Let P be a prime ideal of R and set $\bar{R} = R/P$. Consider an element $p \in P$. By hypothesis, $0 = T(x+p)(x+p)^3 = (T(x)+T(p))(x^3+xp^2+px^2+p^2x+x^2p+xp^2+pxp+p^3)$. Thus, $0 = T(x)(x^3+xp^2+px^2+p^2x+x^2p+xp^2+pxp+p^3) + T(p)(x^3+xp^2+px^2+p^2x+x^2p+xp^2+pxp)$. Hence, $T(p)x^3 \in P$, equivalently $\overline{T(p)x^3} = 0$. By Levitzki's lemma, $\overline{T(p)x} = 0$, and then $\overline{T(p)} = 0$ (since \bar{R} is a prime ring). Thus, $T(P) \subseteq P$, which implies that $T(x+P) = T(x) + P$. Then, the induced map $\bar{T} : R/P \rightarrow R/P$ such that $\bar{T}(\bar{x}) = \overline{T(x)}$ for every $x \in R$, is well defined. Now, since $\bar{T}(\bar{x})\bar{x}^3 = 0$ and $\bar{T}(\bar{x}^4) = 0$, $\bar{T}(\bar{x}^4) = \bar{T}(\bar{x})\bar{x}^3$. This shows, using Lemma 2.2, that $\bar{T}(\bar{xy}) = \bar{T}(\bar{x})\bar{y}$. Therefore, $T(xy) - T(x)y \in P$. Finally, by the semiprimeness of R , we get the desired result. ■

Now we are ready to prove the first main result.

Proof of Theorem 1.3. Let d be the associated (m, n) -Jordan derivation of F . Since R is a semiprime ring, d is a derivation which maps R into $Z(R)$ (by Theorem 1.1). Let us denote $F - d$ by D . Then, we have $(m+n)D(x^2) = (m+n)F(x^2) - (m+n)d(x^2) = 2mF(x)x + 2nxd(x) - 2md(x)x - 2nxd(x) = 2mD(x)x$ for all $x \in R$. Thus

$$(m+n)D(x^2) = 2mD(x)x, \quad x \in R. \quad (2.1)$$

Replacing x with x^2 in (2.1), we get

$$(m+n)D(x^4) = 2mD(x^2)x^2, \quad x \in R. \quad (2.2)$$

Multiplying by $m+n$ and then using (2.1), we get

$$(m+n)^2D(x^4) = 4m^2D(x)x^3, \quad x \in R. \quad (2.3)$$

On the other hand, putting x^2 for y in the relation of Lemma 2.1 and using the fact that D is a generalized (m, n) -Jordan derivation associated with the zero map as an (m, n) -Jordan derivation, we get

$$(m+n)^2D(x^4) = m(n-m)D(x)x^3 + m(3m+n)D(x)x^3 + m(m-n)D(x^2)x^2, \quad x \in R. \quad (2.4)$$

Multiplying both sides in (2.4) by 2 we get

$$2(m+n)^2D(x^4) = 2m(n-m)D(x)x^3 + 2m(3m+n)D(x)x^3 + 2m(m-n)D(x^2)x^2, \quad x \in R. \quad (2.5)$$

Combining (2.2) and (2.5), we get

$$2(m+n)^2D(x^4) = 2m(n-m)D(x)x^3 + 2m(3m+n)D(x)x^3 + (m+n)(m-n)D(x^4), \quad x \in R, \quad (2.6)$$

which gives

$$(m+n)(m+3n)D(x^4) = 4m(m+n)D(x)x^3, \quad x \in R. \quad (2.7)$$

Multiplying both sides in (2.7) by $m+n$, we get

$$(m+n)^2(m+3n)D(x^4) = 4m(m+n)^2D(x)x^3, \quad x \in R. \quad (2.8)$$

Multiplying by $m+3n$ in (2.3), we get

$$(m+n)^2(m+3n)D(x^4) = 4m^2(m+3n)D(x)x^3, \quad x \in R. \quad (2.9)$$

By comparing (2.8) and (2.9), we get

$$4mn(m-n)D(x)x^3 = 0, \quad x \in R. \quad (2.10)$$

Since R is a $2mn|n-m|$ -torsion free ring, $D(x)x^3 = 0$ for all $x \in R$. Applying $D(x)x^3 = 0$ in equation (2.3), we get $(m+n)^2D(x^4) = 0$ for all $x \in R$. By using the torsion free restriction, we have $D(x^4) = 0$ for all $x \in R$. Hence, $D(xy) = D(x)y$ for all $x, y \in R$ (by Lemma 2.5). Applying this in (2.1), yields $(m+n)D(x)x = 2mD(x)x$ for all $x \in R$, equivalently $(m-n)D(x)x = 0$. Since R is an $|m-n|$ -torsion free ring, $D(x)x = 0$ for all $x \in R$. Therefore, by Lemma 2.4, $D = 0$. This completes the proof. \blacksquare

The second main result is proved similarly. Nevertheless, we include a proof for completeness.

Proof of Theorem 1.6. Let T_0 be the associated (m, n) -Jordan centralizer of T . Since R is a semiprime ring, T_0 is a two-sided centralizer (by Theorem 1.4). Let us denote $T - T_0$ by D . Then, we have $(m+n)D(x^2) = (m+n)T(x^2) - (m+n)T_0(x^2) = mT(x)x + nT_0(x)x - mT_0(x)x - nT_0(x)x = mD(x)x$ for all $x \in R$. Thus

$$(m+n)D(x^2) = mD(x)x, \quad x \in R. \quad (2.11)$$

Replacing x with x^2 in (2.11), we get

$$(m+n)D(x^4) = mD(x^2)x^2, \quad x \in R. \quad (2.12)$$

Multiplying by $m + n$ and then using (2.11), we get

$$(m + n)^2 D(x^4) = m^2 D(x)x^3, \quad x \in R. \quad (2.13)$$

On the other hand, if we put $y = x^2$ in the relation of Lemma 2.3, we get

$$2(m + n)^2 D(x^4) = mnD(x)x^3 + m(2m + n)D(x)x^3 - mnD(x^2)x^2, \quad x \in R. \quad (2.14)$$

Multiplying both sides in (2.13) by 2 we get

$$2(m + n)^2 D(x^4) = 2m^2 D(x)x^3, \quad x \in R. \quad (2.15)$$

Combining (2.12) and (2.14), we get

$$2(m+n)^2 D(x^4) = mnD(x)x^3 + m(2m+n)D(x)x^3 - n(m+n)D(x^4), \quad x \in R, \quad (2.16)$$

which implies

$$(m + n)(2m + 3n)D(x^4) = 2m(m + n)D(x)x^3, \quad x \in R. \quad (2.17)$$

Multiplying both sides of above relation by $m + n$, we have

$$(m + n)^2(2m + 3n)D(x^4) = 2m(m + n)^2 D(x)x^3, \quad x \in R. \quad (2.18)$$

Multiplying by $(2m + 3n)$ in (2.13), we get

$$(m + n)^2(2m + 3n)D(x^4) = m^2(2m + 3n)D(x)x^3, \quad x \in R. \quad (2.19)$$

By comparing (2.18) and (2.19), we get

$$mn(2n + m)D(x)x^3 = 0, \quad x \in R. \quad (2.20)$$

Since R is a $mn(2n + m)$ -torsion free ring, $D(x)x^3 = 0$ for all $x \in R$. Applying $D(x)x^3 = 0$ in equation (2.13) and then using $(m + n)$ -torsion freeness of R , we get $D(x^4) = 0$ for all $x \in R$. Moreover, since R is a 2 and a 3-torsion free ring, by Lemma 2.5, we get $D(xy) = D(x)y$ for all $x, y \in R$. Applying this in (2.11), yields $(m + n)D(x)x = mD(x)x$ for all $x \in R$. So $nD(x)x = 0$, which implies that $D(x)x = 0$ for all $x \in R$. Therefore, by Lemma 2.4, $D = 0$. This completes the proof. ■

Acknowledgements. The authors would like to thank Professor Abdellah Mamouni for useful discussions.

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