

# An other approach of the diameter of $\Gamma(R)$ and $\Gamma(R[X])$

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## Abstract

Using the new extension of the zero-divisor graph  $\tilde{\Gamma}(R)$  introduced in [6], we give an approach of the diameter of  $\Gamma(R)$  and  $\Gamma(R[X])$  other than given in [11] thus we give a complete characterization for the possible diameters 1, 2 or 3 of  $\Gamma(R)$  and  $\Gamma(R[x])$ .

## Introduction

The idea of a zero-divisor graph was introduced by I. Beck in [5] while he was mainly interested in colorings. In Beck's work, the graph  $\Gamma_0(R)$  associated with a nontrivial commutative unitary ring  $R$  is the undirected simple graph where the vertices are all elements of  $R$  and two vertices  $x$  and  $y$  are adjacent if and only if  $xy = 0$ .

the study of the interaction between the properties of ring theory and the properties of graph theory begun with the article of D.F. Anderson and P.S. Livingston where they modified the graph considering the zero-divisor graph  $\Gamma(R)$  with vertices in  $Z(R)^* = Z(R) \setminus \{0\}$ , where  $Z(R)$  is the set of zero-divisors of  $R$ , and for distinct  $x, y \in Z(R)^*$ , the vertices  $x$  and  $y$  are adjacent

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if and only if  $xy = 0$  [4]. Also, D. F Anderson and A. Badawi introduced the total graph  $T(\Gamma(R))$  of a commutative ring  $R$  with all elements of  $R$  as vertices and for distinct  $x, y \in R$ , the vertices  $x$  and  $y$  are adjacent if and only if  $x + y \in Z(R)$  [3].

In [6], we introduced a new graph, denoted  $\tilde{\Gamma}(R)$ , as the undirected simple graph whose vertices are the nonzero zero-divisors of  $R$  and for distinct  $x, y \in Z(R)^*$ ,  $x$  and  $y$  are adjacent if and only if  $xy = 0$  or  $x + y \in Z(R)$ . Recall that a path  $P$  in the graph  $G = (V, E)$  is a finite sequence  $(x_0, \dots, x_k)$  of distinct vertices such that for all  $i = 0, \dots, k - 1$ ,  $x_i x_{i+1}$  is an edge. In this case, we said that  $x_0$  and  $x_k$  are linked by  $P = x_0 - x_k$  and the length of  $P$  is  $k$ , i.e., the number of its edges.  $G$  is said to be connected if each pair of distinct vertices belongs to a path. Also, if  $G$  has a path  $x - y$ , then the distance between  $x$  and  $y$ , written  $d_G(x, y)$  or simply  $d(x, y)$  is the least length of a  $x - y$  path. If  $G$  has no such path, then  $d(x, y) = \infty$ . The diameter of  $G$ , denoted  $diam(G)$ , is the greatest distance between any two vertices in  $G$ . A graph  $G$  is complete if each pair of distinct vertices forms an edge, i.e., if  $diam(G) = 1$ .

$R$  is a nontrivial commutative unitary ring and general references for commutative ring theory are [1] and [10].

In [11], T. G. Lucas has studied situations where  $diam(\Gamma(R))$  and  $diam(\Gamma(R[x]))$  are  $= 1, 2$  or  $3$  and gave a complete characterization of these diameter strictly in terms of properties of the ring  $R$ .

In this paper, we give another approach of this problem using the properties of the new graph  $\tilde{\Gamma}(R)$ . In the first section, we begin by showing the link between the non-completeness of  $\tilde{\Gamma}(R)$  and the diameter of  $diam(\Gamma(R))$  and we give a complete characterization of the diameter of  $diam(\Gamma(R))$  using the nature of the ring  $R$ . In the second section, we give some examples illustrating cases where the diameter of  $\Gamma(R)$  is  $1, 2$  or  $3$ . The third section is reserved for characterization of the  $diam(\Gamma(R[X]))$  in terms of the nature of the ring  $R$ .

We recall that  $|Z(R)^*| = 1$  if and only if  $R \simeq \mathbb{Z}_4$  or  $R \simeq \mathbb{Z}[X]/(X^2)$  (cf. Example 2.1, [4]) so we assume, along this paper, that  $R$  is such that  $|Z(R)^*| > 1$ .

# 1 diameter of $\Gamma(R)$

This section is devoted to the study of diameter of  $\Gamma(R)$ . We begin by recalling the Lucas's result:

**Theorem 1.1.** (cf. theorem 2.6, [11]) *Let  $R$  be a ring.*

- (1)  *$\text{diam}(\Gamma(R)) = 0$  if and only if  $R$  is (nonreduced and) isomorphic to either  $\mathbb{Z}_4$  or  $\mathbb{Z}_2[y]/(y^2)$ .*
- (2)  *$\text{diam}(\Gamma(R)) = 1$  if and only if  $xy = 0$  for each distinct pair of zero divisors and  $R$  has at least two nonzero zero divisors.*
- (3)  *$\text{diam}(\Gamma(R)) = 2$  if and only if either (i)  $R$  is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii)  $Z(R)$  is an ideal whose square is not  $(0)$  and each pair of distinct zero divisors has a nonzero annihilator.*
- (4)  *$\text{diam}(\Gamma(R)) = 3$  if and only if there are zero divisors  $a \neq b$  such that  $(0 : (a, b)) = (0)$  and either (i)  $R$  is a reduced ring with more than two minimal primes, or (ii)  $R$  is nonreduced.*

**Remark 1.2.** *As stated above, we assume that  $R$  is such that  $\text{diam}(\Gamma(R)) \neq 0$ , i.e.,  $R \not\cong \mathbb{Z}_4$  and  $R \not\cong \mathbb{Z}_2[X]/(X^2)$ . Also, we recall that  $\text{diam}(\Gamma(R)) \leq 3$  (cf. theorem 2.3, [4]) whose next lemma is an immediate consequence.*

**Lemma 1.3.** *Let  $x, y \in Z(R)^*$ . If  $d_\Gamma(x, y) > 2$ , then  $d_\Gamma(x, y) = 3$ .*

Using the new graph  $\tilde{\Gamma}(R)$ , we obtain some cases where  $\text{diam}(\Gamma(R)) = 3$ :

**Theorem 1.4.** *If  $\tilde{\Gamma}(R)$  is not complete, then  $\text{diam}(\Gamma(R)) = 3$ .*

*Proof.* Since  $\tilde{\Gamma}(R)$  is not complete, then  $\text{diam}(\tilde{\Gamma}(R)) = 2$  ((cf. [6], theorem 2.1), so there exists  $x, y \in Z(R)^*$  such that  $d_{\tilde{\Gamma}}(x, y) = 2$  hence  $xy \neq 0$  and  $x + y \notin Z(R)$  thus  $\text{ann}(x) \cap \text{ann}(y) = (0)$  therefore  $d_\Gamma(x, y) > 2$  and thus, by the previous lemma,  $\text{diam}(\Gamma(R)) = 3$ .  $\square$

**Corollary 1.5.** *If  $Z(R)$  is not an ideal of  $R$  and  $R$  is neither boolean nor (up to isomorphism) a subring of a product of two integral domains, then  $\text{diam}(\Gamma(R)) = 3$ .*

*Proof.* Since  $Z(R)$  is not an ideal of  $R$  and  $R$  is neither boolean nor a subring of a product of two integral domains, then, by theorem 1.7 [7],  $\tilde{\Gamma}(R)$  is not complet and, by the previous theorem,  $diam(\Gamma(R)) = 3$ .  $\square$

**Remark 1.6.** *The previous theorem gives a method to construct graphs  $\Gamma(R)$  of diameter 3: for example,  $diam(\Gamma(\mathbb{Z}_{12})) = 3$  because  $Z(\mathbb{Z}_{12})$  is not an ideal ( $2 + 3 \notin Z(\mathbb{Z}_{12})$ ) and  $\mathbb{Z}_{12}$  is neither boolean ( $\mathbb{Z}_{12}$  is not isomorph to  $\mathbb{Z}_2^n$ ) nor a subring of a product of two integral domains ( $\mathbb{Z}_{12}$  is not reduced).*

We know that  $\tilde{\Gamma}(R)$  is not complete if and only if  $Z(R)$  is not an ideal of  $R$  and  $R$  is neither boolean nor (up to isomorphism) a subring of a product of two integral domains (cf. [7], theorem 1.7) so it is enough to treat the cases where  $\tilde{\Gamma}(R)$  is complete to give a ring characterizations such that  $diam(\Gamma(R)) = 1, 2$  or  $3$ , i.e., the cases where  $Z(R)$  is an ideal of  $R$  or  $R$  is boolean or  $R$  is (up to isomorphism) a subring of a product of two integral domains.

We have the following preliminary lemma:

**Lemma 1.7.**

- (1) *If  $Z(R)^2 = (0)$ , then  $Z(R)$  is an ideal.*
- (2) *Let  $R$  such that  $Z(R)$  is an ideal. If  $Z(R)^2 \neq (0)$ , then there exist a distinct pair of non-zero-divisors  $x, y$  such that  $xy \neq 0$ .*
- (3) *Let  $R$  such that  $Z(R)$  is an ideal. If there exist a pair of zero-divisors  $x, y$  such that  $ann(x, y) = (0)$ , then  $x, y$  is distinct pair of non-zero-divisors such that  $xy \neq 0$ .*

*Proof.* (1) Suppose that  $Z(R)^2 = (0)$  so  $Z(R) \subset Nil(R)$ , where  $Nil(R)$  is the nilradical of  $R$ , then  $Z(R) = Nil(R)$  hence  $Z(R)$  is an ideal.

(2) Let  $x \in Z(R)^*$  such that  $x^2 \neq 0$ . It is clear that if  $2x \neq 0$ , then  $x, -x$  is a distinct pair of non-zero-divisors  $x, y = -x$  such that  $xy \neq 0$ . Suppose that  $2x = 0$  and let  $a \in Z(R)^*$  such that  $ax = 0$ . Let  $y = a + x$  so  $y \in Z(R)$  because  $Z(R)$  is an ideal. Also,  $y \neq x$  and  $yx = (a + x)x = x^2 \neq 0$  then  $x, y$  is a distinct pair of zero-divisors such that  $xy \neq 0$  thus  $diam(\Gamma(R)) > 1$  therefore  $diam(\Gamma(R)) = 2$  because for each pair of zero-divisors  $x, y$ ,  $ann(x, y) \neq (0)$ .

(3) Suppose that there exist a pair of zero-divisors  $x, y$  such that  $ann(x, y) = (0)$  so  $x \neq 0, y \neq 0$  and  $x \neq y$ . Also,  $x + y \in Z(R)$  because  $Z(R)$  is an

ideal so there exist  $a \in R \setminus \{0\}$  such that  $a(x + y) = 0$  then  $ax = -ay$ . We claim that  $xy \neq 0$ , indeed, if  $xy = 0$ , so  $(ax).x = -ayx = 0$  and  $(ax)y = 0$  hence  $ax \in \text{ann}(x, y) = (0)$ . Also,  $(ay)x = 0$  and  $(ay)y = -axy = 0$  then  $ay \in \text{ann}(x, y) = (0)$  therefore  $a \in \text{ann}(x, y) = (0)$ . □

**Proposition 1.8.**

- (1) Let  $R$  such that  $Z(R)$  is an ideal and  $Z(R)^2 \neq (0)$ . If for each distinct pair of zero-divisors  $x, y$ ,  $\text{ann}(x, y) \neq (0)$ , then  $\text{diam}(\Gamma(R)) = 2$ .
- (2) Let  $R$  such that  $Z(R)$  is an ideal. If there exist a pair of zero-divisors  $x, y$  such that  $\text{ann}(x, y) = (0)$ , then  $\text{diam}(\Gamma(R)) = 3$ .

*Proof.* (1) By lemma 1.7, there exist a distinct pair of zero-divisors  $a, b$  such that  $ab \neq 0$  then  $\text{diam}(\Gamma(R)) > 1$ . Let  $x, y \in Z(R)^*$  such that  $d_\Gamma(x, y) > 1$  so  $\text{ann}(x, y) \neq (0)$  hence  $d_\Gamma(x, y) = 2$ .

(2) Suppose that there exist a pair of zero-divisors  $x, y$  such that  $\text{ann}(x, y) = (0)$ , then, by the previous lemma,  $x, y$  is distinct pair of non-zero-divisors such that  $xy \neq 0$  so  $\text{diam}(\Gamma(R)) > 1$  and since  $\text{ann}(x, y) = (0)$ ,  $\text{diam}(\Gamma(R)) > 2$  then  $\text{diam}(\Gamma(R)) = 3$ . □

**Remark 1.9.** Let  $R$  such that  $Z(R)$  is an ideal. By the previous proof, if there exist a pair of zero-divisors  $x, y$  such that  $\text{ann}(x, y) = (0)$  so  $xy \neq 0$  then  $Z(R)^2 \neq 0$ .

**Proposition 1.10.** If  $R$  is (up to isomorphism) a subring of a product of two integral domains and  $R \not\cong \mathbb{Z}_2^2$ , then  $\text{diam}(\Gamma(R)) = 2$ .

*Proof.* Since  $R$  is a subring of a product of two integral domains and  $R$  is not an integral domain, there exists  $a = (a_1, 0), b = (0, a_2) \in Z(R)^*$ .

We claim that  $|Z(R)^*| \geq 3$ , indeed, if  $|Z(R)^*| = 2$ ,  $R \simeq \mathbb{Z}_9$  or  $\mathbb{Z}_2^2$  or  $\mathbb{Z}_3[X]/(X^3)$  then  $R \simeq \mathbb{Z}_9$  or  $\mathbb{Z}_3[x]/(x^3)$  (because  $R \not\cong \mathbb{Z}_2^2$ ). However,  $\mathbb{Z}_9$  and  $\mathbb{Z}_3[x]/(x^3)$  are not reduced but  $R$  is reduced then  $|Z(R)^*| \geq 3$ .

Let  $x \in Z(R)^* \setminus \{a, b\}$  and suppose that  $x = (x_1, 0)$  (the other case is similar). Since  $x \neq a$  and  $ax \neq 0$  so  $\text{diam}(\Gamma(R)) > 1$ . Also, let  $z, t \in Z(R)^*$  such that  $d_{\Gamma(R)}(z, t) > 1$  so we can suppose that  $z = (z_1, 0)$  and  $t = (t_1, 0)$  then  $z - b - t$  hence  $d_\Gamma(z, t) = 2$  and thus  $\text{diam}(\Gamma(R)) = 2$ . □

**Proposition 1.11.** *Let  $R$  be a boolean ring. If  $R \not\cong \mathbb{Z}_2^2$ , then  $\text{diam}(\Gamma(R)) = 3$ .*

*Proof.* Since  $R$  is boolean and  $R \not\cong \mathbb{Z}_2$ , there exists  $e \in R \setminus \{0, 1\}$  such that  $R \simeq Re \oplus R(1 - e)$ . Also, since  $R \not\cong \mathbb{Z}_2^2$ , we can suppose that  $Re \not\cong \mathbb{Z}_2$  thus, since  $Re$  is boolean, there exists  $e' \in Re \setminus \{0, 1\}$  such that  $Re \simeq Re' \oplus R(1 - e')$  therefore we can suppose that  $R \simeq R_1 \oplus R_2 \oplus R_3$ , with  $R_1, R_2, R_3$  boolean rings. Let  $x = (1, 1, 0), y = (1, 0, 1) \in Z(R)^*$  so  $x \neq y$ ,  $xy \neq 0$  and  $\text{ann}(x) \cap \text{ann}(y) = (0)$  then  $d_{\Gamma(R)}(x, y) > 2$  hence, by lemma 1.3,  $\text{diam}(\Gamma(R)) = 3$ .  $\square$

**Theorem 1.12.**

- (1)  $\text{diam}(\Gamma(R)) = 1$  if and only if  $R \simeq \mathbb{Z}_2^2$  or  $Z(R)^2 = (0)$ .
- (2)  $\text{diam}(\Gamma(R)) = 2$  if and only if ( $R$  is (up isomorphism) a subring of a product of two integral domains and  $R \not\cong \mathbb{Z}_2^2$ ) or ( $Z(R)$  is an ideal,  $Z(R)^2 \neq (0)$  and for each distinct pair of zero-divisors  $x, y$ ,  $\text{ann}(x, y) \neq (0)$ ).
- (3)  $\text{diam}(\Gamma(R)) = 3$  if and only if ( $R$  is boolean and  $R \not\cong \mathbb{Z}_2^2$ ) or ( $Z(R)$  is not an ideal and  $R$  is neither boolean nor a subring of a product of two integral domains) or ( $Z(R)$  is an ideal and there there exist a pair of zero-divisors  $x, y$  such that  $\text{ann}(x, y) = (0)$ ).

*Proof.* Suppose that  $Z(R)$  is not an ideal and  $R$  is neither boolean nor a subring of a product of two integral domains. Then, according to theorem 1.7 [7],  $\tilde{\Gamma}(R)$  is not complete and thus, by theorem 1.4,  $\text{diam}(\Gamma(R)) = 3$ .

Suppose that  $R$  is a boolean ring. It is obvious that if  $R \simeq \mathbb{Z}_2^2$ , then  $(\Gamma(R))$  is complete. If  $R \not\cong \mathbb{Z}_2^2$ , then, by proposition 1.11,  $\text{diam}(\Gamma(R)) = 3$ .

Suppose that  $R$  is a subring of a product of two integral domains and  $R \not\cong \mathbb{Z}_2^2$ , then, by proposition 1.10,  $\text{diam}(\Gamma(R)) = 2$ .

Suppose that  $Z(R)$  is an ideal of  $R$ :

It is obvious that if  $Z(R)^2 = (0)$ , then  $\text{diam}(\Gamma(R)) = 1$ .

Suppose that  $Z(R)^2 \neq (0)$  then, by proposition 1.8, if for each distinct pair of zero-divisors  $x, y$ ,  $\text{ann}(x, y) \neq (0)$ ,  $\text{diam}(\Gamma(R)) = 2$ .

If  $Z(R)$  is an ideal and there exist a pair of zero-divisors  $x, y$  such that  $\text{ann}(x, y) = (0)$ , then by proposition 1.8,  $\text{diam}(\Gamma(R)) = 3$ . Also, we recall that, by remark 1.9,  $Z(R)^2 \neq (0)$ .  $\square$

We recall that  $R$  is a McCoy ring (or satisfy the property A) (cf. [9]) if each finitely generated ideal contained in  $Z(R)$  has a nonzero annihilator.

**Corollary 1.13.** *Let  $R$  be a McCoy ring.*

- (1)  $\text{diam}(\Gamma(R)) = 1$  if and only if  $R \simeq \mathbb{Z}_2^2$  or  $Z(R)^2 = (0)$ .
- (2)  $\text{diam}(\Gamma(R)) = 2$  if and only if ( $R$  is (up isomorphism) a subring of a product of two integral domains and  $R \not\simeq \mathbb{Z}_2^2$ ) or ( $Z(R)$  is an ideal,  $Z(R)^2 \neq (0)$ ).
- (3)  $\text{diam}(\Gamma(R)) = 3$  if and only if ( $R$  is boolean and  $R \not\simeq \mathbb{Z}_2^2$ ) or ( $Z(R)$  is not an ideal and  $R$  is neither boolean nor a subring of a product of two integral domains).

*Proof.* Suppose that  $Z(R)$  is an ideal of  $R$  such that  $Z(R)^2 \neq (0)$ . Let a distinct pair of zero-divisors  $x, y$  so  $(x, y) \subset Z(R)$  because  $Z(R)$  is an ideal and since  $R$  is a McCoy ring,  $\text{ann}(x, y) \neq 0$ .  $\square$

**Lemma 1.14.**  *$R$  is a noetherian boolean ring if and only if  $R \simeq \mathbb{Z}_2^n$ .*

*Proof.*  $\Rightarrow$ ) Since  $R$  is boolean, then  $\dim R = 0$  so  $R$  is artinian hence  $R$  has a finite number of maximal ideals  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ . Since  $R$  is boolean,  $R$  is reduced then  $\bigcap_{i=1}^n \mathfrak{m}_i = (0)$  so  $R \simeq \prod_{i=1}^n R/\mathfrak{m}_i$  therefore  $R \simeq \mathbb{Z}_2^n$  because  $R/\mathfrak{m}_i$  are boolean fields. The other implication is obvious.  $\square$

Since a noetherian ring is a McCoy ring (cf. theorem 82, [10]), using the previous lemma, we obtain:

**Corollary 1.15.** *Let  $R$  a noetherian ring.*

- (1)  $\text{diam}(\Gamma(R)) = 1$  if and only if  $R \simeq \mathbb{Z}_2^2$  or  $Z(R)^2 = (0)$ .
- (2)  $\text{diam}(\Gamma(R)) = 2$  if and only if ( $R$  is (up isomorphism) a subring of a product of two integral domains and  $R \not\simeq \mathbb{Z}_2^2$ ) or ( $Z(R)$  is an ideal,  $Z(R)^2 \neq (0)$ ).
- (3)  $\text{diam}(\Gamma(R)) = 3$  if and only if ( $R \simeq \mathbb{Z}_2^n$ , with  $n > 2$ ) or ( $Z(R)$  is not an ideal and  $R$  is neither  $\mathbb{Z}_2^n$ , with  $n > 2$  nor a subring of a product of two integral domains).

Using theorem 2.4 [6], we obtain when  $R$  is a finite ring:

**Corollary 1.16.** *Let  $R$  be a finite ring.*

- (1)  $\text{diam}(\Gamma(R)) = 1$  if and only if  $R \simeq \mathbb{Z}_2^2$  or ( $R$  is local and  $\mathfrak{m}^2 = (0)$ ).
- (2)  $\text{diam}(\Gamma(R)) = 2$  if and only if  $R$  is a product of two fields or ( $R$  is local and  $\mathfrak{m}^2 \neq (0)$ ).
- (3)  $\text{diam}(\Gamma(R)) = 3$  if and only if ( $R \not\simeq \mathbb{Z}_2^n$  and  $R$  is neither a product of two fields nor local) or ( $R \simeq \mathbb{Z}_2^n$ , with  $n > 2$ ).

**Corollary 1.17.** *Let  $n > 1$  a composite integer.*

- (1)  $\text{diam}(\Gamma(\mathbb{Z}_n)) = 0$  if and only if  $n = 4$ .
- (2)  $\text{diam}(\Gamma(\mathbb{Z}_n)) = 1$  if and only if  $n = p^2$  with  $p$  is an odd prime.
- (3)  $\text{diam}(\Gamma(\mathbb{Z}_n)) = 2$  if and only if  $n = p^k$  with  $k > 2$  and  $p$  is prime or  $n$  is a product of two distinct primes.
- (4)  $\text{diam}(\Gamma(\mathbb{Z}_n)) = 3$  if and only if  $n$  is neither a power of a prime number nor a product of two distinct primes.

## 2 examples

In this section, we give examples of the situations described in the theorem. We begin by giving an example where  $\text{diam}(\Gamma(R)) = 1$ .

**Example 2.1.** *Let  $R = \mathbb{R}[X]/(X^2)$ . It is obvious that  $Z(R) = (X + (X^2))$  and  $Z(R)^2 = (0)$  then  $\text{diam}(\Gamma(R)) = 1$ .*

For the case where  $\text{diam}(\Gamma(R)) = 2$ , we give the following two examples:

**Example 2.2.** *Let  $R = \mathbb{Z}^2$  so  $\text{diam}(\Gamma(R)) = 2$ .*

**Example 2.3.** *Let  $R = k[X, Y]/(X^2, XY)$ . It is obvious that the  $Z(R) = (X + (X^2, XY), Y + (X^2, XY))$  is an ideal of  $R$  and since  $Y + (X^2, XY) \in Z(R)$  and  $(Y + (X^2, XY))^2 \neq 0$  in  $R$ ,  $Z(R)^2 \neq (0)$ . Also  $R$  is noetherian so, by corollary 1.14,  $\text{diam}(\Gamma(R)) = 2$ .*

For the case where  $\text{diam}(\Gamma(R)) = 3$ , we give also the following three examples:



**Example 2.4.** Let  $R = \mathbb{Z}_2^n$ , where  $n > 2$  is an integer, so  $R$  is boolean then  $\text{diam}(\Gamma(R)) = 3$ .

**Example 2.5.** Let  $R = \mathbb{Z}^3$ . It is obvious that  $Z(R)$  is not an ideal and  $R$  is neither boolean nor a subring of a product of two integral domains hence  $\text{diam}(\Gamma(R)) = 3$ .

**Example 2.6.** As in [11], we will use a variation of the construction "  $A+B$  " described in [9] and [2] to give an example of a ring  $R$  such that  $Z(R)$  is an ideal and there exist a pair of zero-divisors  $r, s$  such that  $\text{ann}(r, s) = 0$  (then by remark 1.9,  $Z(R)^2 \neq (0)$ ): Let  $M$  the maximal ideal of  $A = k[X, Y]_{(X, Y)}$  and  $\mathcal{P} = \{\mathfrak{p}_\alpha/\alpha \in \Gamma\}$  the set of height one primes of  $A$ . For every  $i = (\alpha, n) \in I = \Gamma \times \mathbb{N}$ , let  $\mathfrak{p}_i = \mathfrak{p}_\alpha$  and  $M_i = M/\mathfrak{p}_i$ . It is obvious that  $B = \bigoplus_{i \in I} M_i$  is a non-unital ring and is a unitary  $A$ -module. As in theorem 2.1 [2], define on  $R = A \times B$ :  $(a, x) + (b, y) = (a+b, x+y)$  and  $(a, x)(b, y) = (ab, ay + bx + xy)$  then  $R$  is a commutative ring with identity  $1_R = (1, 0)$  and is noted  $R = A + B$ .

We claim that  $Z(R) = \{(m, b)/m \in M, b \in B\}$  and consequently  $Z(R)$  is an ideal: Let  $(a, x) \in R$  such that  $a \notin M$  and  $x = (x_i + \mathfrak{p}_i)_{i \in I} \in B$  so  $\forall m \in M$ ,  $a + m \notin M$  and since  $A$  is local and  $M$  is the maximal ideal of  $A$ ,  $a + m$  is unit in  $A$ . For every  $i \in I$ , let  $y_i = -a^{-1}(a + x_i)^{-1}x_i$  so  $y = (y_i + \mathfrak{p}_i) \in B$  and we have  $(a, x)(a^{-1}, y) = 1_R$  hence  $(a, x) \notin Z(R)$ . Conversely, let  $(a, x) \in R$  such that  $a \in M$  and  $x = (x_i + \mathfrak{p}_i)_{i \in I} \in B$ . It follows from the Krull's principal ideal theorem that there exist  $\beta \in \Gamma$  such that  $a \in \mathfrak{p}_\beta$  so there exist  $j \in I$  such that  $a \in \mathfrak{p}_j$  and  $x \in \mathfrak{p}_j$  (because  $\{i \in I/a \in \mathfrak{p}_i\}$  is infinite and  $\{i \in I/x_i \notin \mathfrak{p}_i\}$  is finite). Let  $v \in M \setminus \mathfrak{p}_j$  and  $y_i = \begin{cases} v & \text{si } i = j \\ 0 & \text{si } i \neq j \end{cases}$  so  $y = (y_i + \mathfrak{p}_i) \in B \setminus \{0\}$  and  $(a, x)(0, y) = 0$  thus  $(a, x) \in Z(R)$ .

Also, we claim that there exist  $(r, s) \in Z(R)^2$  such that  $\text{ann}(r, s) = (0)$ : let  $r = (X, 0)$  and  $s = (Y, 0)$ . If  $(a, x) \in \text{ann}(r) \cap \text{ann}(s)$ , where  $a \in A$  and  $x = (x_i + \mathfrak{p}_i)_{i \in I} \in B$ , so  $r(a, x) = s(a, x) = 0$  then  $a = 0$  and  $\forall i \in I$ ,  $Xx_i \in \mathfrak{p}_i$  and  $Yx_i \in \mathfrak{p}_i$  then  $\forall i \in I$ ,  $x_i \in \mathfrak{p}_i$ , if not  $\exists j \in I$  such that  $x_j \notin \mathfrak{p}_j$  so  $M = (X, Y) = \mathfrak{p}_j$ , contradiction, because  $\text{ht}(M) = 2$ . Thus  $a = 0$  and  $x = 0$ .

By the previous theorem, we obtain  $\text{diam}(\Gamma(R)) = 3$ .

### 3 diameter of $\Gamma(R[X])$

Lucas gave a following characterization of the diameter of  $\Gamma(R[X])$  (see theorem 3.4, [11]):

**Theorem 3.1.** *Let  $R$  be a ring.*

- (1)  $\text{diam}(R[X]) \geq 1$ .
- (2)  $\text{diam}(R[X]) = 1$  if and only if  $R$  is a nonreduced ring such that  $Z(R)^2 = (0)$ .
- (3)  $\text{diam}(R[X]) = 2$  if and only if either (i)  $R$  is a reduced ring with exactly two minimal primes, or (ii)  $R$  is a McCoy ring and  $Z(R)$  is an ideal with  $Z(R)^2 \neq (0)$ .
- (4)  $\text{diam}(R[X]) = 3$  if and only if  $R$  is not a reduced ring with exactly two minimal primes and either  $R$  is not a McCoy ring or  $Z(R)$  is not an ideal.

In this section, we will use the results of the study of the graph  $\tilde{\Gamma}(R[X])$  [7] to approach the same problem. We recall that  $R[X]$  is a McCoy ring (cf. Theorem 2.7, [9]). We note also that  $R[X]$  is not boolean and if  $R$  is not an integral domain, then  $|Z(R[X])| > 2$ .

**Lemma 3.2.**  $Z(R[X])^2 = 0$  if and only if  $Z(R)^2 = 0$ .

*Proof.* (1) Since  $Z(R) \subset Z(R[X])$ , if  $Z(R[X])^2 = 0$  then  $Z(R)^2 = 0$ . Conversely, since  $Z(R[X]) \subset (Z(R))[X]$  (cf. Exercise 2, iii), page 13, [1]), if  $Z(R)^2 = 0$ , then  $Z(R[X])^2 = 0$ .  $\square$

Using corollary 1.13 and the previous lemma, we obtain:

**Theorem 3.3.** *Let  $R$  a ring such that  $R$  is not an integral domain.*

- (1)  $\text{diam}(\Gamma(R[X])) = 1$  if and only if  $Z(R)^2 = (0)$ .
- (2)  $\text{diam}(\Gamma(R[X])) = 2$  if and only if ( $R$  is (up isomorphism) a subring of a product of two integral domains or ( $R$  is a McCoy ring and  $Z(R)$  is an ideal such that  $Z(R)^2 \neq (0)$ ).

(3)  $\text{diam}(\Gamma(R[X])) = 3$  if and only if ( $R$  is not a McCoy ring or  $Z(R)$  is not an ideal) and  $R$  is not a subring of a product of two integral domains.

*Proof.* (1) The result is a consequence of the lemma 3.2 and the fact that  $R[X] \not\cong \mathbb{Z}_2^2$ .

(2) By corollary 1.13,  $\text{diam}(\Gamma(R[X])) = 2$  if and only if  $R[X]$  is (up isomorphism) a subring of a product of two integral domains and  $R[X] \not\cong \mathbb{Z}_2^2$  or ( $Z(R[X])$  is an ideal,  $Z(R[X])^2 \neq (0)$ ). It is obvious that  $R[X]$  is (up isomorphism) a subring of a product of two integral domains if and only if  $R$  is (up isomorphism) a subring of a product of two integral domains. On the other hand, by lemma 1.10 [7],  $Z(R[X])$  is an ideal such that  $Z(R[X])^2 \neq (0)$  if and only if  $R$  is a McCoy ring and  $Z(R)$  is an ideal such that  $Z(R)^2 \neq (0)$ .

(3) Also, by corollary 1.13,  $\text{diam}(\Gamma(R[X])) = 3$  if and only if ( $R[X]$  is boolean and  $R[X] \not\cong \mathbb{Z}_2^2$ ) or ( $Z(R[X])$  is not an ideal and  $R[X]$  is neither boolean nor a subring of a product of two integral domains). It is obvious that  $R[X]$  is not boolean and  $R[X]$  is not a subring of a product of two integral domains if and only if  $R$  is not a subring of a product of two integral domains. Also, by lemma 1.10 [7],  $Z(R[X])$  is not an ideal if and only if  $R$  is not a McCoy ring or  $Z(R)$  is not an ideal. □

**Remark 3.4.** We recall that, by lemma 1.9 [7],  $Z(R[X])$  is an ideal of  $R[X]$  if and only if  $Z(R)$  is an ideal of  $R$  and  $R$  is a McCoy ring if and only if for any ideal  $I$  of  $R$  generated by a finite number of zero-divisors,  $\text{ann}(I) \neq (0)$ . If  $R$  is noetherian so  $R$  is a McCoy ring, then  $Z(R[X])$  is an ideal of  $R[X]$  if and only if  $Z(R)$  is an ideal of  $R$ .

**Corollary 3.5.** Let  $R$  a noetherian ring such that  $R$  is not an integral domain.

(1)  $\text{diam}(\Gamma(R[X])) = 1$  if and only if  $Z(R)^2 = (0)$ .

(2)  $\text{diam}(\Gamma(R[X])) = 2$  if and only if ( $R$  is (up isomorphism) a subring of a product of two integral domains or ( $Z(R)$  is an ideal and  $Z(R)^2 \neq (0)$ ).

(3)  $\text{diam}(\Gamma(R[X])) = 3$  if and only if  $Z(R)$  is not an ideal and  $R$  is neither boolean nor a subring of a product of two integral domains.

**Corollary 3.6.** *Let  $R$  a finite ring such that  $R$  is not a field.*

- (1)  *$\text{diam}(\Gamma(R[X])) = 1$  if and only if  $R$  is local and  $\mathfrak{m}^2 = (0)$ .*
- (2)  *$\text{diam}(\Gamma(R[X])) = 2$  if and only if  $R$  is a product of two fields or ( $R$  is local and  $\mathfrak{m}^2 \neq (0)$ ).*
- (3)  *$\text{diam}(\Gamma(R[X])) = 3$  if and only if  $R$  is not local and  $R$  is not a product of two fields.*

**Corollary 3.7.** *Let  $n > 1$  a composite integer.*

- (1)  *$\Gamma(\mathbb{Z}_n[X]) = 1$  if and only if  $n = p^2$  with  $p$  is prime.*
- (2)  *$\text{diam}(\Gamma(\mathbb{Z}_n[X])) = 2$  if and only if  $n$  is a product of two distinct primes or  $n = p^k$  with  $k > 2$  and  $p$  is prime.*
- (3)  *$\text{diam}(\Gamma(\mathbb{Z}_n[X])) = 3$  if and only if  $n$  is neither a power of a prime number nor a product of two distinct primes.*

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