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PARAPRODUCT IN BESOV–MORREY SPACES

YOSHIHIRO SAWANO

ABSTRACT. Recently it turned out that the paraproduct plays the key role in some highly singular partial differential equations. In this note the counterparts for Besov–Morrey spaces are obtained. This note is organized in a self-contained manner. Besov–Morrey spaces, paraproduct

1. INTRODUCTION

In this note we investigate the boundedness property of the pointwise multiplier of the functions in Hölder–Zygmund spaces and Besov–Morrey spaces including the commutators. Starting from the seminal papers [2, 3, 4], we investigate these operators from the viewpoint of harmonic analysis.

To describe our first result, we recall some notation. First, we use the following convention on balls in $\mathbf{R}^n$ here and below: We denote by $B(x,r)$ the *ball centered at x of radius r*. Namely, we write

$$B(x,r) \equiv \{y \in \mathbf{R}^n \,:\, |x-y| < r\}$$

when $x \in \mathbf{R}^n$ and $r > 0$. Given a ball B, we denote by $c(B)$ its *center* and by $r(B)$ its *radius*. We write $B(r)$ instead of $B(o,r)$, where $o \equiv (0,0,\ldots,0)$. Keeping this definition of balls in mind, we define Morrey spaces. Let $1 \le q \le p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{x\in\mathbf{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B(x,r))}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbf{R}^n)$ is the set of all the measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite. We move on to the definition of Besov–Morrey spaces. Choose $\psi \in C^\infty_{\rm c}(\mathbf{R}^n)$ so that

$$\chi_{B(\frac{6}{5})} \le \psi \le \chi_{B(\frac{3}{2})}. \tag{1.1}$$

We write

$$\varphi_0(\xi) = \psi(\xi), \quad \varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi), \quad \psi_j(\xi) = \psi(2^{-j}\xi)$$

for $j \in \mathbf{N}$ and $\xi \in \mathbf{R}^n$.

For $f \in L^1(\mathbf{R}^n)$, define the *Fourier transform* and the *inverse Fourier transform* by:

$$\mathcal{F}f(\xi) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} f(x) {\rm e}^{-ix\cdot\xi} {\rm d}x, \quad \mathcal{F}^{-1}f(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} f(\xi) {\rm e}^{ix\cdot\xi} {\rm d}\xi.$$

Here and below we write $\theta(D)f \equiv \mathcal{F}^{-1}[\theta \cdot \mathcal{F}f]$ for $\theta \in \mathcal{S}(\mathbf{R}^n)$ and $f \in \mathcal{S}'(\mathbf{R}^n)$. It is known that $\theta(D)f \in \mathcal{S}'(\mathbf{R}^n) \cap L^1_{\rm loc}(\mathbf{R}^n)$ and it satisfies

$$\theta(D)f(x) = (2\pi)^{-\frac{n}{2}} \langle f, \mathcal{F}^{-1}\theta(x-\cdot)\rangle$$

for all $x \in \mathbf{R}^n$. We define

$$\|f\|_{\mathcal{N}^s_{pqr}} \equiv \left(\sum_{j=0}^{\infty} (2^{js} \|\varphi_j(D) f\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}}$$

for $f \in \mathcal{S}'(\mathbf{R}^n)$.

Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbf{R}$. The space $\mathcal{N}^s_{pqr}(\mathbf{R}^n)$, which we call the *Besov–Morrey space*, is the set of all $f \in \mathcal{S}'(\mathbf{R}^n)$ for which the norm $\|f\|_{\mathcal{N}^s_{pqr}}$ is finite. The parameter s describes the differential property in terms of Morrey spaces as is indicated by the relations $\mathcal{N}^{s+\varepsilon}_{pqr}(\mathbf{R}^n) \subset \mathcal{N}^s_{pqr}(\mathbf{R}^n)$ and $\partial_j : \mathcal{N}^{s+1}_{pqr}(\mathbf{R}^n) \subset \mathcal{N}^s_{pqr}(\mathbf{R}^n)$ for all $\varepsilon > 0$ and $j = 1, 2, \ldots, n$. It is also clear from the triangle inequality in $\mathcal{M}^p_q(\mathbf{R}^n)$ that $\mathcal{N}^0_{pq1}(\mathbf{R}^n) \subset \mathcal{M}^p_q(\mathbf{R}^n)$. The main results in this note are the following:

Theorem 1.1. *Let $1 \le q_1 \le p_1 < \infty$, $1 \le q_2 \le p_2 < \infty$, $1 \le q \le p < \infty$, $1 \le r \le \infty$, and $s > 0$. Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then for $f \in \mathcal{N}^s_{p_1 q_1 r}(\mathbf{R}^n)$ and $g \in \mathcal{N}^s_{p_2 q_2 r}(\mathbf{R}^n)$ the product $f \cdot g \in \mathcal{N}^s_{pqr}(\mathbf{R}^n)$ makes sense and satisfies

$$\|f \cdot g\|_{\mathcal{N}^s_{pqr}} \le C \|f\|_{\mathcal{N}^s_{p_1 q_1 r}} \|g\|_{\mathcal{N}^s_{p_2 q_2 r}}.$$

Theorem 1.1 is an extension of the inequality

$$\|f \cdot g\|_{\mathcal{M}^p_q} \le C \|f\|_{\mathcal{M}^{p_1}_{q_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2}}$$

for $f \in \mathcal{M}^{p_1}_{q_1}(\mathbf{R}^n)$ and $g \in \mathcal{M}^{p_2}_{q_2}(\mathbf{R}^n)$. The proof of Theorem 1.1 hinges on the paraproduct introduced by Bony [1]. Let $f, g \in \mathcal{S}'(\mathbf{R}^n)$. The (right) paraproduct $f \preceq g$ is defined to be

$$f \preceq g = \sum_{j=2}^{\infty} \psi_{j-2}(D) f \cdot \varphi_j(D) g,$$

while the (left) paraproduct $f \succeq g$ is defined to be

$$f \succeq g = \sum_{j=2}^{\infty} \varphi_j(D) f \cdot \psi_{j-2}(D) g.$$

Furthermore, the resonant operator $f \odot g$ is defined by

$$f \odot g = \sum_{j=0}^{\infty} \varphi_j(D) f \cdot \varphi_j(D) g + \sum_{j=1}^{\infty} \varphi_{j-1}(D) f \cdot \varphi_j(D) g + \sum_{j=1}^{\infty} \varphi_j(D) f \cdot \varphi_{j-1}(D) g.$$

We need some assumptions on f and g to justify these definitions. These three linear operators are key linear operators used in the proof of Theorem 1.1.

Another aim of this paper is to extend the results used in [2, 4], which also use these operators, to the Morrey setting:

Theorem 1.2. *Assume that the parameters α, β, s satisfy*

$$0 < \alpha \le 1, \quad s + \beta < 0 < s + \alpha + \beta.$$

Then for $f \in \mathrm{Lip}_\alpha(\mathbf{R}^n)$, $g \in \mathcal{C}^\beta(\mathbf{R}^n)$ and $h \in \mathcal{N}^s_{pqr}(\mathbf{R}^n)$

$$\|(f \preceq g) \odot h - f(g \odot h)\|_{\mathcal{N}^{s+\alpha+\beta}_{pqr}} \le C \|f\|_{\mathrm{Lip}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{N}^s_{pqr}}.$$

This result is a counterpart to [2, Lemma 2.4].

Here we briefly recall how Besov–Morrey spaces emerged. See [12, 19] for an exhaustive account. The first paper dates back to 1984. In [9] Netrusov considered Besov–Morrey spaces. Later on Kozono and Yamazaki investigated Besov–Morrey spaces and applied them to the Navier–Stokes equations [6]. Mazzucato expanded this application more in [8]. Decompositions of Besov–Morrey spaces can be found in [7, 14, 16]. After that Yang and Yuan defined Besov-type spaces and Triebel–Lizorkin-type spaces in [17, 18]. A close relation between these spaces is pointed out in [15]. Recently more and more is investigated. For example, Haroske and Skrzypczak investigated embedding relation of Besov–Morrrey spaces [5]. One of the important consequence of definining the Besov–Morrey spaces is that we have the embedding

$$\mathcal{N}^s_{pq\infty}(\mathbf{R}^n) \hookrightarrow \mathcal{C}^{s-\frac{n}{p}}(\mathbf{R}^n)$$

for $s > \frac{n}{p}$. See [13].

We organize this paper as follows: Section 2 is devoted to collecting some preliminary facts. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2.

2. Preliminaries

2.1. **Schwartz distributions and the Fourier transform.** Let us recall the notation of multi-indexes to define the Schwartz space $\mathcal{S}(\mathbf{R}^n)$. By "a multi-index", we mean an element in ${\mathbf{N}_0}^n \equiv \{0, 1, 2, \dots\}^n$. In this paper a tacit understanding is that all functions assume their value in $\mathbf{C}$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in {\mathbf{N}_0}^n$ $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, we define $x^\alpha \equiv {x_1}^{\alpha_1}{x_2}^{\alpha_2}\cdots{x_n}^{\alpha_n}$. For a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in {\mathbf{N}_0}^n$ and $f \in C^\infty(\mathbf{R}^n)$, we set

$$\partial^\beta f \equiv \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\left(\frac{\partial}{\partial x_2}\right)^{\beta_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n} f.$$

Definition 2.1 (Schwartz function space $\mathcal{S}(\mathbf{R}^n)$)**.** For multi-indexes $\alpha, \beta \in {\mathbf{N}_0}^n$ and a function φ, write $\varphi_{(\alpha,\beta)}(x) \equiv x^\alpha \partial^\beta \varphi(x)$, $x \in \mathbf{R}^n$ temporarily. The *Schwartz function space* $\mathcal{S}(\mathbf{R}^n)$ is the set of all the functions satisfying

$$\mathcal{S}(\mathbf{R}^n) \equiv \bigcap_{\alpha,\beta\in{\mathbf{N}_0}^n} \left\{\varphi \in C^\infty(\mathbf{R}^n) \,:\, \varphi_{(\alpha,\beta)} \in L^\infty(\mathbf{R}^n)\right\}.$$

The elements in $\mathcal{S}(\mathbf{R}^n)$ are called the *test functions*.

Denote by $\mathcal{S}'(\mathbf{R}^n)$ the set of all continuous linear mappings from $\mathcal{S}(\mathbf{R}^n)$ to $\mathbf{C}$. Denote by $\langle f, \varphi\rangle$ the value of f evaluated at φ; $\langle f, \varphi\rangle \equiv f(\varphi)$.

Note that $\mathcal{S}(\mathbf{R}^n)$ is embedded into $L^1(\mathbf{R}^n)$ and that $\mathcal{F}$ mapsto $\mathcal{S}(\mathbf{R}^n)$ isomorphically to itself. Thus by duality $\mathcal{F}$ mapsto $\mathcal{S}'(\mathbf{R}^n)$ isomorphically to itself.

A function $h \in C^\infty(\mathbf{R}^n)$ is said to *have at most polynomial growth at infinity*, if for all $\alpha \in {\mathbf{N}_0}^n$, there exist $C_\alpha > 0$ and $N_\alpha > 0$ such that:

$$|\partial^\alpha h(x)| \le C_\alpha \langle x\rangle^{N_\alpha}, \quad x \in \mathbf{R}^n. \tag{2.1}$$

Here we are interested in the inclusion:

$$\mathrm{supp}(\mathcal{F}[f\cdot g]) \subset \mathrm{supp}(\mathcal{F}f) + \mathrm{supp}(\mathcal{F}g) \tag{2.2}$$

for $f, g \in \mathcal{S}'(\mathbf{R}^n)$ having at most polynomial growth at infinity. Usually we assume that $\mathcal{F}f$ is compactly supported.

Let Ω be a bounded set in $\mathbf{R}^n$. Denote by $\mathcal{S}'_\Omega(\mathbf{R}^n)$ the set of all distributions whose Fourier transform is contained in the closure $\overline{\Omega}$. Define $\mathcal{S}_\Omega(\mathbf{R}^n) \equiv \mathcal{S}'_\Omega(\mathbf{R}^n) \cap \mathcal{S}(\mathbf{R}^n)$.

Lemma 2.2.

(1) *For all* $F \in C_{\rm c}^\infty({\bf R}^n)$, $G \in \mathcal{S}({\bf R}^n)$,

$$\mathrm{supp}(F * G) \subset \mathrm{supp}(F) + \mathrm{supp}(G). \tag{2.3}$$

(2) *Let* K *be a compact set. Then for all* $f \in \mathcal{S}_K({\bf R}^n)$, $g \in \mathcal{S}({\bf R}^n)$, (2.2) *holds.*

Proof.

(1) The proof of (2.3) is standard: Simply write out the convolution $f * g$ in full in terms of the integral to have

$$\begin{aligned}\{x \in {\bf R}^n \,:\, F * G(x) \neq 0\} &\subset \{x \in {\bf R}^n \,:\, F(x) \neq 0\} + \{x \in {\bf R}^n \,:\, G(x) \neq 0\} \\ &\subset \mathrm{supp}(F) + \mathrm{supp}(G).\end{aligned}$$

Since $\mathrm{supp}(F)$ is compact and $\mathrm{supp}(G)$ is closed, $\mathrm{supp}(F) + \mathrm{supp}(G)$ is a closed set. Thus, taking the closure of the above inclusion, we conclude that (2.3) holds.

(2) Inclusion (2.2) is a consequence of $\mathcal{F}[f \cdot g] = (2\pi)^{-\frac{n}{2}} \mathcal{F}f * \mathcal{F}g$ and the fact that $\mathcal{F}$ maps $\mathcal{S}({\bf R}^n)$ isomorphically.

□

Define the convolution $f * g$ by $f * g(x) \equiv \int_{{\bf R}^n} f(x-y)g(y){\rm d}y$ as long as the integral makes sense.

A band-limited distribution is a distribution whose Fourier transform is compactly supported.

Lemma 2.3. *For all band-limited distributions* $f \in \mathcal{S}'({\bf R}^n)$ *and all functions* $g \in \mathcal{S}({\bf R}^n)$, (2.2) *holds.*

Proof. Let $\tau \in C_{\rm c}^\infty({\bf R}^n)$ be such that $\mathrm{supp}(\tau) \cap (\mathrm{supp}(\mathcal{F}f) + \mathrm{supp}(\mathcal{F}g)) = \emptyset$. We need to show that

$$\langle \mathcal{F}[f \cdot g], \tau \rangle = 0.$$

By the definition of the Fourier transform this amounts to showing:

$$\langle f \cdot g, \mathcal{F}\tau \rangle = 0.$$

Since $g \in \mathcal{S}({\bf R}^n)$, we have

$$\langle f \cdot g, \mathcal{F}\tau \rangle = \langle f, g \cdot \mathcal{F}\tau \rangle$$

from the definition of the pointwise multiplication $f \cdot g \in \mathcal{S}'({\bf R}^n)$ for $f \in \mathcal{S}'({\bf R}^n)$ and $g \in \mathcal{S}({\bf R}^n)$. We note that

$$\mathcal{F}^{-1}[g \cdot \mathcal{F}\tau] = (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}g * \tau.$$

Thus, by the definition of the Fourier transform $\mathcal{F}$ acting on $\mathcal{S}'({\bf R}^n)$

$$\langle f \cdot g, \mathcal{F}\tau \rangle = (2\pi)^{-\frac{n}{2}} \langle \mathcal{F}f, \mathcal{F}^{-1}g * \tau \rangle.$$

From the definition of the Fourier transform $x \in \mathrm{supp}(\mathcal{F}g)$ if and only if $-x \in \mathrm{supp}(\mathcal{F}^{-1}g)$. Since $\mathrm{supp}(\tau) \cap (\mathrm{supp}(\mathcal{F}f) + \mathrm{supp}(\mathcal{F}g)) = \emptyset$, we have

$$\mathrm{supp}(\tau * \mathcal{F}^{-1}g) \cap \mathrm{supp}(f) \subset (\mathrm{supp}(\tau) + \mathrm{supp}(\mathcal{F}^{-1}g)) \cap \mathrm{supp}(f) = \emptyset.$$

thanks to Lemma 2.2. Thus, $\langle f \cdot g, \mathcal{F}\tau \rangle = 0$ and (2.2) holds. □

Corollary 2.4. *For all band-limited* $f, g \in \mathcal{S}({\bf R}^n)$, (2.2) *holds.*

Proof. Let $\tau \in C_c^\infty(\mathbf{R}^n)$ be such that $\mathrm{supp}(\tau) \cap (\mathrm{supp}(\mathcal{F}f) + \mathrm{supp}(\mathcal{F}g)) = \emptyset$. We need to show that $\langle f \cdot g, \mathcal{F}\tau \rangle = 0$. Let $\Phi \in \mathcal{S}(\mathbf{R}^n)$ be such that $\Phi(0) = 1$ and that $\mathrm{supp}(\mathcal{F}\Phi) \subset B(1)$. Then

$$\langle f \cdot g, \mathcal{F}\tau \rangle = \lim_{\varepsilon \downarrow 0} \langle f \cdot g, \Phi(\varepsilon \cdot)^2 \mathcal{F}\tau \rangle,$$

since

$$\lim_{\varepsilon \downarrow 0} \Phi(\varepsilon \cdot)^2 \mathcal{F}\tau = \tau$$

in $\mathcal{S}(\mathbf{R}^n)$. If $\varepsilon > 0$ is chosen so that $\mathrm{supp}(\tau) \cap (\mathrm{supp}(\mathcal{F}f) + \mathrm{supp}(\mathcal{F}g) + \overline{B(2\varepsilon)}) = \emptyset$, then we have

$$\langle \Phi(\varepsilon \cdot) f \cdot \Phi(\varepsilon \cdot) g, \mathcal{F}\tau \rangle = 0$$

thanks to Lemma 2.3, since $\Phi(\varepsilon \cdot) f$ and $\Phi(\varepsilon \cdot) g$ are both band-limited due to Lemma 2.2. Thus, $\langle f \cdot g, \mathcal{F}\tau \rangle = 0$. □

2.2. **Lipschitz spaces and Hölder–Zygmund spaces.** Let $0 < \alpha \le 1$. We let $\mathrm{Lip}^\alpha(\mathbf{R}^n)$ be the set of all bounded continuous functions $f : \mathbf{R}^n \to \mathbf{C}$ for which the quantity $\|f\|_{\mathrm{Lip}^\alpha} \equiv \|f\|_{L^\infty} + \sup_{x,y \in \mathbf{R}^n} |x-y|^{-\alpha} |f(x) - f(y)|$ is finite. Let ψ satisfy (1.1). We write

$$\varphi_0(\xi) = \psi(\xi), \quad \varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi), \quad \psi_j(\xi) = \psi(2^{-j}\xi)$$

for $j \in \mathbf{N}$ and $\xi \in \mathbf{R}^n$ as before. Then the (Besov)–Hölder–Zygmund space $\mathcal{C}^\beta(\mathbf{R}^n)$ with $\beta \in \mathbf{R}$. is defined to be the set of all $f \in \mathcal{S}'(\mathbf{R}^n)$ for which

$$\|f\|_{\mathcal{C}^\beta} = \sup_{j \in \mathbf{N}_0} 2^{j\beta} \|\varphi_j(D) f\|_{L^\infty}$$

is finite. Noteworthy is the fact that $\mathrm{Lip}^\alpha(\mathbf{R}^n)$ and $\mathcal{C}^\alpha(\mathbf{R}^n)$ are isomorphic for all $0 < \alpha < 1$ but that $\mathrm{Lip}^1(\mathbf{R}^n)$ is a proper subset of $\mathcal{C}^1(\mathbf{R}^n)$.

Usually we replace (1.1) by $\chi_{B(1)} \le \psi \le \chi_{B(2)}$. However, if we pose a stronger condition (1.1) on ψ, we can quantify what we are doing. The following is an example of such an attempt.

Example 2.5. Let $j, k, l \in \mathbf{N}$ satisfy $l \ge 2$.

(1) We note that $\varphi_k \cdot \psi_{l-2} \ne 0$ only if $l \ge k$. In this case, we have

$$\mathrm{supp}(\mathcal{F}[\varphi_k(D)\psi_{l-2}(D) f \cdot \varphi_l(D) g]) \subset \overline{B\left(\frac{3}{2} \cdot 2^k\right)} + \overline{B\left(\frac{3}{2} \cdot 2^l\right) \setminus B\left(\frac{3}{5} \cdot 2^l\right)}.$$

(2) Assume $l \ge k+2$. Then since

$$\frac{1}{8} < \frac{3}{5} - \frac{3}{8} < \frac{3}{2} + \frac{3}{8} < 2,$$

$$\mathrm{supp}(\mathcal{F}[\varphi_k(D)\psi_{l-2}(D) f \cdot \varphi_l(D) g]) \subset \overline{B\left(2^{l+1}\right) \setminus B\left(\frac{1}{8} \cdot 2^l\right)}.$$

Consequently,

$$\varphi_j(D)[\varphi_k(D)\psi_{l-2}(D) f \cdot \varphi_l(D) g] \ne 0$$

only if $l-3 \le j+1 \le l+1$ or $l-3 \le j-1 \le l+1$, or equivalently $l-4 \le j \le l+2$.

2.3. **Some estimates in Besov–Morrey spaces.** For the paraproducts, we use the following observation:

Lemma 2.6. *Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbf{R}$. Suppose that we are given a collection $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}^p_q(\mathbf{R}^n)$ satisfying $f_0 \in \mathcal{S}'_{B(8)}(\mathbf{R}^n)$, $f_j \in \mathcal{S}'_{B(2^{j+3}) \setminus B(2^{j-1})}(\mathbf{R}^n)$, $j \in \mathbf{N}$ and*

$$\left(\sum_{j=0}^{\infty} (2^{js} \|f_j\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}} < \infty.$$

Then

$$f = \sum_{j=0}^{\infty} f_j \in \mathcal{N}^s_{pqr}(\mathbf{R}^n)$$

with

$$\|f\|_{\mathcal{N}^s_{pqr}} \le C \left(\sum_{j=0}^{\infty} (2^{js} \|f_j\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}}.$$

Let $j \in \mathbf{Z}$ and $\tau \in \mathcal{S}(\mathbf{R}^n)$. Then define $\tau_j \equiv \tau(2^{-j} \cdot)$.

Proof. Let $\psi, \varphi_j \in C^\infty_{\rm c}(\mathbf{R}^n)$ be as before for each $j \in \mathbf{N}_0$. $\xi \in \mathbf{R}^n$. Then

$$\varphi_k(D) f = \sum_{j=\max(0,k-4)}^{k+4} \varphi_k(D) f_j \quad (k \in \mathbf{N}).$$

Thus,

$$\|\varphi_k(D) f\|_{\mathcal{M}^p_q} \le C \sum_{j=\max(0,k-4)}^{k+4} \|f_j\|_{\mathcal{M}^p_q} \quad (k \in \mathbf{N}).$$

As a consequence

$$\begin{aligned}
\|f\|_{\mathcal{N}^s_{pqr}} &= \left(\sum_{k=0}^{\infty} (2^{ks} \|\varphi_k(D) f\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}} \\
&\le C \left(\sum_{k=4}^{\infty} \left(2^{ks} \sum_{j=\max(0,k-4)}^{k+4} \|f_j\|_{\mathcal{M}^p_q} \right)^r \right)^{\frac{1}{r}} \\
&\le C \sum_{j=0}^{8} \|f_j\|_{\mathcal{M}^p_q} + C \left(\sum_{k=4}^{\infty} \left(2^{ks} \sum_{l=-4}^{4} \|f_{k+l}\|_{\mathcal{M}^p_q} \right)^r \right)^{\frac{1}{r}} \\
&\le C \sum_{j=0}^{4} \|f_j\|_{\mathcal{M}^p_q} + C \sum_{l=-4}^{4} \left(\sum_{k=4}^{\infty} \left(2^{ks} \|f_{k+l}\|_{\mathcal{M}^p_q} \right)^r \right)^{\frac{1}{r}} \\
&\le C \left(\sum_{j=0}^{\infty} (2^{js} \|f_j\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}},
\end{aligned}$$

as required. □

3. Paraproduct

3.1. **Paraproduct.** For the paraproducts, we use the following observation:

Lemma 3.1. *Let* $1 \le q_1 \le p_1 < \infty$, $1 \le r_1 \le \infty$, $1 \le q_2 \le p_2 < \infty$, $1 \le r_2 \le \infty$, $1 \le q \le p < \infty$, $1 \le r \le \infty$, *and* $s_0, s_1, s \in \mathbf{R}$. *Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad s = s_1 + s_2.$$

Suppose that we are given collections $\{f_j\}_{j=1}^\infty, \{g_j\}_{j=1}^\infty \subset \mathcal{M}^p_q(\mathbf{R}^n)$ *satisfying* $f_j \in \mathcal{S}'_{B(2^{j-1})}(\mathbf{R}^n)$, $g_j \in \mathcal{S}'_{B(2^{j+2}) \setminus B(2^j)}(\mathbf{R}^n)$, $j \in \mathbf{N}$ *and*

$$\left(\sum_{j=1}^\infty (2^{js_1}\|f_j\|_{\mathcal{M}^{p_1}_{q_1}})^{r_1}\right)^{\frac{1}{r_1}}, \quad \left(\sum_{j=1}^\infty (2^{js_2}\|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^{r_2}\right)^{\frac{1}{r_2}} < \infty.$$

Then we have

$$\sum_{j=1}^\infty f_j \cdot g_j \in \mathcal{N}^s_{pqr}(\mathbf{R}^n)$$

and satisfies

$$\left\|\sum_{j=1}^\infty f_j \cdot g_j\right\|_{\mathcal{N}^s_{pqr}} \le C \left(\sum_{j=1}^\infty (2^{js_1}\|f_j\|_{\mathcal{M}^{p_1}_{q_1}})^{r_1}\right)^{\frac{1}{r_1}} \left(\sum_{j=1}^\infty (2^{js_2}\|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^{r_2}\right)^{\frac{1}{r_2}}.$$

Proof. Thanks to Corollary 2.4 we have $\operatorname{supp}(f_j \cdot g_j) \subset \overline{B(2^{j+3}) \setminus B(2^{j-1})}$ for all $j \in \mathbf{N}$. Thus by the equivalent expression (see Lemma 2.6) and the Hölder inequality, we have

$$\begin{aligned}
\left\|\sum_{j=1}^\infty f_j \cdot g_j\right\|_{\mathcal{N}^s_{pqr}} &\le C \left(\sum_{j=1}^\infty (2^{js}\|f_j \cdot g_j\|_{\mathcal{M}^p_q})^r\right)^{\frac{1}{r}} \\
&\le C \left(\sum_{j=1}^\infty (2^{j(s_1+s_2)}\|f_j\|_{\mathcal{M}^{p_1}_{q_1}}\|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^r\right)^{\frac{1}{r}} \\
&\le C \left(\sum_{j=1}^\infty (2^{js_1}\|f_j\|_{\mathcal{M}^{p_1}_{q_1}})^{r_1}\right)^{\frac{1}{r_1}} \left(\sum_{j=1}^\infty (2^{js_2}\|f_j\|_{\mathcal{M}^{p_2}_{q_2}})^{r_2}\right)^{\frac{1}{r_2}}.
\end{aligned}$$

□

3.2. **Resonant part.** To handle the resonant part, we use the following lemma. When we prove this type of estimates, we can use the atomic decomposition taking advanatage of the assumption $s > 0$ and $p, q, r \ge 1$. Here we estimate the distributions directly. This corresponds to [2, Lemma A3].

Lemma 3.2. *Let* $1 \le q \le p < \infty$, $1 \le r \le \infty$ *and* $s > 0$. *Suppose that we are given a collection* $\{f_j\}_{j=0}^\infty \subset \mathcal{M}^p_q(\mathbf{R}^n)$ *satisfying* $f_j \in \mathcal{S}'_{B(2^{j+2})}(\mathbf{R}^n)$, $j \in \mathbf{N}_0$ *and*

$$\left(\sum_{j=0}^\infty (2^{js}\|f_j\|_{\mathcal{M}^p_q})^r\right)^{\frac{1}{r}} < \infty.$$

Then

$$\sum_{j=0}^{\infty} f_j \in \mathcal{N}^s_{pqr}(\mathbf{R}^n).$$

Proof. Let $\psi, \varphi_j \in C^\infty_{\rm c}(\mathbf{R}^n)$ be as before for each $j \in \mathbf{N}_0$. We have

$$2^{ks}\sum_{j=0}^{\infty} |\varphi_k(D)f_j| \le \sum_{j=\max(0,k-3)}^{\infty} 2^{(k-j)s}|\varphi_k(D)[2^{js}f_j]|.$$

As a consequence, by the translation invariance of $\mathcal{M}^p_q(\mathbf{R}^n)$ and the equality $\|\mathcal{F}^{-1}\varphi_k\|_{L^1} = \|\mathcal{F}^{-1}\varphi_1\|_{L^1}$ for all $k \in \mathbf{N}$

$$\begin{aligned}\left\|2^{ks}\sum_{j=0}^{\infty}|\varphi_k(D)f_j|\right\|_{\mathcal{M}^p_q} &\le C\sum_{j=\max(0,k-3)}^{\infty} 2^{(k-j)s}(2^{js}\|\varphi_k(D)f_j\|_{\mathcal{M}^p_q})\\ &\le C\sum_{j=\max(0,k-3)}^{\infty} 2^{(k-j)s}(2^{js}\|f_j\|_{\mathcal{M}^p_q}).\end{aligned}$$

Since $s > 0$, by the Hölder inequality

$$\begin{aligned}&\left\|2^{ks}\sum_{j=0}^{\infty}|\varphi_k(D)f_j|\right\|_{\mathcal{M}^p_q}\\ &\le C\sum_{j=\max(0,k-3)}^{\infty} 2^{\frac{1}{2}(k-j)s}2^{\frac{1}{2}(k-j)s}(2^{js}\|f_j\|_{\mathcal{M}^p_q})\\ &\le C\left(\sum_{j=\max(0,k-3)}^{\infty} 2^{\frac{1}{2}(k-j)sr'}\right)^{\frac{1}{r'}}\left(\sum_{j=\max(0,k-3)}^{\infty}(2^{\frac{1}{2}(k-j)s}(2^{js}\|f_j\|_{\mathcal{M}^p_q}))^r\right)^{\frac{1}{r}}\\ &\le C\left(\sum_{j=\max(0,k-3)}^{\infty}(2^{\frac{1}{2}(k-j)s}(2^{js}\|f_j\|_{\mathcal{M}^p_q}))^r\right)^{\frac{1}{r}}.\end{aligned}$$

Thus, if we take the ℓ^r-norm, then we obtain

$$\left\|\sum_{j=0}^{\infty} f_j\right\|_{\mathcal{N}^s_{pqr}} \le C\left(\sum_{j=0}^{\infty}(2^{js}\|f_j\|_{\mathcal{M}^p_q})^r\right)^{\frac{1}{r}}.$$

□

Corollary 3.3. *Let* $1 \le q_1 \le p_1 < \infty$, $1 \le r_1 \le \infty$, $1 \le q_2 \le p_2 < \infty$, $1 \le r_2 \le \infty$, $1 \le q \le p < \infty$, $1 \le r \le \infty$, *and* $s_0, s_1, s \in \mathbf{R}$. *Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad s = s_1 + s_2 > 0.$$

Suppose that we are given collections $\{f_j\}_{j=0}^\infty, \{g_j\}_{j=0}^\infty \subset \mathcal{M}^p_q(\mathbf{R}^n)$ *satisfying* $f_j, g_j \in \mathcal{S}'_{B(2^{j+1})}(\mathbf{R}^n)$, $j \in \mathbf{N}_0$ *and*

$$\left(\sum_{j=0}^{\infty}(2^{js_1}\|f_j\|_{\mathcal{M}^{p_1}_{q_1}})^{r_1}\right)^{\frac{1}{r_1}}, \quad \left(\sum_{j=0}^{\infty}(2^{js_2}\|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^{r_2}\right)^{\frac{1}{r_2}} < \infty$$

Then

$$\sum_{j=0}^{\infty} f_j \cdot g_j \in \mathcal{N}^s_{pqr}(\mathbf{R}^n).$$

and

$$\left\| \sum_{j=0}^{\infty} f_j \cdot g_j \right\|_{\mathcal{N}^s_{pqr}} \le C \left(\sum_{j=0}^{\infty} (2^{js_1} \|f_j\|_{\mathcal{M}^{p_1}_{q_1}})^{r_1} \right)^{\frac{1}{r_1}} \left(\sum_{j=0}^{\infty} (2^{js_2} \|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^{r_2} \right)^{\frac{1}{r_2}}.$$

Proof. In fact, by Corollary 2.4, we see that $f_j \cdot g_j \in \mathcal{S}'_{B(2^{j+2})}(\mathbf{R}^n)$. Thus, invoking Lemma 3.2 and using the Hölder inequality twice, we have

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} f_j \cdot g_j \right\|_{\mathcal{N}^s_{pqr}} &\le C \left(\sum_{j=0}^{\infty} (2^{js} \|f_j \cdot g_j\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}} \\ &\le C \left(\sum_{j=0}^{\infty} (2^{js_1} \|f_j\|_{\mathcal{M}^{p_1}_{q_1}})^{r_1} \right)^{\frac{1}{r_1}} \left(\sum_{j=0}^{\infty} (2^{js_2} \|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^{r_2} \right)^{\frac{1}{r_2}}. \end{aligned}$$

□

3.3. **Conclusion of the proof of Theorem 1.1.** We prove Theorem 1.1 as follows: If we use Lemma 3.1, then we have

$$\begin{aligned} \|f \preceq g\|_{\mathcal{N}^s_{pqr}} &= \left\| \sum_{j=2}^{\infty} \psi_{j-2}(D) f \cdot \varphi_j(D) g \right\|_{\mathcal{N}^s_{pqr}} \\ &\le C \sup_{j \in \mathbf{N}_0} \|\psi_j(D) f\|_{\mathcal{M}^{p_1}_{q_1}} \left(\sum_{j=2}^{\infty} (2^{js} \|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^r \right)^{\frac{1}{r}}. \end{aligned}$$

Since $\psi_j(D) f = (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}\psi_j * f$ and $\mathcal{F}^{-1}\psi_j = 2^{jn}\mathcal{F}^{-1}\psi(2^j \cdot)$, we have

$$\begin{aligned} \|\psi_j(D) f\|_{\mathcal{M}^{p_1}_{q_1}} &= (2\pi)^{-\frac{n}{2}} \|\mathcal{F}^{-1}\psi_j * f\|_{\mathcal{M}^{p_1}_{q_1}} \\ &\le (2\pi)^{-\frac{n}{2}} \|\mathcal{F}^{-1}\psi_j\|_{L^1} \|f\|_{\mathcal{M}^{p_1}_{q_1}} \\ &= (2\pi)^{-\frac{n}{2}} \|\mathcal{F}^{-1}\psi\|_{L^1} \|f\|_{\mathcal{M}^{p_1}_{q_1}}. \end{aligned}$$

Thus,

$$\|f \preceq g\|_{\mathcal{N}^s_{pqr}} \le C \|f\|_{\mathcal{M}^{p_1}_{q_1}} \left(\sum_{j=2}^{\infty} (2^{js} \|g_j\|_{\mathcal{M}^{p_2}_{q_2}})^r \right)^{\frac{1}{r}}.$$

Recall that $s > 0$. Since $\mathcal{M}^{p_1}_{q_1}(\mathbf{R}^n) \supset \mathcal{N}^s_{p_1 q_1 r}(\mathbf{R}^n)$, we have

$$\|f \preceq g\|_{\mathcal{N}^s_{pqr}} \le C \|f\|_{\mathcal{N}^s_{p_1 q_1 r}} \|g\|_{\mathcal{N}^s_{p_2 q_2 r}}.$$

Likewise

$$\|f \succeq g\|_{\mathcal{N}^s_{pqr}} \le C \|f\|_{\mathcal{N}^s_{p_1 q_1 r}} \|g\|_{\mathcal{N}^s_{p_2 q_2 r}}.$$

Meanwhile, we have

$$\|f \odot g\|_{\mathcal{N}^s_{pqr}} \le \|f \odot g\|_{\mathcal{N}^{2s}_{pqr}} \le C \|f\|_{\mathcal{N}^s_{p_1 q_1 2r}} \|g\|_{\mathcal{N}^s_{p_2 q_2 2r}} \le C \|f\|_{\mathcal{N}^s_{p_1 q_1 r}} \|g\|_{\mathcal{N}^s_{p_2 q_2 r}}$$

by Corollary 3.3.

Putting together these observations, we obtain the desired result.

4. Commutator estimate

We recall the following lemma obtained in [2, Lemma 2.2]:

Lemma 4.1. *Let $0 < \alpha \le 1$, $j \in \mathbf{N}_0$, and let $F \in \mathrm{Lip}^\alpha(\mathbf{R}^n)$, $G \in L^\infty(\mathbf{R}^n)$. Then*

$$\|\varphi_j(D)[F \cdot G] - F\varphi_j(D)G\|_{L^\infty} \le C2^{-j\alpha}\|F\|_{\mathrm{Lip}^\alpha}\|G\|_{L^\infty}.$$

This is a slight extension of [2, Lemma 2.2] to the case where $\alpha = 1$. Here for the sake of convenience for readers, we recall the whole proof.

Proof. Since $\varphi_j(D)H(x) = (2\pi)^{-\frac{n}{2}}\mathcal{F}^{-1}\varphi_j * H(x)$ for all $H \in \mathcal{S}'(\mathbf{R}^n)$ which grows polynomially at infinity,

$$\begin{aligned}&\varphi_j(D)[F \cdot G](x) - F(x)\varphi_j(D)G(x)\\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n} 2^{jn}\mathcal{F}^{-1}\varphi(2^j(x-y))(F(y)-F(x))G(y)\,dy.\end{aligned}$$

As a result, letting

$$C = (2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n} |z|^\alpha|\mathcal{F}^{-1}\varphi(z)|\,dz,$$

we have

$$\|\varphi_j(D)[F \cdot G] - F\varphi_j(D)G\|_{L^\infty} \le C2^{-j\alpha}\|F\|_{\mathrm{Lip}^\alpha}\|G\|_{L^\infty},$$

as required. □

Lemma 4.2. *Let $0 < \alpha \le 1$, $j \in \mathbf{N}_0$, and let $F \in \mathrm{Lip}^\alpha(\mathbf{R}^n)$, $G \in L^\infty(\mathbf{R}^n)$. Then we have*

$$\|\varphi_j(D)[F \preceq G] - F\varphi_j(D)G\|_{L^\infty} \le C2^{-j\alpha}\|F\|_{\mathrm{Lip}^\alpha}\|G\|_{L^\infty}.$$

This is also a slight extension of [2, Lemma 2.3] to the case where $\alpha = 1$. Here for the sake of convenience for the readers we supply the proof.

Proof. We assume $j \gg 1$; otherwise we can mimic the argument below and we can readily incorporate the case where j is not so large. We decompose

$$\begin{aligned}&\varphi_j(D)[F \preceq G] - F\varphi_j(D)G\\ &= \sum_{k=j-3}^{j+3}(\varphi_j(D)[F \preceq \varphi_k(D)G] - F\varphi_j(D)\varphi_k(D)G)\\ &= \sum_{k=j-3}^{j+3}(\varphi_j(D)[F \cdot \varphi_k(D)G] - F\varphi_j(D)\varphi_k(D)G - \varphi_j(D)[F \succeq \varphi_k(D)G]).\end{aligned}$$

Let k be fixed. We use Lemma 4.1 to have

$$\|\varphi_j(D)[F \cdot \varphi_k(D)G] - F\varphi_j(D)\varphi_k(D)G\|_{L^\infty} \le C2^{-j\alpha}\|F\|_{\mathrm{Lip}^\alpha}\|G\|_{L^\infty}.$$

Meanwhile, using

$$\varphi_j(D)[F \succeq \varphi_k(D)G] = \sum_{l=j-5}^{j+5}\varphi_j(D)[\varphi_l(D)F \succeq \varphi_k(D)G]$$

for $k \in [j-3, j+3]$, we have

$$\begin{aligned}
\|\varphi_j(D)[F \succeq \varphi_k(D)G]\|_{L^\infty} &\le \sum_{l=j-5}^{j+5} \|\varphi_j(D)[\varphi_l(D)F \succeq \varphi_k(D)G]\|_{L^\infty} \\
&\le C \sum_{l=j-5}^{j+5} \|\varphi_l(D)F \succeq \varphi_k(D)G\|_{L^\infty} \\
&\le C \sup_{l,l' \in \mathbf{N}_0} \|\varphi_l(D)F \cdot \varphi_{l'}(D)G\|_{L^\infty} \\
&\le C 2^{-j\alpha} \|F\|_{\mathrm{Lip}^\alpha} \|G\|_{L^\infty}.
\end{aligned}$$

□

We prove Theorem 1.2 to conclude this note.

Proof. We decompose

$$\begin{aligned}
(f \preceq g) \odot h - f(g \odot h) &= \sum_{j=0}^{\infty} [\varphi_j(D)[f \preceq g] - f \cdot \varphi_j(D)g]\varphi_j(D)h \\
&\quad + \sum_{j=1}^{\infty} (\varphi_{j-1}(D)[f \preceq g] - f\varphi_{j-1}(D)g)\varphi_j(D)h \\
&\quad + \sum_{j=1}^{\infty} [\varphi_j(D)[f \preceq g] - f \cdot \varphi_j(D)g]\varphi_{j-1}(D)h.
\end{aligned}$$

We handle the first term; other two terms are dealt with similarly. We decompose

$$\begin{aligned}
&\sum_{j=0}^{\infty} [\varphi_j(D)[f \preceq g] - f \cdot \varphi_j(D)g]\varphi_j(D)h \\
&= \sum_{j=0}^{\infty} [\varphi_j(D)[\psi_{j+4}(D)f \preceq g] - \psi_{j+4}(D)f \cdot \varphi_j(D)g]\varphi_j(D)h \\
&\quad + \sum_{j=0}^{\infty} \sum_{k=j+5}^{\infty} \varphi_j(D)[\varphi_k(D)f \preceq g] \cdot \varphi_j(D)h \\
&\quad - \sum_{j=0}^{\infty} \sum_{k=j+5}^{\infty} \varphi_k(D)f \cdot \varphi_j(D)g \cdot \varphi_j(D)h.
\end{aligned}$$

Since

$$\|\partial^m[\varphi_j(D)[\psi_{j+4}(D)f \preceq g] - \psi_{j+4}(D)f \cdot \varphi_j(D)g]\|_{L^\infty} = \mathrm{O}(2^{-j(\alpha+\beta-|m|)})$$

for all $m = (m_1, m_2, \ldots, m_n) \in {\mathbf{N}_0}^n$, we have

$$\begin{aligned}
&\left\| \sum_{j=0}^{\infty} [\varphi_j(D)[\psi_{j+4}(D)f \preceq g] - \psi_{j+4}(D)f \cdot \varphi_j(D)g]\varphi_j(D)h \right\|_{\mathcal{N}^{s+\alpha+\beta}_{pqr}} \\
&\le C\|f\|_{\mathrm{Lip}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{N}^s_{pqr}}.
\end{aligned}$$

Using Example 2.5, we estimate the second term:

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{k=j+5}^{\infty}\varphi_j(D)[\varphi_k(D)f \preceq g]\cdot\varphi_j(D)h\\&=\sum_{j=0}^{\infty}\sum_{k=j+5}^{\infty}\varphi_j(D)[\varphi_k(D)\psi_{k-2}(D)f\cdot\varphi_k(D)g]\cdot\varphi_j(D)h\\&\quad+\sum_{j=0}^{\infty}\sum_{k=j+5}^{\infty}\varphi_j(D)[\varphi_k(D)\psi_{k-1}(D)f\cdot\varphi_{k+1}(D)g]\cdot\varphi_j(D)h.\end{aligned}$$

Next, we note that

$$\begin{aligned}&\|\varphi_k(D)f\cdot\varphi_j(D)g\cdot\varphi_j(D)h\|_{\mathcal{M}^p_q}\\&\le C2^{-k\alpha-j(s+\beta)}\|f\|_{\mathrm{Lip}^\alpha}\|g\|_{\mathcal{C}^\beta}\|2^{js}\varphi_j(D)h\|_{\mathcal{M}^p_q}.\end{aligned}$$

Adding this estimate over j,k, we have

$$\begin{aligned}&\left\{\sum_{k=5}^{\infty}\left(2^{k(s+\alpha+\beta)}\left\|\sum_{j=0}^{k-5}\varphi_k(D)f\cdot\varphi_j(D)g\cdot\varphi_j(D)h\right\|_{\mathcal{M}^p_q}\right)^r\right\}^{\frac{1}{r}}\\&\le C\left\{\sum_{k=5}^{\infty}\left(\sum_{j=0}^{k-5}2^{(k-j)(s+\beta)}\|f\|_{\mathrm{Lip}^\alpha}\|g\|_{\mathcal{C}^\beta}\|2^{js}\varphi_j(D)h\|_{\mathcal{M}^p_q}\right)^r\right\}^{\frac{1}{r}}\\&=C\left\{\sum_{k=5}^{\infty}\sum_{j=0}^{k-5}\left(2^{\frac{1}{2}(k-j)(s+\beta)}\|f\|_{\mathrm{Lip}^\alpha}\|g\|_{\mathcal{C}^\beta}\|2^{js}\varphi_j(D)h\|_{\mathcal{M}^p_q}\right)^r\right\}^{\frac{1}{r}}\\&=C\left\{\sum_{j=0}^{\infty}\left(\|f\|_{\mathrm{Lip}^\alpha}\|g\|_{\mathcal{C}^\beta}\|2^{js}\varphi_j(D)h\|_{\mathcal{M}^p_q}\right)^r\right\}^{\frac{1}{r}}\\&=C\|f\|_{\mathrm{Lip}^\alpha}\|g\|_{\mathcal{C}^\beta}\|h\|_{\mathcal{N}^s_{pqr}}.\end{aligned}$$

□

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Department of Mathematical Science,
1-1 Minami-Ohsawa, Hachioji, 192-0397, Tokyo Japan
yoshihiro-sawano@celery.ocn.ne.jp