The fundamental theorem of affine geometry in $(L^0)^n$

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Abstract

Let L^0 be the algebra of equivalence classes of real valued random variables on a given probability space, and $(L^0)^n$ the n-ary Cartesian power of L^0 for each integer $n \geq 2$. We consider $(L^0)^n$ as a free module over L^0 and study affine geometry in $(L^0)^n$. One of our main results states that: an injective mapping $T: (L^0)^n \to (L^0)^n$ which is local and maps each L^0 -line onto an L^0 -line must be an L^0 affine linear mapping. The other main result states that: a bijective mapping $T : (L^0)^n \to (L^0)^n$ which is local and maps each L^0 -line segment onto an L^0 -line segment must be an L^0 -affine linear mapping. These results extend the fundamental theorem of affine geometry from \mathbb{R}^n to $(L^0)^n$.

Keywords: L^0 -module, L^0 -affine linear, the fundamental theorem of affine geometry MSC2010: 14R10, 51A15, 13C13

1 Introduction

The fundamental theorem of affine geometry is a classical and useful result. It states that for an integer $n \geq 2$, if a bijective mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ maps any line to a line, then it must be affine linear. A short proof can be found in Remark 6 of Artstein-Avidan and Milman [\[1\]](#page-9-0).

The fundamental theorem of affine geometry has been generalized and strengthened in numerous ways. Please see Section 5 of Artstein-Avidan and Slomka [\[2\]](#page-9-1) for an account of the various forms and generalizations of the fundamental theorems of affine geometry for \mathbb{R}^n , together with references and other historical remarks, and see Kvirikashvili and Lashkhi [\[9\]](#page-9-2) and the references therein for generalizations of the fundamental theorems of affine geometry for free modules over some kinds of rings and other more general underlying structures.

Let L^0 be the algebra of equivalence classes of real valued random variables on a given probability space, and for any positive integer n , $(L^0)^n = \{(\xi_1, \ldots, \xi_n) : \xi_i \in L^0, i = 1, \ldots, n\}$. Since L^0 and $(L^0)^n$ (endowed with the usual topology of convergence in probability) usually fail to be local convex

spaces, most of mathematicians are not interested in L^0 and $(L^0)^n$ for quite a long time. However, motivated by financial applications and stochastic optimizations, the study of L^0 and $(L^0)^n$ became active in the literature recently. For example, Kardaras [\[7\]](#page-9-3) studied the uniform integrability and the local convexity in L^0 ; Žitković[\[11\]](#page-9-4), Kardaras and Žitković[\[8\]](#page-9-5) considered the forward convex convergence in L^0 's nonnegative orthant L^0_+ ; Drapeau, et. al [\[5\]](#page-9-6) established the Brouwer fixed point theorem in $(L^0)^n$; Wu [\[10\]](#page-9-7) established the Farkas' lemma and Minkowski-Weyl type results in $(L^0)^n$ and Cheridito, et. al [\[3\]](#page-9-8) generalized some classical results from linear algebra, real analysis and convex analysis to $(L^{0})^{n}.$

In this paper, for each $n \geq 2$, we consider $(L^0)^n$ as a free L^0 -module of rank n and study affine geometry in $(L^0)^n$. We extend the fundamental theorem of affine geometry from \mathbb{R}^n to $(L^0)^n$. One of our main results states that: an injective mapping $T : (L^0)^n \to (L^0)^n$ which is local and maps each L^0 -line onto an L^0 -line must be an L^0 -affine linear mapping. The other main result states that: a bijective mapping $T : (L^0)^n \to (L^0)^n$ which is local and maps each L^0 -line segment onto an L^0 -line segment must be an L^0 -affine linear mapping. Besides, we also give an example to show that a bijective mapping $T: (L^0)^n \to (L^0)^n$ which maps L^0 -lines onto L^0 -lines is not necessarily L^0 -affine linear.

2 Basic notations and definitions

Throughout this paper, (Ω, \mathcal{F}, P) will denote a given probability space. Let L^0 be the set of all equivalence classes of real valued random variables on (Ω, \mathcal{F}, P) . Under the usual addition and multiplication operations, L^0 is an algebra. For each $A \in \mathcal{F}$, the equivalence class of A refers to $\tilde{A} = \{B \in \mathcal{F} :$ $P(A \triangle B) = 0$, \tilde{I}_A denotes the equivalence class of the characteristic function I_A , and $I_{\tilde{A}}$ also stands for I_A . Given $\xi \in L^0$, let $\xi^0(\cdot)$ be an arbitrarily chosen representative of ξ . We write $[\xi \neq 0]$ for the equivalence class of the measurable set $\{\omega \in \Omega : \xi^0(\omega) \neq 0\}$. The statement " $\xi \neq 0$ on Ω " means that $\xi^0(\omega) \neq 0$, P-a.s., in other words, ξ is an invertible element of the algebra L^0 . Some other notation like $[\xi = 0]$ or statement like " $\xi > 0$ on Ω " and so on are understood in a similar way.

In this paper, an L^0 -module refers to a left module over the algebra L^0 , and θ always denotes its null element.

Given any positive integer n, denote $(L^0)^n = \{(\xi_1, \ldots, \xi_n) : \xi_i \in L^0, i = 1, \ldots, n\}$, then $(L^0)^n$ is a free L^0 -module of rank n generated by e_i , $i = 1, ..., n$, where e_i is the *i*-th unit vector in $\mathbb{R}^n \subset (L^0)^n$. Given $x = (\xi_1, \ldots, \xi_n), y = (\eta_1, \ldots, \eta_n) \in (L^0)^n$, the L^0 -inner product of x and y is defined to be $\langle x, y \rangle = \sum_{i=1}^n \xi_i \eta_i$, and the L⁰-normed of x is given by $|x| = \sqrt{\langle x, x \rangle}$. We can see that $|x| = 0$ iff $x = \theta$ and $|\xi x| = |\xi||x|$ for every $x \in (L^0)^n$ and $\xi \in L^0$.

Given $x \in (L^0)^n$, if $|x| \neq 0$ on Ω , then x is said to have full support. Obviously, each e_i has full support. If $x \in (L^0)^n$ has full support, then for any $\xi \in L^0$, using the equality $|\xi x| = |\xi||x|$, we see

that $\xi x = \theta$ iff $\xi = 0$; as a result, if ξ and η are elements of L^0 such that $\xi x = \eta x$, then $\xi = \eta$. On the contrary, if $x \in (L^0)^n$ does not have full support, then $A = [x] = 0$ has positive probability and $I_A x = \theta$, immediately $x = I_{A^c} x$, where $A^c = [x] \neq 0$ so that $I_{A^c} = 1 - I_A$, and it is probably that $I_{A^c} \neq 1.$

A group of elements x_1, \ldots, x_k in $(L^0)^n$ are said to be L^0 -independent, if for $\xi_1, \ldots, \xi_k \in L^0$, the equality $\xi_1 x_1 + \cdots + \xi_k x_k = \theta$ implies that $\xi_1 = \cdots = \xi_k = 0$. Since $I_{[|x_1| = 0]} x_1 + \cdots + I_{[|x_k| = 0]} x_k =$ $\theta + \cdots + \theta = \theta$, we see that all elements x_i 's have full support whenever x_1, \ldots, x_k are L^0 -independent. Clearly, e_1, \ldots, e_n are L^0 -independent.

Fix an integer $n \geq 2$. For any nonzero vector $x \in \mathbb{R}^n$, we can find a vector $y \in \mathbb{R}^n$ linearly independent of x, while for a nonzero element $x \in (L^0)^n$, we may not find an element $y \in (L^0)^n$ such that y and x are L^0 -independent, in fact, according to the aforementioned fact, for a nonzero x which does not have full support, there does not exist y which is L^0 -independent of x at all! When $x \in (L^0)^n$ has full support, the existence of $y \in (L^0)^n$ which is L^0 -independent of x is also not obvious. These observations force us to use the base $\{e_1, \ldots, e_n\}$ in the proof of our main result Theorem [1.](#page-3-0)

Definition 1. Let E_1 and E_2 be two L^0 -modules and $T: E_1 \rightarrow E_2$ a mapping.

(1). T is said to be L^0 -linear, if $T(x+y) = T(x)+T(y)$, $\forall x, y \in E_1$, and $T(\xi x) = \xi T(x)$, $\forall x \in E_1, \xi \in L^0$; (2). T is said to be L^0 -affine linear, if $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y), \forall x, y \in E_1, \lambda \in L^0$, equivalently, $T(\cdot) - T(\theta)$ is L^0 -linear;

(3). T is said to be local (or have the local property) if $\tilde{I}_AT(\tilde{I}_{A}x) = \tilde{I}_AT(x), \forall x \in E_1, A \in \mathcal{F}$;

(4). T is said to be stable if for any $x, y \in E_1$ and $A \in \mathcal{F}$, $T(\tilde{I}_A x + \tilde{I}_{A^c} y) = \tilde{I}_A T(x) + \tilde{I}_{A^c} T(y)$.

Proposition 1. Let E_1 and E_2 be two L^0 -modules and $T : E_1 \to E_2$ a mapping. Then the following statements are true:

- (1). If T is L^0 -affine linear, then T must have the local property.
- (2). T has the local property if and only if T is stable.
- (3). If T has the local property and $T(\theta) = \theta$, then $T(\tilde{I}_A x) = \tilde{I}_A T(x)$ for any $x \in E_1$ and $A \in \mathcal{F}$.
- (4). If T is bijective and has the local property, then T^{-1} also has local property.

Proof. (1). Define the L^0 -linear mapping $S : E_1 \to E_2$ by $S(x) = T(x) - T(\theta), \forall x \in E_1$. For any $x \in E$ and $A \in \mathcal{F}$, $\tilde{I}_A T(\tilde{I}_A x) = \tilde{I}_A[S(\tilde{I}_A x) + T(\theta)] = \tilde{I}_A[\tilde{I}_A S(x) + T(\theta)] = \tilde{I}_A[S(x) + T(\theta)] = \tilde{I}_A T(x)$, namely, T has the local property.

(2). If T has the local property, then for any $x, y \in E_1$ and $A \in \mathcal{F}$

$$
T(\tilde{I}_Ax + \tilde{I}_{A^c}y) = \tilde{I}_AT(\tilde{I}_Ax + \tilde{I}_{A^c}y) + \tilde{I}_{A^c}T(\tilde{I}_Ax + \tilde{I}_{A^c}y)
$$

\n
$$
= \tilde{I}_AT(\tilde{I}_A(\tilde{I}_Ax + \tilde{I}_{A^c}y)) + \tilde{I}_{A^c}T(\tilde{I}_{A^c}(\tilde{I}_Ax + \tilde{I}_{A^c}y))
$$

\n
$$
= \tilde{I}_AT(\tilde{I}_Ax) + \tilde{I}_{A^c}T(\tilde{I}_{A^c}y)
$$

\n
$$
= \tilde{I}_AT(x) + \tilde{I}_{A^c}T(y).
$$

Thus, T is stable.

Conversely, if T is stable, then for any $x \in E_1$ and $A \in \mathcal{F}$, $T(\tilde{I}_Ax) = T(\tilde{I}_Ax + \tilde{I}_{A^c}\theta) = \tilde{I}_AT(x) +$ $\tilde{I}_{A} \circ T(\theta)$, immediately we obtain $\tilde{I}_A T(\tilde{I}_A x) = \tilde{I}_A T(x)$, which means that T has the local property.

(3). From (2), $T(\tilde{I}_{A}x) = T(\tilde{I}_{A}x + \tilde{I}_{A}e\theta) = \tilde{I}_{A}T(x) + \tilde{I}_{A}eT(\theta) = \tilde{I}_{A}T(x)$.

(4). Fix $A \in \mathcal{F}$ and $y \in E_2$. Using (2), $T[\tilde{I}_A T^{-1}(y)] = \tilde{I}_A T[T^{-1}(y)] + \tilde{I}_{A^c} T(\theta) = \tilde{I}_A y + \tilde{I}_{A^c} T(\theta)$, and $T[\tilde{I}_AT^{-1}(\tilde{I}_Ay)]=\tilde{I}_AT[T^{-1}(\tilde{I}_Ay)]+\tilde{I}_{A^c}T(\theta)=\tilde{I}_Ay+\tilde{I}_{A^c}T(\theta).$ Then the assumption that T is injective yields that $\tilde{I}_AT^{-1}(y) = \tilde{I}_AT^{-1}(\tilde{I}_Ay)$, exactly meaning T^{-1} also has the local property. \Box

3 Main results

For any two distinct points x, y in $(L^0)^n$, denote $l(x, y) = {\lambda x + (1 - \lambda)y : \lambda \in L^0}$, called the L^0 -line determined by x and y. In \mathbb{R}^n , any two distinct points in a given straight line determine the same straight line, while in $(L^0)^n$, if u, v are two distinct points in the L^0 -line $l(x, y)$, the L^0 -line $l(u, v)$ may be not the same as $l(x, y)$. For instance, let $x \in (L^0)^n$ be a nonzero element, then for any $A \in \mathcal{F}$, \tilde{I}_Ax lies in the L^0 -line $l(\theta, x) = {\lambda x : \lambda \in L^0}$, if $\tilde{I}_A x$ is nonzero, then the L^0 -line $l(\theta, \tilde{I}_A x) = {\tilde{I}_A \lambda x : \lambda \in L^0}$ is probably not the same as $l(\theta, x)$. Thus we should be careful when we handle problems involving L^0 -lines.

It is easy to verify that any injective L^0 -affine linear mapping $T : (L^0)^n \to (L^0)^n$ maps each L^0 -line onto an L^0 -line. Theorem [1](#page-3-0) below states that the converse is also true.

Theorem 1. Fix an integer $n \geq 2$. Let $T : (L^0)^n \to (L^0)^n$ be an injective mapping which is local and maps each L^0 -line onto an L^0 -line, that is to say, for any two distinct points $x, y \in (L^0)^n$, the image of the L^0 -line $l(x, y)$ under the mapping T is $l(u, v)$, where $u = T(x), v = T(y)$, then T must be an L^0 -affine linear mapping.

Proof. Define $S: (L^0)^n \to (L^0)^n$ by $S(x) = T(x) - T(\theta), \forall x \in (L^0)^n$. Note that $S(\theta) = \theta$ and S is also injective. With the assumptions on T, it is easy to check that S is local and maps each L^0 -line onto an L^0 -line. It remains to show that S is L^0 -linear. The proof is composed of 5 steps as below. We point out in advance that (2) and (3) of Proposition [1](#page-2-0) are used frequently.

Step 1. For any L^0 -independent $x, y \in (L^0)^n$, we have $S(x)$ and $S(y)$ are L^0 -independent, and $S(x + y) = S(x) + S(y).$

For any $z \in (L^0)^n$ which has full support, let $A = [|S(z)| = 0]$, then by (3) of Proposition [1,](#page-2-0) $S(I_A z) = I_A S(z) = \theta$. Since S is injective, we obtain that $I_A z = \theta$. Thus $I_A = 0$, implying $S(z)$ has full support.

Suppose $\xi, \eta \in L^0$ satisfy the equality $\xi S(x) + \eta S(y) = \theta$. Since S is injective and maps the L^0 -line $l(\theta, x)$ onto the L^0 -line $l(\theta, S(x))$, there exists $\alpha \in L^0$ such that $\xi S(x) = S(\alpha x)$. Similarly, there exists $\beta \in L^0$ such that $-\eta S(y) = S(\beta y)$. By the injectivity of S we get $\alpha x = \beta y$, then $\alpha = \beta = 0$ follows from the assumption that x, y are L^0 -independent. As a result, $\xi S(x) = -\eta S(y) = \theta$, then using the fact that both $S(x)$ and $S(y)$ have full support, we conclude that $\xi = \eta = 0$, which means that $S(x)$ and $S(y)$ are L^0 -independent.

We then show: there exist $a, b \in L^0$ such that $S(x + y) = aS(x) + bS(y)$. In fact, since $x + y$ lies in the L^0 -line $l(2x, 2y)$ and S maps L^0 -lines to L^0 -lines, thus there exists $\mu \in L^0$ such that $S(x + y) =$ $\mu S(2x) + (1 - \mu)S(2y)$. Since 2x lies in the L^0 -line $l(\theta, x)$ and 2y lines in the L^0 -line $l(\theta, y)$, there exist $\alpha_1, \beta_1 \in L^0$ such that $S(2x) = \alpha_1 S(x), S(2y) = \beta_1 S(y)$, then $a = \mu \alpha_1, b = (1 - \mu)\beta_1$ satisfy the equality $S(x + y) = aS(x) + bS(y).$

It remains to show $a = 1$ and $b = 1$. Let $A = [a-1 \neq 0]$, if by contrary that $a \neq 1$, then A has positive probability and $I_A \neq 0$. According to the notation, there exists $c_1 \in L^0$ such that $I_A[1+c_1(a-1)]=0$. Since the L^0 -line $l(x, x + y) = \{x + cy : c \in L^0\}$ is mapped by S onto the L^0 -line $l(S(x), S(x + y))$, there exists $c_0 \in L^0$ such that $S(x + c_0y) = (1 - c_1)S(x) + c_1S(x + y) = [1 + c_1(a - 1)]S(x) + c_1bS(y)$. Using Proposition [1](#page-2-0) we obtain $S(I_A(x+c_0y)) = I_AS(x+c_0y) = I_Ac_1bS(y)$. Note that there exists some $\xi \in L^0$ such that $\tilde{I}_Ac_1bS(y) = S(\xi y)$, then by the injectivity of S, we get $I_A(x+c_0y) = \xi y$, contradicting to the assumption that x, y are L^0 -independent. Therefore, $a = 1$. Similarly, $b = 1$.

Step 2. For any L^0 -independent $x, y \in (L^0)^n$, we have $S(\xi x + \eta y) = S(\xi x) + S(\eta y), \forall \xi, \eta \in L^0$.

First suppose ξ, η are characteristic functions, that is $\xi = \tilde{I}_A$, $\eta = \tilde{I}_B$, for some $A, B \in \mathcal{F}$. Since $ilde{I}_Ax + \tilde{I}_By = \tilde{I}_{A \cap B}(x+y) + \tilde{I}_{A \setminus B}x + \tilde{I}_{B \setminus A}y$, by the local property, we have $S(\tilde{I}_Ax + \tilde{I}_By) = \tilde{I}_{A \cap B}S(x+y) + \tilde{I}_{B \setminus B}S(x+y)$ $\tilde{I}_{A\setminus B}S(x)+\tilde{I}_{B\setminus A}S(y)=\tilde{I}_{A\cap B}[S(x)+S(y)]+\tilde{I}_{A\setminus B}S(x)+\tilde{I}_{B\setminus A}S(y)=\tilde{I}_AS(x)+\tilde{I}_BS(y)=S(\tilde{I}_Ax)+S(\tilde{I}_By).$

Generally, for any $\xi, \eta \in L^0$, let $A = [\xi \neq 0], B = [\eta \neq 0]$, and take $x_1 = \xi x + I_{A^c} x, y_1 = \eta y + I_{B^c} y$, then $\xi x = I_A x_1, \eta y = I_B y_1$, and x_1, y_1 are L^0 -independent. In fact, if $\alpha, \beta \in L^0$ satisfy the equality $\alpha x_1 + \beta y_1 = \theta$, then since x, y are L⁰-independent, we must have $\alpha(\xi + I_{A^c}) = 0$ and $\beta(\eta + I_{B^c}) = 0$, due to the fact $\xi + I_{A^c} \neq 0$ on Ω and $\eta + I_{B^c} \neq 0$ on Ω , we thus obtain $\alpha = \beta = 0$. Now we have shown that x_1, y_1 are L^0 -independent, then $S(\xi x + \eta y) = S(I_A x_1 + I_B y_1) = S(\tilde{I}_A x_1) + S(\tilde{I}_B y_1) = S(\xi x) + S(\eta y)$.

Step 3. For each $i \in \{1, 2, \ldots, n\}$ and any $\xi, \eta \in L^0$, we have $S(\xi e_i + \eta e_i) = S(\xi e_i) + S(\eta e_i)$.

By symmetry, it suffices to prove the case when $i = 1$.

Since $e_1 - e_2$ and e_2 is obvious L^0 -independent, we get from Step 2 that $S(e_1) = S(e_1 - e_2 + e_2)$ $S(e_1 - e_2) + S(e_2) = S(e_1) + S(-e_2) + S(e_2)$, therefore $S(e_2) + S(-e_2) = \theta$.

Now fix $\xi, \eta \in L^0$, let $A = [\xi + \eta \neq 0]$. Then $x_1 = \xi e_1 + I_{A^c}e_1 + e_2$ and $y_1 = \eta e_1 - e_2$ are L⁰-independent. Indeed, if $\alpha, \beta \in L^0$ satisfy $\alpha x_1 + \beta y_1 = (\alpha \xi + \alpha I_{A^c} + \beta \eta) e_1 + (\alpha - \beta) e_2 = \theta$, then $\alpha\xi + \alpha I_{A^c} + \beta \eta = 0$ and $\alpha - \beta = 0$, equivalently, $\alpha = \beta$ and $\alpha(\xi + \eta + I_{A^c}) = 0$, thus $\alpha = \beta = 0$ follows from the fact that $\xi + \eta + I_{A^c} \neq 0$ on Ω . From Step 2, noting that e_1 and $I_{A^c}e_1 + e_2$ are L^0 -independent, we get $S(x_1) = S(\xi e_1) + S(I_{A^c}e_1 + e_2) = S(\xi e_1) + S(I_{A^c}e_1) + S(e_2)$ and $S(y_1) = S(\eta e_1) + S(-e_2)$. On the other hand, due to the fact that x_1 and y_1 are L^0 -independent, it follows from Step 2 that $S(\xi x + I_{A^c}x + \eta x) = S(x_1 + y_1) = S(x_1) + S(y_1)$. Therefore, using the known fact $S(e_2) + S(-e_2) = \theta$ we get $S(\xi e_1 + I_{A^c}e_1 + \eta e_1) = S(\xi e_1) + S(I_{A^c}e_1) + S(\eta e_1)$. Using the local property, $I_{A^c}S(\xi e_1 + I_{A^c}e_1)$ $I_{A^c}e_1 + \eta e_1 = S[I_{A^c}(\xi e_1 + I_{A^c}e_1 + \eta e_1)] = S(I_{A^c}e_1),$ hence $S(\xi e_1 + \eta e_1) = S[I_{A}(\xi e_1 + I_{A^c}e_1 + \eta e_1)] = S[I_{A^c}e_1 + I_{A^c}e_1 + \eta e_1]$ $I_A S(\xi e_1 + I_{A^c} e_1 + \eta e_1) = S(\xi e_1 + I_{A^c} e_1 + \eta e_1) - I_{A^c} S(\xi e_1 + I_{A^c} e_1 + \eta e_1) = S(\xi e_1) + S(\eta e_1)$.

Step 4. For any $\xi_1, \xi_2, ..., \xi_n \in L^0$, we have $S(\xi_1 e_1 + ... + \xi_n e_n) = S(\xi_1 e_1) + ... + S(\xi_n e_n)$.

Indeed, let $y_1 = \xi_2 e_2 + \cdots + \xi_n e_n$ and set $A = [y_1] \neq 0$, further take $y = I_A y_1 + I_{A^c} e_2$, then e_1, y are L^0 -independent, and $y_1 = I_A y$, thus according to Step 2, $S(\xi_1 e_1 + \cdots + \xi_n e_n) = S(\xi_1 e_1 + \cdots + \xi_n e_n)$ y_1) = $S(\xi_1e_1 + I_Ay) = S(\xi_1e_1) + S(I_Ay) = S(\xi_1e_1) + S(\xi_2e_2 + \cdots + \xi_ne_n)$. By induction, we obtain $S(\xi_1e_1 + \cdots + \xi_ne_n) = S(\xi_1e_1) + \cdots + S(\xi_ne_n).$

Step 5. For each $i \in \{1, 2, \ldots, n\}$ and any $\xi \in L^0$, we have $S(\xi e_i) = \xi S(e_i)$.

Fix an $x \in (L^0)^n$ which has full support. Since ξx lies in the L^0 -line $l(\theta, x)$, there exists $\mu \in L^0$ such that $S(\xi x) = \mu S(x)$. By Step 1, $S(x)$ has full support, thus this μ is unique determined by ξ (and x). Therefore, we can define a mapping $f_x: L^0 \to L^0$ by the relation $S(\xi x) = f_x(\xi)S(x), \forall \xi \in L^0$.

Specially, for each $i \in \{1, 2, ..., n\}$, we have a mapping $f_i: L^0 \to L^0$ such that $S(\xi e_i) = f_i(\xi)S(e_i)$, $\forall \xi \in$ L^0 .

We show that all mappings f_i are indeed the same one.

In fact, by symmetry, it suffices to verify that $f_1 = f_2$. For each $\xi \in L^0$, on one hand, since $e_1 + e_2$ has full support, we have $S(\xi(e_1+e_2)) = f_{e_1+e_2}(\xi)S(e_1+e_2) = f_{e_1+e_2}(\xi)[S(e_1)+S(e_2)],$ where the last equality follows from Step 1. On the other hand, from Step 2, $S(\xi(e_1 + e_2)) = S(\xi e_1) + S(\xi e_2)$ $f_1(\xi)S(e_1) + f_2(\xi)S(e_2)$. Thus we obtain $f_{e_1+e_2}(\xi)[S(e_1) + S(e_2)] = f_1(\xi)S(e_1) + f_2(\xi)S(e_2)$. Since we have known from Step 1 that $S(e_1)$ and $S(e_2)$ are L^0 -independent, thus $f_1(\xi) = f_{e_1+e_2}(\xi) = f_2(\xi)$.

Please note that using a similar argument, for any $\eta \in L^0$ such that $\eta \neq 0$ on Ω , we have $f_{\eta e_1}(\xi)$ $f_2(\xi) = f_1(\xi), \forall \xi \in L^0.$

We proceed to show that $f_1(\xi) = \xi, \forall \xi \in L^0$.

First, it is obvious that $f_1(0) = 0$ and $f_1(1) = 1$. Then by the local property of S, for any $\xi \in L^0$ and $A \in \mathcal{F}$, $S(\tilde{I}_A \xi e_1) = \tilde{I}_A S(\xi e_1)$, we obtain that $f_1(\tilde{I}_A \xi) = \tilde{I}_A f_1(\xi)$, which means that f_1 is local. By Step (3), for any $\xi, \eta \in L^0$, $S(\xi e_1 + \eta e_1) = S(\xi e_1) + S(\eta e_1)$, implying that $f_1(\xi + \eta) = f_1(\xi) + f_1(\eta)$.

Finally, for any $\xi, \eta \in L^0$, choose $\eta_1 \in L^0$ such that $\eta_1 \neq 0$ on Ω and $\eta = I_A \eta_1$, where $A = [\eta \neq 0]$ (for instance, we can take $\eta_1 = I_A \eta + I_{A^c}$, then on one hand $S((\xi \eta)e_1) = f_1(\xi \eta)S(e_1)$, on the other hand, $S((\xi \eta)e_1) = S(\xi I_A \eta_1 e_1) = I_A S(\xi \eta_1 e_1) = I_A f_{\eta_1 e_1}(\xi) S(\eta_1 e_1) = I_A f_1(\xi) f_1(\eta_1) S(e_1) = f_1(\xi) f_1(\eta) S(e_1).$ Noting that $S(e_1)$ has full support, we thus obtain $f_1(\xi \eta) = f_1(\xi) f_1(\eta)$.

To sum up, f_1 satisfies all the conditions (1-4) in Lemma [1](#page-6-0) below, thus $f_1(\xi) = \xi, \forall \xi \in L^0$. Combining Step 4 and Step 5, we conclude that S is L^0 -linear, completing the proof.

 \Box

Lemma 1. Let $\phi: L^0 \to L^0$ be a mapping such that:

- (1). ϕ is local;
- (2). $\phi(\xi + \eta) = \phi(\xi) + \phi(\eta), \forall \xi, \eta \in L^0;$
- (3). $\phi(\xi \eta) = \phi(\xi) \phi(\eta), \forall \xi, \eta \in L^0;$
- (4). $\phi(1) = 1$.

Then ϕ is the identity, namely, $\phi(\xi) = \xi, \forall \xi \in L^0$.

Proof. From (2), $\phi(0) + \phi(1) = \phi(1 + 0) = \phi(1)$, thus $\phi(0) = 0$, then $\phi(\xi - \xi) = \phi(0) = \phi(\xi) + \phi(-\xi)$ yields that $\phi(-\xi) = -\phi(\xi)$ for every $\xi \in L^0$. Since $\phi(1) = 1$, it is easy to deduce: for any integer p, $\phi(p) = p$ and further for any rational number r, $\phi(r) = r$. Now assume $q = \sum_{i=1}^{d} r_i \tilde{I}_{A_i}$ is a simple function in L^0 such that every r_i is a rational number, then by the local property of ϕ , we obtain: $\phi(q) = \sum_{i=1}^{d} \tilde{I}_{A_i} \phi(r_i) = \sum_{i=1}^{d} \tilde{I}_{A_i} r_i = q.$

Let ξ, η be two elements in L^0 with $\xi \geq \eta$, then from (3) we obtain $\phi(\xi - \eta) = \phi(\sqrt{\xi - \eta}\sqrt{\xi - \eta}) =$ $\phi(\sqrt{\xi-\eta})\phi(\sqrt{\xi-\eta}) \geq 0$. Since $\phi(\xi-\eta) = \phi(\xi) + \phi(-\eta) = \phi(\xi) - \phi(\eta)$, it follows that $\phi(\xi) \geq \phi(\eta)$, that is to say, ϕ is monotonically increasing.

For any $\xi \in L^0$, let $q_- = \sum_{i=1}^d r_i \tilde{I}_{A_i}$ and $q_+ = \sum_{j=1}^k t_j \tilde{I}_{B_j}$ be any two simple functions in L^0 such that every r_i and t_j are rational numbers and $q_-\leq \xi\leq q_+$, then using the monotonicity of ϕ , we have $q_ - = \phi(q_-) \leq \phi(\xi) \leq \phi(q_+) = q_+$. Taking all such possible q_- and q_+ , we thus obtain that $\phi(\xi) = \xi$, \Box completing the proof.

Remark 1. The local property appears frequently in the study related to L^0 and $(L^0)^n$, for example, it appears in the intermediate value theorem of L^0 -valued functions (Theorem 1.6 of [\[6\]](#page-9-9)) and the Brouwer fixed point theorem in $(L^0)^n$ (Theorem 2.3 in [\[5\]](#page-9-6), where the local property appears in a slightly more general form which is equivalent to be stable in Definition [1\)](#page-2-1).

In the following, we give an example which shows that a bijective mapping $T : (L^0)^n \to (L^0)^n$ which maps any L^0 -line onto an L^0 -line may not have the local property, thus according to Proposition [1,](#page-2-0) this mapping T is not L^0 -affine linear.

Example 1. Let θ : $(\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, P)$ be an isomorphism, that is to say, θ is bijective and both θ and θ^{-1} are measure-preserving. Then θ induces a bijection $\sigma : L^0 \to L^0$ through $\xi \mapsto$ the equivalence

class of $\xi^0(\theta(\cdot))$, where ξ^0 is a representative of $\xi \in L^0$. Further, for each positive integer n, θ induces a bijection $T: (L^0)^n \to (L^0)^n$ through $(\xi_1,\ldots,\xi_n) \mapsto (\sigma(\xi_1),\ldots,\sigma(\xi_n))$. Then it is straightforward to check that T maps each L^0 -line onto an L^0 -line. However, if θ is not the identity mapping, T is probably not local.

As follows is a more concrete example.

Let $\Omega = [0, 1),$ $\mathcal{F} = \mathcal{B}([0, 1)),$ namely the Borel σ -algebra of $[0, 1),$ and P the Lebesgue measure. Define $\theta : [0,1) \to [0,1)$ by $\theta(\omega) = \omega + \frac{1}{2}$ for $\omega \in [0,\frac{1}{2})$, and $\theta(\omega) = \omega - \frac{1}{2}$ for $\omega \in [\frac{1}{2},1)$. We show the induced mapping $T: (L^0)^n \to (L^0)^n$ is not local. Let $A = [0, \frac{1}{2}), B = [\frac{1}{2}, 1),$ then $\sigma(\tilde{I}_A) = \tilde{I}_B$, therefore for each $x \in (L^0)^n$, we have $T(\tilde{I}_A x) = \tilde{I}_B T(x)$, specially $T(\tilde{I}_A e_1) = \tilde{I}_B T(e_1) = \tilde{I}_B e_1$, it follows that $ilde{I}_AT(\tilde{I}_Ae_1)=\theta \neq \tilde{I}_Ae_1=\tilde{I}_AT(e_1).$ Thus T is not local.

Remark 2. Since L^0 is a commutative algebra and $L^0 \neq \{0\}$, it follows from Thereom 2.6 in [\[4\]](#page-9-10) that L^0 is an IB-ring (see [\[9\]](#page-9-2) for the meaning of this notation), then applying Theorem 1 in [9] to $(L^0)^n$ gives: for $n \geq 2$, if $T : (L^0)^n \to (L^0)^n$ with $T(\theta) = \theta$ is a collineation preserving parallelism, that is to say, T is a bijection such that the images of collinear points under T are themselves collinear and T preserves parallelism(see [\[9\]](#page-9-2) for this notion), then there exists an isomorphism $\sigma : L^0 \to L^0$ such that T is a σ -semilinear isomorphism, namely $T(x + y) = T(x) + T(y), \forall x, y \in (L^0)^n$ and $T(\xi x) =$ $\sigma(\xi)T(x), \forall x \in (L^0)^n, \xi \in L^0$. Since bijection and preserving parallelism are not premise conditions in our Theorem [1](#page-3-0) and our theorem [1](#page-3-0) require the local property instead, one can see that our Theorem [1](#page-3-0) is not a special case of Theorem 1 in [\[9\]](#page-9-2). We also would like to point out that although T in Example [1](#page-6-1) is not an L^0 -linear mapping, it is indeed a σ -semilinear isomorphism.

For any two distinct $x, y \in (L^0)^n$, denote $[x, y] = \{ \mu x + (1 - \mu)y : \mu \in L^0, 0 \le \mu \le 1 \}$, called the L^0 -line segment between x and y. In the end of this paper, we discuss self-mappings on $(L^0)^n$ which map L^0 -line segments to L^0 -line segments.

Proposition 2. Suppose that $T : (L^0)^n \to (L^0)^n$ is a bijection which is local and maps each L^0 -line segment onto an L^0 -line segment, that is to say, for any two distinct $x, y \in (L^0)^n$, the image of the L^0 -line segment $[x, y]$ is the L^0 -line segment $[Tx, Ty]$, then T maps each L^0 -line onto an L^0 -line.

Proof. Without loss of generality, we can assume $T(\theta) = \theta$, otherwise we make a translation. Let x, y be any two elements in $(L^0)^n$ such that $y \neq \theta$. Since T is a bijection, we can see that T^{-1} also maps each L^0 -line segment onto an L^0 -line segment, and T^{-1} is local by Proposition [1,](#page-2-0) thus we only need to show that each point z in the L^0 -line $l(x, x + y) = \{x + \lambda y : \lambda \in L^0\}$ will be mapped into the L^0 -line $l(T(x), T(x + y)) = {\lambda T(x) + (1 - \lambda)T(x + y) : \lambda \in L^0}.$

We first show that: for each $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, $z = x + ky$ will be mapped into $l(T(x), T(x +$ $(y)).$

First assume that y has full support. (1) The cases $k = 0$ and $k = 1$ are obvious. (2) Fix a $k \in \{2, 3, 4, ...\}$. Since $x + y = (1 - \frac{1}{k})x + \frac{1}{k}(x + ky) \in [x, x + ky]$ and T maps an L^0 -line segment onto an L^0 -line segment, there exists $\mu \in L^0$ with $0 \le \mu \le 1$ such that $T(x+y) = (1-\mu)T(x) + \mu T(x+ky)$. Let $A = [\mu = 0]$, then $I_A \mu = 0$. By the local property and Proposition [1,](#page-2-0) $T(I_A(x + y)) = I_A T(x + y) =$ $I_A[(1 - \mu)T(x) + \mu T(x + ky)] = I_A T(x) = T(I_A x)$. Since T is a bijection, we obtain $I_A(x + y) = I_A x$. Then the assumption y has full support implies $I_A = 0$, equivalently, $\mu > 0$ on Ω . As a result, $T(x+ky) = \frac{1}{\mu}T(x+y) + (1-\frac{1}{\mu})T(x) \in l(T(x),T(x+y))$. (3) Fix a $k \in \{-1,-2,-3,\ldots\}$. Since $x = \frac{1}{1-k}(x+ky) + (1-\frac{1}{1-k})(x+y) \in [x+ky, x+y]$, there exists $\mu \in L^0$ with $0 \le \mu \le 1$ such that $T(x) = \mu T(x + ky) + (1 - \mu)T(x + y)$, by a similar argument we deduce that $\mu > 0$ on Ω , then $T(x + ky) = \frac{1}{\mu}T(x) + (1 - \frac{1}{\mu})T(x + y) \in l(T(x), T(x + y)).$

Now for a general nonzero y. Take $y_1 = I_A y + I_{A^c} e_1$, where $A = ||y|| \neq 0$, we see that y_1 has full support and $y = I_A y_1$. Fix any $k \in \mathbb{Z}$, we have proved that there exists $\mu \in L^0$ such that $T(x+ky_1) = \mu T(x) + (1-\mu)T(x+y_1)$. Using Proposition [1,](#page-2-0) $T(x+ky) = T[I_A(x+ky)] + T[I_{A^c}(x+ky)] =$ $T[I_A(x+ky_1)]+T(I_{A^c}x) = I_A[\mu T(x)+(1-\mu)T(x+y_1)]+I_{A^c}T(x) = (I_A\mu + I_{A^c})T(x)+I_A(1-\mu)T(x+y),$ implying that $T(x + ky) \in l(T(x), T(x + y)).$

We then show that: for each $\lambda \in L^0$, $z = x + \lambda y$ will be mapped into $l(T(x), T(x + y))$.

For $k = 1, 2, ...,$ let $A_k = [k-1] \leq |\lambda| < k$, then $x + \lambda I_{A_k} y = (\frac{1}{2} - \frac{\lambda}{2k} I_{A_k})(x - ky) + (\frac{1}{2} + \frac{\lambda}{2k} I_{A_k})(x + ky)$ belongs to $[x - ky, x + ky]$, consequently, $T(x + \lambda I_{A_k}y) \in [T(x - ky), T(x + ky)] \subset l(T(x), T(x + y))$, where the last inclusion follows from Claim 1 that both the two endpoints of the L^0 -line segment belong to the L^0 -line. By the notation, for each positive integer k, there exists $\mu_k \in L^0$ such that $T(x + \lambda I_{A_k}y) = \mu_k T(x) + (1 - \mu_k)T(x + y)$. By the local property of T, we have $I_{A_k}T(x + \lambda y) =$ $I_{A_k}T(x+\lambda I_{A_k}y)=I_{A_k}[\mu_kT(x)+(1-\mu_k)T(x+y)]$ for each k. Let $\mu=\sum_{k=1}^{\infty}I_{A_k}\mu_k$, then $T(x+\lambda y)=$ $\sum_{k=1}^{\infty} I_{A_k} T(x + \lambda y) = \sum_{k=1}^{\infty} I_{A_k} [\mu_k T(x) + (1 - \mu_k) T(x + y)] = \mu T(x) + (1 - \mu) T(x + y)$, which means that $T(x + \lambda y) \in l(Tx, T(x + y)).$ \Box

Remark 3. In the proof of Proposition [2,](#page-7-0) the local property plays an important role, we wonder whether the assumption that T has the local property can be removed or not? That is, if $T : (L^0)^n \to (L^0)^n$ is a bijection and maps each L^0 -line segment onto an L^0 -line segment, then can we deduce that T maps each L^0 -line onto an L^0 -line?

Combining Theorem [1](#page-3-0) and Proposition [2,](#page-7-0) we immediately obtain:

Theorem 2. Fix an integer $n \geq 2$. If $T : (L^0)^n \to (L^0)^n$ is a bijection which is local and maps each L^0 -line segment onto an L^0 -line segment, then T must be an L^0 -affine linear mapping.

Theorem [2](#page-8-0) will be used in our forthcoming study to give representations of fully order preserving and fully order reversing operators acting on the set of L^0 -lower semi-continuous L^0 -convex functions

on $(L^0)^n$, and on general complete random normed modules.

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References

- [1] S. Artstein-Avidan, V. Milman, The concept of duality in convex analysis, and the characterization of the Legendre transform, Ann. Math., 2009, 169(2): 661-674.
- [2] S. Artstein-Avidan, B.A.Slomka, The fundamental theorems of affine and projective geometry revisited, Commun. Contemp. Math., 19(5), 1650059 (2017).
- [3] P. Cheridito, M. Kupper, N. Vogelpoth, Conditional analysis on R^d , in: Set Optimization and Applications - State of the Art, Springer, 2015, pp. 179-211.
- [4] P. M. Cohn, Some remarks on the invariant basis property, Topology, 5(1966) 215-228.
- [5] S. Drapeau, M. Karliczek, M. Kupper, M. Streckfuß, Brouwer fixed point theorem in $(L^0)^d$, Fixed Point Theory Appl., 301(1) (2013).
- [6] T. X. Guo, X. L. Zeng, An $L^0(\mathcal{F}, R)$ -valued function's intermediate value theorem and its applications to random uniform convexity. Acta Math. Sin. (Engl. Ser.) 28(5)(2012) 909-924.
- [7] C.Kardaras, Uniform integrability and local convexity in L^0 , J. Funct. Anal., 266 (2014) 1913-1927.
- [8] C.Kardaras, G. Žitković, Forward–convex convergence in probability of sequences of nonnegative random variables, Proc. Amer. Math. Soc., 141(3) (2013) 919-929.
- [9] T. G. Kvirikashvili, A. A. Lashkhi, Geometrical maps in ring affine geometries, J. Math. Sci., 186(5) (2012) 759-765.
- [10] M. Z. Wu, Farkas' lemma in random locally convex modules and Minkowski-Weyl type results in $L^0(\mathcal{F}, R^n)$, J. Math. Anal. Appl., 404(2)(2013) 300-309.
- [11] G. Žitković, Convex compactness and its applications, Math. Financ. Econ. $3(1)$ (2010) 1-12.