# On q-tensor product of Cuntz algebras

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To 75-th birthday of our teacher Yurii S. Samoilenko

#### Abstract

We consider  $C^*$ -algebra  $\mathcal{E}_{n,m}^q$ , which is a q-twist of two Cuntz-Toeplitz algebras. For the case  $|q| < 1$  we give an explicit formula, which untwists the  $q$ -deformation, thus showing that the isomorphism class of  $\mathcal{E}_{n,m}^q$  does not depend on q. For the case  $|q|=1$  we give an explicit description of all ideals in  $\mathcal{E}_{n,m}^q$ . In particular,  $\mathcal{E}_{n,m}^q$  contains a unique largest ideal  $\mathcal{M}_q$ . Then we identify  $\mathcal{E}_{n,m}^q/\mathcal{M}_q$  with the Rieffel deformation of  $\mathcal{O}_n \otimes \mathcal{O}_m$  and use a K-theoretical argument to show that the isomorphism class does not depend on  $q$ .

Key Words: Cuntz-Toeplitz algebra, Rieffel's deformation, q-deformation, Fock representation, K-theory.

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### 1 Introduction

1. Since early 80-th, a wide study of non-classical models of mathematical physics, quantum group theory and noncommutative probability (see e.g.,  $[5, 18, 20, 36, 38, 50]$  gave rise to a number of papers on operator algebras generated by various deformed commutation relations, see [6, 32, 37] etc. A general approach to the study of these relations has been provided by the framework of quadratic ∗-algebras allowing Wick ordering (Wick algebras), see [27]. The class of Wick algebras includes, among others, deformations of canonical commutation relations of quantum mechanics, some quantum groups and quantum homogeneous spaces, see e.g., [19, 33, 46, 41]. On the other hand, one can consider Wick algebras as a deformation of Cuntz-Toeplitz algebra, see [10, 14, 27].

Let  $\{T_{ij}^{kl}, i, j, k, l = \overline{1,d}\} \subset \mathbb{C}, T_{ij}^{kl} = \overline{T}_{ji}^{lk}$ . Wick algebra  $W(T)$ , see [27], is the ∗-algebra generated by elements  $a_j$ ,  $a_j^*$ ,  $j = 1$ , d subject to the relations

$$
a_i^* a_j = \delta_{ij} \mathbf{1} + \sum_{k,l=1}^d T_{ij}^{kl} a_l a_k^*.
$$

It was shown in [27] that properties of  $W(T)$  depend on a self-adjoint operator T called the operator of coefficients of  $W(T)$ . Namely, let  $\mathcal{H} = \mathbb{C}^d$  and  $e_1$ ,  $\ldots$ ,  $e_d$  be the standard orthonormal basis of  $H$ . Construct

$$
T: \mathfrak{R}^{\otimes 2} \to \mathfrak{R}^{\otimes 2}, \quad Te_k \otimes e_l = \sum_{i,j=1}^d T_{ik}^{lj} e_i \otimes e_j.
$$

Notice that the subalgebra of  $W(T)$  generated by  $\{a_j\}_{j=1}^d$  is free and can be identified with the full tensor algebra  $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$  via

$$
a_{i_1}\dots a_{i_k}\mapsto e_{i_1}\otimes\dots\otimes e_{i_k}\in\mathfrak{H}^{\otimes k}.
$$

**Definition 1.** The Fock representation of  $W(T)$  is the unique irreducible  $\ast$ -representation π<sub>F,T</sub> determined by a cyclic vector Ω,  $||Ω|| = 1$ , such that

$$
\pi_{F,T}(a_j^*)\Omega = 0, \ j = \overline{1, d}.
$$

The problem of existence of  $\pi_{FT}$  is non-trivial and is one of the central problems in representation theory of Wick algebras. Some sufficient conditions are collected in the following theorem, see [6, 24, 27].

**Theorem 1.** The Fock representation  $\pi_{F,T}$  of  $W(T)$  exists if one of the conditions below is satisfied

- The operator of coefficients  $T > 0$ ;
- $||T|| < \sqrt{2} 1;$
- T is braided, i.e.  $(1 \otimes T)(T \otimes 1)(1 \otimes T) = (T \otimes 1)(1 \otimes T)(1 \otimes T)$ on  $\mathfrak{H}^{\otimes 3}$  and  $||T|| \leq 1$ . Moreover, if  $||T|| < 1$  then  $\pi_{F,T}$  is a faithful representation of  $W(T)$  and  $||\pi_{F,T}(a_j)|| < (1 - ||T||)^{-\frac{1}{2}}$ . If  $||T|| = 1$ , one can not guarantee boundedness of  $\pi_{FT}$  and in this case ker  $\pi_{FT}$  is a  $∗-ideal J<sub>2</sub> generated as a ∗-ideal by ker(1+T). Hence π<sub>F,T</sub> is a faithful$ representation of  $W(T)/\mathfrak{I}_2$ .

Another important question in the theory of Wick algebras is the question of stability of isomorphism classes of  $W(T) = C^{*}(W(T))$  for the case  $||T|| <$ 1. The following problem was posed in [26].

Conjecture 1. Let  $T: \mathfrak{K}^{\otimes 2} \to \mathfrak{K}^{\otimes 2}$  be a self-adjoint braided operator and  $||T|| < 1$ . Then  $W(T) \simeq W(0)$ .

In particular, the authors of [26] have shown that the conjecture holds for the case  $||T|| < \sqrt{2} - 1$ , for more results on the subject see [14], [29].

Consider the case  $T = 0$  in a few more details. If  $d = \dim \mathcal{H} = 1$ , then  $W(0)$  is generated by a single isometry s,  $s^*s = 1$ . In this case the universal  $C^*$ -algebra  $\mathcal E$  of  $W(0)$  exists and is isomorphic to the  $C^*$ -algebra generated by the unilateral shift S in  $l_2(\mathbb{Z}_+)$ . Notice also that  $\pi_{F,0}(s) = S$ , so the Fock representation of the C<sup>\*</sup>-algebra  $\mathcal E$  is faithful. The ideal  $\mathcal I$  in  $\mathcal E$ , generated by 1 − ss<sup>\*</sup> is isomorphic to the algebra of compact operators and  $\mathcal{E}/\mathcal{I} \simeq C(S^1)$ , see [9]. When  $d \geq 2$ ,  $W(0)$  is generated by  $s_j$ ,  $s_j^*$ , such that

$$
s_i^* s_j = \delta_{ij} \mathbf{1}, \quad i, j = \overline{1, d}.
$$

The Fock representation  $\pi_{F,d}$  acts on  $\mathcal{F} := \mathcal{F}_d$  as follows

$$
\pi_{F,d}(s_j)\Omega = e_j, \quad \pi_{F,d}(s_j)e_{i_1} \otimes \cdots \otimes e_{i_k} = e_j \otimes e_{i_1} \otimes \cdots \otimes e_{i_k}, \ k \ge 1, \n\pi_{F,d}(s_j^*)\Omega = 0, \quad \pi_{F,d}(s_j^*)e_{i_1} \otimes \cdots \otimes e_{i_k} = \delta_{ji_1}e_{i_2} \otimes \cdots \otimes e_{i_k}, \ k \ge 1.
$$

The universal C<sup>\*</sup>-algebra generated by  $W(0)$  with  $d \geq 2$  exists and is called the Cuntz-Toeplitz agebra  $\mathcal{O}_d^{(0)}$  $d_{d}^{(0)}$ . It is isomorphic to  $C^*(\pi_{F,d}(W(0)))$ , so the Fock representation of  $\mathcal{O}_d^{(0)}$  $\frac{d^{(0)}}{d}$  is faithful, see [10]. Further, the ideal J generated

by  $1 - \sum_{j=1}^{d} s_j s_j^*$  is the unique largest ideal in  $\mathcal{O}_d^{(0)}$  $\frac{d^{(0)}}{d}$ . It is isomorphic to the algebra of compact operators on  $\mathcal{F}_d$ . The quotient  $\mathcal{O}_d^{(0)}$  $\frac{d^{(0)}}{d}$ /J is called the Cuntz algebra  $\mathcal{O}_d$ . It is nuclear (as well as  $\mathcal{O}_d^{(0)}$  $\binom{00}{d}$ , simple and purely infinite, see [10] for more details.

2. In this paper we study the C<sup>\*</sup>-algebras  $\mathcal{E}_{n,m}^q$  generated by Wick algebras with operator of coefficients T described as follows. Let  $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^m$ ,  $|q| \leq 1$ and

$$
Tu_1 \otimes u_2 = 0, \quad Tv_1 \otimes v_2 = 0, \quad u_1, u_2 \in \mathbb{C}^n, \ v_1, v_2 \in \mathbb{C}^m,
$$
  

$$
Tu \otimes v = qv \otimes u, \quad Tv \otimes u = \overline{q}u \otimes v, \quad u \in \mathbb{C}^n, \ v \in \mathbb{C}^m.
$$

We denote the corresponding Wick algebra by  $WE_{n,m}^q$ . Notice that T satisfies the braid relation and  $||T|| = |q| \leq 1$  for any  $n, m \in \mathbb{N}$ . In particular, the Fock representation  $\pi_{F,q}$  exists for  $|q| \leq 1$  and is faithful on  $WE_{n,m}^q$  for  $|q| < 1$ .

The case  $n = 1$ ,  $m = 1$  was studied by various authors. Namely,  $WE_{1,1}^q$ is generated by isometries  $s_1$ ,  $s_2$  subject to the relations

$$
s_1^* s_2 = q s_2 s_1^*.
$$

It is easy to see that the corresponding universal  $C^*$ -algebra  $\mathcal{E}_1^q$  $_{1,1}^q$  exists for any  $|q| \leq 1$ .

If  $|q| < 1$ , the main result of [25] states that  $\mathcal{E}_{1,1}^q \simeq \mathcal{E}_{1,1}^{(0)} = \mathcal{O}_2^{(0)}$  $2^{(0)}$  for any  $|q| < 1$  In particular the Fock representation of  $\mathcal{E}_1^q$  $_{1,1}^q$  is faithful.

The case  $|q| = 1$  was studied in [30, 40, 47]. In this situation the additional relation

$$
s_2s_1 = qs_1s_2
$$

holds in  $\mathcal{E}_1^q$  $_{1,1}^q$ . It was shown that  $\mathcal{E}_1^q$  $_{1,1}^q$  is nuclear for any  $|q|=1$ . Let  $\mathcal{M}_q$  be the ideal generated by the projections  $1 - s_1 s_1^*$  and  $1 - s_2 s_2^*$ . Then  $\mathcal{E}_{1,1}^q / \mathcal{M}_q \simeq \mathcal{A}_q$ , where  $A_q$  is the non-commutative torus, see [42],

$$
\mathcal{A}_q = C^*(u_1, u_2 \mid u_1^* u_1 = u_1 u_1^* = 1, u_2^* u_2 = u_2 u_2^* = 1, u_2^* u_1 = q u_1 u_2^*).
$$

If q is not a root of unity, then the corresponding non-commutative torus  $A_q$ is simple and  $\mathcal{M}_q$  is the unique largest ideal in  $\mathcal{E}_1^q$  $_{1,1}^q$ . Let us stress that unlike the case  $|q| < 1$ , the C<sup>\*</sup>-isomorphism class of  $\mathcal{E}_1^q$  $_{1,1}^q$  is "unstable" with respect to q. Namely,  $\mathcal{E}_{1,1}^{q_1} \simeq \mathcal{E}_{1,1}^{q_2}$  $_{1,1}^{q_2}$  iff  $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$ , see [30, 40, 47].

One can consider another higher-dimensional analog of  $\mathcal{E}_1^q$  $_{1,1}^q$ . For a set  ${q_{ij}}_{i,j=1}^d$  of complex numbers such that  $|q_{ij}| \leq 1$ ,  $q_{ij} = \overline{q}_{ji}$ ,  $q_{ii} = 1$ , and

 $d > 2$ , one can consider a C<sup>\*</sup>-algebra  $\mathcal{E}_{\{q_{ij}\}}$ , generated by  $s_j$ ,  $s_j^*$ ,  $j = \overline{1,d}$ subject to the relations

$$
s_j^* s_j = 1, \quad s_i^* s_j = q_{ij} s_j s_i^*.
$$

The case  $|q_{ij}| < 1$  was considered in [34], where it was proved that  $\mathcal{E}_{\{q_{ij}\}}$ is nuclear and the Fock representation is faithful. It turned out that the fixed point  $C^*$ -subalgebra of  $\mathcal{E}_{\{q_{ij}\}}$  with respect to the canonical action of  $\mathbb{T}^d$ is an AF-algebra and is independent of  $\{q_{ij}\}\$ . However the conjecture that  $\mathcal{E}_{\{q_{ij}\}} \simeq \mathcal{E}_{\{0\}}$  remains open.

The case  $|q_{ij}| = 1$  was studied in [23, 30, 40]. It was shown that  $\mathcal{E}_{\{q_{ij}\}}$  is nuclear for any such family  ${q_{ij}}$  and the Fock representation is faithful. For more details on ideal structure and representation theory see [23, 30].

We focus on the study of  $\mathcal{E}_{n,m}^q$  with  $n,m \geq 2$  (the case  $n = 1, m \geq 2$  will be considered separately, see [49]). It is generated by isometries  $\{s_j\}_{j=1}^n$ , and  ${t_r}_{r=1}^m$ , satisfying commutation relations of the following form

$$
s_i^* s_j = 0, \quad 1 \le i \ne j \le n,
$$
  
\n
$$
t_r^* t_s = 0, \quad 0 \le r \ne s \le m,
$$
  
\n
$$
s_j^* t_r = q t_r s_j^*, \quad 0 \le j \le n, \ 0 \le r \le m.
$$
  
\n(1)

The analysis is separated into two conceptually different cases,  $|q| < 1$  and  $|q| = 1.$ 

If  $|q|$  < 1, we show that  $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0 = \mathcal{O}_{n+m}^{(0)}$ , where the latter is the Cuntz-Toeplitz algebra with  $n + m$  generators.

For the case  $|q| = 1$ , we prove that  $\mathcal{E}_{n,m}^q$  is nuclear, contains a unique largest ideal  $\mathcal{M}_q$ , and the quotient  $\mathcal{O}_n \otimes_q \mathcal{O}_m := \mathcal{E}_{n,m}^q/\mathcal{M}_q$  is simple and purely infinite for any  $q$  specified above. Then we use the Kirchberg-Phillips classification Theorem, see [31, 39], to get one of our main results. Namely we show that

$$
\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m
$$

for any  $q \in \mathbb{C}$ ,  $|q| = 1$ . Further we prove, that the Fock representation of  $\mathcal{E}_{n,m}^q$ is faithful for any  $|q| = 1$  and use this fact to prove that  $\mathcal{E}_{n,m}^q$  is isomorphic to the Rieffel deformation of  $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ . Next we show that the isomorphism class of  $\mathcal{M}_q$  is independent on q and consider  $\mathcal{E}_{n,m}^q$  as an (essential) extension of  $\mathcal{O}_n \otimes \mathcal{O}_m$  by  $\mathcal{M}_q$  and study the corresponding Ext group. In particular, if  $gcd(n-1, m-1) = 1$ , this group is zero. Thus in this case,  $\mathcal{E}_{n,m}^q$  and  $\mathcal{E}_{n,m}^1$ both determine the zero class in  $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q)$ . We stress that unlike

the case of extensions by compacts, one can not immediately deduce that two trivial essential extensions are isomorphic. So the problem of isomorphism  $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^1$  remains open for further investigations.

3. Recall how the algebras generated by isometries discussed above, are related to algebras of deformed canonical commutation relations.

We start with the case of one degree of freedom. The algebra  $G_q$  of  $q$ -deformed canonical commutation relations, see  $[2, 36]$ , is generated by elements  $a, a^*$  such that

$$
a^*a - qaa^* = 1,
$$

where  $q \in [-1, 1]$ . It is known, see [26], that the universal  $C^*$ -algebra  $\mathcal{G}_q$ generated by  $G_q$  exists for  $q \in [-1,1)$  and  $\mathcal{G}_q \simeq \mathcal{E}$  for any  $q \in (-1,1)$ .

The algebra  $G_{q,d}$  of quon commutation relations with d degrees of freedom was introduced and studied in [5, 18, 20, 50]. Namely,  $G_{q,d}$  is generated by  $a_j, a_j^*, j = 1, d$ , subject to the commutation relations

$$
a_j^* a_i = \delta_{ij} \mathbf{1} + q a_i a_j^*, \quad i, j = \overline{1, d}, \quad q \in (0, 1).
$$

Notice that the operator T, corresponding to  $G_{q,d}$  has the form

$$
Te_i\otimes e_j = qe_j\otimes e_i
$$

so it is a braided contraction with  $||T|| = q$ . In particular, for  $q < \sqrt{2} - 1$ one has  $\mathcal{G}_{q,d} \simeq \mathcal{O}_d^0$ , where  $\mathcal{G}_{q,d}$  is the  $C^*$ -algebra generated by  $G_{q,d}$ .

A multiparameter version of quons was considered in [6, 37, 38]. The corresponding ∗-algebra  $G_{\{q_{ij}\}}$ ,  $q_{ij} = \overline{q}_{ji}$ ,  $|q_{ij}| \leq 1$ ,  $i, j = \overline{1, d}$ , is generated by

$$
a_i^* a_j = \delta_{ij} \mathbf{1} + q_{ij} a_j a_i^*, \quad i, j = 1, d.
$$

The operator T acts as  $Te_i \otimes e_j = q_{ij}e_j \otimes e_i$ , so it is a braided contraction as well. For  $|q_{ij}| < \sqrt{2}-1$  we get  $\mathcal{G}_{\{q_{ij}\}} \simeq \mathcal{O}_d^0$ . However, if  $|q_{ij}| = 1$  for all  $i \neq j$ , and  $|q_{ii}| < 1$ , then  $\mathcal{G}_{\{q_{ij}\}} \simeq \mathcal{E}_{\{q_{ij}\}}$ , see [40].

Take  $k \in (0,1)$  and  $q \in \mathbb{C}$ ,  $|q| = 1$ . Construct  $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^m$ ,  $n, m \ge 2$ and define  $T: \mathfrak{H}^{\otimes 2} \to \mathfrak{H}^{\otimes 2}$  as follows

$$
Tu_1 \otimes u_2 = k u_2 \otimes u_1, \quad \text{if either } u_1, u_2 \in \mathbb{C}^n \text{ or } u_1, u_2 \in \mathbb{C}^m
$$

$$
Tu \otimes v = q v \otimes u, \quad \text{if } u \in \mathbb{C}^n, v \in \mathbb{C}^m.
$$

Denote the corresponding Wick algebra by  $WE_{n,m}^{q,k}$  and its universal  $C^*$ algebra by  $\mathcal{E}_{n,m}^{q,k}$ . This C<sup>\*</sup>-algebra is generated by  $s_j$ ,  $t_r$ ,  $j = \overline{1,n}$ ,  $r = \overline{1,m}$ , subject to the relations

$$
s_i^* s_j = \delta_{ij} \mathbf{1} + k s_j s_i^*,
$$
  
\n
$$
t_r^* t_l = \delta_{rl} \mathbf{1} + k t_l t_r^*,
$$
  
\n
$$
s_j^* t_r = q t_r s_j^*, \ t_r s_j = q s_j t_r.
$$
\n(2)

Relations (2) can be regarded as an example of system considered in [4] in the case of finite degrees of freedom. Applying the general stability result, we get that  $\mathcal{E}_{n,m}^{q,k} \simeq \mathcal{E}_{n,m}^q$  for  $k < \sqrt{2} - 1$ .

Notice that for  $k = \pm 1$  we get a discrete analogue of commutation relations for generalized statistics introduced in [35].

# 2 The case  $|q| < 1$

We start with some lemmas. Let  $\Lambda_n$  denote the set of all words in alphabet  $\{1, n\}$ . For any non-empty  $\mu = (\mu_1, \ldots, \mu_k)$ , and a family of elements  $b_1$ ,  $\ldots, b_n$ , we denote by  $b_\mu$  the product  $b_{\mu_1} \cdots b_{\mu_k}$ ; we also put  $b_\emptyset = 1$ . In this section we assume that any word  $\mu$  belongs to  $\Lambda_n$ .

**Lemma 1.** Let  $Q = \sum_{i=1}^{n} s_i s_i^*$ , then

$$
\sum_{|\mu|=k} s_{\mu} Q s_{\mu}^* = \sum_{|\nu|=k+1} s_{\nu} s_{\nu}^*.
$$

Proof. Straightforward.

**Lemma 2.** For any  $x \in \mathcal{E}_{n,m}^q$  one has

$$
\Bigl\|\sum_{|\mu|=k} s_\mu xs_\mu^*\Bigr\|\leq \|x\|.
$$

*Proof.* 1. First prove the claim for positive x. In this case one has  $0 \le x \le$  $||x||$ **1**. Hence 0 ≤  $s_{\mu}xs_{\mu}^{*}$  ≤  $||x||s_{\mu}s_{\mu}^{*}$ , and

$$
\left\|\sum_{|\mu|=k} s_{\mu} x s_{\mu}^*\right\| \leq \|x\| \cdot \left\|\sum_{|\mu|=k} s_{\mu} s_{\mu}^*\right\|.
$$

Note that  $s^*_{\mu}s_{\lambda} = \delta_{\mu\lambda}, \mu, \lambda \in \Lambda_n, |\mu| = |\lambda| = k$ , implying that  $\{s_{\mu}s^*_{\mu} \mid |\mu| = k\}$ form a family of pairwise orthogonal projections. Hence  $\|\sum_{|\mu|=k} s_{\mu} s_{\mu}^* \| = 1$ , and the statement for positive  $x$  is proved.



2. For any  $x \in \mathcal{E}_{n,m}^q$ , write  $A = \sum_{|\mu|=k} s_{\mu} x s_{\mu}^*$ , then  $A^* = \sum_{|\mu|=k} s_{\mu} x s_{\mu}^*$ and  $\overline{\phantom{0}}$ 

$$
A^*A = \sum_{|\mu|=k} s_{\mu} x^* x s_{\mu}^*.
$$

Then by the proved above,

$$
||A||^2 = ||A^*A|| \le ||x^*x|| = ||x||^2.
$$

Construct  $t_l = (1 - Q)t_l, l = \overline{1, m}.$ 

Lemma 3. The following commutation relations hold

$$
s_i^*\tilde{t}_l = 0, \quad i = \overline{1, n}, \quad l = \overline{1, m},
$$
  
\n
$$
\tilde{t}_r^*\tilde{t}_l = 0, \quad l \neq r \quad l, r = \overline{1, m},
$$
  
\n
$$
\tilde{t}_r^*\tilde{t}_r = 1 - |q|^2 Q > 0, \quad r = \overline{1, m}.
$$

*Proof.* We have  $s_i^*(1 - Q) = 0$ , implying that  $s_i^* t_l = 0$  for any  $i = \overline{1, n}$ , and  $l = \overline{1, m}.$ 

Further,

$$
\widetilde{t}_r^* \widetilde{t}_l = t_r^* (1 - Q) t_l = t_r^* t_l - \sum_{i=1}^n t_r^* s_i s_i^* t_l = \delta_{rl} 1 - \sum_{i=1}^n |q|^2 s_i t_r^* t_l s_i^* =
$$
\n
$$
= \delta_{rl} (1 - |q|^2 Q).
$$

**Proposition 1.** For any  $r = \overline{1, m}$ , one has

$$
t_r = \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k s_{\mu} \tilde{t}_r s_{\mu}^*.
$$

In particular, the family  $\{s_i, \tilde{t}_r, i = \overline{1,n}, r = \overline{1,m}\}\$  generates  $\mathcal{E}_{n,m}^q$ . *Proof.* Put  $M_k^r = \sum_{|\mu|=k} q^k s_\mu \tilde{t}_r s_\mu^*, k \in \mathbb{Z}_+$ . Then

$$
M_0^r = t_r - Qt_r = t_r - \sum_{|\mu|=1} s_{\mu} s_{\mu}^* t_r,
$$

and

$$
M_k^r = \sum_{|\mu|=k} q^k s_{\mu} (1 - Q) t_r s_{\mu}^* = \sum_{|\mu|=k} s_{\mu} (1 - Q) s_{\mu}^* t_r =
$$
  
= 
$$
\sum_{|\mu|=k} s_{\mu} s_{\mu}^* t_r - \sum_{|\mu|=k+1} s_{\mu} s_{\mu}^* t_r.
$$

Then

$$
S_N^r = \sum_{k=0}^N M_k^r = t_r - \sum_{|\mu|=N+1} s_{\mu} s_{\mu}^* t_r = t_r - q^{N+1} \sum_{|\mu|=N+1} s_{\mu} t_r s_{\mu}^*.
$$

Since  $\|\sum_{|\mu|=N+1} s_{\mu}t_{r}s_{\mu}^{*}\| \leq ||t_{r}|| = 1$  one has that  $S_{N}^{r} \to t_{r}$  in  $\mathcal{E}_{n,m}^{q}$  as  $N \to$ ∞.

Suppose that  $\mathcal{E}_{n,m}^q$  is realised by Hilbert space operators. Consider the left polar decomposition  $\tilde{t}_r = \hat{t}_r \cdot c_r$ , where  $c_r^2 = \tilde{t}_r^* \tilde{t}_r = 1 - |q|^2 Q > 0$ , implying that  $\widehat{t}_r$  is an isometry and

$$
\widehat{t}_r = \widetilde{t}_r c_r^{-1} \in \mathcal{E}_{n,m}^q, \quad r = \overline{1, m}.
$$

Lemma 4. The following commutation relations hold

$$
s_i^* \hat{t}_r = 0, \quad i = \overline{1, n}, \ r = \overline{1, m},
$$
  

$$
\hat{t}_r^* \hat{t}_l = \delta_{rl} \mathbf{1}, \quad r, l = \overline{1, m}.
$$

*Proof.* Indeed, for any  $i = \overline{1, n}$ , and  $r = \overline{1, m}$ . one has

$$
s_i^* \widehat{t}_r = s_i^* \widetilde{t}_r \, c_r^{-1} = 0,
$$

and

$$
\widehat{t}_r^* \widehat{t}_l = c_r^{-1} \widetilde{t}_r^* \widetilde{t}_l c_r^{-1} = 0, \quad r \neq l.
$$

Summing up the results stated above, we get the following

**Theorem 2.** Let  $\hat{t}_r = (1 - Q)t_r(1 - |q|^2 Q)^{-\frac{1}{2}}$ ,  $r = \overline{1, m}$ . Then the family  $\{s_i, \hat{t}_r\}_{i=1}^n$  m generates  $\mathcal{E}^q_{n,m}$ , and

$$
s_i^* s_j = \delta_{ij} \mathbf{1}, \quad t_r^* t_l = \delta_{rl} \mathbf{1}, \quad s_i^* t_r = 0, \quad i, j = \overline{1, n}, \ r, l = \overline{1, m}.
$$

*Proof.* It remains to note that  $\tilde{t}_r = \hat{t}_r(1 - |q|^2 Q)^{\frac{1}{2}}$ , so  $\tilde{t}_r \in C^*(\hat{t}_r, Q)$ , so by Proposition 1 the elements  $s_i$ ,  $\hat{t}_r$ ,  $i = \overline{1, n}$ ,  $r = \overline{1, m}$ , generate  $\mathcal{E}_{n,m}^q$ .

**Corollary 1.** Denote by  $v_i$ ,  $i = \overline{1,n} + m$ , the isometries generating  $\mathcal{E}_{n,m}^0 =$  $\mathcal{O}_{n+m}^0$ . Then Theorem 2 implies that the correspondence

$$
v_i \mapsto s_i, \ i = \overline{1, n}, \quad v_{n+r} \mapsto \widehat{t}_r, \ r = \overline{1, m},
$$

extends uniquely to a surjective homomorphism  $\varphi: \mathcal{E}_{n,m}^0 \to \mathcal{E}_{n,m}^q$ .

Our next aim is to construct the inverse homomorphism  $\psi: \mathcal{E}^q_{n,m} \to \mathcal{E}^0_{n,m}$ . To do it, put

$$
\widetilde{Q} = \sum_{i=1}^n v_i v_i^* \quad \widetilde{w}_r = v_{n+r} (1 - |q|^2 \widetilde{Q})^{\frac{1}{2}}, \qquad r = \overline{1, m}.
$$

Then  $\widetilde{w}_r^*\widetilde{w}_r = 1 - |q|^2 Q$ , and  $\widetilde{w}_r^*\widetilde{w}_l = 0$  if  $r \neq l$ ,  $r, l = \overline{1,m}$ . Construct

$$
w_r = \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k v_{\mu} \widetilde{w}_r v_{\mu}^*, \quad r = \overline{1, m},
$$

where  $\mu$  runs over  $\Lambda_n$ , and set as above  $v_{\mu} = v_{\mu_1} \cdots v_{\mu_k}$ . Note that the series above converges with respect to norm in  $\mathcal{E}_{n,m}^0$ .

Lemma 5. The following commutation relations hold

$$
w_r^* w_l = \delta_{rl} \mathbf{1}, \quad v_i^* w_r = q w_r v_i^*, \quad i = \overline{1, n}, \ r, l = \overline{1, m}.
$$

*Proof.* First we note that  $v_i^* \tilde{w}_r = 0$ , and  $\tilde{w}_r^* v_i = 0$  for any  $i = \overline{1, n}$ , and  $j = \overline{1, m}$ , implying that

 $v_{\delta}^* \widetilde{w}_r = 0$ ,  $\widetilde{w}_r^* v_{\delta} = 0$ , for any nonempty  $\delta \in \Lambda_n$ ,  $r = \overline{1,m}$ .

Let  $|\lambda| \neq |\mu|, \lambda, \mu \in \Lambda_n$ . If  $|\lambda| > |\mu|$ , then  $\lambda = \widehat{\lambda}\gamma$  with  $|\lambda| = |\mu|$  and

$$
v_{\lambda}^* v_{\mu} = \delta_{\widehat{\lambda}\mu} v_{\gamma}^*.
$$

Otherwise  $\mu = \widehat{\mu}\beta$ ,  $|\widehat{\mu}| = |\lambda|$  and

$$
v_{\lambda}^* v_{\mu} = \delta_{\lambda \widehat{\mu}} v_{\beta}.
$$

So, if  $|\lambda| > |\mu|$  one has

$$
v_{\lambda}\widetilde{w}_r^* v_{\lambda}^* v_{\mu}\widetilde{w}_r v_{\mu}^* = \delta_{\widehat{\lambda}\mu} v_{\lambda}\widetilde{w}_r^* v_{\gamma}^* \widetilde{w}_r v_{\mu} = 0,
$$

and if  $|\mu| > |\lambda|$ , then

$$
v_{\lambda}\widetilde{w}_r^*v_{\lambda}^*v_{\mu}\widetilde{w}_r v_{\mu}^* = \delta_{\lambda\widehat{\mu}}v_{\lambda}\widetilde{w}_r^*v_{\beta}\widetilde{w}_r v_{\mu} = 0.
$$

Since  $v^*_{\mu}v_{\lambda} = \delta_{\mu\lambda}\mathbf{1}$ , if  $|\mu| = |\lambda|$ , one has

$$
w_r^* w_r = \lim_{N \to \infty} \left( \sum_{k=0}^N \sum_{|\lambda|=k} |q|^k v_\lambda \widetilde{w}_r^* v_\lambda^* \right) \cdot \left( \sum_{l=0}^N \sum_{|\mu|=l} |q|^l v_\mu \widetilde{w}_r v_\mu^* \right)
$$
  
\n
$$
= \lim_{N \to \infty} \sum_{k,l=0}^N \sum_{|\lambda|=k, |\mu|=l} |q|^{k+l} v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* = \lim_{N \to \infty} \sum_{k=0}^N \sum_{|\lambda|, |\mu|=k, |\lambda| \leq k} |q|^{2k} v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* \newline = \lim_{N \to \infty} \sum_{k=0}^N \sum_{|\mu|=k} |q|^{2k} v_\mu \widetilde{w}_r^* \widetilde{w}_r v_\mu^* = \lim_{N \to \infty} \sum_{k=0}^N \sum_{|\mu|=k} |q|^{2k} v_\mu (1 - |q|^2 \widetilde{Q}^2) v_\mu^* \newline = \lim_{N \to \infty} \sum_{k=0}^N \left( \sum_{|\mu|=k} |q|^{2k} v_\mu v_\mu^* - \sum_{|\mu|=k+1} |q|^{2k+2} v_\mu v_\mu^* \right)
$$
  
\n
$$
= \lim_{N \to \infty} \left( 1 - |q|^{2N+2} \sum_{|\mu|=N+1} v_\mu v_\mu^* \right) = 1.
$$

Since  $\widetilde{w}_r^* \widetilde{w}_l = 0$ ,  $r \neq l$ , the same arguments as above imply that  $w_r^* w_l = 0$ ,  $r \neq l$ .

For any non-empty  $\mu \in \Lambda_n$  write  $\sigma(\mu) = \emptyset$  if  $|\mu| = 1$ , and  $\sigma(\mu) =$  $(\mu_2, \ldots, \mu_k)$  if  $|\mu| = k > 1$ . Further, for any  $i = \overline{1, n}, r = \overline{1, m}$  one has

$$
v_i^* w_r = \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k s_i^* v_{\mu} \widetilde{w}_r v_{\mu}^* = v_i^* \widetilde{w}_r + \sum_{k=1}^{\infty} \sum_{|\mu|=k} q^k \delta_{i\mu_1} v_{\sigma(\mu)} \widetilde{w}_r v_{\sigma(\mu)}^* v_i^*
$$
  
=  $q \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k v_{\mu} \widetilde{w}_r v_{\mu}^* v_i^* = q w_r v_i^*.$ 

**Lemma 6.** For any  $r = \overline{1,m}$ , one has  $\widetilde{w}_r = (1 - \widetilde{Q})w_r$ . *Proof.* First note that  $(1 - \widetilde{Q})v_i = 0$ ,  $i = \overline{1, n}$ , implies that

$$
(\mathbf{1} - \widetilde{Q})v_{\mu} = 0, \quad |\mu| \in \Lambda_n, \ \mu \neq \emptyset.
$$

Then

$$
(1 - \widetilde{Q})w_r = (1 - \widetilde{Q})\left(\sum_{k=0} \sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*\right)
$$
  
=  $(1 - \widetilde{Q})\widetilde{w}_r + \sum_{k=1} \sum_{|\mu|=k} q^k (1 - \widetilde{Q})v_\mu \widetilde{w}_r v_\mu^* = (1 - \widetilde{Q})\widetilde{w}_r.$ 

To complete the proof it remains to note that  $\tilde{Q}v_{n+r} = 0$ ,  $r = \overline{1,m}$ . So,

$$
\widetilde{Q}\widetilde{w}_r = \widetilde{Q}v_{n+r}(1-|q|^2\widetilde{Q})^{\frac{1}{2}} = 0.
$$

 $\Box$ 

 $\Box$ 

**Theorem 3.** Let  $v_i$ ,  $i = \overline{1,n} + m$ , be the isometries generating  $\mathcal{E}_{n,m}^0$ , and  $\widetilde{Q} = \sum_{i=1}^n v_i v_i^*$ . Put

$$
\widetilde{w}_r = v_{n+r} (1 - |q|^2 \widetilde{Q})^{\frac{1}{2}}
$$
 and  $w_r = \sum_{k=0} \sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*$ .

Then

$$
v_i^* v_j = \delta_{ij} \mathbf{1}, \quad w_r^* w_l = \delta_{rl} \mathbf{1}, \quad v_i^* w_r = q w_r v_i^*, \quad i, j = \overline{1, n}, \ r, l = \overline{1, m}.
$$

Moreover, the family  $\{v_i, w_r\}_{i=1}^n$  generates  $\mathcal{E}^0_{n,m}$ .

Proof. We need to prove only the last statement of the theorem. We have

$$
v_{n+r} = \widetilde{w}_r (1 - |q|^2 \widetilde{Q})^{-\frac{1}{2}} = (1 - \widetilde{Q}) w_r (1 - |q|^2 \widetilde{Q})^{-\frac{1}{2}} \in C^*(w_r, v_i, i = \overline{1, n}).
$$

Hence  $v_i$ ,  $w_r$ ,  $i = \overline{1, n}$ ,  $r = \overline{1, m}$ , generate  $\mathcal{E}_{n,m}^0$ .

**Corollary 2.** The statement of Theorem 3 and the universal property of  $\mathcal{E}_{n,m}^q$ imply the existence of a surjective homomorphism  $\psi: \mathcal{E}^q_{n,m} \to \mathcal{E}^0_{n,m}$  defined by

$$
\psi(s_i) = v_i, \quad \psi(t_r) = w_r, \quad i = \overline{1, n}, \ r = \overline{1, m}.
$$

Now we are ready to formulate the main result of this section.

**Theorem 4.** For any  $q \in \mathbb{C}$ ,  $|q| < 1$ , one has an isomorphism  $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0$ .

*Proof.* In Theorem 2, we constructed the surjective homomorphism  $\varphi: \mathcal{E}_{n,m}^0 \to$  $\mathcal{E}_{n,m}^q$  defined by

$$
\varphi(v_i) = s_i, \quad \varphi(v_{n+r}) = \hat{t}_r, \quad i = \overline{1, n}, \ r = \overline{1, m}.
$$

Show that  $\psi: \mathcal{E}_{n,m}^q \to \mathcal{E}_{n,m}^0$  from Corollary 2 is the inverse of  $\varphi$ . Indeed, the equalities  $\psi(s_i) = v_i$ ,  $i = 1, n$ , imply that

$$
\psi(\mathbf{1} - Q) = \mathbf{1} - \widetilde{Q}.
$$

Then, since  $\psi(t_r) = w_r$ , we get

$$
\psi(\widetilde{t}_r)=\psi((1-Q)t_r)=(1-\widetilde{Q})w_r=\widetilde{w}_r, \quad r=\overline{1,m},
$$

and

$$
\psi(\widehat{t}_r) = \psi(\widetilde{t}_r(1-|q|^2Q)^{-\frac{1}{2}}) = \widetilde{w}_r(1-|q|^2\widetilde{Q})^{-\frac{1}{2}} = v_{n+r}, \quad r = \overline{1,m}.
$$

So,  $\psi \varphi(v_i) = \psi(s_i) = v_i$ ,  $\psi \varphi(v_{n+r}) = \psi(t_r) = v_{n+r}$ ,  $i = 1, n, r = 1, m$ , and

$$
\psi\varphi=id_{\mathcal{E}_{n,m}^0}.
$$

Show that  $\varphi \psi = id_{\mathcal{E}_{n,m}^q}$ . Indeed,

$$
\varphi(\widetilde{w}_r) = \varphi(v_{n+r}(1-|q|^2\widetilde{Q})^{\frac{1}{2}}) = \widehat{t}_r(1-|q|^2Q)^{\frac{1}{2}} = \widetilde{t}_r, \quad r = \overline{1,d}.
$$

Then for any  $r = \overline{1,m}$ , one has

$$
\varphi(w_r) = \sum_{k=0} \sum_{|\mu|=k} q^k \varphi(v_\mu) \varphi(\widetilde{w}_r) \varphi v_\mu^* = \sum_{k=0} \sum_{|\mu|=k} q^k s_\mu \widetilde{t}_r s_\mu^* = t_r.
$$
  
So,  $\varphi \psi(s_i) = \varphi(v_i) = s_i$ ,  $\varphi \psi(t_r) = \varphi(w_r) = t_r$ ,  $i = \overline{1, n}$ ,  $r = \overline{1, m}$ .

# 3 The case  $|q|=1$

In this section, we discuss the case  $|q| = 1$ . Notice that for  $|q| = 1$ , the relations in  $\mathcal{E}_{n,m}^q$  imply that  $t_j s_i = q s_i t_j$ ,  $i = \overline{1,n}$ ,  $j = \overline{1,m}$ . Indeed, for  $B_{ij} = t_j s_i - q s_i t_j$  we have directly  $B_{ij}^* B_{ij} = 0$ .

### 3.1 Auxiliary results

In this subsection we collect some general facts about  $C^*$ -dynamical systems, crossed products and Rieffel deformations which we will use in our considerations.

#### 3.1.1 Fixed point subalgebras

First we recall how properties of a fixed point subalgebra of a  $C^*$ -algebra with an action of a compact group are related to properties of the whole algebra.

**Definition 2.** Let A be a  $C^*$ -algebra with an action  $\gamma$  of a compact group G. A fixed point subalgebra  $A^{\gamma}$  is a subset of all  $a \in A$  such that  $\gamma_g(a) = a$ for all  $q \in G$ .

Notice that for every action of a compact group  $G$  on a  $C^*$ -algebra  $A$  one can construct a faithful conditional expectation  $E_{\gamma}: A \to A^{\gamma}$  onto the fixed point subalgebra, given by

$$
E_{\gamma}(a) = \int_{G} \gamma_{g}(a) d\lambda,
$$

where  $\lambda$  is the Haar measure on G.

A homomorphism  $\varphi: A \to B$  between  $C^*$ -algebras with actions  $\alpha$  and  $\beta$ of a compact group  $G$  is called equivariant if

$$
\varphi \circ \alpha_g = \beta_g \circ \varphi \text{ for any } g \in G.
$$

- **Proposition 2** ([7], Section 4.5, Theorem 1, 2). 1. Let  $\gamma$  be an action of a compact group  $G$  on a  $C^*$ -algebra  $A$ . Then  $A$  is nuclear if and only if  $A^{\gamma}$  is nuclear.
	- 2. Let  $\varphi: A \to B$  be an equivariant  $\ast$ -homomorphism. Then  $\varphi$  is injective on A if and only if  $\varphi$  is injective on  $A^{\alpha}$ .

#### 3.1.2 Crossed products

Given a locally compact group G and a  $C^*$ -algebra A with a G-action  $\alpha$ , consider the full crossed product  $C^*$ -algebra  $A \rtimes_{\alpha} G$ , see [48]. One has two natural embeddings into the multiplier algebra  $M(A \rtimes_{\alpha} G)$ ,

$$
i_A: A \to M(A \rtimes_{\alpha} G), i_G: G \to M(A \rtimes_{\alpha} G),
$$
  

$$
(i_A(a)f)(s) = af(s), (i_G(t)f)(s) = \alpha_t(f(t^{-1}s)), t, s \in G, a \in A,
$$

for  $f \in C_c(G, A)$ .

*Remark* 1. Obviously,  $i_G(s)$  is a unitary element of  $M(A \rtimes_{\alpha} G)$  for any  $s \in G$ . Recall that  $i_G$  determines the following homomorphism denoted also by  $i_G$ 

$$
i_G\colon C^*(G)\to M(A\rtimes_\alpha G)
$$

defined by

$$
i_G(f) = \int_G f(s)i_G(s)d\lambda(s),
$$

where  $\lambda$  is the left Haar measure on  $G$ .

Notice that for any  $g \in C_c(G, A)$  one has

$$
(i_G(f)g)(t) = f \cdot_\alpha g,
$$

where  $\cdot_{\alpha}$  denotes the product in  $A \rtimes_{\alpha} G$ . In particular, when A is unital we can identify  $i_G(f)$  with  $f \cdot_\alpha \mathbf{1}_A$ , and in fact  $i_G$  maps  $C^*(G)$  into  $A \rtimes_\alpha G$ . Also notice that

$$
i_G(t)i_A(a)i_G(t)^{-1} = i_A(\alpha_t(a)) \in M(A \rtimes_{\alpha} G).
$$

If  $\varphi$  is an equivariant homomorphism between C<sup>\*</sup>-algebras A with a G action  $\alpha$  and  $B$  with a G-action  $\beta$ , then one can define the homomorphism

$$
\varphi \rtimes G : A \rtimes_{\alpha} G \to B \rtimes_{\beta} G, \ (\varphi \rtimes G)(f)(t) = \varphi(f(t)), \quad f \in C_c(G, A).
$$

Let A be a unital  $C^*$ -algebra with G-action  $\alpha$ . Then  $\iota_A : \mathbb{C} \to A$ ,

$$
\iota_A(\lambda)=\lambda\mathbf{1}_A,
$$

is an equivariant homomorphism, where G acts trivially on  $\mathbb{C}$ . Since  $\mathbb{C} \rtimes G =$  $C^*(G)$ , one has that

$$
\iota_A \rtimes G \colon C^*(G) \to A \rtimes_{\alpha} G.
$$

In fact, in this case we have

$$
\iota_A \rtimes G = i_G,\tag{3}
$$

where  $i_G: C^*(G) \to A \rtimes_{\alpha} G$  is described in Remark 1. Indeed, for any  $g \in G_c(G, A)$  one has

$$
(i_G(f) \cdot_\alpha g)(s) = \int_G f(t)\alpha_t(g(t^{-1}s))dt
$$
  
= 
$$
\int_G f(t)1_A \alpha_t(g(t^{-1}s))dt
$$
  
= 
$$
((f(\cdot)1_A) \cdot_\alpha g)(s) = ((\iota_A \rtimes G)(f) \cdot_\alpha g)(s),
$$

implying  $i_G(f) = (\iota_A \rtimes G)(f)$  for any  $f \in C^*(G)$ .

#### 3.1.3 Rieffel's deformation

Below, we recall some basic facts on Rieffel's deformations. Given a  $C^*$ algebra A equipped with an action  $\alpha$  of  $\mathbb{R}^n$  and a skew symmetric matrix  $\Theta \in M_n(\mathbb{R})$ , one can construct the Rieffel deformation of A, denoted by  $A_{\Theta}$ , see [1, 43]. In particular the elements  $a \in A$  such that  $x \mapsto \alpha_x(a) \in A$  $C^{\infty}(\mathbb{R}^n, A)$  form a dense subset  $A_{\infty}$  in  $A_{\Theta}$  and for any  $a, b \in A_{\infty}$  their product in  $A_{\Theta}$  is given by the following oscillatory integral (see [43]):

$$
a \cdot_{\Theta} b := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \alpha_{\Theta(x)}(a) \alpha_y(b) e^{2\pi i \langle x, y \rangle} dx dy, \tag{4}
$$

where  $\langle \cdot, \cdot \rangle$  is a scalar product in  $\mathbb{R}^n$ .

In what follows, we will be interested in periodic actions of  $\mathbb{R}^n$ , i.e., we assume that  $\alpha$  is an action of  $\mathbb{T}^n$ . Given a character  $\chi \in \mathbb{T}^n \simeq \mathbb{Z}^n$ , consider

$$
A_{\chi} = \{ a \in A : \alpha_z(a) = \chi(z)a \text{ for every } z \in \mathbb{T}^n \}.
$$

Then

$$
A = \overline{\bigoplus_{\chi \in \mathbb{Z}^n} A_{\chi}},
$$

where some terms could be equal to zero, and  $A_{\chi_1} \cdot A_{\chi_2} \subset A_{\chi_1 + \chi_2}, A_{\chi}^* = A_{-\chi}$ . So,  $A_{\chi}$ ,  $\chi \in \mathbb{Z}^n$ , can be treated as homogeneous components of  $\mathbb{Z}^n$ -grading on A. Conversely, any  $\mathbb{Z}^n$ -grading of A determines an action of  $\mathbb{T}^n$  on A: for  $a \in A_p$  we let  $\alpha_t(a) = e^{2\pi i \langle t, p \rangle} a$  (see, e.g., [48]).

For a periodic action  $\alpha$  of  $\mathbb{R}^n$  on a  $C^*$ -algebra A and a skew-symmetric matrix  $\Theta \in M_n(\mathbb{R})$ , construct the Rieffel deformation  $A_{\Theta}$ . Notice that all homogeneous elements belong to  $A_{\infty}$ . Apply formula (4) to  $a \in A_p$ ,  $b \in A_q$ :

$$
a \cdot_{\Theta} b = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i \langle \Theta(x), p \rangle} a e^{2\pi i \langle y, q \rangle} b e^{2\pi i \langle x, y \rangle} dx dy
$$
  
\n
$$
= a \cdot b \int_{\mathbb{R}^n} e^{2\pi i \langle y, q \rangle} \int_{\mathbb{R}^n} e^{2\pi i \langle x, -\Theta(p) \rangle} e^{2\pi i \langle x, y \rangle} dx dy
$$
  
\n
$$
= a \cdot b \int_{\mathbb{R}^n} e^{2\pi i \langle y, q \rangle} \delta_{y-\Theta(p)} dy
$$
  
\n
$$
= e^{2\pi i \langle \Theta(p), q \rangle} a \cdot b.
$$

Thus, given  $a \in A_p$  and  $b \in A_q$  one has

$$
a \cdot_{\Theta} b = e^{2\pi i \langle \Theta(p), q \rangle} a \cdot b. \tag{5}
$$

Remark 2. Notice that  $A_{\Theta}$  also possesses a  $\mathbb{Z}^n$ -grading such that  $(A_{\Theta})_p = A_p$ for every  $p \in \mathbb{Z}^n$ . Due to (5), we have  $a \cdot_{\Theta} b = a \cdot b$  for any  $a, b \in A_{\pm p}$ ,  $p \in \mathbb{Z}^n$ . Indeed, for any skew symmetric  $\Theta \in M_n(\mathbb{R}^n)$  and  $p \in \mathbb{Z}^n$ , one has  $\langle \Theta p , \pm p \rangle = 0$ . The involution on  $(A_{\Theta})_p$  coincides with the involution on  $A_p$ .

Consider a  $C^*$ -dynamical system  $(A, \mathbb{T}^n, \alpha)$ , and its covariant representation  $(\pi, U)$  on a Hilbert space  $\mathcal{H}$ . For any  $p \in \mathbb{Z}^n \simeq \mathbb{T}^n$ , put

$$
\mathcal{H}_p = \{ h \in \mathcal{H} \mid U_t h = e^{2\pi i \langle t, p \rangle} h \}.
$$

Then  $\mathcal{H} = \bigoplus_{p \in \mathbb{Z}^n} \mathcal{H}_p$  (see [48]).

**Proposition 3** ([8], Theorem 2.8). Let  $(\pi, U)$  be a covariant representation of  $(A, \mathbb{T}^n, \alpha)$  on a Hilbert space  $\mathfrak{H}$ . Then one can define a representation  $\pi_{\Theta}$ of  $A_{\Theta}$  as follows:

$$
\pi_{\Theta}(a)\xi = e^{2\pi i \langle \Theta(p), q \rangle} \pi(a)\xi,
$$

for every  $\xi \in \mathcal{H}_q$ ,  $a \in A_p$ ,  $p, q \in \mathbb{Z}^n$ . Moreover,  $\pi_{\Theta}$  is faithful if and only if  $\pi$  is faithful.

It is known that Rieffel's deformation can be embedded into  $M(A \rtimes_{\alpha} \mathbb{R}^n)$ , but for the periodic actions we have an explicit description of this embedding.

**Proposition 4** ([45], Lemma 3.1.1). The following mapping defines an embedding

$$
i_{A_{\Theta}}: A_{\Theta} \to M(A \rtimes_{\alpha} \mathbb{R}^n), \ i_{A_{\Theta}}(a_p) = i_A(a_p)i_{\mathbb{R}^n}(-\Theta(p)),
$$

where  $p \in \mathbb{Z}^n$  and  $a_p$  is homogeneous of degree p.

**Proposition 5** ([28], Proposition 3.2 and [45], Section 3.1). Let  $(A, \mathbb{R}^n, \alpha)$ be a  $C^*$ -dynamical system with periodic  $\alpha$  and unital A. Put  $A_{\Theta}$  to be the Rieffel deformation of A. There exist a periodic action  $\alpha^{\Theta}$  of  $\mathbb{R}^n$  on  $A_{\Theta}$  and an isomorphism  $\Psi: A_{\Theta} \rtimes_{\alpha} {\Bbb R}^n \to A \rtimes_{\alpha} {\Bbb R}^n$  such that the following diagram is commutative



Namely,  $\alpha^{\Theta}(a) = \alpha(a)$  holds for any  $a \in A_p$ ,  $p \in \mathbb{Z}^n$ . Then it is easy to verify that  $i_{A_{\Theta}}: A_{\Theta} \to M(A \rtimes_{\alpha} \mathbb{R}^n)$  with  $i_{\mathbb{R}^n}: \mathbb{R}^n \to M(A \rtimes_{\alpha} \mathbb{R}^n)$  determine a covariant representation of  $(A_{\Theta}, \mathbb{R}^n, \alpha^{\Theta})$  in  $M(A \rtimes_{\alpha} \mathbb{R}^n)$ . Hence, by the universal property of crossed product we get the corresponding homomorphism

$$
\Psi\colon A_{\Theta}\rtimes_{\alpha^{\Theta}}\mathbb{R}^n\to M(A\rtimes_{\alpha}\mathbb{R}^n).
$$

In fact, the range of  $\Psi$  coincides with  $A \rtimes_{\alpha} \mathbb{R}^n$  and  $\Psi$  defines an isomorphism

$$
\Psi: A_{\Theta} \rtimes_{\alpha} {\Theta} \mathbb{R}^n \to A \rtimes_{\alpha} \mathbb{R}^n, \tag{6}
$$

see [28, 45] for more detailed considerations.

The following propositions shows that Rieffel's deformation inherits properties of the non-deformed counterpart.

**Proposition 6** ([28], Theorem 3.10). A  $C^*$ -algebra  $A_{\Theta}$  is nuclear if and only if A is nuclear.

**Proposition 7** ([28], Theorem 3.13). For a  $C^*$ -algebra A one has

$$
K_0(A_{\Theta}) = K_0(A)
$$
 and  $K_1(A_{\Theta}) = K_1(A)$ .

#### 3.1.4 Rieffel's deformation of a tensor product

In this part we apply Rieffel's deformation procedure to a tensor product of two nuclear unital  $C^*$ -algebras equipped with an action of  $\mathbb{T}$ .

Let A, B be  $C^*$ -algebras with actions  $\alpha$  and  $\beta$  of  $\mathbb{T}$ . Then there is a natural action  $\alpha \otimes \beta$  of  $\mathbb{T}^2$  on  $A \otimes B$  defined as

$$
(\alpha \otimes \beta)_{\varphi_1, \varphi_2}(a \otimes b) = \alpha_{\varphi_1}(a) \otimes \beta_{\varphi_2}(b).
$$

Consider the induced gradings on A and B:

$$
A = \bigoplus_{p_1 \in \mathbb{Z}} A_{p_1}, \quad B = \bigoplus_{p_2 \in \mathbb{Z}} B_{p_2}.
$$

Then the corresponding grading on  $A \otimes B$  is

$$
A \otimes B := \bigoplus_{(p_1,p_2)^t \in \mathbb{Z}^2} A_{p_1} \otimes B_{p_2}.
$$

In particular,  $a \otimes \mathbf{1} \in (A \otimes B)_{(p_1,0)^t}$  and  $\mathbf{1} \otimes b \in (A \otimes B)_{(0,p_2)^t}$ , where  $a \in A_{p_1}$ and  $b \in B_{p_2}$ .

Given  $q = e^{2\pi i \varphi_0}$ , consider

$$
\Theta_q = \begin{pmatrix} 0 & \frac{\varphi_0}{2} \\ -\frac{\varphi_0}{2} & 0 \end{pmatrix} . \tag{7}
$$

We construct the Rieffel deformation  $(A \otimes B)_{\Theta_q}$ .

Proposition 8. One has the following homomorphisms

$$
\eta_A \colon A \to (A \otimes B)_{\Theta_q}, \ \eta_A(a) = a \otimes \mathbf{1},
$$
  

$$
\eta_B \colon B \to (A \otimes B)_{\Theta_q}, \ \eta_B(b) = \mathbf{1} \otimes b,
$$

such that for homogeneous elements  $a \in A_{p_1}$  and  $b \in B_{p_2}$  it holds

$$
\eta_B(b) \cdot_{\Theta_q} \eta_A(a) = e^{2\pi i p_1 p_2 \varphi_0} \eta_A(a) \cdot_{\Theta_q} \eta_B(b).
$$

*Proof.* Recall that  $\mathbb{Z}^2$ -homogeneous components of  $A \otimes B$  and  $(A \otimes B)_{\Theta_q}$ coincide and will be considered the same. Let  $e_1 = (1,0)<sup>t</sup>$ ,  $e_2 = (0,1)<sup>t</sup>$ .

Given  $a \in A_p$ , we have

$$
\eta_A(a)=a\otimes \mathbf{1}\in ((A\otimes B)_{\Theta_q})_{p\,e_1},
$$

implying that

$$
\eta_A(a)^* = a^* \otimes \mathbf{1} \in ((A \otimes B)_{\Theta_q})_{-p\,e_1}.
$$

Let  $a_1 \in A_{p_1}$  and  $a_2 \in A_{p_2}$ . Then

$$
\eta_A(a_1) \cdot_{\Theta_q} \eta_A(a_2) = e^{2\pi i \langle p_1 \Theta_q(e_1), p_2 e_1 \rangle} (a_1 \otimes 1)(a_2 \otimes 1) = a_1 a_2 \otimes 1 = \eta_A(a_1 a_2).
$$

Thus  $\eta_A$  is a homomorphism. The arguments for  $\eta_B$  are the same.

Given  $a \in A_{p_1}$  and  $b \in B_{p_2}$ , one has

$$
\eta_A(a) \cdot_{\Theta_q} \eta_B(b) = e^{2\pi i \langle \Theta_q(p_1 e_1), p_2 e_2 \rangle}(a \otimes 1)(1 \otimes b) = e^{-\pi i p_1 p_2 \varphi_0} a \otimes b,
$$
  

$$
\eta_B(b) \cdot_{\Theta_q} \eta_A(a) = e^{2\pi i \langle \Theta_q(p_2 e_2), p_1 e_1 \rangle}(1 \otimes b)(a \otimes 1) = e^{\pi i p_1 p_2 \varphi_0} a \otimes b,
$$

implying that

$$
\eta_B(b) \cdot_{\Theta_q} \eta_A(a) = e^{2\pi i p_1 p_2 \varphi_0} \eta_A(a) \cdot_{\Theta_q} \eta_B(b).
$$

 $\Box$ 

#### 3.2 Fock representation of  $\mathcal{E}_n^q$  $\tilde{n},m$

In this part we show that the Fock representation of  $\mathcal{E}_{n,m}^q$  is faithful, and apply this result to show that  $\mathcal{E}_{n,m}^q$  is isomorphic to the Rieffel deformation  $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$ , where  $\Theta_q$  is specified in (7).

**Definition 3.** The Fock representation of  $\mathcal{E}_{n,m}^q$  is the unique up to unitary equivalence irreducible  $*$ -representation  $\pi_F^q$  determined by the action on vacuum vector  $\Omega$ ,  $||\Omega|| = 1$ ,

$$
\pi_F^q(s_j^*)\Omega = 0
$$
,  $\pi_F^q(t_r^*)\Omega = 0$ ,  $j = \overline{1, n}$ ,  $r = \overline{1, m}$ .

Denote by  $\pi_{F,n}$  the Fock representation of  $\mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$  acting on the space

$$
\mathcal{F}_n = \mathfrak{T}(\mathcal{H}_n) = \mathbb{C}\Omega \oplus \bigoplus_{d=1}^{\infty} \mathfrak{H}_n^{\otimes d}, \quad \mathcal{H}_n = \mathbb{C}^n,
$$

described by formulas

$$
\pi_{F,n}(s_j)\Omega = e_j, \quad \pi_{F,n}(s_j)e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d} = e_j \otimes e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d},
$$
  

$$
\pi_{F,n}(s_j^*)\Omega = 0, \quad \pi_{F,n}(s_j^*)e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} = \delta_{ji_1}e_{i_2} \otimes \cdots \otimes e_{i_d}, \quad d \in \mathbb{N},
$$

where  $e_1, \ldots, e_n$  is the standard orthonormal basis of  $\mathcal{H}_n$ . Notice that  $\pi_{F,n}$ is the unique irreducible faithful representations of  $\mathcal{O}_n^{(0)}$ , see for example [26].

Recall that the Fock representation of  $\mathcal{E}_{n,m}^q$  exists for any  $q \in \mathbb{C}$ ,  $|q| \leq 1$ . For  $|q| = 1$ , one has  $||T|| = 1$ , and the kernel of the Fock representation of the Wick algebra  $WE_{n,m}^q$  coincides with the ∗-ideal  $\mathcal{I}_2$  generated by ker $(1+T)$ , see Introduction. In our case,

$$
\mathcal{I}_2 = \langle t_r s_j - q s_j t_r, \ j = \overline{1, n}, \ r = \overline{1, n} \rangle.
$$

Denote by  $E_{n,m}^q$  the quotient  $WE_{n,m}^q/\mathcal{I}_2$ . Obviously,  $\mathcal{E}_{n,m}^q = C^*(E_{n,m}^q)$ . So one has the following corollary of Theorem 1.

**Proposition 9.** The Fock representation of  $\mathcal{E}_{n,m}^q$  exists and is faithful on the  $\ast$ -*subalgebra*  $E^q_{n,m}$  ⊂  $\mathcal{E}^q_{n,m}$ .

Below we give an explicit formula for  $\pi_F(s_i)$ ,  $\pi_F(t_r)$ . Consider the Fock representations  $\pi_{F,n}$  and  $\pi_{F,m}$  of \*-subalgebras  $C^*(\{s_1,\ldots,s_n\}) = \mathcal{O}_n^{(0)} \subset$  $\mathcal{E}_{n,m}^q$  and  $C^*(\{t_1,\ldots,t_m\}) = \mathcal{O}_m^{(0)} \subset \mathcal{E}_{n,m}^q$  respectively. Denote by  $\Omega_n \in \mathcal{F}_n$ and  $\Omega_m \in \mathcal{F}_m$  the corresponding vacuum vectors.

**Theorem 5.** The Fock representation  $\pi^q$  $_{F}^{q}$  of  $\mathcal{E}_{n,m}^{q}$  acts on the space  $\mathcal{F}=$  $\mathfrak{F}_n \otimes \mathfrak{F}_m$  as follows

$$
\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes d_m(q^{-\frac{1}{2}}), \quad j = \overline{1, n},
$$
  

$$
\pi_F^q(t_r) = d_n(q^{\frac{1}{2}}) \otimes \pi_{F,m}(t_r), \quad r = \overline{1, m},
$$

where  $d_k(\lambda)$  acts on  $\mathcal{F}_k$ ,  $k = n, m$  by

$$
d_k(\lambda)\Omega_k = \Omega_k, \quad d_k(\lambda)X = \lambda^l X, \quad X \in \mathcal{H}_k^{\otimes l}, \quad l \in \mathbb{N}.
$$

Proof. It is a direct calculation to verify that the operators defined above satisfy the relations of  $\mathcal{E}_{n,m}^q$ . Since  $\pi_{F,k}$  is irreducible on  $\mathcal{F}_k$ ,  $k = m, n$ , the representation  $\pi^q$  $F_F^q$  is irreducible on  $\mathcal{F}_n \otimes \mathcal{F}_m$ . Finally put  $\Omega = \Omega_n \otimes \Omega_m$ , then obviously

$$
\pi_F^q(s_j^*)\Omega = 0
$$
, and  $\pi_F^q(t_r^*)\Omega = 0$ ,  $j = \overline{1, n}, r = \overline{1, m}$ 

Thus  $\pi^q$  $_{F}^{q}$  is the Fock representation of  $\mathcal{E}_{n,m}^{q}$ .

Remark 3. In some cases, it will be more convenient to present the operators of the Fock representation of  $\mathcal{E}_{n,m}^q$  in one of the alternative forms,

$$
\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m}, \quad j = \overline{1, n},
$$
  

$$
\pi_F^q(t_r) = d_n(q) \otimes \pi_{F,m}(t_r), \quad r = \overline{1, m},
$$

or

$$
\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes d_m(q^{-1}), \quad j = \overline{1, n},
$$
  

$$
\pi_F^q(t_r) = \mathbf{1}_{\mathcal{F}_n} \otimes \pi_{F,m}(t_r), \quad r = \overline{1, m},
$$

which are obviously unitary equivalent to the one presented in the statement above.

Consider the action  $\alpha$  of  $\mathbb{T}^2$  on  $\mathcal{E}_{n,m}^q$ ,

$$
\alpha_{\varphi_1,\varphi_2}(s_i) = e^{2\pi i \varphi_1} s_i, \quad \alpha_{\varphi_1,\varphi_2}(t_r) = e^{2\pi i \varphi_2} t_r.
$$

Recall, see Section 3.1.1, that the conditional expectation, associated to  $\alpha$  is denoted by  $E_{\alpha}$ .

**Proposition 10.** The fixed point C<sup>\*</sup>-subalgebra  $(\mathcal{E}_{n,m}^q)^{\alpha}$  is an AF-algebra.

 $\Box$ 

*Proof.* The family  $\{s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*, \ \mu_1, \nu_1 \in \Lambda_n, \ \mu_2, \nu_2 \in \Lambda_m\}$  is dense in  $\mathcal{E}_{n,m}^q$ , thus the family  $\{E_{\alpha}(s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*), \mu_1, \nu_1 \in \Lambda_n, \mu_2, \nu_2 \in \Lambda_m\}$  is dense in  $(\mathcal{E}_{n,m}^q)^{\alpha}$ . Further,

$$
E_{\alpha}(s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*)=0, \text{ if } |\mu_1|\neq |\nu_1| \text{ or } |\mu_2|\neq |\nu_2|,
$$

and  $E_{\alpha}(s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*)=s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*$  otherwise. Hence

 $(\mathcal{E}_{n,m}^q)^{\alpha} = c.l.s.\{s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*, |\mu_1| = |\nu_1|, |\mu_2| = |\nu_2|, \ \mu_1, \nu_1 \in \Lambda_n, \ \mu_2, \nu_2 \in \Lambda_m\}.$ Put  $\mathcal{A}_{1,0}^0 = \mathbb{C}$ ,

$$
\mathcal{A}_{1,0}^{k_1} = c.l.s.\{s_{\mu_1}s_{\nu_1}^*, |\mu_1| = |\nu_1| = k_1, \ \mu_1, \nu_1 \in \Lambda_n\}, \quad k_1 \in \mathbb{N},
$$

and  $\mathcal{A}_{2,0}^0 = \mathbb{C}$ ,

$$
\mathcal{A}_{2,0}^{k_2} = c.l.s.\{t_{\mu_2}t_{\nu_2}^*, |\mu_2| = |\nu_2| = k_2, \ \mu_1, \nu_1 \in \Lambda_n\}, \quad k_2 \in \mathbb{N}.
$$

It is easy to see that  $xy = yx, x \in \mathcal{A}_{1,0}^{k_1}, y \in \mathcal{A}_{2,0}^{k_2}$ . Let

$$
\mathcal{A}_k^{\alpha} = \sum_{k_1 + k_2 = k} \mathcal{A}_{1,0}^{k_1} \cdot \mathcal{A}_{2,0}^{k_2}.
$$

Evidently  $\mathcal{A}_k^{\alpha}$  is a finite-dimensional subalgebra in  $(\mathcal{E}_{n,m}^q)^{\alpha}$  for any  $k \in \mathbb{Z}_+$ and

$$
({\mathcal E}^q_{n,m})^\alpha=\overline{\bigcup_{k\in{\mathbb Z}_+}{\mathcal A}^\alpha_k}.
$$

*Remark* 4. Define unitary operators  $U_{\varphi_1,\varphi_2}$ ,  $(\varphi_1,\varphi_2) \in \mathbb{T}^2$  on  $\mathcal{F}_n \otimes \mathcal{F}_m$  as follows:

$$
U_{\varphi_1,\varphi_2} = d_n(e^{2\pi i \varphi_1}) \otimes d_m(e^{2\pi i \varphi_2}).
$$

Then  $(\pi^q_\mu)$  $F$ ,  $U_{\varphi_1,\varphi_2}$ ) is a covariant representation of  $(\mathcal{E}_{n,m}^q, \mathbb{T}^2, \alpha)$ .

**Theorem 6.** The Fock representation  $\pi^q$  $_{F}^{q}$  of  $\mathcal{E}_{m,n}^{q}$  is faithful.

*Proof.* Consider the action  $\alpha^{\pi}$  of  $\mathbb{T}^2$  on  $\pi^q$  $_{F}^{q}(\mathcal{E}_{n,m}^{q})$  induced by the action  $\alpha$  on  $\mathcal{E}_{n,m}^q$ 

$$
\alpha_{\varphi_1, \varphi_2}^{\pi} \pi_F^q(s_j) = e^{2\pi i \varphi_1} \pi_F^q(s_j) := S_{j, \varphi_1, \varphi_2},
$$
  

$$
\alpha_{\varphi_1, \varphi_2}^{\pi} \pi_F^q(t_r) = e^{2\pi i \varphi_2} \pi_F^q(t_r) := T_{r, \varphi_1, \varphi_2}.
$$

To see that  $\alpha_{\varphi_1,\varphi_2}^q$  is an automorphism of  $\pi_f^q$  $_{F}^{q}(\mathcal{E}_{n,m}^{q})$  we notice that the operators  $S_{j,\varphi_1,\varphi_2}$ ,  $T_{r,\varphi_1,\varphi_2}$ , satisfy the defining relations in  $\mathcal{E}_{n,m}^q$ , and

$$
S_{j,\varphi_1,\varphi_2}^*\Omega = T_{r,\varphi_1,\varphi_2}^*\Omega = 0.
$$

Evidently the family  $\{S_{j,\varphi_1,\varphi_2}, S_{j,\varphi_1,\varphi_2}^*, T_{r,\varphi_1,\varphi_2}, T_{r,\varphi_1,\varphi_2}^* \}_{j=1}^n$  is irreducible and therefore defines the Fock representation of  $\mathcal{E}_{n,m}^q$ . Thus, by the uniqueness of the Fock representation, there exists a unitary  $V_{\varphi_1,\varphi_2}$  on  $\mathcal{F} = \mathcal{F}_n \otimes \mathcal{F}_m$ such that for any  $j = \overline{1, n}, r = \overline{1, m}$ , one has

$$
S_{j,\varphi_1,\varphi_2} = \mathrm{Ad}(V_{\varphi_1,\varphi_2}) \circ \pi_F^q(S_j), \quad T_{r,\varphi_1,\varphi_2} = \mathrm{Ad}(V_{\varphi_1,\varphi_2}) \circ \pi_F^q(T_r),
$$

implying that  $\alpha_{\varphi_1,\varphi_2}^{\pi}$  is an automorphism of  $\pi_F^q$  $_{F}^{q}(\mathcal{E}_{n,m}^{q})$  for any  $(\varphi_1, \varphi_2) \in \mathbb{T}^2$ . Evidently,

$$
\pi^q_F\colon \mathcal{E}^q_{n,m} \to \pi^q_F(\mathcal{E}^q_{n,m})
$$

is equivariant with respect to  $\alpha$  and  $\alpha^{\pi}$ .

By Proposition 2, the representation  $\pi_\mu^q$  $\frac{q}{F}$  is faithful on  $\mathcal{E}_{n,m}^q$  if and only if it is faithful on  $(\mathcal{E}_{n,m}^q)^{\alpha}$ . Further, by Proposition 10,

$$
(\mathcal{E}_{n,m}^q)^{\alpha} = \overline{\bigcup_{k \in \mathbb{Z}_+} \mathcal{A}_k^{\alpha}}.
$$

Evidently  $\mathcal{A}_{k}^{\alpha} \subset E_{n,m}^{q}$ ,  $k \in \mathbb{Z}_{+}$ . Hence by Proposition 9,  $\pi_{k}^{q}$  $\frac{q}{F}$  is faithful on  $\mathcal{A}_{k}^{\alpha}$ for any  $k \in \mathbb{Z}_+$ . It is an easy exercise to show that a representation of an AF-algebra is injective if and only if it is injective on the finite-dimensional subalgebras.  $\Box$ 

The next step is to construct a representation of  $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$  corresponding to the Fock representation  $\pi_{F,n} \otimes \pi_{F,m}$  of  $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ .

The pair  $(\pi_{F,n} \otimes \pi_{F,m}, U_{\varphi_1,\varphi_2})$  determines a covariant representation of  $(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0, \mathbb{T}^2, \alpha)$ , where as above

$$
\alpha_{\varphi_1,\varphi_2}(s_j \otimes \mathbf{1}) = e^{2\pi i \varphi_1}(s_j \otimes \mathbf{1}), \quad \alpha_{\varphi_1,\varphi_2}(\mathbf{1} \otimes t_r) = e^{2\pi i \varphi_2}(\mathbf{1} \otimes t_r).
$$

Notice that for  $p = (p_1, p_2)^t \in \mathbb{Z}_+^2$ , the subspace  $\mathcal{H}_n^{\otimes p_1} \otimes \mathcal{H}_m^{\otimes p_2}$  is the  $(p_1, p_2)^t$ . homogeneous component of  $\mathcal F$  related to the action of  $U_{\varphi_1,\varphi_2}$ , and  $(\mathcal F)_p = \{0\}$ for any  $p \in \mathbb{Z}^2 \setminus \mathbb{Z}_+^2$ .

Recall also that  $\hat{s}_j = s_j \otimes \mathbf{1}$  is contained in  $e_1 = (1, 0)^t$ -homogeneous component and  $\hat{t}_r = \mathbf{1} \otimes t_r$  is in  $e_2 = (0, 1)^t$ -homogeneous component with respect to  $\alpha$ . Now one can apply Proposition 3. Namely, given  $\xi = \xi_1 \otimes \xi_2 \in$  $\mathfrak{H}^{\otimes p_{1}}_{n}\otimes \mathfrak{H}^{\otimes p_{2}}_{m}$  one gets

$$
(\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\widehat{s}_j) \xi = e^{2\pi i \langle \Theta_q e_1, p \rangle} \pi_{F,n} \otimes \pi_{F,m}(\widehat{s}_j) \xi =
$$
  
=  $\pi_{F,n}(s_j) \xi_1 \otimes e^{-\pi i p_2 \varphi_0} \xi_2 = (\pi_{F,n}(s_j) \otimes d_m(q^{-\frac{1}{2}})) \xi,$ 

and

$$
(\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\hat{t}_r) \xi = e^{2\pi i \langle \Theta_q e_2, p \rangle} \pi_{F,n} \otimes \pi_{F,m}(\hat{t}_r) \xi =
$$
  
=  $e^{\pi i p_1 \varphi_0} \xi_1 \otimes \pi_{F,m}(t_r) \xi_2 = (d_n(q^{\frac{1}{2}}) \otimes \pi_{F,m}(t_r)) \xi.$ 

Notice that for any  $j = \overline{1, n}$ , and  $r = \overline{1, m}$ ,

$$
(\pi_{F,n}\otimes \pi_{F,m})_{\Theta_q}(\widehat{s}_j^*)\Omega=0, \quad (\pi_{F,n}\otimes \pi_{F,m})_{\Theta_q}(\widehat{t}_r^*)\Omega=0.
$$

**Theorem 7.** For any  $q \in \mathbb{C}$ ,  $|q| = 1$ , the C<sup>\*</sup>-algebra  $\mathcal{E}_{n,m}^q$  is isomorphic to  $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}.$ 

*Proof.* Proposition 8 implies that elements  $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q} \ni \hat{s}_j = s_j \otimes \mathbf{1}$  and  $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q} \ni \hat{t}_r = \mathbf{1} \otimes t_r$  satisfy

$$
\widehat{s}_j^*\widehat{s}_i=\delta_{ij}\mathbf{1}\otimes \mathbf{1},\quad \widehat{t}_r^*\widehat{t}_s=\delta_{rs}\mathbf{1}\otimes \mathbf{1},\quad \widehat{t}_r^*\widehat{s}_j=q\widehat{s}_j\widehat{t}_r^*.
$$

Hence, by the universal property one can construct a surjective homomorphism  $\Phi: \mathcal{E}_{n,m}^q \to (\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(\bar{0})})_{\Theta_q}$  defined by

$$
\Phi(s_j) = \widehat{s}_j, \quad \Phi(t_r) = \widehat{t}_r, \quad j = \overline{1, n}, \ r = \overline{1, m}.
$$

Notice that due to the considerations above,  $\pi_F^q = (\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q} \circ \Phi$ . Since  $\pi^q_{\overline{F}}$ <sup>q</sup> is faithful representation of  $\mathcal{E}_{n,m}^q$ , we deduce that  $\Phi$  is injective.

The nuclearity of  $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$  and Proposition 6 immediately imply the following

**Corollary 3.** The C<sup>\*</sup>-algebra  $\mathcal{E}_{n,m}^q$  is nuclear for any  $q \in \mathbb{C}$ ,  $|q| = 1$ .

The nuclearity of  $\mathcal{E}_{n,m}^q$  can also be shown using more explicit arguments. One can use the standard trick of untwisting the  $q$ -deformation in the crossed product, which clarifies informally the nature of isomorphism (6). Namely,

for  $q = e^{2\pi i \varphi_0}$  consider the action  $\alpha_q$  of  $\mathbb{Z}$  on  $\mathcal{E}_{n,m}^q$  defined on the generators as

$$
\alpha_q^k(s_j) = e^{\pi i k \varphi_0} s_j, \quad \alpha_q^k(t_r) = e^{-\pi i k \varphi_0} t_r, \quad j = 1, \dots, n, \ r = 1, \dots, m, \ k \in \mathbb{Z}.
$$

Denote by the same symbol the similar action on  $\mathcal{E}_{n,m}^1 \simeq \mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ . Here we denote by  $\widetilde{s}_j$  and  $\widetilde{t}_r$  the generators of  $\mathcal{E}^1_{n,m}$ .

**Proposition 11.** For any  $\varphi_0 \in [0,1)$ , one has an isomorphism  $\mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z} \simeq$  ${\mathcal E}^1_{n,m}\rtimes_{\alpha_q}\mathbb{Z}.$ 

*Proof.* Recall that  $\mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$  is generated as a  $C^*$ -algebra by elements  $\widetilde{s}_j$ ,  $\widetilde{t}_r$ and a unitary  $u$ , such that the following relations satisfied

$$
u\widetilde{s}_j u^* = e^{i\pi\varphi_0} \widetilde{s}_j, \quad u\widetilde{t}_r u^* = e^{-i\pi\varphi_0} \widetilde{t}_r, \quad j = \overline{1, n}, \ r = \overline{1, m}.
$$

Put  $\widehat{s}_j = \widetilde{s}_j u$  and  $\widehat{t}_r = \widetilde{t}_r u$ . Obviously,  $\widehat{s}_j$ ,  $\widehat{t}_r$  and u generate  $\mathcal{E}^1_{n,m} \rtimes_{\alpha_q} \mathbb{Z}$ . Further,

$$
\widehat{s}_j^* \widehat{s}_k = \delta_{jk} \mathbf{1}, \quad \widehat{t}_r^* \widehat{t}_l = \delta_{rl} \mathbf{1}
$$

and

$$
\widehat{s}_j \widehat{t}_r = \widetilde{s}_j u \widetilde{t}_r u = e^{-i\pi\varphi_0} \widetilde{s}_j \widetilde{t}_r u^2 = e^{-i\pi\varphi_0} \widetilde{t}_r \widetilde{s}_j u^2 = e^{-2\pi i\varphi_0} \widetilde{t}_r u \widetilde{s}_j u = \overline{q} \widehat{s}_j \widehat{t}_r.
$$

In a similar way we get  $\hat{s}^*_{j}t_r = qt_r\hat{s}^*_{j}$ ,  $j = 1, n, r = 1, m$ . Finally

$$
u\widehat{s}_j u^* = e^{i\pi\varphi_0}\widehat{s}_j, \quad u\widehat{t}_r u^* = e^{-i\pi\varphi_0}\widehat{t}_r.
$$

Hence the correspondence

$$
s_j \mapsto \widehat{s}_j, \quad t_j \mapsto \widehat{t}_j, \quad u \mapsto u,
$$

determines a homomorphism  $\Phi_q: \mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z} \to \mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$ . The inverse is constructed evidently.  $\Box$ 

Let us show the nuclearity of  $\mathcal{E}_{n,m}^q$  again. Indeed,  $\mathcal{E}_{n,m}^1 = \mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ is nuclear. Then so is the crossed product  $\mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$ . Then due to the isomorphism above,  $\mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z}$  is nuclear, implying the nuclearity of  $\mathcal{E}_{n,m}^q$ , see [3].

We finish this part by an analog of the well-known Wold decomposition theorem for a single isometry. Recall that

$$
Q = \sum_{j=1}^{n} s_j s_j^*, \ P = \sum_{r=1}^{m} t_r t_r^*.
$$

**Theorem 8** (Generalised Wold decomposition). Let  $\pi: \mathcal{E}_{n,m}^q \to \mathbb{B}(\mathcal{H})$  be a ∗-representation. Then

$$
\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3\oplus\mathfrak{H}_4,
$$

where each  $\mathfrak{H}_j$ ,  $j = 1, 2, 3, 4$ , is invariant with respect to  $\pi$ , and for  $\pi_j = \pi$   $\upharpoonright_{\mathfrak{H}_j}$ one has

- $\mathcal{H}_1 = \mathcal{F} \otimes \mathcal{K}$  for some Hilbert space  $\mathcal{K}$ , and  $\pi_1 = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$ ;
- $\pi_2(1-Q) = 0$ ,  $\pi_2(1-P) \neq 0$ ;
- $\pi_3(1-P) = 0$ ,  $\pi_3(1-Q) \neq 0$ ;
- $\pi_4(1-Q) = 0$ ,  $\pi_4(1-P) = 0$ ;

where any of  $\mathcal{H}_j$ ,  $j = 1, 2, 3, 4$ , could be zero.

*Proof.* We will use the fact that any representation of  $\mathcal{O}_n^{(0)}$  is a direct sum of a multiple of the Fock representation and a representation of  $\mathcal{O}_n$ .

So, restrict  $\pi$  to  $\mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$ , and decompose  $\mathcal{H} = \mathcal{H}_F \oplus \mathcal{H}_F^{\perp}$ , where

$$
\pi(\mathbf{1}-Q)_{|\mathcal{H}_F^\perp}=0,
$$

and  $\pi(\mathcal{O}_n^0)_{|\mathcal{H}_F}$  is a multiple of the Fock representation. Denote

$$
S_j := \pi(s_j) \upharpoonright_{\mathcal{H}_F}, \quad Q := \pi(Q) \upharpoonright_{\mathcal{H}_F}.
$$

Since

$$
\mathcal{H}_F = \bigoplus_{\lambda \in \Lambda_n} S_{\lambda}(\ker Q),
$$

it is invariant with respect to  $\pi(t_r)$ ,  $\pi(t_r^*)$ ,  $r = 1, m$ . Indeed,  $t_r Q = Qt_r$ in  $\mathcal{E}_{n,m}^q$ , implying the invariance of ker Q with respect to  $\pi(t_r)$  and  $\pi(t_r^*)$ . Denote ker Q by G and  $T_r := \pi(t_r) \restriction_{\mathcal{G}}$ . Then

$$
\pi(t_r)S_\lambda\xi = q^{|\lambda|}S_\lambda\pi(t_r)\xi = q^{|\lambda|}S_\lambda T_r\xi, \quad \xi \in \mathcal{G}.
$$

Thus  $\mathcal{H}_F \simeq \mathcal{F}_n \otimes \mathcal{G}$  with

$$
\pi(s_j) \upharpoonright_{\mathfrak{R}_F} = \pi_{F,n}(s_j) \otimes \mathbf{1}_g, \quad \pi(t_r) \upharpoonright_{\mathfrak{R}_F} = d_n(q) \otimes T_r, \quad j = \overline{1,n}, \ r = \overline{1,m},
$$

where the family  $\{T_r\}$  determines a \*-representation  $\tilde{\pi}$  of  $\mathcal{O}_m^{(0)}$  on  $\mathcal{G}$ .

Further, decompose  $\mathcal{G}$  as  $\mathcal{G} = \mathcal{G}_F \oplus \mathcal{G}_F^{\perp}$  into an orthogonal sum of subspaces invariant with respect to  $\widetilde{\pi}$ , where  $\mathcal{G}_F = \mathcal{F}_m \otimes \mathcal{K}$ ,

$$
\widetilde{\pi}_{\mathcal{G}_F}(t_r) = \pi_{F,m}(t_r) \otimes \mathbf{1}_{\mathcal{K}}, \quad r = \overline{1, m}, \text{ and } \widetilde{\pi} \restriction_{\mathcal{G}_F^{\perp}} (\mathbf{1} - P) = 0.
$$

Thus  $\mathcal{H}_F = (\mathcal{F}_n \otimes \mathcal{F}_m \otimes \mathcal{K}) \oplus (\mathcal{F}_n \otimes \mathcal{G}_F^{\perp})$  and

$$
\pi_{\mathfrak{H}_F}(s_j) = (\pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathfrak{F}_m} \otimes \mathbf{1}_K) \oplus (\pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathfrak{H}_F^{\perp}}), \quad j = \overline{1,n},
$$
  

$$
\pi_{\mathfrak{H}_F}(t_r) = (d_n(q) \otimes \pi_{F,m}(t_r) \otimes \mathbf{1}_K) \oplus (d_n(q) \otimes \widetilde{\pi}_{|\mathfrak{G}_F^{\perp}}(t_r)), \quad r = \overline{1,m}.
$$

Put  $\mathcal{H}_1 = \mathcal{F}_n \otimes \mathcal{F}_m \otimes \mathcal{K} = \mathcal{F} \otimes \mathcal{K}$  and notice that that  $\pi \upharpoonright_{\mathcal{H}_1} = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$ , see Remark 3. Put  $\mathfrak{H}_3 = \mathfrak{F}_n \otimes \mathfrak{G}_F^{\perp}$  and  $\pi_3 = \pi \upharpoonright_{\mathfrak{H}_3}$  i.e.,

$$
\pi_3(s_j)=\pi_{F,n}(s_j)\otimes\mathbf{1}_{\mathcal{G}_F^\perp},\quad \pi_3(t_r)=d_n(q)\otimes \widetilde{\pi}_{|\mathcal{G}_F^\perp}(t_r),\quad j=\overline{1,n},\;r=\overline{1,m}.
$$

Evidently,  $\pi_3(1 - P) = 0$  and  $\pi_3(1 - Q) \neq 0$ .

Finally, applying similar arguments to the invariant subspace  $\mathcal{H}_F^{\perp}$  one can show that there exists a decomposition

$$
\mathfrak{K}_F^\perp=\mathfrak{K}_2\oplus\mathfrak{K}_4
$$

into the orthogonal sum of invariant subspaces, where

•  $\mathcal{H}_2 = \mathcal{F}_m \otimes \mathcal{L}$  and

 $\pi_2(s_j) := \pi \upharpoonright_{\mathcal{H}_2} (s_j) = d_m(\overline{q}) \otimes \widehat{\pi}(s_j), \quad \pi_2(t_r) := \pi \upharpoonright_{\mathcal{H}_2} (t_r) = \pi_{F,m}(t_r) \otimes \mathbf{1}_{\mathcal{L}},$ 

for a representation  $\hat{\pi}$  of  $\mathcal{O}_n$ . Evidently,  $\pi_2(1 - Q) = 0$ ,  $\pi_2(1 - P) \neq 0$ .

• For  $\pi_4 := \pi \upharpoonright_{\mathcal{H}_4}$  one has

$$
\pi_4(\mathbf{1} - Q) = 0, \quad \pi_4(\mathbf{1} - P) = 0.
$$

 $\Box$ 

#### 3.3 Ideals in  $\mathcal{E}_n^q$  $\tilde{n},\!m$

In this part, we give a complete description of ideals in  $\mathcal{E}_{n,m}^q$ , and prove their independence on the deformation parameter q.

For

$$
Q = \sum_{j=1}^{n} s_j s_j^*, \quad P = \sum_{r=1}^{m} t_r t_r^*.
$$

we consider two-sided ideals,  $\mathcal{M}_q$  generated by 1 – P and 1 – Q,  $\mathcal{I}_1^q$  generated by  $1 - Q$ ,  $\mathcal{I}_2^q$  $Q_2^q$  generated by  $1 - P$ , and  $\mathcal{I}_q$  generated by  $(1 - Q)(1 - P)$ . Evidently,

$$
\mathcal{I}_q = \mathcal{I}_1^q \cap \mathcal{I}_2^q = \mathcal{I}_1^q \cdot \mathcal{I}_2^q.
$$

Below we will show that any ideal in  $\mathcal{E}_{n,m}^q$  coincides with the one listed above.

To clarify the structure of  $\mathfrak{I}_1^q$  $_1^q$ ,  $\mathfrak{I}_2^q$  and  $\mathfrak{I}_q$ , we use the construction of twisted tensor product of a certain  $C^*$ -algebra with the algebra of compact operators K, see [47]. We give a brief review of the construction, adapted to our situation.

Recall that the  $C^*$ -algebra K can be considered as a universal  $C^*$ -algebra generated by a closed linear span of elements  $e_{\mu\nu}$ ,  $\mu, \nu \in \Lambda_m$  subject to the relations

$$
e_{\mu_1\nu_1}e_{\mu_2\nu_2} = \delta_{\mu_2\nu_1}e_{\mu_1\nu_2}, \quad e_{\mu_1\nu_1}^* = e_{\nu_1\mu_1}, \quad \nu_i, \mu_i \in \Lambda_m,
$$

here  $e_{\emptyset} := e_{\emptyset \emptyset}$  is a minimal projection.

**Definition 4.** Let  $A$  be a  $C^*$ -algebra,

$$
\alpha = \{ \alpha_{\mu}, \ \mu \in \Lambda_m \} \subset \text{Aut}(A), \ \text{where } \alpha_{\emptyset} = \text{id}_A,
$$

and  $e_{\mu\nu}$ ,  $\mu, \nu \in \Lambda_m$  be the generators of K specified above. Construct the universal  $C^*$ -algebra

$$
\langle A, \mathbb{K} \rangle_{\alpha} = C^*(a \in A, e_{\mu\nu} \in \mathbb{K} \mid a e_{\mu\nu} = e_{\mu\nu} \alpha_{\nu}^{-1}(\alpha_{\mu}(a)).
$$

We define  $A \otimes_{\alpha} \mathbb{K}$  as a subalgebra of  $\langle A, \mathbb{K} \rangle_{\alpha}$  generated by  $ax, a \in A$  $\langle A, \mathbb{K} \rangle_\alpha, x \in \mathbb{K} \subset \langle A, \mathbb{K} \rangle_\alpha.$ 

Notice that  $\langle A, \mathbb{K} \rangle_\alpha$  exists for any C<sup>\*</sup>-algebra A and family  $\alpha \subset \text{Aut}(A)$ , see [47].

Remark 5.

1. Let  $x_{\mu} = e_{\mu\emptyset}$ . Then  $ax_{\mu} = x_{\mu}\alpha_{\mu}(a)$ ,  $ax_{\mu}^{*} = x_{\mu}^{*}\alpha_{\mu}^{-1}(a)$ ,  $a \in A$ , compare with  $|47|$ .

2. For any  $a \in A$  one has  $e_{\mu\nu} a = \alpha_{\mu}^{-1}(\alpha_{\nu}(a))e_{\mu\nu}$  implying that

$$
(ae_{\mu\nu})^* = \alpha_\mu^{-1}(\alpha_\nu(a))e_{\nu\mu}.
$$

3. For any  $a_1, a_2 \in A$  one has  $(a_1e_{\mu_1\nu_1})(a_2e_{\mu_2\nu_2}) = \delta_{\nu_1\mu_2}a_1\alpha_{\mu_1}^{-1}(\alpha_{\mu_2}(a_2))e_{\mu_1\nu_2}$ .

**Proposition 12** ([47]). Let A be a C<sup>\*</sup>-algebra and

 $\alpha = {\alpha_{\mu}, \ \mu \in \Lambda_m} \subset \text{Aut}(A)$  with  $\alpha_{\emptyset} = \text{id}_A$ .

Then the correspondence

 $ae_{\mu\nu} \mapsto \alpha_{\mu}(a) \otimes e_{\mu\nu}, \quad a \in A, \ \mu, \nu \in \Lambda_m$ 

extends by linearity and continuity to an isomorphism

$$
\Delta_{\alpha} \colon A \otimes_{\alpha} \mathbb{K} \to A \otimes \mathbb{K},
$$

where  $\Delta_{\alpha}^{-1}$  is constructed via the correspondence

$$
a \otimes e_{\mu\nu} \mapsto \alpha_{\mu}^{-1}(a)e_{\mu\nu}, \quad a \in A, \ \mu, \nu \in \Lambda_m.
$$

*Remark* 6. For  $x_{\mu} = e_{\mu\emptyset}$ ,  $\mu \in \Lambda_m$  one has, see [47],

$$
\Delta_{\alpha}(ax_{\mu}) = \alpha_{\mu}(a) \otimes x_{\mu}, \quad \Delta_{\alpha}(ax_{\mu}^{*}) = a \otimes x_{\mu}^{*}.
$$

The following functorial property of  $\otimes_{\alpha} \mathbb{K}$  can be derived easily. Consider

$$
\alpha=(\alpha_\mu)_{\mu\in\Lambda_m}\subset\operatorname{\mathsf{Aut}}(A),\ \beta=(\beta_\mu)_{\mu\in\Lambda_m}\subset\operatorname{\mathsf{Aut}}(B).
$$

Suppose  $\varphi: A \to B$  is equivariant, i.e.  $\varphi(\alpha_\mu(a)) = \beta_\mu(\varphi(a))$  for any  $a \in A$ and  $\mu \in \Lambda_m$ . Then one can define the homomorphism

$$
\varphi \otimes_{\alpha}^{\beta}: A \otimes_{\alpha} \mathbb{K} \to B \otimes_{\beta} \mathbb{K}, \quad \varphi \otimes_{\alpha}^{\beta} (ak) = \varphi(a)k, \quad a \in A, \ k \in \mathbb{K},
$$

making the following diagram commutative

$$
A \otimes_{\alpha} \mathbb{K} \xrightarrow{\varphi \otimes_{\alpha}^{\beta}} B \otimes_{\beta} \mathbb{K}
$$
  
\n
$$
\downarrow^{\Delta_{\alpha}} \qquad \qquad \downarrow^{\Delta_{\beta}} \qquad (8)
$$
  
\n
$$
A \otimes \mathbb{K} \xrightarrow{\varphi \otimes id_{\mathbb{K}}} B \otimes \mathbb{K}
$$

Namely, it is easy to verify that

$$
(\Delta_{\beta}^{-1}\circ(\varphi\otimes id_{\mathbb{K}})\circ\Delta_{\alpha})(ae_{\mu\nu})=\varphi(a)e_{\mu\nu}=\varphi\otimes_{\alpha}^{\beta}(ae_{\mu\nu}),\quad a\in A,\ \mu,\nu\in\Lambda_{m}.
$$

An important consequence of the commutativity of the diagram above is exactness of the functor  $\otimes_{\alpha} \mathbb{K}$ . Let

$$
\beta = (\beta_{\mu})_{\mu \in \Lambda_m} \subset \text{Aut}(B), \ \alpha = (\alpha_{\mu})_{\mu \in \Lambda_m} \subset \text{Aut}(A), \ \gamma = (\gamma_{\mu})_{\mu \in \Lambda_m} \subset \text{Aut}(C)
$$

and consider a short exact sequence

 $0 \longrightarrow B \xrightarrow{\varphi_1} A \xrightarrow{\varphi_2} C \longrightarrow 0$ 

where  $\varphi_1$ ,  $\varphi_2$  are equivariant homomorphisms. Then one has the following short exact sequence

$$
0 \longrightarrow B \otimes_{\beta} \mathbb{K} \xrightarrow{\varphi_1 \otimes \alpha} A \otimes_{\alpha} \mathbb{K} \xrightarrow{\varphi_2 \otimes \alpha} C \otimes_{\gamma} \mathbb{K} \longrightarrow 0
$$

Now we are ready to study the structure of the ideals  $\mathcal{I}_1^q$  $^{q}_{1}$ ,  $\mathfrak{I}_{2}^{q}$  $_2^q, \mathcal{I}_q \subset \mathcal{E}_{n,m}^q.$ We start with  $J_1^q$  $_1^q$ . Notice that

$$
\mathcal{I}_1^q = c.l.s. \{ t_{\mu_2} t_{\nu_2}^* s_{\mu_1} (1 - Q) s_{\nu_1}^*, \ \mu_1, \nu_1 \in \Lambda_n, \ \mu_2, \nu_2 \in \Lambda_m \}.
$$

Put  $E_{\mu_1\nu_1} = s_{\mu_1}(1-Q)s_{\nu_1}^*, \mu_1, \nu_1 \in \Lambda_n$ . Then  $E_{\mu_1\nu_1}$  satisfy the relations for matrix units generating K. Moreover, c.l.s.  $\{E_{\mu\nu}, \mu, \nu \in \Lambda_n\}$  is an ideal in  $\mathcal{O}_n^{(0)}$  isomorphic to K.

Consider the family  $\alpha^q = (\alpha_\mu)_{\mu \in \Lambda_n} \subset \text{Aut}(\mathcal{O}_m^{(0)})$  defined as

$$
\alpha_{\mu}(t_r) = q^{|\mu|} t_r, \quad \alpha_{\mu}(t_r^*) = q^{-|\mu|} t_r^*, \quad \mu \in \Lambda_n, \ r = \overline{1, m}.
$$

**Proposition 13.** The correspondence  $ae_{\mu\nu} \mapsto aE_{\mu\nu}$ ,  $a \in \mathcal{O}_m^{(0)}$ ,  $\mu, \nu \in \Lambda_n$ , extends to an isomorphism

$$
\Delta_{q,1} \colon \mathcal{O}_{m}^{(0)} \otimes_{\alpha^{q}} \mathbb{K} \to \mathcal{I}_{1}^{q}.
$$

*Proof.* We note that for any  $\mu_1, \nu_1 \in \Lambda_n$  and  $\mu_2, \nu_2 \in \Lambda_m$  one has

$$
t_{\mu_2}t_{\nu_2}^*E_{\mu_1\nu_1} = q^{(|\nu_1|-|\mu_1|)(|\mu_2|-|\nu_2|)}E_{\mu_1\nu_1}t_{\mu_2}t_{\nu_2}^* = E_{\mu_1\nu_1}\alpha_{\nu_1}^{-1}(\alpha_{\mu_1}(t_{\mu_2}t_{\nu_2}^*)).
$$

Thus, due to the universal property of  $\langle \mathcal{O}_m^{(0)}, \mathbb{K} \rangle_{\alpha^q}$ , the correspondence

$$
ae_{\mu\nu} \mapsto aE_{\mu\nu}
$$

determines a surjective homomorphism  $\Delta_{q,1} : \mathcal{O}_{m}^{(0)} \otimes_{\alpha_q} \mathbb{K} \to \mathcal{I}_1^q$  $\frac{q}{1}$ .

It remains to show that  $\Delta_{q,1}$  is injective. Since the Fock representation of  $\mathcal{E}_{n,m}^q$  is faithful, we can identify  $\mathcal{I}_1^q$  with  $\pi_\beta^q$  ${}^q_F$  $({\mathcal I}^q_1)$  $_{1}^{q}$ ). It will be convenient for us to use the following form of the Fock representation, see Remark 3,

$$
\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m} := S_j \otimes \mathbf{1}_{\mathcal{F}_m}, \ j = \overline{1, n},
$$
  

$$
\pi_F^q(t_r) = d_n(q) \otimes \pi_{F,m}(t_r) := d_n(q) \otimes T_r, \ r = \overline{1, m}.
$$

In particular, for any  $\mu_1, \nu_1 \in \Lambda_n$ ,  $\mu_2, \nu_2 \in \Lambda_m$ 

$$
\pi_F^q(t_{\mu_2}t_{\nu_2}^*E_{\mu_1\nu_1})=d_n(q^{|\mu_2|-|\nu_2|})S_{\mu_1}(\mathbf{1}-Q)S_{\nu_1}\otimes T_{\mu_2}T_{\nu_2}^*.
$$

Consider  $\Delta_{q,1} \circ \Delta_{\alpha^q}^{-1} : \mathcal{O}_m^{(0)} \otimes \mathbb{K} \to \pi^q$  ${}^q_F$ ( $\mathfrak{I}^q_1$  $_{1}^{q}$ ). We intend to show that

$$
\Delta_{q,1} \circ \Delta_{\alpha^q}^{-1} = \pi_F^1,
$$

where  $\pi_F^1$  is the restriction of the Fock representation of  $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$  to  $\mathbb{K} \otimes \mathcal{O}_m^{(0)}$ , and K is generated by  $E_{\mu\nu}$  specified above. Notice that the family

$$
\{t_{\mu_2}t_{\nu_2}^*\otimes E_{\mu_1\nu_1}, \ \mu_1,\nu_1\in\Lambda_n, \ \mu_2,\nu_2\in\Lambda_m\}
$$

generates  $\mathcal{O}_m^{(0)} \otimes \mathbb{K}$ . Then

$$
\Delta_{\alpha}^{-1}(t_{\mu_2}t_{\nu_2}^* \otimes E_{\mu_1\nu_1}) = \alpha_{\mu_1}^{-1}(t_{\mu_2}t_{\nu_2}^*)e_{\mu_1\nu_1} = q^{-|\mu_1|(|\mu_2|-|\nu_2|)}t_{\mu_2}t_{\nu_2}^*e_{\mu_1\nu_1},
$$

and

$$
\Delta_{q,1} \circ \Delta_{\alpha^q}^{-1} (t_{\mu_2} t_{\nu_2}^* \otimes E_{\mu_1 \nu_1}) = q^{-|\mu_1|(|\mu_2| - |\nu_2|)} \pi_F^q (t_{\mu_2} t_{\nu_2}^* E_{\mu_1 \nu_1})
$$
  
\n
$$
= q^{-|\mu_1|(|\mu_2| - |\nu_2|)} d_n(q^{|\mu_2| - |\nu_2|}) S_{\mu_1} (1 - Q) S_{\nu_1}^* \otimes T_{\mu_2} T_{\nu_2}^*
$$
  
\n
$$
= q^{-|\mu_1|(|\mu_2| - |\nu_2|)} q^{|\mu_1|(|\mu_2| - |\nu_2|)} S_{\mu_1} d_n(q^{|\mu_2| - |\nu_2|}) (1 - Q) S_{\nu_1}^* \otimes T_{\mu_2} T_{\nu_2}^*
$$
  
\n
$$
= S_{\mu_1} (1 - Q) S_{\nu_1}^* \otimes T_{\mu_2} T_{\nu_2}^* = \pi_F^1 (E_{\mu_1 \nu_1} \otimes t_{\mu_2} t_{\nu_2}^*),
$$

where we used relations  $d_n(\lambda)S_j = \lambda S_j d_n(\lambda)$ ,  $j = \overline{1,n}$ ,  $\lambda \in \mathbb{C}$ , and the obvious fact that

$$
d_n(\lambda)(1-Q)=1-Q.
$$

To complete the proof we recall that  $\pi_F^1$  is a faithful representation of  $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^0$ , so its restriction to  $\mathbb{K} \otimes \mathcal{O}_m^{(0)}$  is also faithful, implying the injectivity of  $\Delta_q$ .  $\Box$ 

*Remark* 7. Evidently,  $\mathcal{I}_q$  is a closed linear span of the family

$$
\{t_{\mu_2}(1-P)t_{\nu_2}^*s_{\mu_1}(1-Q)s_{\nu_1}^*, \ \mu_1,\nu_1\in\Lambda_n, \ \mu_2,\nu_2\in\Lambda_m\}\subset\mathfrak{I}_1^q.
$$

Moreover,  $c.l.s.\{t_{\mu_2}(1-P)t_{\nu_2}^*, \mu_2, \nu_2 \in \Lambda_m\} = \mathbb{K} \subset \mathcal{O}_m^{(0)}$ . It is easy to see that

$$
\alpha_{\mu}(t_{\mu_2}(1-P)t_{\nu_2}^*) = q^{|\mu|(|\mu_2|-|\nu_2|)}t_{\mu_2}(1-P)t_{\nu_2}^*,
$$

so every  $\alpha_{\mu} \in \alpha^q$  can be regarded as an element of  $\text{Aut}(\mathbb{K})$ .

A moment reflection and Proposition 13 give the following corollary

**Proposition 14.** Restriction of  $\Delta_{q,1}$  to  $\mathbb{K} \otimes_{\alpha^q} \mathbb{K} \subset \mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K}$  gives an isomorphism

$$
\Delta_{q,1}\colon \mathbb{K} \otimes_{\alpha^q} \mathbb{K} \to \mathbf{I}_q.
$$

To deal with  $\mathcal{I}_2^q$ <sup>q</sup><sub>2</sub>, we consider the family  $\beta^q = {\beta_\mu, \ \mu \in \Lambda_m} \subset \text{Aut}(0_n^{(0)})$ defined as

$$
\beta_{\mu}(s_j) = q^{-|\mu|} s_j, \ \beta_{\mu}(s_j^*) = q^{|\mu|} s_j^*, \ j = \overline{1, n}.
$$

**Proposition 15.** One has an isomorphism  $\Delta_{q,2}$ :  $\mathcal{O}_n^{(0)} \otimes_{\beta^q} \mathbb{K} \to \mathcal{I}_2^q$  $\frac{q}{2}$ .

Obviously,  $\Delta_{q,2}$  induces the isomorphism  $\mathbb{K} \otimes_{\beta^q} \mathbb{K} \simeq \mathcal{I}_q$ , where the first term is an ideal in  $\mathcal{O}_n^{(0)}$  and the second in  $\mathcal{O}_m^{(0)}$  respectively.

Write

$$
\varepsilon_n \colon \mathbb{K} \to \mathcal{O}_n^{(0)}, \quad \varepsilon_m \colon \mathbb{K} \to \mathcal{O}_m^{(0)},
$$

for the canonical embeddings and

$$
q_n: \mathcal{O}_n^{(0)} \to \mathcal{O}_n, \quad q_m: \mathcal{O}_m^{(0)} \to \mathcal{O}_m,
$$

for the quotient maps. Let also

$$
\varepsilon_{q,j}\colon \mathfrak{I}_q\to \mathfrak{I}_j^q, \quad j=1,2,
$$

be the embeddings and

$$
\pi_{q,j} : \mathcal{I}_j^q \to \mathcal{I}_j^q / \mathcal{I}_q, \quad j = 1, 2,
$$

the quotient maps. Notice also that the families  $\alpha^q \subset \text{Aut}(\mathcal{O}_m^{(0)})$ ,  $\beta^q \subset$  $\mathsf{Aut}(\mathcal{O}_n^{(0)})$  determine families of automorphisms of  $\mathcal{O}_m$  and  $\mathcal{O}_n$  respectively, also denoted by  $\alpha^q$  and  $\beta^q$ .

Theorem 9. One has the following isomorphism of extensions

$$
\begin{array}{ccc}\n0 & \longrightarrow & \mathcal{I}_q & \xrightarrow{\varepsilon_{q,1}} & \mathcal{I}_1^q & \xrightarrow{\pi_{q,1}} & \mathcal{I}_1^q / \mathcal{I}_q & \longrightarrow & 0 \\
& & \downarrow & \Delta_{\alpha q} \circ \Delta_{q,1}^{-1} & \downarrow & \Delta_{\alpha q} \circ \Delta_{q,1}^{-1} & \downarrow & \sim \\
0 & \longrightarrow & \mathbb{K} \otimes \mathbb{K} & \xrightarrow{\varepsilon_m \otimes id_{\mathbb{K}}} & \mathcal{O}_m^0 \otimes \mathbb{K} & \xrightarrow{\underline{q_m} \otimes id_{\mathbb{K}}} & \mathcal{O}_m \otimes \mathbb{K} & \longrightarrow & 0\n\end{array}
$$

and

$$
\begin{array}{ccc}\n0 & \xrightarrow{\qquad \qquad } \mathcal{I}_q \xrightarrow{\qquad \qquad } \mathcal{I}_2^q \xrightarrow{\qquad \qquad } \mathcal{I}_2^q/\mathcal{I}_q \xrightarrow{\qquad \qquad } 0 \\
& \downarrow^{\Delta_{\beta^q} \circ \Delta_{q,2}^{-1}} & \downarrow^{\Delta_{\beta^q} \circ \Delta_{q,2}^{-1}} & \downarrow^{\Delta_{\beta^q}} \\
0 & \xrightarrow{\qquad \qquad } \mathbb{K} \otimes \mathbb{K} \xrightarrow{\varepsilon_n \otimes id_{\mathbb{K}}} \mathcal{O}_n^0 \otimes \mathbb{K} \xrightarrow{q_n \otimes id_{\mathbb{K}}} \mathcal{O}_n \otimes \mathbb{K} \xrightarrow{\qquad \qquad } 0\n\end{array}
$$

Proof. Indeed, each row in diagram (9) below is exact and every non-dashed vertical arrow is an isomorphism. The bottom left and bottom right squares are commutative due to (8). The top left square is commutative due to the arguments in the proof of Proposition 13 combined with Remark 7. Hence there exists a unique isomomorphism

$$
\Phi_{q,1} : \mathfrak{I}_1^q / \mathfrak{I}_q \to \mathcal{O}_m \otimes_{\alpha^q} \mathbb{K},
$$

making the diagram (9) commutative

$$
0 \longrightarrow \mathcal{I}_{q} \xrightarrow{\varepsilon_{q,1}} \mathcal{I}_{1}^{q} \xrightarrow{\pi_{q,1}} \mathcal{I}_{1}^{q}/\mathcal{I}_{q} \longrightarrow 0
$$
  
\n
$$
\downarrow \Delta_{q,1}^{-1} \qquad \qquad \downarrow \Delta_{q,1}^{-1} \qquad \qquad \downarrow \Phi_{q,1}
$$
  
\n
$$
0 \longrightarrow \mathbb{K} \otimes_{\alpha^{q}} \mathbb{K} \xrightarrow{\varepsilon_{m} \otimes_{\alpha^{q}}^{\alpha^{q}}} \mathcal{O}_{m}^{0} \otimes_{\alpha^{q}} \mathbb{K} \xrightarrow{q_{m} \otimes_{\alpha^{q}}^{\alpha^{q}}} \mathcal{O}_{m} \otimes_{\alpha^{q}} \mathbb{K} \longrightarrow 0
$$
  
\n
$$
\downarrow \Delta_{\alpha^{q}} \qquad \qquad \downarrow \Delta_{\alpha^{q}} \qquad \qquad \downarrow \Delta_{\alpha^{q}}
$$
  
\n
$$
0 \longrightarrow \mathbb{K} \otimes \mathbb{K} \xrightarrow{\varepsilon_{m} \otimes id_{\mathbb{K}}} \mathcal{O}_{m}^{0} \otimes \mathbb{K} \xrightarrow{q_{m} \otimes id_{\mathbb{K}}} \mathcal{O}_{m} \otimes \mathbb{K} \longrightarrow 0
$$
  
\n
$$
(9)
$$

 $\Box$ 

The proof for  $\mathfrak{I}_2^q$  $_2^q$  is similar.

> The following Lemma follows from the fact that  $\mathcal{M}_q = \mathcal{I}_1^q + \mathcal{I}_2^q$  $\frac{q}{2}$ .

Lemma 7.

$$
\mathcal{M}_q/\mathcal{I}_q \simeq \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_m \otimes \mathbb{K} \oplus \mathcal{O}_n \otimes \mathbb{K}.
$$

Theorem 9 implies that  $\mathcal{I}_q, \mathcal{I}_1^q$  $_1^q$ ,  $\mathfrak{I}_2^q$  are stable  $C^*$ -algebras. It follows from [44], Proposition 6.12, that an extension of a stable  $C^*$ -algebra by K is also stable. Thus, Lemma 7 implies immediately the following important corollary.

**Corollary 4.** For any  $q \in \mathbb{C}$ ,  $|q| = 1$ , the C<sup>\*</sup>-algebra  $\mathcal{M}_q$  is stable.

Denote the Calkin algebra by  $Q$ . Recall that for  $C^*$ -algebras A and B the isomorphism

$$
\operatorname{\mathsf{Ext}}(A \oplus B, \mathbb{K}) \simeq \operatorname{\mathsf{Ext}}(A, \mathbb{K}) \oplus \operatorname{\mathsf{Ext}}(B, \mathbb{K})
$$

is given as follows. Let

$$
\iota_1: A \to A \oplus B
$$
,  $\iota_1(a) = (a, 0)$ ,  $\iota_2: B \to A \oplus B$ ,  $\iota_2(b) = (0, b)$ .

For a Busby invariant  $\tau:A\oplus B\to Q$  define

$$
\mathsf{F}:\mathsf{Ext}(A\oplus B,\mathbb{K})\to\mathsf{Ext}(A,\mathbb{K})\oplus\mathsf{Ext}(B,\mathbb{K}),\quad \mathsf{F}(\tau)=(\tau\circ\iota_1,\tau\circ\iota_2).
$$

It can be shown, see [21], that F determines a group isomorphism.

Remark 8. Consider an extension

$$
0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{10}
$$

Let  $i: B \to M(B)$  be the canonical embedding. Define  $\beta$  to be the unique map such that

$$
\beta(e)i(b) = i(eb), \quad \text{for every } b \in B, \ e \in E.
$$

Then the Busby invariant  $\tau$  is the unique map which makes the diagram commute.

$$
0 \longrightarrow B \xrightarrow{i} M(B) \longrightarrow M(B)/B \longrightarrow 0
$$
  

$$
\parallel \qquad \qquad \beta \uparrow \qquad \qquad \tau \uparrow
$$
  

$$
0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0
$$

We will use both notations  $[E]$  and  $[\tau]$  in order to denote the class of the extension (10) in  $\textsf{Ext}(A, B)$ .

Let  $[\mathcal{M}_q] \in \mathsf{Ext}(\mathcal{I}_1^q)$  $\frac{q}{1}/\mathfrak{I}_q\oplus \mathfrak{I}_2^q$  $\mathcal{Z}_2^q/\mathcal{I}_q, \mathcal{I}_q), [\mathcal{I}_1^q]$  $\left[ \begin{smallmatrix} q \\ 1 \end{smallmatrix} \right] \in \mathsf{Ext}(\mathfrak{I}_1^q)$  $\binom{q}{1}$  $\mathcal{I}_q$ ,  $\mathcal{I}_q$ ),  $\left[\mathcal{I}_2^q\right]$  $\mathcal{C}_2^q$   $\in$  Ext $(\mathcal{I}_2^q)$  $_2^q/\mathfrak{I}_q, \mathfrak{I}_q)$ respectively be the classes of the following extensions

$$
0 \to \mathcal{I}_q \to \mathcal{M}_q \to \mathcal{I}_1^q / \mathcal{I}_q \oplus \mathcal{I}_2^q / \mathcal{I}_q \to 0,
$$
  
\n
$$
0 \to \mathcal{I}_q \to \mathcal{I}_1^q \to \mathcal{I}_1^q / \mathcal{I}_q \to 0,
$$
  
\n
$$
0 \to \mathcal{I}_q \to \mathcal{I}_2^q \to \mathcal{I}_2^q / \mathcal{I}_q \to 0.
$$

Lemma 8.

$$
[\mathcal{M}_q] = ([\mathfrak{I}_1^q],[\mathfrak{I}_2^q]) \in \text{Ext}(\mathfrak{I}_1^q/\mathfrak{I}_q,\mathfrak{I}_q) \oplus \text{Ext}(\mathfrak{I}_2^q/\mathfrak{I}_q,\mathfrak{I}_q) \simeq \text{Ext}(\mathfrak{I}_1^q/\mathfrak{I}_q \oplus \mathfrak{I}_2^q/\mathfrak{I}_q,\mathfrak{I}_q).
$$

Proof. Consider the following morphism of extensions:



Here

$$
\beta_1: \mathfrak{I}_1^q \to M(\mathfrak{I}_q), \quad \beta_2: \mathfrak{M}_q \to M(\mathfrak{I}_q),
$$

are homomorphisms introduced in Remark 8, the arrow

$$
j_1\colon \mathfrak{I}_1^q\hookrightarrow \mathfrak{M}_q
$$

is the inclusion, and the arrow

$$
\iota_1: \mathfrak{I}_1^q/\mathfrak{I}_q \rightarrow \mathfrak{I}_1^q/\mathfrak{I}_q \oplus \mathfrak{I}_2^q/\mathfrak{I}_q
$$

has the form  $\iota_1(x) = (x, 0)$ .

Notice that for every  $b \in \mathcal{I}_q$  and  $x \in \mathcal{I}_1^q$  one has

$$
(\beta_2 \circ j_1)(x)i(b) = i(j_1(x)b) = i(xb) = \beta_1(x)i(b).
$$

By the uniqueness of  $\beta_1$ , we get  $\beta_2 \circ j_1 = \beta_1$ . Thus the following diagram commutes

$$
\begin{array}{ccc}\n\mathfrak{I}_{1}^{q} & \xrightarrow{\beta_{1}} & M(\mathfrak{I}_{q}) \\
\downarrow & & \parallel \\
\mathfrak{M}_{q} & \xrightarrow{\beta_{2}} & M(\mathfrak{I}_{q})\n\end{array}
$$

Further, Remark 8 implies that for Busby invariants  $\tau_{\mathfrak{I}_1^q}$  and  $\tau_{\mathfrak{M}_q}$  the squares below are commutative

$$
M(\mathcal{I}_q) \longrightarrow M(\mathcal{I}_q)/\mathcal{I}_q \longrightarrow M(\mathcal{I}_q)/\mathcal{I}_q
$$
  
\n
$$
\beta_1 \uparrow \qquad \qquad \tau_{\mathcal{I}_1^q} \uparrow \qquad \qquad \beta_2 \uparrow \qquad \qquad \tau_{\mathcal{M}_q} \uparrow
$$
  
\n
$$
\mathcal{I}_1^q \longrightarrow \qquad \qquad \mathcal{I}_1^q/\mathcal{I}_q \longrightarrow \qquad \qquad \mathcal{M}_q \longrightarrow \qquad \qquad \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q
$$

Hence the square

$$
\begin{array}{ccc}\n\mathfrak{I}_{1}^{q}/\mathfrak{I}_{q} & \xrightarrow{\tau_{\mathfrak{I}_{1}^{q}}} & M(\mathfrak{I}_{q})/\mathfrak{I}_{q} \\
\downarrow^{\iota_{1}} & & \parallel \\
\mathfrak{I}_{1}^{q}/\mathfrak{I}_{q} \oplus \mathfrak{I}_{2}^{q}/\mathfrak{I}_{q} & \xrightarrow{\tau_{\mathcal{M}_{q}}} & M(\mathfrak{I}_{q})/\mathfrak{I}_{q}\n\end{array}
$$

is also commutative. Thus,  $\tau_{\mathfrak{I}^q_1} = \tau_{\mathfrak{M}_q} \circ \iota_1$ . By the same arguments we get  $\tau_{\mathfrak{I}_2^q} = \tau_{\mathfrak{M}_q} \circ \iota_2$ , where

$$
\iota_2 \colon \mathfrak{I}_2^q / \mathfrak{I}_q \to \mathfrak{I}_1^q / \mathfrak{I}_q \oplus \mathfrak{I}_2^q / \mathfrak{I}_q, \quad \iota_2(y) = (0, y).
$$

Thus

$$
[\tau_{\mathcal{M}_q}] = ([\tau_{\mathcal{M}_q} \circ \iota_1], [\tau_{\mathcal{M}_q} \circ \iota_2]) = ([\tau_{\mathcal{I}_1^q}], [\tau_{\mathcal{I}_2^q}]).
$$

 $\Box$ 

In the following theorem we give a description of all ideals in  $\mathcal{E}_{n,m}^q$ .

**Theorem 10.** Any ideal  $J \subset \mathcal{E}_{n,m}^q$  coincides with one of  $\mathcal{I}_q$ ,  $\mathcal{I}_1^q$  $\frac{q}{1}$ ,  $\mathfrak{I}_{2}^{q}$  $_2^q$ ,  $\mathcal{M}_q$ .

*Proof.* First we notice that  $\mathcal{I}_1^q/\mathcal{I}_q \simeq \mathcal{O}_m \otimes \mathbb{K}$ ,  $\mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_n \otimes \mathbb{K}$  are simple. Hence for any ideal  $\mathcal{J}$  such that  $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{I}_1^q$  $\frac{q}{1}$  or  $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{I}_2^q$  $q_2^q$ , one has  $\mathcal{J} = \mathcal{I}_q$ , or  $\mathcal{J} = \mathcal{I}_1^q$  $_1^q$ , or  $\mathcal{J} = \mathcal{I}_2^q$  $\frac{q}{2}$ .

Further, using the fact that  $\mathcal{M}_q = \mathcal{I}_1^q + \mathcal{I}_2^q$  $Q_2^q$  and  $\mathcal{I}_q = \mathcal{I}_1^q \cap \mathcal{I}_2^q$  we get

$$
\mathcal{M}_q/\mathcal{I}_1^q \simeq \mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_n \otimes \mathbb{K}.
$$

So if  $\mathcal{I}^q \subseteq \mathcal{J} \subseteq \mathcal{M}_q$ , then again either  $\mathcal{J} = \mathcal{I}_1^q$  $_1^q$  or  $\mathcal{J} = \mathcal{M}_q$ .

Below, see Theorem 13, we show that  $\mathcal{E}_{n,m}^q/\mathcal{M}_q$  is simple and purely infinite. In particular,  $\mathcal{M}_q$  contains any ideal in  $\mathcal{E}_{n,m}^q$ , see Corollary 7.

Let  $\mathcal{J} \subset \mathcal{E}_{n,m}^q$  be an ideal and  $\pi$  be a representation of  $\mathcal{E}_{n,m}^q$  such that ker  $\pi = \mathcal{J}$ . Notice that the Fock component  $\pi_1$  in the Wold decomposition of  $\pi$  is zero. Thus, by Theorem 8,

$$
\pi = \pi_2 \oplus \pi_3 \oplus \pi_4, \tag{11}
$$

and  $\mathcal{J} = \ker \pi = \ker \pi_2 \cap \ker \pi_3 \cap \ker \pi_4$ . Let us describe these kernels. Suppose that the component  $\pi_2$  is non-zero. Since  $\pi_2(1 - Q) = 0$  and  $\pi_2(1 - P) \neq 0$ , we have

$$
\mathfrak{I}_1^q \subseteq \ker \pi_2 \subsetneq \mathcal{M}_q,
$$

implying ker  $\pi_2 = \mathcal{I}_1^q$  $\frac{q}{1}$ . Using the same arguments, one can deduce that if the component  $\pi_3$  is non-zero, then ker  $\pi_3 = \mathcal{I}_2^q$  $_2^q$ , and if  $\pi_4$  is non-zero, then ker  $\pi_4 = \mathcal{M}_q$ .

Finally, if in (11)  $\pi_2$  and  $\pi_3$  are non-zero then  $\mathcal{J} = \ker \pi = \mathcal{J}_q$ . If either  $\pi_2 \neq 0$  and  $\pi_3 = 0$  or  $\pi_3 \neq 0$  and  $\pi_2 = 0$ , then either  $\mathcal{J} = \mathcal{J}_1^q$  $_1^q$  or  $\mathcal{J} = \mathcal{J}_2^q$  $\frac{q}{2}$ . In the case  $\pi_2 = 0$  and  $\pi_3 = 0$  one has  $\mathcal{J} = \ker \pi_4 = \mathcal{M}_q$ . □

**Corollary 5.** All ideals in  $\mathcal{E}_{n,m}^q$  are essential. The ideal  $\mathcal{I}^q$  is the unique minimal ideal.

In particular, the extension

$$
0 \to \mathcal{I}_q \to \mathcal{M}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \to 0
$$

is essential. Indeed, the ideal  $\mathbb{K} = \mathcal{I}_q \subset \mathcal{E}_{n,m}^q$  is the unique minimal ideal. Since an ideal of an ideal in a  $C^*$ -algebra is an ideal in the whole algebra,  $\mathcal{I}_q$ is the unique minimal ideal in  $\mathcal{M}_q$ , thus it is essential in  $\mathcal{M}_q$ .

The following proposition is a corollary of Voiculescu's Theorem, see Theorem 15.12.3 of [3].

**Proposition 16.** Let  $E_1, E_2$  be two essential extensions of a nuclear  $C^*$ algebra A by K. If  $[E_1] = [E_2] \in \text{Ext}(A, K)$  then  $E_1 \simeq E_2$ .

**Theorem 11.** For any  $q \in \mathbb{C}$ ,  $|q| = 1$ , one has  $\mathcal{M}_q \simeq \mathcal{M}_1$ .

*Proof.* By Theorem 9,  $[\mathcal{I}_1^q]$  $\mathbb{E}[\mathbb{E}(\mathbb{O}_m \otimes \mathbb{K}, \mathbb{K}), \text{ and } [\mathbb{I}_2^q]$  $\mathcal{L}_2^q$   $\in$  Ext $(\mathcal{O}_n \otimes \mathbb{K}, \mathbb{K})$ do not depend on q. By Lemma 8,  $[\mathcal{M}_q]$  does not depend on q. Thus by Corollary 5 and Proposition 16,  $\mathcal{M}_q \simeq \mathcal{M}_1$ .  $\Box$ 

### 3.4 Simplicity and pure infiniteness of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$

The next step is to show that the quotient  $\mathcal{O}_n \otimes_q \mathcal{O}_m = \mathcal{E}_{n,m}^q/\mathcal{M}_q$ , being nuclear, is also simple and purely infinite.

It is easy to see that

$$
\mathcal{M}_q = c.l.s.\{s_{\mu_1}t_{\nu_1}(\mathbf{1} - P)^{\varepsilon_1}(\mathbf{1} - Q)^{\varepsilon_2}t_{\nu_2}^*s_{\mu_2}^*\},
$$

where  $\mu_j \in \Lambda_n$ ,  $\nu_j \in \Lambda_m$ ,  $j = 1, 2$ , and  $\varepsilon_j \in \{0, 1\}$ ,  $\varepsilon_1 + \varepsilon_2 \neq 0$ .

We denote the generators of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  in the same way as generators of  $\mathcal{E}_{n,m}^q$ . Notice, that for any  $k \in \mathbb{N}$ , the following relations hold in  $\mathcal{O}_n \otimes_q \mathcal{O}_m$ 

$$
\sum_{\lambda\in\Lambda_n, \, |\lambda|=k} s_\lambda s_\lambda^*=1, \quad \sum_{\nu\in\Lambda_m, \, |\nu|=k} t_\nu t_\nu^*=1,
$$

and

$$
\mathcal{O}_n \otimes_q \mathcal{O}_m = c.l.s.\{s_{\mu_1}s_{\mu_2}^*t_{\nu_1}t_{\nu_2}^*, \ \mu_i \in \Lambda_n, \ \nu_j \in \Lambda_m\}.
$$

Consider the action  $\alpha$  of  $\mathbb{T}^2$  on  $\mathcal{O}_n \otimes_q \mathcal{O}_m$ ,

$$
\alpha_{\varphi_1,\varphi_2}(s_j) = e^{2\pi i \varphi_1} s_j, \quad \alpha_{\varphi_1,\varphi_2}(t_r) = e^{2\pi i \varphi_2} t_r, \quad j = \overline{1,n}, \ r = \overline{1,m}.
$$

Construct the corresponding faithful conditional expectation  $E_{\alpha}$ , and denote by  $A_q$  the fixed point  $C^*$ -algebra of  $(\mathcal{O}_n \otimes_q \mathcal{O}_m)^\alpha$ , see Section 3.1.1. Similarily to the case of  $\mathcal{E}_{n,m}^q$ , one has

$$
E_{\alpha}(s_{\mu_1}s_{\mu_2}^*t_{\nu_1}t_{\nu_2}^*) = 0, \quad \text{if either } |\mu_1| \neq |\mu_2| \text{ or } |\nu_1| \neq |\nu_2|,
$$
  

$$
E_{\alpha}(s_{\mu_1}s_{\mu_2}^*t_{\nu_1}t_{\nu_2}^*) = s_{\mu_1}s_{\mu_2}^*t_{\nu_1}t_{\nu_2}^* \quad \text{if } |\mu_1| = |\mu_2| \text{ and } |\nu_1| = |\nu_2|.
$$

Lemma 9. If  $\nu_1, \nu_2 \in \Lambda_m$ , then

$$
s_j t_{\nu_1} t_{\nu_2}^* = \overline{q}^{|\nu_1| - |\nu_2|} t_{\nu_1} t_{\nu_2}^* s_j, \quad s_j^* t_{\nu_1} t_{\nu_2}^* = q^{|\nu_1| - |\nu_2|} t_{\nu_1} t_{\nu_2}^* s_j^*, \quad j = \overline{1, n}.
$$

If  $\mu_1, \mu_2 \in \Lambda_n$ , then

$$
t_i s_{\mu_1} s_{\mu_2}^* = q^{|\mu_1| - |\mu_2|} s_{\mu_1} s_{\mu_2}^* t_i, \quad t_i^* s_{\mu_1} s_{\mu_2}^* = \overline{q}^{|\mu_1| - |\mu_2|} s_{\mu_1} s_{\mu_2}^* t_i^*, \quad i = \overline{1, m}.
$$

As in the proof of Proposition 10, denote

$$
\mathcal{A}_1^0 = \mathbb{C}, \quad \mathcal{A}_1^k = \text{span}\{s_{\mu_1}s_{\mu_2}^*, \ |\mu_1| = |\mu_2| = k, \ \mu_i \in \Lambda_n\}, \quad k \in \mathbb{N},
$$
  

$$
\mathcal{A}_2^0 = \mathbb{C}, \quad \mathcal{A}_2^k = \text{span}\{t_{\nu_1}t_{\nu_2}^*, \ |\nu_1| = |\nu_2| = k, \ \nu_i \in \Lambda_m\}, \quad k \in \mathbb{N}.
$$

Recall also that  $\mathcal{A}_1^k \simeq M_{n^k}(\mathbb{C})$  and  $\mathcal{A}_2^k \simeq M_{m^k}(\mathbb{C})$ , see [10].

Put  $\mathcal{A}_q^0 := \mathbb{C}$ ,

$$
\mathcal{A}_q^k:=\sum_{k_1+k_2=k}\mathcal{A}_1^{k_1}\cdot\mathcal{A}_2^{k_2},
$$

and set

$$
\mathcal{A}_q = \overline{\bigcup_{k \in \mathbb{Z}_+} \mathcal{A}_q^k}.
$$

By Lemma 9, for any  $x \in \mathcal{A}_1^k$  and  $y \in \mathcal{A}_2^l$  one has  $xy = yx$ . Thus  $\mathcal{A}_q$  is an AF-subalgebra in  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  and  $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$  for any  $q_1, q_2 \in \mathbb{C}, |q_1| = |q_2| = 1$ .

To prove pure infiniteness of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  we essentially follow Chapter V.4 of [13].

Denote by  $Fin_q^k$  the span of monomials  $s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^*$  such that

$$
\max\{|\mu_1|, |\mu_2|\} + \max\{|\nu_1|, |\nu_2|\} \le k.
$$

**Proposition 17.** For any  $k \in \mathbb{N}$  there exists an isometry  $w_k \in \mathcal{O}_n \otimes_q \mathcal{O}_m$ such that

$$
E_{\alpha}(x) = w_k^* x w_k, \quad \text{for any } x \in Fin_q^k,
$$

and  $w_k^* y w_k = y$  for any  $y \in A_q^k$ .

*Proof.* Let  $s_{\gamma} = s_1^{2k} s_2$  and  $t_{\gamma} = t_1^{2k} t_2$ . Consider the isometries

$$
w_{k,1} = \sum_{|\delta|=k, \delta \in \Lambda_n} s_{\delta} s_{\gamma} s_{\delta}^*,
$$

and

$$
w_{k,2} = \sum_{|\lambda|=k, \lambda \in \Lambda_m} t_{\lambda} t_{\gamma} t_{\lambda}^*.
$$

Then, see Lemma V.4.5 of [13],

$$
w_{k,1}^* s_{\mu_1} s_{\mu_2}^* w_{k,1} = 0
$$
, if  $|\mu_1| \neq |\mu_2|, |\mu_i| \leq k, \mu_i \in \Lambda_n$ ,

and

$$
w_{k,1}^* s_{\mu_1} s_{\mu_2}^* w_{k,1} = s_{\mu_1} s_{\mu_2}^*, \quad \text{if } |\mu_1| = |\mu_2|, \ |\mu_i| \le k, \mu_i \in \Lambda_n.
$$

Analogously,

$$
w_{k,2}^* t_{\nu_1} t_{\nu_2}^* w_{k,2} = 0
$$
, if  $|\nu_1| \neq |\nu_2|, |\nu_i| \leq k, \nu_i \in \Lambda_m$ ,

and

$$
w_{k,2}^* t_{\nu_1} t_{\nu_2}^* w_{k,2} = t_{\nu_1} t_{\nu_2}^*,
$$
 if  $|\nu_1| = |\nu_2|, |\nu_i| \leq k, \nu_i \in \Lambda_m.$ 

By Lemma 9 we get

$$
w_{k,1}t_{\nu_1}t_{\nu_2}^* = \overline{q}^{(|\nu_1|-|\nu_2|)(2k+1)}t_{\nu_1}t_{\nu_2}^*w_{k,1},
$$
  
\n
$$
w_{k,1}^*t_{\nu_1}t_{\nu_2}^* = q^{(|\nu_1|-|\nu_2|)(2k+1)}t_{\nu_1}t_{\nu_2}^*w_{k,1}^*,
$$
  
\n
$$
w_{k,2}s_{\mu_1}s_{\mu_2}^* = q^{(|\mu_1|-|\mu_2|)(2k+1)}s_{\mu_1}s_{\mu_2}^*w_{k,2},
$$
  
\n
$$
w_{k,2}^*s_{\mu_1}s_{\mu_2}^* = \overline{q}^{(|\mu_1|-|\mu_2|)(2k+1)}s_{\mu_1}s_{\mu_2}^*w_{k,2}^*.
$$

Then

$$
w_{k,2}w_{k,1} = q^{(2k+1)^2}w_{k,1}w_{k,2}, \quad w_{k,2}^*w_{k,1} = \overline{q}^{(2k+1)^2}w_{k,1}w_{k,2}^*
$$

.

Let  $w_k = w_{k,2}w_{k,1}$ . Evidently  $w_k$  is an isometry. Then for any  $|\mu_i| \leq k$  and  $|\nu_i| \leq k$  one has

$$
w_k^* s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^* w_k = q^{((|v_1| - |v_2|) - (|\mu_1| - |\mu_2|))(2k+1)} w_{k,1}^* s_{\mu_1} s_{\mu_2}^* w_{k,1} w_{k,2}^* t_{\nu_1} t_{\nu_2}^* w_{k,2},
$$

implying that for any  $|\mu_i| \leq k$ , and  $|\nu_i| \leq k$ ,

$$
w_k^* s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^* w_k = 0
$$
, if  $|\mu_1| \neq |\mu_2|$  or  $|\nu_1| \neq |\nu_2|$ ,

and

$$
w_k^* s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^* w_k = s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^*,
$$
 if  $|\mu_1| = |\mu_2|$  and  $|\nu_1| = |\nu_2|$ .

Hence, for any  $x \in Fin_q^k$  one has  $w_k^* xw_k = E_\alpha(x)$  and  $w_k^* yw_k = y$  for  $y \in$  $\mathcal{A}_q^k.$ 

Remark 9. Since  $A_q^k$  is finite-dimensional, it is a direct sum of full matrix algebras, where matrix units are represented by  $s_{\mu_1} t_{\nu_1} t_{\nu_2}^* s_{\mu_2}^*$ ,  $|\mu_1| = |\mu_2|$ ,  $|\nu_1| = |\nu_2|$  and  $|\mu_1| + |\nu_1| = k$ . In particular, any minimal projection in  $A_q^k$  is unitary equivalent in  $A_q^k$  to a "matrix-unit projection" having form  $s_{\mu_1}t_{\nu_1}t_{\nu_1}^*s_{\mu_1}^*$  with  $|\mu_1| + |\nu_1| = k$ . So any minimal projection in  $\mathcal{A}_q^k$  has the form  $u u^*$  for some isometry  $u \in \mathcal{E}_{n,m}^q$ .

The following statement is the main result of this Subsection.

**Theorem 12.** For any non-zero  $x \in \mathcal{O}_n \otimes_q \mathcal{O}_m$  with  $|q| = 1$ , there exist  $a, b \in \mathcal{O}_n \otimes_{\alpha} \mathcal{O}_m$  such that  $axb = 1$ .

Proof. The proof repeats the arguments of the proof of Theorem V.4.6 in [13]. We present it here for the reader's convenience.

Let  $\mathcal{O}_n^q \otimes \mathcal{O}_m^q \ni x \neq 0$ . Then  $x^*x > 0$  and  $E_\alpha(x^*x) > 0$ . After normalisation of x we can suppose that  $||E_{\alpha}(x^*x)|| = 1$ . Find  $k \in \mathbb{N}$  and  $y = y^* \in Fin_q^k$ such that  $||x^*x - y|| < \frac{1}{4}$  $\frac{1}{4}$ . Since  $E_{\alpha}$  is a contraction, one has

$$
||E_{\alpha}(x^*x) - E_{\alpha}(y)|| < \frac{1}{4}
$$
 and  $||E_{\alpha}(y)|| > \frac{3}{4}$ .

Further,  $w_k^* y w_k = E_\alpha(y)$ . Since  $E_\alpha(y) = E_\alpha(y)^* \in \mathcal{A}_q^k$ , by the spectral theorem for a self-adjoint operator on a finite-dimensional Hilbert space, there exists a minimal projection  $p \in \mathcal{A}_q^k$ , such that

$$
pE_{\alpha}(y) = E_{\alpha}(y)p = ||E_{\alpha}(y)|| \cdot p.
$$

As noted above,  $p = u u^*$  for an isometry  $u \in \mathcal{O}_n \otimes_q \mathcal{O}_m$ . Put

$$
z = ||E_{\alpha}(y)||^{-\frac{1}{2}}u^*pw_k^*.
$$

Then  $||z|| < \frac{2}{\sqrt{2}}$  $\frac{1}{3}$ , and

$$
zyz^* = ||E_{\alpha}(y)||^{-1}u^*pw_k^*yw_kpu = ||E_{\alpha}(y)||^{-1}u^*pE_{\alpha}(y)pu
$$
  
=  $||E_{\alpha}(y)||^{-1}||E_{\alpha}(y)||u^*pu = u^*uu^*u = 1.$ 

Then

$$
||1 - zx^*xz^*|| = ||zyz^* - zx^*xz^*|| \le ||z||^2 \cdot ||y - x^*x|| < \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3}.
$$

Hence  $zx^*xz^*$  is invertible in  $\mathcal{O}_n \otimes_q \mathcal{O}_m$ . Let  $c \in \mathcal{O}_n \otimes_q \mathcal{O}_m$  satisfies  $czx^*xz^* = \mathbf{1}$ , then for  $a = czx^*$  and  $b = z^*$  one has  $axb = 1$ .

The following corollary is immediate.

**Theorem 13.** The C<sup>\*</sup>-algebra  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  is nuclear, simple and purely infinite.

Given  $q = e^{2\pi i \varphi_0}$ , consider

$$
\Theta_q = \begin{pmatrix} 0 & \frac{\varphi_0}{2} \\ -\frac{\varphi_0}{2} & 0 \end{pmatrix},\tag{12}
$$

and construct the Rieffel deformation  $(0_n \otimes 0_m)_{\Theta_q}$ .

Corollary 6. The following isomorphism holds:

$$
\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq (\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}.
$$

*Proof.* As in the proof of Theorem 7, the universal property of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$ implies that the correspondence

$$
s_j \mapsto s_j \otimes \mathbf{1}, \quad t_r \mapsto \mathbf{1} \otimes t_r, \quad j = \overline{1, n}, \ r = \overline{1, m},
$$

extends to a surjective homomorphism  $\Phi: \mathcal{O}_n \otimes_q \mathcal{O}_m \to (\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}$ . Finally, the simplicity of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  implies that  $\Phi$  is an isomorphism.  $\Box$ 

Remark 10. The isomorphism established in Corollary 6 is equivariant with respect to the introduced above actions of  $\mathbb{T}^2$  on  $\mathcal{O} \otimes_q \mathcal{O}_m$  and  $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}$ respectively.

The simplicity of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  implies that  $\mathcal{M}_q \subset \mathcal{E}_{n,m}^q$  is the largest ideal.

### **Corollary 7.** The ideal  $\mathcal{M}_q \subset \mathcal{E}_{n,m}^q$  is the unique largest ideal.

*Proof.* Let  $\eta: \mathcal{E}_{n,m}^q \to \mathcal{O}_n \otimes_q \mathcal{O}_m$  be the quotient homomorphism. Suppose that  $\mathcal{J} \subset \mathcal{E}_{n,m}^q$  is a two-sided \*-ideal. Due to the simplicity of  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  we have that either  $\eta(\mathcal{J}) = \{0\}$  and  $\mathcal{J} \subset \mathcal{M}_q$ , or  $\eta(\mathcal{J}) = \mathcal{O}_n \otimes_q \mathcal{O}_m$ . In the latter case,  $1 + x \in \mathcal{J}$  for a certain  $x \in \mathcal{M}_q$ . For any  $0 < \varepsilon < 1$ , choose  $N_{\varepsilon} \in \mathbb{N}$ , such that for

x<sup>ε</sup> = X ε1,ε2∈{0,1}, ε1+ε26=0 X µ1,µ2∈Λn, |µ<sup>j</sup> |≤N<sup>ε</sup> X ν1,ν2∈Λm, |νj|≤N<sup>ε</sup> Ψ (ε1,ε2) µ1,µ2ν1ν<sup>2</sup> s<sup>µ</sup><sup>1</sup> tν1 (1 − P) ε1 (1 − Q) ε2 t ∗ ν2 sµ ∗ 2 ∈ M<sup>q</sup>

one has  $||x - x_{\varepsilon}|| < \varepsilon$ . Notice that for any  $\mu \in \Lambda_n$ ,  $\nu \in \Lambda_m$  with  $|\mu|, |\nu| > N_{\varepsilon}$ one has  $s^*_{\mu} t^*_{\nu} x_{\varepsilon} = 0$ .

Fix  $\mu \in \Lambda_n$  and  $\nu \in \Lambda_m$ ,  $|\mu| = |\nu| > N_{\varepsilon}$ , then

$$
y_{\varepsilon}=s_{\mu}^*t_{\nu}^*(1-x)t_{\nu}s_{\mu}=1-s_{\mu}^*t_{\nu}^*(x-x_{\varepsilon})t_{\nu}s_{\mu}\in\mathcal{J}.
$$

Thus  $||s_{\mu}^* t_{\nu}^*(x - x_{\varepsilon}) t_{\nu} s_{\mu}|| < \varepsilon$  implies that  $y_{\varepsilon}$  is invertible, so  $1 \in \mathcal{J}$ .

 $\Box$ 

## 3.5 The isomorphism  $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$

In this section we prove the main result of Section 3. Namely, we show that for each  $q, |q| = 1$ ,

$$
\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m.
$$

In [17], the authors have shown that for every  $C^*$ -algebra A with an action  $\alpha$  of R, there exists a KK-isomorphism  $t_{\alpha} \in KK_1(A, A \rtimes_{\alpha} \mathbb{R})$ . This  $t_{\alpha}$  is a generalization of the Connes-Thom isomorphisms for K-theory. Below we will denote by  $\circ: KK(A, B) \times KK(B, C) \rightarrow KK(A, C)$  the Kasparov product, and by  $\boxtimes : KK(A, B) \times KK(C, D) \rightarrow KK(A \otimes C, B \otimes D)$  the exterior tensor product. Given a homomorphism  $\phi: A \to B$ , put  $[\phi] \in KK(A, B)$  to be the induced KK-morphism. For more details see [3, 22].

We list some properties of  $t_{\alpha}$  that will be used below.

- 1. Inverse of  $t_{\alpha}$  is given by  $t_{\hat{\alpha}}$ , where  $\hat{\alpha}$  is the dual action.
- 2. If  $A = \mathbb{C}$  with the trivial action of  $\mathbb{R}$ , then the corresponding element

$$
t_1 \in KK_1(\mathbb{C}, C_0(\mathbb{R})) \simeq \mathbb{Z}
$$

is the generator of the group.

3. Let  $\phi: (A, \alpha) \to (B, \beta)$  be an equivariant homomorphism. Then the following diagram commutes in KK-theory

$$
A \xrightarrow{t_{\alpha}} A \rtimes_{\alpha} \mathbb{R}
$$
  

$$
\downarrow^{\phi} \qquad \qquad \downarrow^{\phi \rtimes \mathbb{R}}
$$
  

$$
B \xrightarrow{t_{\beta}} B \rtimes_{\beta} \mathbb{R}
$$

4. Let  $\beta$  be an action of R on B. For the action  $\gamma = id_A \otimes \beta$  on  $A \otimes B$ we have

$$
t_{\gamma} = \mathbf{1}_A \boxtimes t_{\beta}.
$$

We will need the classification result by Kirchberg and Philips:

**Theorem 14** ([31], Corollary 4.2.2). Let A and B be separable nuclear unital purely infinite simple  $C^*$ -algebras, and suppose that there exists an invertible element  $\eta \in KK(A, B)$ , such that  $[\iota_A] \circ \eta = [\iota_B]$ , where  $\iota_A : \mathbb{C} \to A$  is defined by  $\iota_A(1) = \mathbf{1}_A$ , and  $\iota_B : \mathbb{C} \to B$  is defined by  $\iota_B(1) = \mathbf{1}_B$ . Then A and B are isomorphic.

**Theorem 15.** The C<sup>\*</sup>-algebras  $\mathcal{O}_n \otimes_q \mathcal{O}_m$  and  $\mathcal{O}_n \otimes \mathcal{O}_m$  are isomorphic for any  $|q|=1$ .

*Proof.* Throughout the proof we will distinguish between the actions of  $\mathbb{T}^2$ on  $\mathcal{O}_n \otimes \mathcal{O}_m$  and on  $\mathcal{O}_n \otimes_q \mathcal{O}_m$ , denoting the latter by  $\alpha^q$ . Due to Theorem 13, the both algebras are separable nuclear unital simple and purely infinite.

Further, Corollary 6, Proposition 5, and Remark 10 yield the isomorphism

 $\Psi: (\mathbb{O}_n \otimes \mathbb{O}_m) \rtimes_\alpha \mathbb{R}^2 \to (\mathbb{O}_n \otimes_q \mathbb{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2.$ 

Decompose the crossed products as follows:

$$
(\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_{\alpha} \mathbb{R}^2 \simeq (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_{\alpha_1} \mathbb{R} \rtimes_{\alpha_2} \mathbb{R},
$$
  

$$
(\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2 \simeq (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha_1^q} \mathbb{R} \rtimes_{\alpha_2^q} \mathbb{R}.
$$

Define

$$
t_{\alpha} = t_{\alpha_1} \circ (\mathbf{1}_{C_0(\mathbb{R})} \boxtimes t_{\alpha_2}) \in KK(\mathcal{O}_n \otimes \mathcal{O}_m, (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_{\alpha} \mathbb{R}^2),
$$
  

$$
t_{\alpha^q} = t_{\alpha_1^q} \circ (\mathbf{1}_{C_0(\mathbb{R})} \boxtimes t_{\alpha_2^q}) \in KK(\mathcal{O}_n \otimes_q \mathcal{O}_m, (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2),
$$

Then

$$
\eta = t_{\alpha^q} \circ [\Psi] \circ t_{\alpha}^{-1} \in KK(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{O}_n \otimes \mathcal{O}_m)
$$

is a KK-isomorphism. The property  $[\iota_{\mathcal{O}_n \otimes_{\mathcal{O}} \mathcal{O}_m}] \circ \eta = [\iota_{\mathcal{O}_n \otimes \mathcal{O}_m}]$  follows from the commutativity of the following diagram



Remark 11. After our paper was submitted, we were informed by Prof. M. Weber that in unpublished part of his PhD thesis he studied a multiparameter twisted tensor product of Cuntz algebras and obtained independently the analog of Theorem 15.

#### 3.6 Computation of Ext for  $\mathcal{E}_n^q$  $\tilde{n},\!m$

Here we show that  $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) = 0$  if  $\gcd(n-1, m-1) = 1$ . We use the isomorphism  $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$ ,  $|q|=1$ .

Recall the notion of UCT property for KK-theory, see [3].

**Definition 5.** Suppose A and B are separable nuclear  $C^*$ -algebras. We say that a pair  $(A, B)$  satisfies the Universal Coefficient Theorem (UCT), if the following sequences are exact,  $j \in \mathbb{Z}_2$ ,

$$
0 \to \bigoplus_{i \in \mathbb{Z}_2} Ext^1_{\mathbb{Z}}(K_{i+1}(A), K_{i+j}(B)) \to KK_j(A, B) \to \bigoplus_{i \in \mathbb{Z}_2} Hom(K_i(A), K_{i+j}(B)) \to 0.
$$

We say that A satisfies UCT if  $(A, B)$  satisfies UCT for every B.

It is known that  $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$  satisfies UCT. The following statement is an easy consequence of the Kunneth formula.

**Theorem 16.** Let  $d = \gcd(n-1, m-1)$ . Then

$$
K_0(\mathcal{O}_n \otimes_q \mathcal{O}_m) \simeq \mathbb{Z}/d\mathbb{Z}, \quad K_1(\mathcal{O}_n \otimes_q \mathcal{O}_m) \simeq \mathbb{Z}/d\mathbb{Z},
$$

Proof. The Kunneth formula for K-theory, see [3] Theorem 23.1.3, gives the following short exact sequences,  $j \in \mathbb{Z}_2$ ,

$$
0 \to \bigoplus_{i \in \mathbb{Z}_2} K_i(\mathcal{O}_n) \otimes_{\mathbb{Z}} K_{i+j}(\mathcal{O}_m) \to K_j(\mathcal{O}_n \otimes \mathcal{O}_m) \to \bigoplus_{i \in \mathbb{Z}_2} Tor_1^{\mathbb{Z}}(K_i(\mathcal{O}_n), K_{i+j+1}(\mathcal{O}_m)) \to 0
$$

It is a well known fact in homological algebra, see [15], that for an abelian group  $A$ 

$$
Tor_1^{\mathbb{Z}}(A,\mathbb{Z}/d\mathbb{Z}) \simeq Ann_A(d) = \{a \in A \mid da = 0\}.
$$

In particular,

$$
Tor_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/\gcd(n, m)\mathbb{Z}.
$$

Recall that, see [11],

$$
K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}, \quad K_1(\mathcal{O}_n) = 0.
$$

Hence, for  $\mathcal{O}_n \otimes \mathcal{O}_m$ , one has the following short exact sequences:

$$
0 \to \mathbb{Z}/(n-1)\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(m-1)\mathbb{Z} \to K_0(\mathbb{O}_n \otimes \mathbb{O}_m) \to 0 \to 0,
$$
  

$$
0 \to 0 \to K_1(\mathbb{O}_n \otimes \mathbb{O}_m) \to \mathbb{Z}/d\mathbb{Z} \to 0.
$$

Next step is to compute the K-theory of  $\mathcal{M}_q$ .

Theorem 17. Let  $d = \gcd(n-1, m-1)$ . Then

$$
K_0(\mathcal{M}_q)\simeq \mathbb{Z}/d\mathbb{Z}\oplus \mathbb{Z},\ K_1(\mathcal{M}_q)\simeq 0.
$$

Proof. By Theorem 7, Proposition 7, and [11], Proposition 3.9,

$$
K_0(\mathcal{E}_{n,m}^q) = K_0((\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}) = K_0(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0) = \mathbb{Z},
$$
  

$$
K_1(\mathcal{E}_{n,m}^q) = K_1((\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}) = K_1(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0) = 0.
$$

Applying the 6-term exact sequence for

$$
0 \to \mathbb{K} \to \mathcal{M}_q \to \mathcal{O}_n \otimes \mathbb{K} \oplus \mathcal{O}_m \otimes \mathbb{K} \to 0,
$$

we get

$$
\mathbb{Z} \longrightarrow K_0(\mathcal{M}_q) \longrightarrow \mathbb{Z}/(n-1)\mathbb{Z} \oplus \mathbb{Z}/(m-1)\mathbb{Z}
$$
\n
$$
\uparrow \qquad \qquad \downarrow
$$
\n
$$
0 \longleftarrow K_1(\mathcal{M}_q) \longleftarrow 0
$$

Then  $K_1(\mathcal{M}_q) = 0$ , and elementary properties of finitely generated abelian groups imply that

$$
K_0(\mathcal{M}_q)=\mathbb{Z}\oplus \textsf{Tors},
$$

where Tors is a direct sum of finite cyclic groups.

Further, the following exact sequence

$$
0\longrightarrow \mathcal{M}_q\longrightarrow \mathcal{E}_{n,m}^q\rightarrow \mathcal{O}_n\otimes_q \mathcal{O}_m\longrightarrow 0
$$

gives

$$
K_0(\mathcal{M}_q) \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}
$$
  
\n
$$
i \uparrow \qquad \qquad \downarrow
$$
  
\n
$$
\mathbb{Z}/d\mathbb{Z} \longleftarrow 0 \longleftarrow 0
$$

The map  $p: K_0(\mathcal{M}_q) \simeq \mathbb{Z} \oplus \mathsf{Tors} \to \mathbb{Z}$  has form  $p = (p_1, p_2)$ , where

$$
p_1\colon \mathbb{Z}\to \mathbb{Z},\quad p_2\colon \textsf{Tors}\to \mathbb{Z}.
$$

Evidently,  $p_2 = 0$ , and  $p \neq 0$  implies that ker  $p_1 = \{0\}$ . Thus,

$$
\ker p = \text{Tors} = \text{Im}(i) \simeq \mathbb{Z}/d\mathbb{Z}.
$$

**Theorem 18.** Let  $d = \gcd(n-1, m-1) = 1$ . Then  $\text{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) = 0$ .

*Proof.* Recall that for nuclear  $C^*$ -algebras  $\text{Ext}(A, B) \simeq KK_1(A, B)$ .

We use the sequence from Definition 5 for  $j = 1, A = \mathcal{O}_n \otimes_q \mathcal{O}_m$  and  $B = \mathcal{M}_q$ :

$$
0 \to \bigoplus_{i \in \mathbb{Z}_2} Ext^1_{\mathbb{Z}}(K_i(A), K_i(B)) \to KK_1(A, B) \to \bigoplus_{i \in \mathbb{Z}_2} Hom(K_i(A), K_{i+1}(B)).
$$

Since  $K_0(A) = K_1(A) = \mathbb{Z}/d\mathbb{Z}$  and  $K_0(B) = \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ ,  $K_1(B) = 0$ , one has

$$
Hom(K_0(A), K_1(B)) = 0, \quad Hom(K_1(A), K_0(B)) = \mathbb{Z}/d\mathbb{Z},
$$

and, see  $[15]$ ,

$$
Ext^1_{\mathbb{Z}}(K_0(A), K_0(B)) = \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}, \quad Ext^1_{\mathbb{Z}}(K_1(A), K_1(B)) = 0.
$$

Hence the following sequence is exact

$$
0 \to \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \to KK_1(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) \to \mathbb{Z}/d\mathbb{Z} \to 0.
$$

 $\Box$ 

By Theorem 18, for the case of  $gcd(n-1, m-1) = 1$  one can immediately deduce that extension classes of

$$
0\to \mathcal{M}_q\to \mathcal{E}^q_{n,m}\to \mathcal{O}_n\otimes_q \mathcal{O}_m\to 0,
$$

and

$$
0 \to \mathcal{M}_1 \to \mathcal{E}^1_{n,m} \to \mathcal{O}_n \otimes \mathcal{O}_m \to 0,
$$

coincide in  $\mathsf{Ext}(\mathcal{O}_n \otimes \mathcal{O}_m, \mathcal{M}_1)$  and are trivial. These extensions are essential, however in general case one does not have an immediate generalization of Proposition 16. Thus the study of the problem whether  $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^1$  would require further investigations, see [12, 16].

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## References

- [1] Andersson, A.: Index pairings for  $\mathbb{R}^n$ -actions and Rieffel deformations (2017). URL http://arxiv.org/abs/1406.4078v5; http://arxiv.org/pdf/1406.4078v5
- [2] Biedenharn, L.C.: The quantum group  $SU<sub>q</sub>(2)$  and a q-analogue of the boson operators. J. Phys. A  $22(18)$ , L873–L878 (1989). URL http://stacks.iop.org/0305-4470/22/L873
- [3] Blackadar, B.: K-theory for operator algebras, Mathematical Sciences Research Institute Publications, vol. 5. Springer-Verlag, New York (1986). DOI 10.1007/978-1-4613-9572-0. URL https://doi.org/10.1007/978-1-4613-9572-0
- [4] Bo˙zejko, M., Lytvynov, E., Wysocza´nski, J.: Fock representations of Q-deformed commutation relations. J. Math. Phys.

58(7), 073501, 19 (2017). DOI 10.1063/1.4991671. URL https://doi.org/10.1063/1.4991671

- [5] Bożejko, M., Speicher, R.: An example of a generalized Brownian motion. Comm. Math. Phys. 137(3), 519–531 (1991). URL http://projecteuclid.org/euclid.cmp/1104202738
- [6] Bo˙zejko, M., Speicher, R.: Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. Math. Ann. 300(1), 97–120 (1994). DOI 10.1007/BF01450478. URL https://doi.org/10.1007/BF01450478
- [7] Brown, N.P., Ozawa, N.: C<sup>\*</sup>-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, vol. 88. American Mathematical Society, Providence, RI (2008). DOI 10.1090/gsm/088. URL https://doi.org/10.1090/gsm/088
- [8] Buchholz, D., Lechner, G., Summers, S.J.: Warped convolutions, Rieffel deformations and the construction of quantum field theories. Comm. Math. Phys. 304(1), 95–123 (2011). DOI 10.1007/s00220-010-1137-1. URL https://doi.org/10.1007/s00220-010-1137-1
- [9] Coburn, L.A.: The  $C^*$ -algebra generated by an isometry. Bull. Amer. Math. Soc. 73, 722–726 (1967). DOI 10.1090/S0002-9904-1967-11845-7. URL https://doi.org/10.1090/S0002-9904-1967-11845-7
- $[10]$  Cuntz, J.: Simple  $C^*$ -algebras generated by isometries. Comm. Math. Phys. 57(2), 173–185 (1977). URL http://projecteuclid.org/euclid.cmp/1103901288
- [11] Cuntz, J.: K-theory for certain C ∗ Ann. of Math. (2) 113(1), 181–197 (1981). DOI 10.2307/1971137. URL https://doi.org/10.2307/1971137
- [12] Dadarlat, M., Meyer, R.: E-theory for -algebras over topological spaces. J. Funct. Anal. 263(1), 216– 247 (2012). DOI 10.1016/j.jfa.2012.03.022. URL https://doi.org/10.1016/j.jfa.2012.03.022
- [13] Davidson, K.R.: C<sup>\*</sup>-algebras by example, Fields Institute Monographs, vol. 6. American Mathematical Society, Providence, RI (1996). DOI 10.1090/fim/006. URL https://doi.org/10.1090/fim/006
- [14] Dykema, K., Nica, A.: On the Fock representation of the q-commutation relations. J. Reine Angew. Math. 440, 201–212 (1993)
- [15] Eilenberg, S., MacLane, S.: Group extensions and homology. Ann. of Math. (2) 43, 757–831 (1942). DOI 10.2307/1968966. URL https://doi.org/10.2307/1968966
- [16] Eilers, S., Restorff, G., Ruiz, E.: The ordered K-theory of a full extension. Canad. J. Math. 66(3), 596–625 (2014). DOI 10.4153/CJM-2013- 015-7. URL https://doi.org/10.4153/CJM-2013-015-7
- [17] Fack, T., Skandalis, G.: Connes' analogue of the Thom isomorphism for the Kasparov groups. Invent. Math.  $64(1)$ ,  $7-14$  (1981). DOI 10.1007/BF01393931. URL https://doi.org/10.1007/BF01393931
- [18] Fivel, D.I.: Errata: "Interpolation between Fermi and Bose statistics using generalized commutators" [Phys. Rev. Lett. 65 (1990), no. 27, 3361–3364; MR1084327 (91m:81108)]. Phys. Rev. Lett. 69(13), 2020 (1992). DOI 10.1103/PhysRevLett.69.2020. URL https://doi.org/10.1103/PhysRevLett.69.2020
- [19] Giselsson, O.: The Universal C<sup>\*</sup>-Algebra of the Quantum Matrix Ball and its Irreducible ∗-Representations (2018). URL http://arxiv.org/abs/1801.10608v2; http://arxiv.org/pdf/1801.10608v2
- [20] Greenberg, O.W.: Particles with small violations of Fermi or Bose statistics. Phys. Rev. D  $(3)$  43(12), 4111– 4120 (1991). DOI 10.1103/PhysRevD.43.4111. URL https://doi.org/10.1103/PhysRevD.43.4111
- [21] Higson, N., Roe, J.: Analytic K-homology. Oxford Mathematical Monographs. Oxford University Press, Oxford (2000). Oxford Science Publications
- [22] Jensen, K.K., Thomsen, K.: Elements of KK-theory. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA (1991).

DOI 10.1007/978-1-4612-0449-7. URL https://doi.org/10.1007/978-1- 4612-0449-7

- [23] de Jeu, M., Pinto, P.R.: The structure of doubly non-commuting isometries (2018). URL https://arxiv.org/abs/1801.09716
- [24] Jørgensen, P.E.T., Proskurin, D.P., Samoĭlenko, Y.S.: The kernel of Fock representations of Wick algebras with braided operator of coefficients. Pacific J. Math. 198(1), 109–122 (2001). DOI 10.2140/pjm.2001.198.109. URL https://doi.org/10.2140/pjm.2001.198.109
- [25] Jørgensen, P.E.T., Proskurin, D.P., Samoĭlenko, Y.S.: ∗ algebras generated by pairs of q-commuting isometries. J. Phys. A 38(12), 2669–2680 (2005). DOI 10.1088/0305-4470/38/12/009. URL https://doi.org/10.1088/0305-4470/38/12/009
- [26] Jorgensen, P.E.T., Schmitt, L.M., Werner, R.F.: q-canonical commutation relations and stability of the Cuntz algebra. Pacific J. Math. 165(1), 131–151 (1994). URL http://projecteuclid.org/euclid.pjm/1102621916
- [27] Jorgensen, P.E.T., Schmitt, L.M., Werner, R.F.: Positive representations of general commutation relations allowing Wick ordering. J. Funct. Anal. 134(1), 33–99 (1995). DOI 10.1006/jfan.1995.1139. URL https://doi.org/10.1006/jfan.1995.1139
- [28] Kasprzak, P.: Rieffel deformation via crossed products. J. Funct. Anal. 257(5), 1288–1332 (2009). DOI 10.1016/j.jfa.2009.05.013. URL https://doi.org/10.1016/j.jfa.2009.05.013
- [29] Kennedy, M., Nica, A.: Exactness of the Fock space representation of the q-commutation relations. Comm. Math. Phys. 308(1), 115–132 (2011). DOI 10.1007/s00220-011-1323-9. URL https://doi.org/10.1007/s00220-011-1323-9
- [30] Kim, C.S., Proskurin, D.P., Iksanov, A.M., Kabluchko, Z.A.: The generalized CCR: representations and enveloping  $C^*$ -algebra. Rev. Math. Phys. 15(4), 313–338 (2003). DOI 10.1142/S0129055X03001618. URL https://doi.org/10.1142/S0129055X03001618
- [31] Kirchberg, E.: The classification of purely infinite  $C^*$ -algebras using Kasparov's theory. Preprint (1994)
- [32] Klimek, S., Lesniewski, A.: A two-parameter quantum deformation of the unit disc. J. Funct. Anal. 115(1), 1–23 (1993). DOI 10.1006/jfan.1993.1078. URL https://doi.org/10.1006/jfan.1993.1078
- [33] Klimyk, A., Schmüdgen, K.: Quantum groups and their representations. Texts and Monographs in Physics. Springer-Verlag, Berlin (1997). DOI 10.1007/978-3-642-60896-4. URL https://doi.org/10.1007/978-3- 642-60896-4
- [34] Kuzmin, A., Pochekai, N.: Faithfulness of the Fock representation of the  $C^*$ -algebra generated by  $q_{ij}$ -commuting isometries. J. Operator Theory 80(1), 77–93 (2018)
- [35] Liguori, A., Mintchev, M.: Fock representations of quantum fields with generalized statistics. Comm. Math. Phys.  $169(3)$ ,  $635-652(1995)$ . URL http://projecteuclid.org/euclid.cmp/1104272858
- [36] Macfarlane, A.J.: On  $q$ -analogues of the quantum harmonic oscillator and the quantum group  $SU(2)_{q}$ . J. Phys. A  $22(21)$ , 4581-4588 (1989). URL http://stacks.iop.org/0305-4470/22/4581
- [37] Marcinek, W.: On commutation relations for quons. Rep. Math. Phys. 41(2), 155–172 (1998). DOI 10.1016/S0034-4877(98)80173-0. URL https://doi.org/10.1016/S0034-4877(98)80173-0
- [38] Meljanac, S., Perica, A.: Generalized quon statistics. Modern Phys. Lett. A 9(35), 3293–3299 (1994). DOI 10.1142/S0217732394003117. URL https://doi.org/10.1142/S0217732394003117
- [39] Phillips, N.C.: A classification theorem for nuclear purely infinite simple C ∗ -algebras. Doc. Math. 5, 49–114 (2000)
- [40] Proskurin, D.: Stability of a special class of  $q_{ij}$ -CCR and extensions of higher-dimensional noncommutative tori. Lett. Math. Phys. 52(2), 165–175 (2000). DOI 10.1023/A:1007668304707. URL https://doi.org/10.1023/A:1007668304707
- [41] Pusz, W., Woronowicz, S.L.: Twisted second quantization. Rep. Math. Phys. 27(2), 231–257 (1989). DOI 10.1016/0034-4877(89)90006-2. URL https://doi.org/10.1016/0034-4877(89)90006-2
- [42] Rieffel,  $M.A.:$  $C^*$ -algebras associated with irrational rotations. Pacific J. Math. 93(2), 415–429 (1981). URL http://projecteuclid.org/euclid.pjm/1102736269
- [43] Rieffel, M.A.: Deformation quantization for actions of  $\mathbb{R}^d$ . Mem. Amer. Math. Soc. 106(506), x+93 (1993). DOI 10.1090/memo/0506. URL https://doi.org/10.1090/memo/0506
- [44] Rørdam, M.: Stable C<sup>\*</sup>-algebras. In: Operator algebras and applications, Adv. Stud. Pure Math., vol. 38, pp. 177–199. Math. Soc. Japan, Tokyo (2004)
- [45] Sangha, A.: Cocycle Deformations of Operator Algebras and Noncommutative Geometry. Ph.D. thesis, University of Oslo (2013)
- [46] Vaksman, L.L.: Quantum bounded symmetric domains, Translations of Mathematical Monographs, vol. 238. American Mathematical Society, Providence, RI (2010). Translated from the Russian manuscript and with a foreword by Olga Bershtein and Sergey D. Sinelshchikov
- [47] Weber,  $M$ .: On  $C^*$ -algebras generated by isometries with twisted commutation relations. J. Funct. Anal. 264(8), 1975–2004 (2013). DOI 10.1016/j.jfa.2013.02.001. URL https://doi.org/10.1016/j.jfa.2013.02.001
- [48] Williams, D.P.: Crossed products of C<sup>\*</sup>-algebras, Mathematical Surveys and Monographs, vol. 134. American Mathematical Society, Providence, RI (2007). DOI 10.1090/surv/134. URL https://doi.org/10.1090/surv/134
- [49] Yakymiv, R.: On isometries satisfying twisted commutaion relations. In preparation
- [50] Zagier, D.: Realizability of a model in infinite statistics. Comm. Math. Phys.  $147(1)$ ,  $199-210$  (1992). URL http://projecteuclid.org/euclid.cmp/1104250533