

# HYPERRIGID GENERATORS IN $C^*$ -ALGEBRAS

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ABSTRACT. In this article, we show that, if  $S \in \mathcal{B}(H)$  is irreducible and essential unitary, then  $\{S, SS^*\}$  is a hyperrigid generator for the unital  $C^*$ -algebra  $\mathcal{T}$  generated by  $\{S, SS^*\}$ . We prove that, if  $T$  is an operator in  $\mathcal{B}(H)$  that generates an unital  $C^*$ -algebra  $\mathcal{A}$  then  $\{T, T^*T, TT^*\}$  is a hyperrigid generator for  $\mathcal{A}$ . As a corollary it follows that, if  $T \in \mathcal{B}(H)$  is normal then  $\{T, TT^*\}$  is hyperrigid generator for the unital  $C^*$ -algebra generated by  $T$  and if  $T \in \mathcal{B}(H)$  is unitary then  $\{T\}$  is hyperrigid generator for the  $C^*$ -algebra generated by  $T$ . We show that if  $V \in \mathcal{B}(H)$  is an isometry (not unitary) that generates the  $C^*$ -algebra  $\mathcal{A}$  then the minimal generating set  $\{V\}$  is not hyperrigid for  $\mathcal{A}$ .

## 1. INTRODUCTION

The classical theorems of Korovkin impressed several mathematicians since their discovery for the simplicity and the potential. Positive approximation process play a fundamental role in the approximation theory and it appears in a very natural way in several problems dealing with the approximation of continuous functions and qualitative properties such as monotonicity, convexity, shape preservation and so on.

Korovkin [12] made an assertion that, if a sequence of positive linear maps  $\phi_n : C[0, 1] \rightarrow C[0, 1]$ ,  $n = 1, 2, 3, \dots$ , has the property

$$\lim_{n \rightarrow \infty} \|\phi_n(f_k) - f_k\| = 0, \quad k = 0, 1, 2,$$

for the three functions  $f_0(x) = 1$ ,  $f_1(x) = x$ ,  $f_2(x) = x^2$  then

$$\lim_{n \rightarrow \infty} \|\phi_n(f) - f\| = 0, \quad \forall f \in C[0, 1].$$

The set  $\{1, x, x^2\}$  is called a *Korovkin set* or *test set*. Korovkin [12] showed that, the set  $\{1, x\}$  is not a Korovkin set. Therefore, the set  $\{1, x, x^2\}$  is a minimal set to satisfy the above assertion.

Korovkin's theorem generated considerable activity among researchers in approximation theory. The generalizations make essential use of the *Choquet boundary* in one way or another. Saskin [14] proved a remarkable theorem. Let  $G$  be a subset of  $C(X)$  that separates points of compact Hausdorff space  $X$  and contains constant function 1. Then  $G$  is a Korovkin set in  $C(X)$  if and only if the Choquet boundary  $\partial G$  of  $G$  is  $X$ . That is  $\partial G = X$ .

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Arveson [4] initiated the study of noncommutative approximation theory focusing on the question: How does one determine whether a set of generators of a  $C^*$ -algebra is *hypercyclic*? Arveson [4] introduced a noncommutative counterpart of Korovkin set as follows:

**Definition 1.1.** A finite or countably infinite set  $\mathcal{G}$  of generators of a  $C^*$ -algebra  $\mathcal{A}$  is said to be *hypercyclic* if for every faithful representation  $\mathcal{A} \subseteq \mathcal{B}(H)$  of  $\mathcal{A}$  on a Hilbert space  $H$  and every sequence of unital completely positive (UCP) maps  $\phi_n : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ ,  $n = 1, 2, \dots$ ,

$$\lim_{n \rightarrow \infty} \|\phi_n(g) - g\| = 0, \forall g \in \mathcal{G} \implies \lim_{n \rightarrow \infty} \|\phi_n(a) - a\| = 0, \forall a \in \mathcal{A}.$$

Note that, a set  $\mathcal{G}$  is hypercyclic if and only if  $\mathcal{G} \cup \mathcal{G}^*$  is hypercyclic if and only if the linear span of  $\mathcal{G}$  is hypercyclic. If  $\mathcal{A}$  is unital, then  $\mathcal{G}$  is hypercyclic if and only if  $\mathcal{G} \cup \{1\}$  is hypercyclic [15, Proposition 2.1].

The following characterization of hypercyclic operator systems due to Arveson [4] is more of a workable definition of hypercyclicity of operator systems.

**Theorem 1.2.** [4, Theorem 2.1] *Let  $\mathcal{S}$  be a separable operator system generating the  $C^*$ -algebra  $\mathcal{A} = C^*(\mathcal{S})$  then  $\mathcal{S}$  is hypercyclic if and only if every nondegenerate representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$  on a separable Hilbert space,  $\pi|_{\mathcal{S}}$  has the unique extension property in the sense that the only unital completely positive (UCP) map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  that satisfies  $\phi|_{\mathcal{S}} = \pi|_{\mathcal{S}}$  is  $\phi = \pi$  itself.*

The interesting examples of hypercyclic generators are obtained by a direct application of the above criterion. Arveson [4] established the noncommutative strengthening of a classical approximation-theoretic result of Korovkin.

**Theorem 1.3.** [4, Theorem 3.1] *Let  $X \in \mathcal{B}(H)$  be a self adjoint operator with atleast 3 points in its spectrum and let  $\mathcal{A}$  be the  $C^*$ -algebra generated by  $X$ . Then*

- (i)  $\mathcal{G} = \{X, X^2\}$  is a hypercyclic generator for  $\mathcal{A}$ , while
- (ii)  $\mathcal{G}_0 = \{X\}$  is not hypercyclic generator for  $\mathcal{A}$ .

**Theorem 1.4.** [4, Theorem 3.3] *Let  $V \in \mathcal{B}(H)$  be an isometry that generates a  $C^*$ -algebra  $\mathcal{A}$ . Then  $\mathcal{G} = \{V, VV^*\}$  is hypercyclic generator for  $\mathcal{A}$ .*

Arveson [4] essentially used the noncommutative Choquet boundary. He found the hypercyclic generator for compact operators  $\mathcal{K}(H)$ .

**Theorem 1.5.** [4, Theorem 8.1] *Let  $V \in \mathcal{B}(H)$  be an irreducible compact operator with cartesian decomposition  $V = A + iB$ , where  $A$  is a finite rank positive operator and  $B$  is essential with  $\text{Ker}B = \{0\}$ . Then*

- (i)  $\mathcal{G} = \{V, V^2\}$  is hypercyclic generator for  $C^*$ -algebra  $\mathcal{K}(H)$  of compact operators. In particular every sequence of unital completely positive maps  $\phi_n : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  for which

$$\lim_{n \rightarrow \infty} \|\phi_n(V) - V\| = \lim_{n \rightarrow \infty} \|\phi_n(V^2) - V^2\| = 0,$$

one has

$$\lim_{n \rightarrow \infty} \|\phi_n(K) - K\| = 0$$

for every compact operator  $K \in \mathcal{B}(H)$ .

(ii) *The smaller generating set  $\mathcal{G}_0 = \{V\}$  of  $\mathcal{K}(H)$  is not hyperrigid.*

Let  $S = (S_1, \dots, S_d)$  denote the compression of the  $d$ -shift to the complement of a homogeneous ideal  $I$  of  $\mathbb{C}[z_1, \dots, z_d]$ . Following the remark above, Kennedy and Shalit [9, Theorem 4.12] (see also [10]) proved that, if homogeneous ideals are sufficiently non-trivial then  $S$  is essentially normal if and only if it is hyperrigid as the generating set of a  $C^*$ -algebra.

The main purpose of this paper is to find the minimal hyperrigid generators for  $C^*$ -algebras. We show that, if  $S \in \mathcal{B}(H)$  is irreducible and is an essential unitary and  $\mathcal{G} = \{S, SS^*\}$ . Let  $\mathcal{T} = C^*(\mathcal{G})$  be the unital  $C^*$ -algebra generated by  $\mathcal{G}$ . Then  $\mathcal{G}$  is a hyperrigid generator for  $\mathcal{T}$ . We prove that, if  $T$  is an operator in  $\mathcal{B}(H)$  that generates a unital  $C^*$ -algebra  $\mathcal{A}$  and  $\mathcal{G} = \{T, T^*T, TT^*\}$ , then  $\mathcal{G}$  is a hyperrigid generators for the unital  $C^*$ -algebra  $\mathcal{A}$ . As a corollary it follows that, if  $T$  be a normal operator in  $\mathcal{B}(H)$  that generate a unital  $C^*$ -algebra  $\mathcal{A}$  and let  $\mathcal{G} = \{T, TT^*\}$ . Then  $\mathcal{G}$  is hyperrigid generator for unital  $C^*$ -algebra  $\mathcal{A}$ . If  $T$  be an unitary operator in  $\mathcal{B}(H)$  that generate a  $C^*$ -algebra  $\mathcal{A}$  and let  $\mathcal{G} = \{T\}$ . Then  $\mathcal{G}$  is hyperrigid generator for  $C^*$ -algebra  $\mathcal{A}$ . We show that, if  $V \in \mathcal{B}(H)$  be an isometry (not unitary) that generates a  $C^*$ -algebra  $\mathcal{A}$ . Then the minimal generating set  $\mathcal{G}_0 = \{V\}$  is not hyperrigid generator for  $C^*$ -algebra  $\mathcal{A}$ .

## 2. PRELIMINARIES

Here, we recall the necessary definitions, conventions and notations.

Let  $H$  be a separable complex Hilbert space and let  $\mathcal{B}(H)$  be the set of all bounded linear operators on  $H$ . A *operator system*  $\mathcal{S}$  in a  $C^*$ -algebra  $\mathcal{A}$  is a self-adjoint linear subspace of  $\mathcal{A}$  containing the identity of  $\mathcal{A}$ . A *operator algebra*  $\mathcal{A}_0$  in a  $C^*$ -algebra  $\mathcal{A}$  is a unital subalgebra of  $\mathcal{A}$ . Given a linear map  $\phi$  from a  $C^*$ -algebra  $\mathcal{A}$  into a  $C^*$ -algebra  $\mathcal{B}$  we can define a family of maps  $\phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$  given by  $\phi_n([a_{ij}]) = [\phi(a_{ij})]$ ,  $n \in \mathbb{N}$ . We say that  $\phi$  is *completely bounded* (CB) if  $\|\phi\|_{\text{CB}} = \sup_{n \geq 1} \|\phi_n\| < \infty$ . We say that  $\phi$  is *completely contractive* (CC) if  $\|\phi\|_{\text{CB}} \leq 1$  and that  $\phi$  is *completely isometric* if  $\phi_n$  is isometric for all  $n \geq 1$ . We say that  $\phi$  is *completely positive* (CP) if  $\phi_n$  is positive for all  $n \geq 1$ , and that  $\phi$  is *unital completely positive* (UCP) if in addition  $\phi(1) = 1$ .

**Definition 2.1.** Let  $\mathcal{S}$  be an operator system that generates a  $C^*$ -algebra  $\mathcal{A}$ . A unital completely positive map  $\phi : \mathcal{S} \rightarrow \mathcal{B}(H)$  is said to have the *unique extension property* if it has a unique extension to a UCP map  $\tilde{\phi} : \mathcal{A} \rightarrow \mathcal{B}(H)$

The boundary representations of  $\mathcal{A}$  for  $\mathcal{S}$ , which were introduced by Arveson [1], are precisely the irreducible representations  $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$  with the property that the restriction  $\pi|_{\mathcal{S}}$  has the unique extension property. The existence of boundary representations was an open question for some time. Arveson [3] proved the existence of boundary representations for separable  $C^*$ -algebras. Davidson and Kennedy [7] settled the existence of boundary representations for general  $C^*$ -algebras.

Arveson [4] tried to prove the non-commutative analogue of Saskin's theorem [14] using theory of noncommutative Choquet boundary for unital completely positive maps on  $C^*$ -algebras and noncommutative counterpart of the Korovkin's

set which is the hyperrigid set. Arveson [4] proved that if the separable operator system is hyperrigid in the  $C^*$ -algebra then every irreducible representation of  $C^*$ -algebra is a boundary representation for the operator system. The converse to this result is called *hyperrigidity conjecture*: that is, if every irreducible representation of a  $C^*$ -algebra is a boundary representation for a separable operator system then the operator system is hyperrigid.

Arveson [4] showed that the hyperrigidity conjecture is true for  $C^*$ -algebras with countable spectrum. Kleski [11] established the hyperrigidity conjecture for all type-I  $C^*$ -algebras with additional assumptions on the co-domain. Davidson and Kennedy [8] proved the conjecture for function systems. Clouatre [6] established the hyperrigidity conjecture with assumption of unperforated. The hyperrigidity conjecture is still open for general  $C^*$ -algebras. Namboodiri, Pramod, Shankar and Vijayarajan [13] approached the hyperrigidity conjecture with weaker notions. They got the partial answers.

### 3. ESSENTIAL UNITARY AND HYPERRIGIDITY

Let  $\mathcal{B}(H)$  be the algebra of bounded linear operators on a separable complex Hilbert space  $H$  and  $\mathcal{K}(H)$  ideal of compact operators on  $H$ . Let  $\pi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)/\mathcal{K}(H)$  be the natural surjection onto the Calkin algebra  $\mathcal{B}(H)/\mathcal{K}(H)$ . The operator  $T \in \mathcal{B}(H)$  is called essentially normal if  $\pi(T)$  is normal in the Clakin algebra, or equivalently,  $T^*T - TT^*$  is compact. The operator  $S \in \mathcal{B}(H)$  is called essentially unitary if  $\pi(S)$  is unitary in the Clakin algebra, or equivalently,  $I - S^*S$  and  $I - SS^*$  are compact. The above definitions can be found in [5].

Here, we will have the following assumptions to proceed. Let  $S$  be a irreducible and essential unitary but not unitary operator in  $\mathcal{B}(H)$  and let  $\mathcal{G} = \{S, SS^*\}$ . Let  $\mathcal{S}$  be a operator system generated by  $\mathcal{G}$ . Let  $\mathcal{T} = C^*(\mathcal{G})$  be the unital  $C^*$ -algebra generated by  $\mathcal{G}$ . The unital  $C^*$ -algebra  $\mathcal{T}$  contains the compact operators  $\mathcal{K}(H)$ .

A representation  $\rho : \mathcal{T} \rightarrow \mathcal{B}(H)$  is said to be singular representation if it annihilates the compact operators  $\mathcal{K}(H)$ .

**Lemma 3.1.** *Let  $\rho : \mathcal{T} \rightarrow \mathcal{B}(H)$  be a representation, and let  $\pi : \mathcal{T} \rightarrow \mathcal{B}(K)$  be a representation such that  $\pi|_{\mathcal{S}}$  is a dilation of  $\rho|_{\mathcal{S}}$ . Then the subspace  $H$  is coinvariant for  $\pi(\mathcal{S})$ .*

*Proof.* With respect to the decomposition  $K = H \oplus H^\perp$ . By assumption we have

$$\pi(S) = \begin{pmatrix} \rho(S) & X \\ Y & Z \end{pmatrix}$$

Note that  $X = P_H \pi(S)|_{H^\perp}$ . We must prove that  $X = 0$ . By assumption,

$$\begin{pmatrix} \rho(SS^*) & X_0 \\ Y_0 & Z_0 \end{pmatrix} = \pi(SS^*) = \pi(S)\pi(S)^* = \begin{pmatrix} \rho(S) & X \\ Y & Z \end{pmatrix} \begin{pmatrix} \rho(S)^* & Y^* \\ X^* & Z^* \end{pmatrix}.$$

We get,

$$\rho(SS^*) = \rho(S)\rho(S)^* + XX^*$$

Therefore,  $XX^* = 0$ , and hence  $X = 0$ . □

**Proposition 3.2.** *Suppose that  $S$  is irreducible and essential unitary and  $\mathcal{G} = \{S, SS^*\}$ . Let  $\mathcal{S}$  be a operator system generated by  $\mathcal{G}$  and  $\mathcal{T} = C^*(\mathcal{G})$ . Let  $\rho : \mathcal{T} \rightarrow \mathcal{B}(H)$  be a singular representation. Then the restriction  $\rho|_{\mathcal{S}}$  has unique extension property.*

*Proof.* We will use the fact that a UCP map  $\phi'$  has the unique extension property if and only if  $\phi'$  is *maximal*, meaning that every UCP map that dilates  $\phi'$  contains as a direct summand [3, Proposition 2.4].

Let  $K$  be a Hilbert space properly containing  $H$ . Let  $\pi : \mathcal{T} \rightarrow \mathcal{B}(K)$  be a representation such that the restriction  $\pi|_{\mathcal{S}}$  is a dilation of  $\rho|_{\mathcal{S}}$ . To show that the restriction  $\rho|_{\mathcal{S}}$  has unique extension property, it is enough to show that the dilation  $\pi$  is trivial, that is,  $\pi|_{\mathcal{S}} = \rho|_{\mathcal{S}} \oplus \phi$  for some UCP map  $\phi$ .

Using the Lemma 3.1, we can decompose  $K = H \oplus H^\perp$  and write

$$\pi(S) = \begin{pmatrix} \rho(S) & 0 \\ Y & Z \end{pmatrix}.$$

Since  $\rho$  is singular,  $\rho(S)$  is unitary, so it cannot be dilated to a compression. Therefore the dilation  $\pi$  must be trivial. □

**Proposition 3.3.** *Suppose that  $S$  is irreducible and essential unitary and  $\mathcal{G} = \{S, SS^*\}$ . Let  $\mathcal{S}$  be a operator system generated by  $\mathcal{G}$  and  $\mathcal{T} = C^*(\mathcal{G})$ . Then the identity representation of  $\mathcal{T}$  is a boundary representation for  $\mathcal{S}$ .*

*Proof.* Since  $S$  is irreducible and essential unitary. The unital  $C^*$ -algebra generated by  $\mathcal{G}$  contains the compact operators, that is,  $\mathcal{K}(H) \subseteq \mathcal{T} = C^*(\mathcal{G})$ . The operator system  $\mathcal{S} \subset \mathcal{T}$  is irreducible and contains the identity operator. By our assumption,  $0 \neq \mathcal{K} = I - SS^* \in \mathcal{S}$  is a compact operator, we have  $\|\mathcal{K} - \mathcal{K}\| < \|\mathcal{K}\|$ . Therefore, the quotient map  $q : \mathcal{B}(H) \rightarrow \mathcal{B}(H)/\mathcal{K}(H)$  is not completely isometric on  $\mathcal{S}$ . Hence by boundary theorem of Arveson [2, Theorem 2.1.1], identity representation of  $\mathcal{T}$  is a boundary representation for  $\mathcal{S}$ . □

**Theorem 3.4.** *Let  $S$  be an irreducible and essential unitary and  $\mathcal{G} = \{S, SS^*\}$ . Let  $\mathcal{T} = C^*(\mathcal{G})$  be the unital  $C^*$ -algebra generated by  $\mathcal{G}$ . Then  $\mathcal{G}$  is a hyperrigid generator for  $\mathcal{T}$ .*

*Proof.* Let  $\mathcal{S}$  be the operator system generated by  $\mathcal{G}$ . Note that  $\mathcal{G}$  is hyperrigid if and only if  $\mathcal{S}$  is hyperrigid. By Theorem 1.2, it suffices to show that for every nondegenerate representation  $\rho$  of  $\mathcal{T}$ ,  $\rho|_{\mathcal{S}}$  has the unique extension property.

The Proposition 3.2 implies that every singular nondegenerate representation  $\pi$  of  $\mathcal{T}$ ,  $\pi|_{\mathcal{S}}$  has the unique extension property. By Proposition 3.3, the restriction of the identity representation of  $\mathcal{T}$  to  $\mathcal{S}$  has the unique extension property. Since every nondegenerate representation of  $\mathcal{T}$  splits as the direct sum of a multiple of the identity representation and a singular nondegenerate representation and by [4, Proposition 4.4] the unique extension property passes to direct sums. Hence every nondegenerate representation of  $\mathcal{T}$  restricted to  $\mathcal{S}$  has the unique extension property. □

**Example 3.5.** Let  $H$  be a Hilbert space having an orthonormal basis  $\{e_n : n \geq 0\}$ . The unilateral shift  $S$  is defined by  $Se_n = e_{n+1}$ . The  $C^*$ -algebra

$\mathcal{T}$  generated by  $S$  is called the Toeplitz  $C^*$ -algebra. Observe that  $I - S^*S$  and  $I - SS^*$  are compact, therefore  $S$  is essential unitary. Also,  $S$  is irreducible. The Toeplitz  $C^*$ -algebra  $\mathcal{T}$  contains the compact operators  $\mathcal{K}(H)$ . We know that the set  $\{S, SS^*\}$  also generates the Toeplitz  $C^*$ -algebra  $\mathcal{T}$ . Hence, by Theorem 3.4, The set  $\{S, SS^*\}$  is hyperrigid generator for Toeplitz  $C^*$ -algebra  $\mathcal{T}$ .

#### 4. HYPERRIGID GENERATORS

The main purpose of this section is to find the hyperrigid generators for the  $C^*$ -algebras generated by a single operator.

**Theorem 4.1.** *Let  $T$  be an operator in  $\mathcal{B}(H)$  that generate a unital  $C^*$ -algebra  $\mathcal{A}$  and let  $\mathcal{G} = \{T, T^*T, TT^*\}$ . Then  $\mathcal{G}$  is hyperrigid generators for unital  $C^*$ -algebra  $\mathcal{A}$ .*

*Proof.* Let  $\mathcal{S}$  be the operator system generated by  $\mathcal{G}$ . By Theorem 1.2, it suffices to show that for every nondegenerate representation  $\pi$  of  $\mathcal{A}$ ,  $\pi|_{\mathcal{S}}$  has the unique extension property.

Let  $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$  be a nondegenerate representation. Let  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  be a UCP map satisfying  $\phi(T) = \pi(T)$ ,  $\phi(T^*T) = \pi(T^*T)$  and  $\phi(TT^*) = \pi(TT^*)$ . We have to show that  $\phi = \pi$  on  $\mathcal{A}$ .

Using Stinespring theorem, we can express  $\phi$  in the form

$$\phi(S) = V^*\sigma(S)V, \quad \forall S \in \mathcal{A}.$$

Where  $\sigma$  is a representation of  $A$  on a Hilbert space  $K$ ,  $V : H \rightarrow K$  is an isometry, and which is minimal in the sense that  $\overline{\sigma(\mathcal{A})VH} = K$ .

We first claim that  $\sigma(T)V = V\pi(T)$ , We have

$$V^*\sigma(T)^*VV^*\sigma(T)V = \phi(T)^*\phi(T) = \pi(T)^*\pi(T) = \pi(T^*T)$$

Hence,

$$\begin{aligned} V^*\sigma(T)^*(1 - VV^*)\sigma(T)V &= V^*\sigma(T)^*\sigma(T)V - V^*\sigma(T)^*VV^*\sigma(T)V \\ &= V^*\sigma(T^*T)V - \pi(T)^*\pi(T) \\ &= \pi(T^*T) - \pi(T^*T) = 0. \end{aligned}$$

$\sigma(T)$  leaves  $VH$  invariant. Therefore  $\sigma(T)V = VV^*\sigma(T)V = V\phi(T) = V\pi(T)$ .

$$\begin{aligned} VV^*\sigma(T)(1_K - VV^*)\sigma(T)^*VV^* &= VV^*\sigma(T)\sigma(T)^*VV^* \\ &\quad - VV^*\sigma(T)VV^*\sigma(T)^*VV^* \\ &= VV^*\sigma(TT^*)VV^* - V\pi(T)\pi(T)^*V^* \\ &= V\pi(TT^*)V^* - V\pi(TT^*)V^* = 0. \end{aligned}$$

Hence  $(1_K - VV^*)\sigma(T)^*VV^* = 0$ , we conclude that  $VH$  is invariant under both  $\sigma(T)$  and  $\sigma(T)^*$ . Since  $\mathcal{A}$  is generated by  $T$  it follows that  $\sigma(\mathcal{A})VH \subseteq VH$ . By minimality we must have  $VH = K$ , which implies that  $V$  is unitary and therefore  $\phi(S) = V^{-1}\sigma(S)V$  is a representation. Since  $\phi$  agrees with  $\pi$  on a generating set. Therefore  $\phi = \pi$  on  $\mathcal{A}$ .  $\square$

**Corollary 4.2.** *Let  $T$  be a normal operator in  $\mathcal{B}(H)$  that generate a unital  $C^*$ -algebra  $\mathcal{A}$  and let  $\mathcal{G} = \{T, TT^*\}$ . Then  $\mathcal{G}$  is hyperrigid generator for unital  $C^*$ -algebra  $\mathcal{A}$ .*

**Corollary 4.3.** *Let  $T$  be an unitary operator in  $\mathcal{B}(H)$  that generate a  $C^*$ -algebra  $\mathcal{A}$  and let  $\mathcal{G} = \{T\}$ . Then  $\mathcal{G}$  is hyperrigid generator for  $C^*$ -algebra  $\mathcal{A}$ .*

**Proposition 4.4.** *Let  $V \in \mathcal{B}(H)$  be an isometry (not unitary) that generates a  $C^*$ -algebra  $\mathcal{A}$ . Then*

- (i)  $\mathcal{G} = \{V, VV^*\}$  is hyperrigid generator for  $\mathcal{A}$ .
- (ii) The smaller generating set  $\mathcal{G}_0 = \{V\}$  is not hyperrigid.

*Proof.* (i) follows from the Theorem 1.4. Now we will prove (ii), let  $\mathcal{S}$  be the operator system generated by  $\mathcal{G}_0$ . Let  $Id$  denote the identity representation of a  $C^*$ -algebra  $\mathcal{A}$ . Let  $V^*Id(\cdot)V$  be a completely positive map on the  $C^*$ -algebra  $\mathcal{A}$ . We have  $V^*IdV|_{\mathcal{S}} = Id|_{\mathcal{S}}$ , but

$$V^*Id(VV^*)V = I \neq VV^* = Id(VV^*).$$

This implies that  $Id$  representation restricted to  $\mathcal{S}$  has two UCP map extensions  $V^*IdV$  and  $Id$ . Therefore the nondegenerate representation  $Id|_{\mathcal{S}}$  does not have unique extension property. Using the Theorem 1.2,  $\mathcal{S}$  is not hyperrigid operator system in a  $C^*$ -algebra  $\mathcal{A}$ . This will imply that  $\mathcal{G}_0$  is not hyperrigid in  $\mathcal{A}$ .  $\square$

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## REFERENCES

1. W. B. Arveson, *Subalgebras of  $C^*$ -algebras*, Acta Math. 123 (1969), 141–224.
2. W. B. Arveson, *Subalgebras of  $C^*$ -algebras, II*. Acta Math. 128 (1972), no. 3-4, 271–308.
3. W. B. Arveson, *The noncommutative Choquet boundary*, J. Amer. Math. Soc. 21 (2008), no. 4, 1065–1084.
4. W. B. Arveson, *The noncommutative Choquet boundary II: Hyperrigidity*, Israel J. Math. 184 (2011), 349–385.
5. L. Brown, R. Douglas and P. Fillmore, *Unitary equivalence modulo the compact operators and extensions of  $C^*$ -algebras*, Proc. conference on Operator theory, Halifax, NS, Lect. Notes Math. 3445, Springer Verlag, Berlin, 1973.
6. R. Clouatre, *Unperforated pairs of operator spaces and hyperrigidity of operator systems*, Canad. J. Math. To appear.
7. K. R. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke Math. J. 164 (2015), 2989–3004.
8. K. R. Davidson and M. Kennedy, *Choquet order and hyperrigidity for function systems*, arXiv:1608.02334v1, To appear.
9. M. Kennedy, O. M. Shalit, *Essential normality, essential norms and hyperrigidity*, J. Funct. Anal. 268 (2015), no. 10, 2990–3016.
10. M. Kennedy, O. M. Shalit, *Corrigendum to "Essential normality, essential norms and hyperrigidity" [J. Funct. Anal. 268 (2015), 2990–3016]*, J. Funct. Anal. 270 (2016), no. 7, 2812–2815.

11. C. Klesky, *Korovkin-type properties for completely positive maps*, Illinois J. Math. 58 (2014), no. 4, 1107-1116.
12. P. P. Korovkin, *Linear operators and approximation theory*, Hindustan publishing corp., Delhi, 1960.
13. M. N. N. Namboodiri, S. Pramod, P. Shankar and A. K. Vijayarajan, *Quasi hyperrigidity and weak peak points for non-commutative operator systems*, Proc. Indian Acad. Sci. Math. Sci. To appear.
14. Y.A. Saksin, *Korovkin systems in spaces of continuous functions*, Amer. Math. Soc. Transl. 54 (1966), no. 2, 125-144.
15. G. Salomon, *Hyperrigid subsets of graph  $C^*$ -algebras and the property of rigidity at zero*, Preprint arXiv:1709.00554 (2017).

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