Answers to some questions about Zadeh's extension on metric spaces $\stackrel{\Leftrightarrow}{\sim}$

Xinxing Wu^{a,b}, Xu Zhang^{c,*}, Guanrong Chen^b

^aSchool of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China ^bDepartment of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, China ^cDepartment of Mathematics, Shandong University, Weihai, Shandong 264209, China

Abstract

This paper shows that there exists a contraction whose Zadeh's extension is not a contraction under the Skorokhod metric, answering negatively Problems 5.8 and 5.12 posted in [5, Jardón, Sánchez, and Sanchis, Some questions about Zadeh's extension on metric spaces, Fuzzy Sets and Systems, 2018].

Keywords: Contraction; fuzzy set; Skorokhod metric; Zadeh's extension. 2010 MSC: 03E72, 54H20.

Zadeh's extension principle is the soul of the fuzzy set theory, and is the base for the concepts of fuzzy numbers and fuzzy arithmetic. The Skorokhod topology is defined on the space of functions from the unit interval to the real line, where these functions are right continuous and their left limits exist. This topology is used in the study of the convergence of the probability measures, the central limit theorems and many other results in stochastic processes [1, 4]. There is a close relation between this topology and fuzzy numbers. The Skorokhod metric, induced by the Skorokhod topology, is used in the study of fuzzy numbers [5, 6].

Throughout this paper, denote $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{Z}^+ = \{0, 1, 2, ...\}$. A dynamical system is a pair (X, f), where X is a metric space with a metric d and $f : X \to X$ is a continuous map. Let K(X) be the hyperspace on X, i.e., the space of non-empty compact subsets of X with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max\left\{\max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y)\right\}$$
$$= \inf\left\{\varepsilon > 0 : A \subset B^{\varepsilon} \text{ and } B \subset A^{\varepsilon}\right\},$$

for $A, B \in K(X)$, where A^{ε} is the ε -neighbourhood of the set A.

A fuzzy set u in the space X is a function $u: X \to I$, where I = [0, 1]. Given a fuzzy set u, its α -cuts (or α -level sets) $[u]_{\alpha}$ ($\alpha \in (0, 1]$) and support $[u]_0$ are defined respectively by

$$[u]_{\alpha} = u^{-1}([\alpha, 1]) = \{ x \in X : u(x) \ge \alpha \},\$$

and

$$[u]_0 = \overline{\{x \in X : u(x) > 0\}}.$$

^{*}This work was supported by the National Natural Science Foundation of China (Nos. 11601449 and 11701328), the National Nature Science Foundation of China (Key Program) (No. 51534006), Science and Technology Innovation Team of Education Department of Sichuan for Dynamical System and its Applications (No. 18TD0013), Youth Science and Technology Innovation Team of Southwest Petroleum University for Nonlinear Systems (No. 2017CXTD02), scientific research starting project of Southwest Petroleum University (No. 2015QHZ029), Shandong Provincial Natural Science Foundation, China (Grant ZR2017QA006), and Young Scholars Program of Shandong University, Weihai (No. 2017WHWLJH09).

^{*}Corresponding author

Email addresses: wuxinxing5201314@163.com (Xinxing Wu), xu_zhang_sdu@mail.sdu.edu.cn (Xu Zhang), gchen@ee.cityu.edu.hk (Guanrong Chen)

Clearly, $[u]_0 = \bigcup_{\alpha \in (0,1]} [u]_{\alpha}$. Let $\mathfrak{F}(X)$ be the set of all upper semi-continuous fuzzy sets $u : X \to I$, satisfying that $[u]_0$ is compact and $[u]_1 \neq \emptyset$. Define a *level-wise metric* d_{∞} on $\mathfrak{F}(X)$ by

$$d_{\infty}(u,v) = \sup\{d_H([u]_{\alpha}, [v]_{\alpha}) : \alpha \in I\}, \quad \forall u, v \in \mathfrak{F}(X).$$

$$(1)$$

For the level-wise metric d_{∞} , the following result shows that the supports are not essential for the calculation of d_{∞} .

Proposition 1. For any $u, v \in \mathfrak{F}(X)$, $d_{\infty}(u, v) = \sup\{d_H([u]_{\alpha}, [v]_{\alpha}) : \alpha \in (0, 1]\}$.

Proof. Since for any $\alpha, \beta \in (0,1]$ with $\alpha < \beta$, $[u]_{\alpha} \supset [u]_{\beta}$ and $[u]_{\alpha} \subset [u]_{0}$, it follows that $\lim_{\alpha \to 0^{+}} d_{H}([u]_{\alpha}, [u]_{0})$ exists, denoted by ξ . It can be shown that $\xi = 0$. In fact, if $\xi > 0$, then for any $\alpha \in (0,1]$, $d_{H}([u]_{\alpha}, [u]_{0}) > \frac{\xi}{2}$. The compactness of $[u]_{0}$ implies that there exist several points $x_{1}, x_{2}, \ldots, x_{n} \in [u]_{0}$ such that $\bigcup_{i=1}^{n} B(x_{i}, \frac{\xi}{8}) \supset [u]_{0}$, where $B(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$. Applying $[u]_{0} = \overline{\bigcup_{\alpha \in (0,1]} [u]_{\alpha}}$ yields that for any $1 \le i \le n$, $B(x_{i}, \frac{\xi}{8}) \cap (\bigcup_{\alpha \in (0,1]} [u]_{\alpha}) \neq \emptyset$, i.e., there exist $\alpha_{i} \in (0,1]$ and z_{i} such that $z_{i} \in B(x_{i}, \frac{\xi}{8}) \cap [u]_{\alpha_{i}}$. Take $\alpha = \min\{\alpha_{i} : 1 \le i \le n\}$. Clearly, $\{z_{i} : 1 \le i \le n\} \subset [u]_{\alpha}$. For any $x \in [u]_{0}$, there exists $1 \le i \le n$ such that $x \in B(x_{i}, \frac{\xi}{8})$, implying that $d(x, z_{i}) \le d(x, x_{i}) + d(x_{i}, z_{i}) < \frac{\xi}{4}$, i.e., $[u]_{0} \subset ([u]_{\alpha})^{\frac{\xi}{4}}$, where $([u]_{\alpha})^{\frac{\xi}{4}}$ is the $\frac{\xi}{4}$ -neighborhood of $[u]_{\alpha}$. Thus,

$$\xi = d_H([u]_0, [u]_\alpha) \le \frac{\xi}{4},$$

which is a contradiction.

Let $\eta = \sup\{d_H([u]_\alpha, [v]_\alpha) : \alpha \in (0, 1]\}$. For any $\alpha \in (0, 1]$, one has

$$d_H([u]_0, [v]_0) \le d_H([u]_0, [u]_\alpha) + d_H([u]_\alpha, [v]_\alpha) + d_H([v]_\alpha, [v]_0) \le d_H([u]_0, [u]_\alpha) + \eta + d_H([v]_\alpha, [v]_0),$$

implying that

$$d_H([u]_0, [v]_0) \le \eta + \lim_{\alpha \to 0^+} (d_H([u]_0, [u]_\alpha) + d_H([v]_\alpha, [v]_0)) = \eta.$$

Therefore, $d_{\infty}(u, v) = \eta$.

Remark 2. In (1), the value of α is taken from the whole interval I = [0, 1].

Zadeh's extension of a dynamical system (X, f) is a map $\widetilde{f} : \mathfrak{F}(X) \to \mathfrak{F}(X)$ defined by

$$\widetilde{f}(u)(x) = \begin{cases} 0, & f^{-1}(x) = \emptyset, \\ \sup\{u(z) : z \in f^{-1}(x)\}, & f^{-1}(x) \neq \emptyset. \end{cases}$$

Let $f_1, f_2, \ldots, f_n : X \to X$ be continuous maps. Define $F : \mathfrak{F}(X) \to \mathfrak{F}(X)$ by

$$[F(u)]_{\alpha} = [\widetilde{f}_1(u)]_{\alpha} \cup \dots \cup [\widetilde{f}_n(u)]_{\alpha}, \quad \forall u \in \mathfrak{F}(X), \ \alpha \in I.$$

Let $\operatorname{Hom}(I)$ be the family of all strictly increasing homeomorphisms from I onto itself. For any $t \in \operatorname{Hom}(I)$ and $u \in \mathfrak{F}(X)$, denote $tu = t \circ u$ for convenience. Clearly, $tu \in \mathfrak{F}(X)$. Given a metric space (X, d), the *Skorokhod metric* d_0 on $\mathfrak{F}(X)$ is defined as follows [6]:

$$d_0(u,v) = \inf \left\{ \varepsilon : \exists t \in \operatorname{Hom}(I) \text{ such that } \sup \{ |t(x) - x| : x \in I \} \le \varepsilon \text{ and } d_\infty(u,tv) \le \varepsilon \right\}.$$

Recently, Jardón et al. [5] proved that both \tilde{f} and F are contractive if the previous dynamical systems are contractive under the level-wise metric d_{∞} , and they proposed the following two questions. For more recent results on Zadeh's extension and g-fuzzification, refer to [2, 3, 7, 8, 9, 10] and some references therein.

Question 3. [5, Problem 5.8] Let (X, d) be a metric space and $f : X \to X$ be a contraction. Is Zadeh's extension $\tilde{f} : (\mathfrak{F}(X), d_0) \to (\mathfrak{F}(X), d_0)$ a contraction?

Question 4. [5, Problem 5.12] If (X, d) is a metric space and $f_1, \ldots, f_n : X \to X$ are contractions. Is $F : (\mathfrak{F}(X), d_0) \to (\mathfrak{F}(X), d_0)$ a contraction?

This paper constructs a contraction on I whose Zadeh's extension is not contractive under the Skorokhod metric d_0 , answering negatively Questions 3 and 4 above (see Example 7, Theorem 8, and Remark 9 below).

Lemma 5. [5, Proposition 3.1] Let X be a Hausdorff space. If $f : X \to X$ is a continuous function, then for any $u \in \mathfrak{F}(X)$ and any $\alpha \in I$, one has

(1)
$$[f(u)]_{\alpha} = f([u]_{\alpha});$$

(2) $[tu]_{\alpha} = [u]_{t^{-1}(\alpha)}$ for $t \in \text{Hom}(I)$.

Lemma 6. Let X be a metric space and $f: X \to X$ be a contraction. Then, for any $u, v \in \mathfrak{F}(X)$, $d_0(\tilde{f}(u), \tilde{f}(v)) \leq d_0(u, v)$.

Proof. Let $\lambda \in [0,1)$ be a contraction factor of f. Applying [5, Proposition 5.7], one has that for any $t \in \text{Hom}(I)$,

$$d_{\infty}(f(u), tf(v)) = \sup\{d_{H}([f(u)]_{\alpha}, [tf(v)]_{\alpha}) : \alpha \in I\}$$

$$= \sup\{d_{H}(f([u]_{\alpha}), f([v]_{t^{-1}(\alpha)})) : \alpha \in I\}$$

$$\leq \sup\{\lambda \cdot d_{H}([u]_{\alpha}, [v]_{t^{-1}(\alpha)}) : \alpha \in I\}$$

$$= \lambda \cdot d_{\infty}(u, tv).$$

This implies that $d_0(\tilde{f}(u), \tilde{f}(v)) \leq d_0(u, v)$.

Example 7. Construct a function $t: I \to I$ as follows:

$$t(x) = \begin{cases} \frac{a}{a-\frac{1}{4}}x, & x \in [0, a - \frac{1}{4}], \\ \frac{1}{2}(x-a+\frac{1}{4})+a, & x \in [a-\frac{1}{4}, a + \frac{1}{4}], \\ x, & x \in [a+\frac{1}{4}, 1]. \end{cases}$$

Take $a = \frac{3}{8}$, so that

$$t(x) = \begin{cases} 3x, & x \in [0, \frac{1}{8}], \\ \frac{1}{2}x + \frac{5}{16}, & x \in [\frac{1}{8}, \frac{5}{8}], \\ x, & x \in [\frac{5}{8}, 1]. \end{cases}$$

It is easy to see that $t \in \text{Hom}(I)$ and

$$\sup\{|t(x) - x| : x \in I\} = \frac{1}{4}.$$
(2)

Take $a_0 = a$, $a_1 = a_0 + \frac{1-a}{2}$, ..., $a_{n+1} = a_n + \frac{1-a}{2^{n+1}}$, and $b_n = t^{-1}(a_n)$ for all $n \in \mathbb{N}$, and choose $a'_0 = a$, $a'_1 = a + \frac{1}{2}$, $a'_2 = a'_1 + \frac{1-(a+\frac{1}{2})}{2}$, ..., $a'_{n+1} = a'_n + \frac{1-(a+\frac{1}{2})}{2^n}$ for all $n \ge 2$. It can be verified that

- (a) $b_n = a_n$ for all $n \ge 1$; (b) $b_1 = \frac{11}{16} > \frac{5}{8} = a + \frac{1}{4}$; (c) $\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n a_n = 1$
- (c) $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} a'_n = 1.$



Figure 1: The illustration diagram of the function t(x)

Meanwhile, take $b'_0 = a - \frac{1}{4} = \frac{1}{8}, b'_1 = a + \frac{1}{2} = \frac{7}{8}, b'_2 = b'_1 + \frac{1 - (a + \frac{1}{2})}{2}, \dots, b'_{n+1} = b'_n + \frac{1 - (a + \frac{1}{2})}{2^n}$ for all $n \ge 2$. Define $u : I \to I$ and $v : I \to I$ by

$$u(x) = \begin{cases} 0, & x \in [0, a'_0), \\ a_n, & x \in [a'_n, a'_{n+1}) \text{ and } n \in \mathbb{Z}^+, \\ 1, & x = 1, \end{cases}$$

and

$$v(x) = \begin{cases} 0, & x \in [0, b'_0), \\ a - \frac{1}{4}, & x \in [b'_0, b'_1), \\ a + \frac{1}{4}, & x \in [b'_1, b'_2), \\ b_{n-1}, & x \in [b'_n, b'_{n+1}) \text{ and } n \ge 2, \\ 1, & x = 1, \end{cases}$$

respectively.



Figure 2: The illustration diagram of the construction of the function u(x)



Figure 3: The illustration diagram of the construction of the function v(x)

Clearly, $u, v \in \mathfrak{F}(I)$ and u, v are monotonically increasing (applying (b)). According to the constructions of u and v, it can be verified that

- (i) $u^{-1}(1) = v^{-1}(1) = \{1\};$
- (ii) $[u]_0 = [a, 1];$
- (iii) $[v]_0 = [a \frac{1}{4}, 1];$
- (iv) $u^{-1}([x,1]) = [a,1]$ for any $x \in (0,a]$;
- (v) $u^{-1}([x,1]) = [a'_{n+1},1]$ for any $x \in (a_n, a_{n+1}], n \in \mathbb{Z}^+;$
- (vi) $v^{-1}([x,1]) = [a \frac{1}{4}, 1]$ for any $x \in (0, a \frac{1}{4}];$
- (vii) $v^{-1}([x,1]) = [a + \frac{1}{2}, 1]$ for any $x \in (a \frac{1}{4}, a + \frac{1}{4}];$
- (viii) $v^{-1}([x,1]) = [b'_2,1]$ for any $x \in (a + \frac{1}{4}, b_1];$
- (ix) $v^{-1}([x,1]) = [b'_{n+2},1]$ for any $x \in (b_n, b_{n+1}], n \in \mathbb{N}$.

Claim 1. $d_0(u, v) = \frac{1}{4}$.

Fix any $0 < \varepsilon < \frac{1}{4}$. For any $t' \in \text{Hom}(I)$ with $\sup\{|t'(x) - x| : x \in I\} \leq \varepsilon$, noting that $|t'(t'^{-1}(a)) - t'^{-1}(a)| \leq \varepsilon$, i.e., $a - \frac{1}{4} < a - \varepsilon < t'^{-1}(a) < a + \varepsilon < a + \frac{1}{4}$, it follows from (iv) and (vii) that

$$d_{\infty}(u, t'v) \ge d_{H}([u]_{a}, [t'v]_{a}) = d_{H}([u]_{a}, [v]_{t'^{-1}(a)})$$

= $d_{H}(u^{-1}([a, 1]), v^{-1}([t'^{-1}(a), 1]))$
= $d_{H}\left([a, 1], \left[a + \frac{1}{2}, 1\right]\right) = \frac{1}{2}.$

This implies that

$$d_0(u,v) \ge \frac{1}{4}.\tag{3}$$

(1) For any $x \in (0, a]$, from $t^{-1}(x) \in (0, a - \frac{1}{4}]$, (iv) and (vi), it follows that

$$d_H([u]_x, [tv]_x) = d_H(u^{-1}([x,1]), v^{-1}([t^{-1}(x),1])) = d_H\left([a,1], \left[a - \frac{1}{4}, 1\right]\right) = \frac{1}{4}.$$

(2) For any $x \in (a, a_1]$, from $t^{-1}(x) \in (t^{-1}(a), t^{-1}(a_1)] = (a - \frac{1}{4}, b_1]$, (v), (vii), and (viii), it follows that

$$d_H([u]_x, [tv]_x) = d_H(u^{-1}([x, 1]), v^{-1}([t^{-1}(x), 1]))$$

$$\leq \max\left\{ d_H\left([a'_1, 1], \left[a + \frac{1}{2}, 1\right]\right), d_H([a'_1, 1], [b'_2, 1]) \right\}$$

$$\leq \frac{1}{4}.$$

(3) For any $x \in (a_n, a_{n+1}]$ and any $n \in \mathbb{N}$, from $t^{-1}(x) \in (t^{-1}(a_n), t^{-1}(a_{n+1})] = (b_n, b_{n+1}]$, (v), and (ix), it follows that

$$d_H([u]_x, [tv]_x) = d_H(a'_{n+1}, b'_{n+2}) < \frac{1}{4}$$

Applying Proposition 1, one has that

$$d_{\infty}(u, tv) = \sup \left\{ d_H([u]_x, [tv]_x) : x \in (0, 1] \right\} = \frac{1}{4}.$$

This, together with (3), implies that

$$d_0(u,v) = \frac{1}{4}$$

For any $\lambda \in [\frac{1}{2}, 1)$, define $f_{\lambda} : I \to I$ by $f(x) = \lambda x$ for all $x \in I$. Clearly, f_{λ} is a contraction.

Claim 2. $d_0(\widetilde{f}_{\lambda}(u), \widetilde{f}_{\lambda}(v)) = d_0(u, v) = \frac{1}{4}$, and thus \widetilde{f}_{λ} is not a contraction.

For any $t' \in \text{Hom}(I)$ with $\sup \{|t'(x) - x| : x \in I\} < \frac{1}{4}$, from $a - \frac{1}{4} < t'^{-1}(a) < a + \frac{1}{4}$ and Lemma 5, it follows that

$$\begin{aligned} d_{\infty}(f_{\lambda}(u), t'f_{\lambda}(v)) &\geq d_{H}([f_{\lambda}(u)]_{a}, [t'f_{\lambda}(v)]_{a}) = d_{H}(f_{\lambda}([u]_{a}), f_{\lambda}([v]_{t'^{-1}(a)})) \\ &= d_{H}\left(f_{\lambda}([a, 1]), f_{\lambda}\left(\left[a + \frac{1}{2}, 1\right]\right)\right) = d_{H}\left([\lambda \cdot a, \lambda], \left[\lambda \cdot a + \frac{\lambda}{2}, \lambda\right]\right) \\ &= \frac{\lambda}{2} \geq \frac{1}{4}, \end{aligned}$$

implying that

$$d_0(\widetilde{f}_{\lambda}(u),\widetilde{f}_{\lambda}(v)) \ge \frac{1}{4}.$$

This, together with Lemma 6, implies that

$$d_0(\widetilde{f}_\lambda(u),\widetilde{f}_\lambda(v)) = \frac{1}{4}.$$

Theorem 8. There exists a contraction (I, f) such that its Zadeh's extension $(\mathfrak{F}(I), \tilde{f})$ is not a contraction under the Skorokhod metric d_0 .

Remark 9. (1) Theorem 8 shows that the answer to Question 3 is negative.

(2) Choose $f_1, f_2 : I \to I$ as $f_1 = \frac{1}{2}x$ and $f_2(x) = \frac{3}{4}x$ for all $x \in I$. For any $t' \in \text{Hom}(I)$ with $\sup\{|t'(x) - x| : x \in I\} < \frac{1}{4}$, it can be verified that

$$\begin{aligned} d_{\infty}(F(u), t'F(v)) &\geq d_{H}([F(u)]_{a}, [t'F(v)]_{t'^{-1}(a)}) = d_{H}(F([u]_{a}), F([v]_{t'^{-1}(a)})) \\ &= d_{H}\left(f_{1}([a, 1]) \cup f_{2}([a, 1]), f_{1}\left(\left[a + \frac{1}{2}, 1\right]\right) \cup f_{2}\left(\left[a + \frac{1}{2}, 1\right]\right)\right) \\ &= d_{H}\left(\left[\frac{3}{16}, \frac{3}{4}\right], \left[\frac{7}{16}, \frac{1}{2}\right] \cup \left[\frac{21}{32}, \frac{3}{4}\right]\right) = \frac{1}{4}, \end{aligned}$$

implying that

$$d_0(F(u), F(v)) \ge \frac{1}{4} = d_0(u, v).$$

Therefore, $F : (\mathfrak{F}(I), d_0) \to (\mathfrak{F}(I), d_0)$ is not a contraction. This gives a negative answer to Question 4 as well.

References

References

- [1] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.
- [2] J. P. Boroński, J. Kupka, The topology and dynamics of the hyperspaces of normal fuzzy sets and their inverse limit spaces, Fuzzy Sets and Systems **321** (2017), 90–100.
- [3] J. J. Font, D. Sanchis, M. Sanchis, Completeness, metrizability and compactness inspaces of fuzzy-numbervalued functions, Fuzzy Sets and Systems **353** (2018) 124–136.
- [4] J. Jacod, A. N. Shirayaev, Limit Theorems for Stochastic Processes, Springer, New York, 1987.
- [5] D. Jardón, I. Sánchez, M. Sanchis, Some questions about Zadeh's extension on metric spaces, Fuzzy Sets and Systems 2018, https://doi.org/10.1016/j.fss.2018.10.019.
- S. Y. Joo, Y. K. Kim, The Skorokhod topology on space of fuzzy numbers, Fuzzy Sets and Systems 111 (2000), 497–501.
- [7] J. Kupka, On approximations of Zadeh's extension principle, Fuzzy Sets and Systems 283 (2016) 26-39.
- [8] X. Wu, G. Chen, Sensitivity and transitivity of fuzzified dynamical systems, Inform. Sci. 396 (2017), 14–23.
- X. Wu, X. Ding, T. Lu, J. Wang, Topological dynamics of Zadeh's extension on upper semicontinuous fuzzy sets, Int. J. Bifurcation and Chaos 27 (2017), 1750165 (13 pages).
- [10] X. Wu, L. Wang, J. Liang, The chain properties and Li-Yorke sensitivity of Zadeh's extension on the space of upper semi-continuous fuzzy sets, Iran. J. Fuzzy Syst. accepted for publication.