CRITERION OF REALITY OF ZEROS IN A POLYNOMIAL SEQUENCE SATISFYING A THREE-TERM RECURRENCE RELATION

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ABSTRACT. This paper establishes the necessary and sufficient conditions for the reality of all the zeros in a polynomial sequence $\{P_i\}_{i=1}^{\infty}$ generated by a three-term recurrence relation $P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$ with the standard initial conditions $P_0(x) = 1, P_{-1}(x) = 0$, where $Q_1(x)$ and $Q_2(x)$ are arbitrary real polynomials.

1. INTRODUCTION

Asymptotic root distributions for sequences of univariate polynomials has been a topic of study in Mathematics for many decades [7]. In particular, sequences of polynomials with all real zeros are important in many branches of Mathematics such as Analysis, Probability theory, Combinatorics and Geometry. Polynomials with real zeros are of interest due to several nice properties they possess. For example, if a polynomial $P(x) = \sum_{i=0}^{n} b_i x^i$ is real-rooted and has nonnegative coefficients, then the sequence $\{b_i\}_{i=0}^{n}$ is log-concave, i.e $b_i^2 \geq b_{i+1}b_{i-1}$ for all $1 \leq j < n$ [6]. This log-concavity implies that $\{b_i\}_{i=0}^{n}$ is unimodal, whereby the sequence increases to a greatest value (or possibly two consecutive equal values) and then decreases [6]. In addition, polynomials with real zeros are closed with respect to differentiation and the zeros of derivatives interlace with the zeros of the polynomial.

In this paper, we discuss some cases of the problem under which conditions polynomials satisfying a finite linear recurrence relation have all real roots. Our general set-up is as follows. Fix complex-valued polynomials $Q_1(x), \ldots, Q_k(x)$ and consider a linear recurrence relation of length $k+1$ of the form

$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) + \cdots + Q_k(x)P_{i-k}(x) = 0, \ i = 1, 2, \ldots \quad (1.1)$$

with the standard initial conditions

$$P_0(x) = 1, P_{-1}(x) = P_{-2}(x) = \cdots = P_{-k}(x) = 0. \quad (1.2)$$

The generating function for the polynomial sequence $\{P_i\}_{i=1}^{\infty}$ of this recurrence is given by

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + Q_1(x)t + Q_2(x)t^2 + \cdots + Q_k(x)t^k}. \quad (1.3)$$

One of the well-known results in this area is a description of the accumulation set for the zeros of $P_i(x)$ when $i \to \infty$ provided by the theorem by Beraha, Kahane and Weiss, see [8]. It asserts that the support of the limiting root-counting measure coincides with the following set. Let $Q_1, \ldots, Q_k$ be complex polynomials as described in Equation 1.1 above and define a curve $\Gamma_Q \subset \mathbb{C}$ consisting of all values of x such that the characteristic equation

$$1 + Q_1(x)t + Q_2(x)t^2 + \cdots + Q_k(x)t^k = 0 \tag{1.4}$$

has at least two roots t_1, t_2 for which

(a) $|t_1| = |t_2|$;

(b) $|t_1|$ is the minimum among the absolute values of all roots.

Theorem 1. [8] *Suppose that $\{P_i(x)\}$ satisfies (1.1), (1.2) and (1.4). Suppose further that $\{P_i(x)\}$ satisfies no recurson of order less than k and that there does not exist a constant $\omega \in \mathbb{C}$ of unit modulus for which $t_r = \omega t_s$ for some $r \neq s$. Then the zeros of $P_i(x)$ accumulate along the curve Γ_Q as $i \to \infty$.*

This result only provides a description of the asymptotic behaviour of the roots of $P_i(x)$. However recently Tran [2] has found a number of cases when the zeros of $P_i(x)$ actually lie on the limiting curve Γ_Q for all or sufficiently large i. In particular, he has proven the following results.

Theorem 2. [2] *Let $\{P_i(x)\}$ be a polynomial sequence whose generating function is*

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + Q_1(x)t + Q_2(x)t^2}, \tag{1.5}$$

where $Q_1(x)$ and $Q_2(x)$ are polynomials in x with complex coefficients. All the zeros of every polynomial in the sequence $\{P_i(x)\}$ which satisfy $Q_2(x) \neq 0$ lie on the curve Γ_Q defined by

$$\mathrm{Im}\left(\frac{Q_1^2(x)}{Q_2(x)}\right) = 0 \text{ and } 0 \leq \mathrm{Re}\left(\frac{Q_1^2(x)}{Q_2(x)}\right) \leq 4 \tag{1.6}$$

Moreover these roots become dense in Γ_Q when $i \to \infty$.

Theorem 2 covers the special case of (1.1) and (1.2) for the polynomials generated by the recurrence

$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0.$$

with the standard initial conditions $P_0(x) = 1$ and $P_{-1}(x) = 0$.

A more general result of Tran is as follows.

Theorem 3. [3] *Let $\{P_i(x)\}$ be a polynomial sequence whose generating function is*

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + Q_1(x)t + Q_2(x)t^k}, \tag{1.7}$$

where $Q_1(x)$ and $Q_2(x)$ are polynomials in x with complex coefficients. Then there exists a constant $C = C(k)$ such that for all $i > C$, all the zeros of $P_i(x)$ which satisfy $Q_2(x) \neq 0$ lie on the curve Γ_Q defined by

$$\mathrm{Im}\left(\frac{Q_1^k(x)}{Q_2(x)}\right) = 0 \ and \ 0 \leq (-1)^n \mathrm{Re}\left(\frac{Q_1^2(x)}{Q_2(x)}\right) \leq \frac{k^k}{(k-1)^k}. \tag{1.8}$$

Moreover, these zeros become dense in Γ_Q when $i \to \infty$.

In the present paper, we want to characterise a situation that gives rise to polynomial sequences with only real zeros.

Problem 1. *In the above notation, consider the recurrence relation*

$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0 \tag{1.9}$$

with the standard initial conditions,

$$P_0(x) = 1, P_{-1}(x) = 0, \tag{1.10}$$

where $Q_1(x)$ and $Q_2(x)$ are arbitrary real polynomials. Give necessary and sufficient conditions on $(Q_1(x), Q_2(x))$ for all the zeros of $P_i(x)$ to be real for all i.

To formulate our main result, we need to look at the curve defined by the first condition in (1.6). We shall view $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$, the extended complex plane and $\mathbb{R}P^1$ as the union of the Euclidean line in $\mathbb{C}$ with $\{\infty\}$. Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be the rational function defined by $f = \frac{Q_1^2(x)}{Q_2(x)}$ where $Q_1(x)$ and $Q_2(x)$ are real polynomials. Denote by $\widetilde{\Gamma_Q} \subset \mathbb{C}P^1$ the curve given by $\mathrm{Im}(f) = 0$, that is

$$\widetilde{\Gamma_Q} = f^{-1}(\mathbb{R}P^1).$$

The curve $\widetilde{\Gamma_Q}$ is the preimage of $\mathbb{R}P^1$. We note that for real polynomials $Q_1(x)$ and $Q_2(x)$, the curve $\widehat{\Gamma_Q}$ contains Γ_Q since the real interval $[0, 4] \subset \mathbb{R}P^1$.

Lemma 4. *The curve $\widetilde{\Gamma_Q}$ has the following properties:*

(a) $\widetilde{\Gamma_Q} \supset \mathbb{R}P^1$;

(b) $\widetilde{\Gamma_Q}$ *is invariant under complex conjugation;*

(c) *except $\mathbb{R}P^1$, $\widetilde{\Gamma_Q}$ might contain ovals disjoint from $\mathbb{R}P^1$ (which come in complex-conjugate pairs) and ovals crossing $\mathbb{R}P^1$ which are mapped to themselves by complex conjugation;*

(d) *the intersection points of the second type of ovals with $\mathbb{R}P^1$ are exactly the real critical points of f.*

Figures 1 and 2 illustrate the properties of $\widetilde{\Gamma_Q}$ claimed in Lemma 4 (a) to (d). The main result of the present paper is as follows.

Theorem 5. *Let $\{P_i(x)\}$ be a sequence of polynomials whose generating function is*

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + Q_1(x)t + Q_2(x)t^2}, \tag{1.11}$$
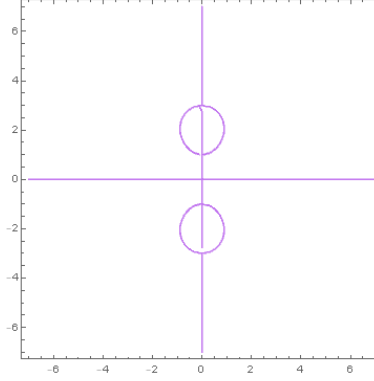
FIGURE 1. The curve $\widetilde{\Gamma_Q}$ for $f = \frac{Q_1^2(x)}{Q_2(x)}$ where $Q_1(x) = x^2 + 1$ and $Q_2(x) = x^2 + 6$.
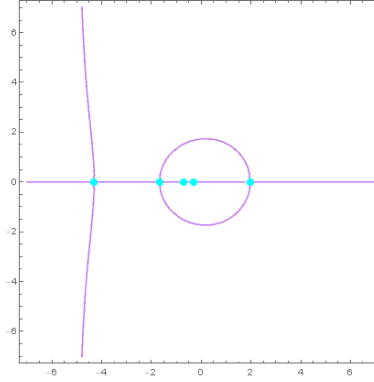

FIGURE 2. The curve $\widetilde{\Gamma_Q}$ for $f = \frac{Q_1^2(x)}{Q_2(x)}$ where $Q_1(x) = x^2 + 5x + 3$ and $Q_2(x) = 5x^2 - 1$. The green points are the real critical points of f.

where $Q_1(x)$ and $Q_2(x)$ are arbitrary coprime real polynomials in x. All the zeros of $P_i(x) \; \forall \; i$ are real if and only if the following conditions are satisfied:

(a) *The polynomial $Q_1(x)$ must have all real and simple zeros.*

(b) *No ovals γ of $\widetilde{\Gamma_Q}$ disjoint from $\mathbb{R}P^1$ should exist.*

(c) *All the zeros of the discriminant $D(x)$ of the characteristic equation $1 + Q_1(x)t + Q_2(x)t^2$ must be real.*

(d) *No real critical values of f should belong to the interval $(0, 4)$.*

(e) *The polynomial $Q_2(x)$ must be non-negative at the zeros of $Q_1(x)$.*

Remark. The situation where $Q_1(x)$ and $Q_2(x)$ have a common real zero is not interesting to consider since from $P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$, such a zero would necessarily be a zero of $P_i(x)$ for all i.

2. PROOFS

Let us begin with the following definitions and remarks;

Definition 1. For a non-constant rational function $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are coprime polynomials, the degree of $R(x)$ is defined as the maximum of the degrees of $P(x)$ and $Q(x)$.

Equivalently, the degree of $R(x)$ is the number of distinct preimages of any generic point.

Definition 2. A point x_0 is called a critical point of $R(x)$ if $R(x)$ fails to be injective in a neighbourhood around x_0, that is, $R'(x_0) = 0$. A critical value of $R(x)$ is the image of a critical point. The order of a critical point x_0 of $R(x)$ is the order of the zero of $R'(x)$ at x_0.

Now, if d is the degree of $R(x)$ and if w is not a critical value then $R^{-1}(w) = \{x_1, x_2, \ldots, x_d\}$ with $x_i \neq x_j$ for all $i \neq j$. Since the points x_j are non-critical then there is a neighbourhood around each of them such that $R(x)$ is injective on that neigbourhood. The function $R(x)$ has $2d - 2$ critical points in $\mathbb{C}P^1$ counting multiplicities. This follows from the fact that in the finite complex plane, $\deg(R') = \deg(P'Q - Q'P) = \deg(P) + \deg(Q) - 1$ while the order of the critical point at infinity is $|\deg(P) - \deg(Q)| - 1$, [4].

Definition 3. Given a pair (P, Q) of polynomials, define its Wronskian as
$$W(P, Q) = P'Q - Q'P.$$
One interesting thing about the Wronskian is that if P and Q are coprime, then the zeros of $W(P, Q)$ are exactly the critical points of the rational map $R = P/Q$. In [9] we find that if P and Q have all real, simple and interlacing zeros, then all zeros of $W(P, Q)$ are non-real. In addition, if we know that α is a zero of R of multiplicity ≥ 2, then α is also a (multiple) zero of the Wronskian. More information about the Wronski map can be found in [1, 9].

Now we start our proof of Theorem 5.

Proof of Theorem 5. (a) From the initial conditions $P_0(x) = 1$ and $P_{-1}(x) = 0$, substituting these conditions in the recurrence relation
$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$$

gives for $i = 1$,

$$P_1(x) + Q_1(x)P_0(x) + Q_2(x)P_{-1}(x) = 0$$

or

$$P_1(x) = -Q_1(x).$$

Therefore $Q_1(x)$ must have all its zeros real since we require all the zeros of $P_i(x)$ to be real for all i and in particular for $i = 1$. These zeros must be simple (see part (e) for the justification).

(b) Suppose there exists an oval γ of $\widetilde{\Gamma_Q}$ which does not intersect $\mathbb{R}P^1$. From Lemma 4 (c), γ is the type one oval contained in $\widetilde{\Gamma_Q}$. We note that all the points on γ and its interior are of the form $z = x + iy$ where $x, y \in \mathbb{R}, y \neq 0$ and this is a connected component with γ as its boundary. This component is mapped by f onto the half plane with degree ≥ 1 depending on the number of critical points of f it strictly contains. The boundary γ of the component, is mapped onto $\mathbb{R}P^1$ (the boundary of the half plane). In particular, the image $f(\gamma)$ covers the interval $[0, 4] \subset \mathbb{R}P^1$. Therefore $\Gamma_Q = f^{-1}([0, 4])$ must contain an arc of the boundary γ. From Theorem 2 [2], all the zeros of $P_i(x)$ are contained in Γ_Q for all i and are dense in Γ_Q. Now since we require all the zeros of $P_i(x)$ to be real, it must be that $\Gamma_Q \subseteq \mathbb{R}P^1$. This is not possible as we already have that Γ_Q contains an arc of γ yet this arc is not contained in $\mathbb{R}P^1$, hence a contradiction.

(c) It is known [5] that the endpoints of Γ_Q are the points where $t_1 = t_2$ (see the notation in Theorem 2). In our case equation (1.4) has degree 2 in t. Therefore every x, for which the roots of (1.4) coincide belongs to Γ_Q. These x are exactly the zeros of $D(x) = Q_1^2(x) - 4Q_2(x)$. Since we require that $\Gamma_Q \subset \mathbb{R}P^1$, all the zeros of $D(x)$ must be real.

(d) Suppose there exists a critical value $w \in (0, 4)$. Then there must exist a real critical point x_c such that $f(x_c) = w$. Clearly, x_c is a point on Γ_Q. It is known [4] that a point $x_c \in \mathbb{C}P^1$ is a critical point of order k for a rational function $R(x)$ if and only if there are open sets U containing x_c and V containing $w = R(x_c)$ such that each $w_0 \in V, w \neq w_0$ has exactly $k + 1$ distinct preimages in U.

   In our scenario let x_c be such a critical point of order k for $f(x)$ and V be the real interval $(w - \epsilon, w + \epsilon)$ for sufficiently small $\epsilon > 0$. Then $V \subset (0, 4)$. Note that since x_c is a critical point of order k for $f(x)$ and any point $z \in V, z \neq w$ has exactly $k + 1$ distinct preimages in U, we have at x_c, locally $U = f^{-1}(V)$ the preimage of V consists of $k + 1$ distinct curves (arcs) with a common intersection only at x_c. One of these curves is a line segment on the real line while the remaining d curves are complex i.e apart from x_c, points on these d curves are of the form $z = x + iy$ where $x, y \in \mathbb{R}, y \neq 0$.

   Now since complex arcs are formed in the preimage of $V \subset (0, 4)$, then some of the zeros of $P_i(x)$ will be contained in the complex arcs since all the zeros of $P_i(x)$ are contained in $\Gamma_Q = f^{-1}([0, 4] \supset V)$ and are dense there as $i \to \infty$. This contradicts our requirement that Γ_Q be contained in $\mathbb{R}P^1$. Therefore in order to have all the real roots of $P_i(x)$ for all i, it must be that no real critical values of f be in the real interval $(0, 4)$. Otherwise the condition that $\Gamma_Q \subset \mathbb{R}P^1$ cannot hold.

(e) Let x_0 be a zero of $Q_1(x)$, i.e $Q_1(x_0) = 0$. Note that $Q_2(x)$ and $Q_1(x)$ do not have a common zero since they are coprime. Therefore at the point x_0, we have $Q_2(x_0) \neq 0$. It remains to show that $Q_2(x_0) > 0$. We note that all the zeros of

$Q_1(x)$ are critical points of f. In addition, they belong to Γ_Q because at x_0, we have that $f(x_0) = Q_1^2(x_0)/Q_2(x_0) = 0$ therefore both the real and the imaginary part of f vanish, hence satisfying (1.6) of Theorem 2. Suppose that x_0 is a simple critical point of f and let $Q_2 > 0$ at x_0. Then locally $f = Q_1^2/Q_2 \geq 0$ on the interval in $\mathbb{R}$ and if $Q_2 < 0$ at x_0, then locally $f = Q_1^2/Q_2 \geq 0$ on the complex arc, i.e there exists an interval $I \subset [0, 4]$ such that $f^{-1}(I)$ contains a complex arc. On the other hand if x_0 is a critical point of order > 1, then locally $f = Q_1^2/Q_2 \geq 0$ on some complex arc irrespective of whether $Q_2 > 0$ or $Q_2 < 0$. However, it is known that the zeros of $P_i(x)$ are contained in $\Gamma_Q = f^{-1}[0, 4]$ and are dense there as $i \to \infty$, and so for the reality of all the zeros of $P_i(x)$ we require that $\Gamma_Q \subset \mathbb{R}P^1$. This is not possible if $Q_2 < 0$ at simple zeros of $Q_1(x)$ or when zeros of Q_1 have multiplicity > 1 since in either case there will be some zeros of $P_i(x)$ on the complex arc which is a contradiction. Therefore the polynomial $Q_2(x)$ must be non-negative at the zeros of $Q_1(x)$ as a necessary condition for the reality of all the zeros of $P_i(x)$ for all i. Furthermore all the zeros of $Q_1(x)$ must be simple otherwise as explained above some zeros of $P_i(x)$ would be on the complex arc contradicting the reality of all the zeros of $P_i(x)$ for all i. (This last part settles part (a) of the Theorem where we require all the zeros of $P_i(x)$ to be simple). $\qquad \square$

Remark. Each of the conditions of Threorem 5 (a) to (e) is only a necessary (and not a sufficient) condition for the reality of all the zeros of $P_i(x)$. To guarantee reality of all the zeros of $P_i(x)$ for all i, all the five conditions must be satisfied simultaneously. Below are some of the examples illustrating this observation.

Example 1. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + (-x^2 + 2x)t + (5x^2 - 1)t^2}. \qquad (2.1)$$

The corresponding f is given by

$$f(x) = \frac{(-x^2 + 2x)^2}{5x^2 - 1}.$$

The zeros of $Q_1(x)$ are 0 and 2 which are real and simple (Theorem 5(a)). There are no ovals disjoint from $\mathbb{R}P^1$ (Theorem 5(b)). The discriminant $D(x) = x^4 - 4x^3 - 16x^2 + 4$ has only real zeros (Theorem 5(c)). These are $-2.39337, -0.54374, 0.47570$ and 6.46141 (corrected to 5 dp and are indicated by red points in Figure 3). However as seen from Figure 3, not all the zeros of $P_{100}(x)$ are real. This shows that the above three conditions satisfied are not each sufficient to guarantee that all the zeros of $P_i(x)$ will be real for all i.

Example 2. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + (2x^2 - 8x + 6)t + (-5x^3 + 37x^2 - 43x - 21)t^2}. \qquad (2.2)$$
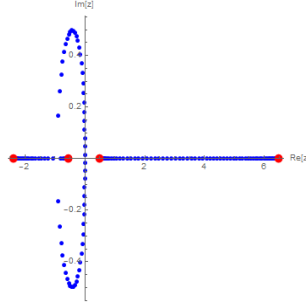
FIGURE 3. The zeros of $P_{100}(x)$ for the generating function $1/(1+ (-x^2 + 2x)t + (5x^2 - 1)t^2)$.

The corresponding f is given by

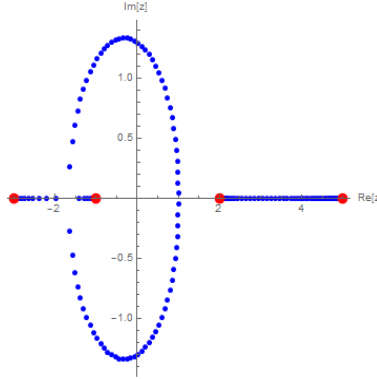$$f(x) = \frac{(2x^2 - 8x + 6)^2}{-5x^3 + 37x^2 - 43x - 21}.$$

FIGURE 4. The zeros of $P_{20}(x)$ for the generating function $1/(1+ (2x^2 - 8x + 6)t + (-5x^3 + 37x^2 - 43x - 21)t^2)$.

The zeros of $Q_1(x)$ are 1 and 3 which are real and simple. Also there are no ovals disjoint from $\mathbb{R}P^1$. The discriminant $D(x) = x^4 - 3x^3 - 15x^2 + 19x + 30$ has only real zeros, i.e $x = -3, x = -1, x = 2$ and $x = 5$. However f has a critical value of $3.50783 \in (0, 4)$ corresponding to the critical point -1.66437 corrected to 5 dp, hence the condition of Theorem 5(d) is violated. Consequently, some of the zeros of P_{100} are not real (see Figure 4). The first three conditions of Theorem 5 are satisfied but not the fourth condition. Therefore having no critical value in $(0, 4)$ is indeed a necessary condition for the reality of all the zeros of $P_i(x)$.

Example 3. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + (2x^2 - 8x + 6)t + (x^4 - 8x^3 + 21x^2 - 14x - 16)t^2}. \qquad (2.3)$$

The corresponding f is given by

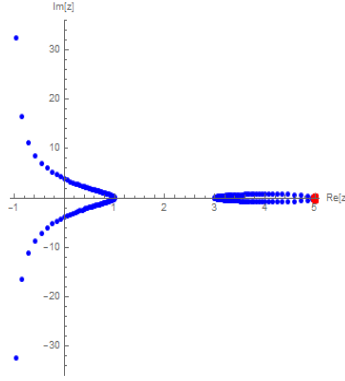$$f(x) = \frac{(2x^2 - 8x + 6)^2}{x^4 - 8x^3 + 21x^2 - 14x - 16}.$$

FIGURE 5. The zeros of $P_{100}(x)$ for the generating function $1/(1 + (2x^2 - 8x + 6)t + (x^4 - 8x^3 + 21x^2 - 14x - 16)t^2)$.

The zeros of $Q_1(x)$ are 1 and 3 which are real and simple. Also there are no ovals disjoint from $\mathbb{R}P^1$. The discriminant $D(x) = 4x^2 - 40x + 100$ has only real zeros, i.e $x = 5$. In addition f has no critical value of in the real interval $(0, 4)$. Thus conditions of Theorem 5 (a) to (d) are satisfied. Note that on the zeros of Q_1 we have $Q_2(1) = -16 \not> 0$ and $Q_2(3) = -4 \not> 0$. Thus condition (e) of Theorem 5 is violated. Consequently some of the zeros of P_{100} are not real as seen in Figure 5.

Example 4. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + (x^2 - 2x - 5)t + x^2 t^2}. \qquad (2.4)$$

The corresponding f is given by

$$f(x) = \frac{(x^2 - 2x - 5)^2}{x^2}.$$

In this case, all the five conditions of Theorem 5 have been satisfied and as seen in Figure 6, all the zeros of $P_{200}(x)$ are real. We used $i = 200$, but any arbitrary value of $i \in \mathbb{N}^+$ could work. The red points are the zeros of the discriminant and these are the endpoints of the intervals where all the zeros of $P_i(x)$ for all i are located, that is, all the zeros of $P_i(x)$ for all i are supported on the real axis and the support is a union of two disjoint real intervals given by $[-\sqrt{5}, -1] \cup [\sqrt{5}, 5] \subset \mathbb{R}P^1$. The zeros of $P_i(x)$ are dense in this support as $i \to \infty$.
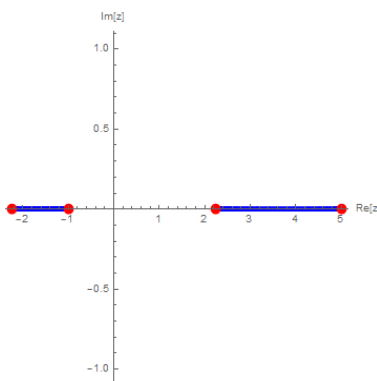
FIGURE 6. The zeros of $P_{200}(x)$ when the generating function is $1/(1 + (x^2 - 2x - 5)t + x^2t^2)$.

3. FINAL REMARKS

Describe similar conditions guaranteeing reality of roots for all polynomials in the context of Theorem 3.

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