A NOTE ON SPANNING TREES WITH A SPECIFIED DEGREE SEQUENCE

María Elena Martínez-Cuero

Departamento de Matemáticas
Universidad Autónoma Metropolitana-Iztapalapa
sherlyroses@hotmail.com

Eduardo Rivera-Campo

Departamento de Matemáticas Universidad Autónoma Metropolitana-Iztapalapa erc@xanum.uam.mx

ABSTRACT. We give an Ore-Type condition sufficient for a graph G to have a spanning tree with a specified degree sequence.

Keywords: Spanning Tree, Degree sequence, Arboreal.

1. Introduction

- O. Ore [3] proved that if G is a graph with n vertices such that $d(u) + d(v) \ge n 1$ for each pair u, v of non-adjacent vertices, then G contains a hamiltonian path. This result has been generalized in many directions.
- S. Win [5] showed that if $r \geq 2$ is an integer and G is a connected graph with n vertices such that $d(u_1) + d(u_2) + \cdots + d(u_r) \geq n 1$ for each set of r independent vertices of G, then G has a spanning tree with maximum degree at most r.

Years later, H. Broersma and H. Tuinstra [2] showed that if $s \geq 2$ is an integer and G is a connected graph with n vertices such that $d(u) + d(v) \geq n - s + 1$ for each pair u, v of non-adjacent vertices, then G contains a spanning tree with at most s vertices with degree 1.

E. Rivera-Campo [4] gave a condition on the graph G that bounds the degree of each vertex in a certain spanning tree T of G and the number of vertices of T with degree 1.

Theorem 1. Let n, k, and d_1, d_2, \ldots, d_n be integers with $1 \le k \le n-1$ and $2 \le d_1 \le d_2 \le \cdots \le d_n \le n-1$. If G is a k-connected graph with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$ such

Partially supported by Conacyt, México.

that $d(u) + d(v) \ge n - 1 - \sum_{j=1}^{n} (d_j - 2)$ for each pair u, v of non-adjacent vertices, then G contains a spanning tree T with at most $2 + \sum_{j=1}^{n} (d_j - 2)$ vertices with degree 1 and such that $d_T(w_j) \le d_j$ for $j = 1, 2, \ldots, n$.

Let n be a positive integer. An arboreal sequence is a sequence of positive integers d_1, d_2, \ldots, d_n such that $\sum_{j=1}^n d_j = 2(n-1)$. It is well known that a sequence $\sigma = d_1, d_2, \ldots, d_n$ is arboreal if and only if there is a tree whose vertices have degrees d_1, d_2, \ldots, d_n .

Let $\sigma = d_1, d_2, \ldots, d_n$ be an arboreal sequence and G be a labelled graph with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$. A spanning tree T of G has degree sequence σ if $d_T(w_i) = d_i$ for $i = 1, 2, \ldots, n$. In this note we prove the following result:

Theorem 2. Let $n \geq 4$ be an integer and G be a labelled graph with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$. If $d(u) + d(v) \geq \frac{3n-1}{2}$ for each pair u, v of non-adjacent vertices, then G contains a spanning tree T with degree sequence σ for each arboreal sequence $\sigma = d_1, d_2, \ldots, d_n$ with $1 \leq d_i \leq 3$ for $i = 1, 2, \ldots, n$.

For each positive integer k let $X_k = \{x_1, x_2, \dots, x_k\}$, $Y_k = \{y_1, y_2, \dots, y_k\}$ and $Z_k = \{z_1, z_2, \dots, z_{2k+2}\}$ be pairwise disjoint sets of vertices and let G_k be the complete graph with vertex set $X_k \cup Y_k \cup Z_k$ with all edges $x_i y_i$, $1 \le i, j \le k$ removed. See Fig. 1 for the case k = 2.

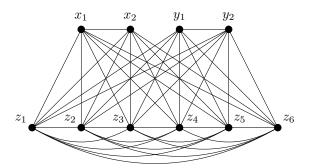


FIGURE 1. Graph G_2

We claim that the graph G_k contains no spanning tree T such that $d_T(x_i) = d_T(y_i) = 3$ for i = 1, 2, ..., k and $d_T(z_j) = 1$ for j = 1, 2, ..., 2k + 2; for if T is such a tree, then $T - Z_k$ would be a spanning tree of the subgraph $G[X_k \cup Y_k]$ of G, induced by the set $X_k \cup Y_k$, which is not possible since $G[X_k \cup Y_k]$ is not connected. On the other hand, if u and v are non-adjacent vertices of G_k , without loss of generality we may assume $u \in X_k$ and $v \in Y_k$. Therefore

$$d_{G_k}(u) + d_{G_k}(v) = 2((k-1) + (2k+2)) = 6k + 2 = \frac{3n-2}{2}$$

where n = 4k+2 is the number of vertices of G_k . This shows that the degree-sum condition in Theorem 2 is tight.

Whenever possible we follow the notation of J. A. Bondy and U. S. R. Murty [1].

2. Proof of Theorem 2

Suppose the result is false. Then for certain integer $n \geq 4$ and certain arboreal sequence $\sigma = d_1, d_2, \ldots, d_n$ with $1 \leq d_i \leq 3$ for $i = 1, 2, \ldots, n$ there exists a counterexample. That is a labelled graph G with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$ such that G contains no spanning tree with degree sequence σ while $d(u) + d(v) \geq \frac{3n-1}{2}$ for each pair u, v of non-adjacent vertices of G. We choose G with the maximum possible number of edges while remaining a counter example with n vertices.

Since σ is an arboreal sequence of order n, a counterexample cannot be a complete graph of order n. Let u, v be non-adjacent vertices of G. By the choice of G, the graph G + uv is not a counterexample and contains a spanning tree T with degree sequence σ . Therefore G contains a spanning forest F = T - uv with exactly two components T_u and T_v with $u \in V(T_u)$ and $v \in V(T_v)$ such that $d_F(u) = d_i - 1$, $d_F(v) = d_j - 1$ and $d_f(w_r) = d_r$ for each vertex $u \neq w_r \neq v$, where i and j are such that $u = w_i$ and $v = w_j$.

Orient the edges of F in such a way that T_u and T_v become outdirected trees $\overrightarrow{T_u}$ and $\overrightarrow{T_v}$ with roots u and v, respectively. For each vertex $u \neq w \neq v$ let w^- denote the unique vertex of G such that the edge w^-w is oriented from w^- to w in \overrightarrow{F} . Let

$$A_u = \{y^- \in V(T_u) : uy \in E(G)\}, B_u = \{x \in V(T_u) : vx \in E(G)\},$$

$$A_v = \{y^- \in V(T_v) : vy \in E(G)\} \text{ and } B_v = \{x \in V(T_v) : ux \in E(G)\}.$$

Notice that $|A_u \cap B_u| = 0$, for if $z^- \in A_u \cap B_u$, then $(F - z^- z) + \{uz, vz^-\}$ would be a spanning tree of G with degree sequence σ , which is not possible (See Fig. 2). Analogously $|A_v \cap B_v| = 0$ and therefore

$$|A_u| + |B_u| = |A_u \cup B_u| \le n_u$$
 and $|A_v| + |B_v| = |A_v \cup B_v| \le n_v$,

where n_u and n_v are the number of vertices of T_u and T_v , respectively.

In an abuse of notation, for each vertex x of G we denote by $d_u(x)$ and $d_v(x)$ the number of vertices of T_u and T_v , respectively, which are adjacent to x in G. Clearly

$$|B_u| = d_u(v)$$
 and $|B_v| = d_v(u)$.

Also notice that

$$|A_u| \ge \frac{d_u(u)}{2}$$
 and $|A_v| \ge \frac{d_v(v)}{2}$

since the out-degree of each vertex of \overrightarrow{F} is at most 2. Then

$$\frac{d_u(u)}{2} + d_u(v) \le |A_u| + |B_u| \le n_u \text{ and } \frac{d_v(v)}{2} + d_v(u) \le |A_v| + |B_v| \le n_v.$$

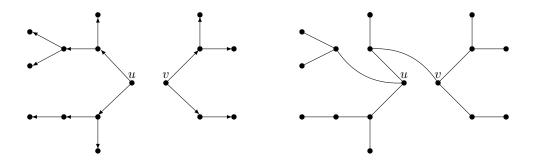


FIGURE 2. Forest F with $z^- \in A_u \cap B_u$ (left). Tree $(F - z^- z) + \{uz, vz^-\}$ (right).

Therefore

$$d_u(u) + 2d_u(v) \le 2n_u$$
 and $d_v(v) + 2d_v(u) \le 2n_v$.

Since u is not adjacent to u and v is not adjacent to v in G, $d_u(u) \le n_u - 1$ and $d_v(v) \le n_v - 1$. Adding these to the previous inequalities we obtain

$$2d_u(u) + 2d_v(v) \le 3n_u - 1$$
 and $2d_v(v) + 2d_v(u) \le 3n_v - 1$.

These imply

$$2d(u) + 2d(v) = (2d_u(u) + 2d_v(u)) + (2d_v(v) + 2d_u(v)) \le 3(n_u + n_v) - 2 = 3n - 2,$$

which is not possible since $d(u) + d(v) \ge \frac{3n-1}{2}$.

3. FINAL REMARKS

With the same approach, we can prove the following generalization of Theorem 2.

Theorem 3. Let $r \geq 2$ be an integer and G be a labelled graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $n \geq r+1$. If $d(u) + d(v) \geq \frac{(2r-3)n - (2r-5)}{r-1}$ for each pair u, v of non-adjacent vertices, then G has a spanning tree T with degree sequence σ for each arboreal sequence $\sigma = d_1, d_2, \ldots, d_n$ with $1 \leq d_i \leq r$ for $i = 1, 2, \ldots, n$.

Let $r \geq 2$ be an integer. For each positive integer k let $X_k = \{x_1, x_2, \dots, x_k\}$, $Y_k = \{y_1, y_2, \dots, y_k\}$ and $Z_{k,r} = \{z_1, z_2, \dots, z_{2k(r-2)+2}\}$ be pairwise disjoint vertex sets and let $G_{k,r}$ be the complete graph with vertex set $X_k \cup Y_k \cup Z_{k,r}$ with all edges $x_i y_i, 1 \leq i, j \leq k$ removed. As for the graphs $G_k = G_{k,3}$ in the introduction, we claim that the graphs $G_{k,r}$ show that the degree-sum condition in Theorem 3 is also tight.

MARIA ELENA MARTÍNEZ-CUERO AND EDUARDO RIVERA-CAMPO

REFERENCES

- [1] Bondy. J. A.; Murty, U. S. R.: *Graph Theory with Applications*, The McMillan Press, (1976).
- [2] Broersma, H.; Tuinstra, H.: Independence tees and Hamilton cycles, J. Graph Theory. 29 (1998), 227 237.
- [3] Ore, O.: Note on Hamilton Circuits, American Mathematical Monthly. 67 (1960), 55.
- [4] Rivera-Campo, E.: Spanning trees with small degrees and few leaves, *Applied Mathematics Letters*. 25 (2012), 1444 1446.
- [5] Win, S.: Existenz von Gerütsen Mit vorgeschriebenem Maximalgraden in Graphen, Abh. Math. Seminar Univ. Hamburg 43 (1975), 263 267.