VACUUM DISTRIBUTION, NORM AND SPECTRAL PROPERTIES FOR SUMS OF MONOTONE POSITION OPERATORS

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Abstract. We investigate the spectrum for partial sums of m position (or gaussian) operators on monotone Fock space based on $\ell^2(\mathbb{N})$. In the basic case of the first consecutive operators, we prove it coincides with the support of the vacuum distribution. Thus, the right endpoint of the support gives their norm. In the general case, we get the last property for norm still holds. As the single position operator has the vacuum symmetric Bernoulli law, and the whole of them is a monotone independent family of random variables, the vacuum distribution for partial sums of n operators can be seen as the monotone binomial with n trials. It is a discrete measure supported on a finite set, and we exhibit recurrence formulas to compute its atoms and probability function as well. Moreover, lower and upper bounds for the right endpoints of the supports are given.

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1. introduction

Position operators on Fock spaces are the self-adjoint part of creators or annihilators with the same test function. In non commutative probability they are also called generalised gaussian operators, and in the monotone case [8, 12, 13] are the most natural examples of monotone independent random variables [15]. As a consequence, their partial sums, up to usual rescaling, weakly converge in the vacuum state to the standard (i.e. centered with unit variance) arcsine law, namely the probability distribution with density $\nu(\mathrm{d}x)=\frac{1}{\pi\sqrt{2-x^2}}\mathrm{d}x$ on $(-\sqrt{2},\sqrt{2})$.

Many results have been obtained in the last years in the monotone kingdom, such as monotone convolution and monotone central limit

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theorems [14, 15, 5], monotone cumulants and monotone infinite divisibility [10, 11], and the list above is far to be complete. Monotone Fock spaces, as prominent examples of interacting Fock spaces, were first investigated in [12], whereas in [1] the author highlighted the relations between monotone creation and annihilation operators and Pusz-Woronowicz twisted operators [16]. More recently, the study of distributional symmetries on monotone stochastic processes built on the concrete C^*-algebra of creation and annihilation operators on monotone Fock space was started in [4, 3]. The basic idea of monotone Fock spaces is a suitable deformation of the usual nth scalar product on the nth particle space of the full Fock space. Namely, the new scalar product is induced by the orthogonal projection onto the linear space spanned by some increasingly ordered (w.r.t. a linear order on the index set) elements of the canonical basis of the full Fock space. As a special case of the so-called Yang-Baxter-Hecke quantisation [1], monotone creation and annihilation operators sometimes exhibit common features with the q-deformed case, with $-1 < q < 1$ (see, e.g. [2]). As an example, the reader is referred to [7]. The situation radically changes for monotone stochastic processes invariant under some distributional symmetries, which behave in a completely different way [4, 3]. Furthermore, in the q-deformed case, the vacuum vector is separating for the von Neumann algebra generated by all of the gaussian operators, whereas in the monotone case it was proved in [4] that the commutant for the same algebra is trivial. As a consequence, even for a single position operator, one cannot directly deduce that the support of the moments distribution in the vacuum state covers the whole spectrum, the latter condition being equivalent to the faithfulness of the vector state. Up to our knowledge, the spectral properties for sums of monotone position operators have not yet been investigated. Here we present a path to achieve information on the spectrum.

Namely, after denoting $s_i := a(e_i) + a^\dagger(e_i)$, $i \in \mathbb{N}$ the gaussian operator on the monotone Fock space built on $\ell^2(\mathbb{N})$, we prove that the Radon measure induced by the vacuum vector on the spectrum of the unital commutative C^*-algebra generated by $S_m := \sum_{i=1}^m s_i$, is *basic* [9] for any m. This property in particular entails the above measure is supported on the whole Gelfand spectrum. As the latter results to be homeomorphic to the spectrum $\sigma(S_m)$ of S_m, it turns out $\sigma(S_m)$ is covered by the support of the vacuum law. Consequently, one figures out that the norm of S_m is exactly the right endpoint of the support. Since arbitrary sums of m position operators are identically distributed in the vacuum state, one naturally wonders if even they share their norm with S_m. Although it is not immediate, we give an affirmative answer.

The above arguments therefore lead us to investigate firstly the vacuum distribution of S_m. Recall that any s_i is endowed with the symmetric Bernoulli law in the vacuum and, as previously mentioned, the collection of such operators is a family of monotone independent random variables. This suggests that the measure of any partial sum can be viewed as the *monotone binomial* distribution. Here, using the monotone convolution, we highlight that any law is a symmetric measure supported on a finite subset of the reals, and give recurrence formulas for computing weights and atoms.

The paper is organised as follows. After some preliminary results comprised in Section 2, monotone binomial laws represent the main argument of Section 3. Using some results of [11], in Proposition 3.1 we show recurrence formulas for atoms and probability functions, and the section ends with an estimation of the right endpoints, say r_n, of the supports. Although affected by a small error, this directly gives the size of the support, avoiding the longer recurrence formula, and as a biproduct, it allows us to achieve the sequence $\left(\frac{r_n}{\sqrt{n}}\right)_n$ converges (increasingly from the left) to $\sqrt{2}$, a consistent result with the monotone Central Limit Theorem [15]. Finally, in Section 4 we state the main theorem, i.e. the Radon measure defined by the vacuum vector is basic on the spectrum of the unital C^*-algebra $\mathcal{S}_m$ generated by any S_m. As previously noticed, this entails that the spectrum of S_m and the support of the vacuum distribution coincide, and in particular provides the value of the norm. The proof is obtained after showing the vacuum is a cyclic vector for the commutant of $\mathcal{S}_m$ for any m. This crucial property is not generally verified when one handles with a general sum involving m monotone position operators, as shown in the paper. Nevertheless, a suitable operator direct sum decomposition gives the norm in this case, which turns out to be equal to that of S_m, as one naturally foresees. The paper ends with an appendix where we briefly show how the method of moments generating function induces a nice relation among the atoms of the vacuum law of S_m and those of the distributions of $S_1, \dots, S_{m-1}$. The result, presented in Proposition 5.3, refines an existing one in [11] based on monotone convolution, and is added here for the convenience of the reader.

2. PRELIMINARIES

In this section we mainly recall some definitions and features which will used throughout the paper.

2.1. **basic measures on Gelfand spectrum.** Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $\mathfrak{A}$ an abelian C^*-algebra of

operators on $\mathcal{H}$. Recall the spectrum $\mathrm{sp}(\mathfrak{A})$ of $\mathfrak{A}$ is the $*$-weakly locally compact (compact if $\mathfrak{A}$ is unital) space of characters. If $C_0(\mathrm{sp}(\mathfrak{A}))$ is the collection of continuous complex-valued functions on $\mathrm{sp}(\mathfrak{A})$ vanishing at infinity and $f \in C_0(\mathrm{sp}(\mathfrak{A}))$, we get T_f as the unique element in $\mathfrak{A}$ realising the *Gelfand isomorphism*, *i.e.* the normed $*$-algebra isomorphism between $C_0(\mathrm{sp}(\mathfrak{A}))$ and $\mathfrak{A}$ such that $\varphi(T_f) = f(\varphi)$ (see, *e.g.* [9] for details). For any $f \in C_0(\mathrm{sp}(\mathfrak{A}))$ and $x, y \in \mathcal{H}$, we denote by $\nu_{x,y}$ the spectral measure such that $\nu_{x,y}(f) := \langle T_f x, y\rangle$. Notice that when $\mathfrak{A}$ is unital, one replaces in the above lines $C_0(\mathrm{sp}(\mathfrak{A}))$ with the algebra $C(\mathrm{sp}(\mathfrak{A}))$ of continuous functions on $\mathrm{sp}(\mathfrak{A})$.
A (Radon) positive measure μ on $\mathrm{sp}(\mathfrak{A})$ is called *basic* [9] if for any subset in $\mathrm{sp}(\mathfrak{A})$ to be locally μ-negligible, it is necessary and sufficient to be locally $\nu_{x,x}$-negligible for all $x \in \mathcal{H}$.
If μ is a basic measure, any other basic measure on $\mathrm{sp}(\mathfrak{A})$ is measure equivalent to μ (*i.e.* they are mutually absolutely continuous). Moreover, since the union of the supports of the $\nu_{x,x}$ is dense in $\mathrm{sp}(\mathfrak{A})$ and any $\nu_{x,x}$ is absolutely continuous w.r.t μ, in particular one has that μ is supported on the whole spectrum of $\mathfrak{A}$.

2.2. **monotone independence.** Let μ be a probability measure defined on the Borel σ-field over $\mathbb{R}$. The moment sequence associated with μ is denoted by $(m_n(\mu))_{n\geq 1}$. Recall that for each $z \in \mathbb{C}$

$$\mathcal{M}_\mu(z) := \sum_{n=0}^{\infty} z^n m_n(\mu)$$

is called moment generating function, which is considered as a formal power series if the series is not absolutely convergent.

From now on $\mathbb{C}^+$ and $\mathbb{C}^-$ will be the the upper and lower complex half-planes, respectively. The Cauchy transform of μ is defined as

$$\mathcal{G}_\mu(z) := \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x},$$

i.e.

$$\mathcal{G}_\mu(z) = \frac{1}{z}\mathcal{M}_\mu\left(\frac{1}{z}\right).$$

The map

$$H_\mu(z) := \frac{1}{\mathcal{G}_\mu(z)}$$

is called the reciprocal Cauchy transform of μ. $\mathcal{G}_\mu(z)$ is analytic in $\mathbb{C} \setminus \mathrm{supp}(\mu)$, and since $\mathcal{G}_\mu(\overline{z}) = \overline{\mathcal{G}_\mu(z)}$, we can restrict its domain on $\mathbb{C}^+ \cup \mathbb{R}$, where it uniquely determines μ.

The reciprocal Cauchy transform $H_\mu(z)$ plays an important role when one has to compute the distribution of a sum of monotone independent random variables [15], as we will see below.

Recall that an algebraic probability space is a pair $(\mathfrak{A}, \varphi)$, where $\mathfrak{A}$ is a unital $*$-algebra and φ a state on $\mathfrak{A}$, i.e. a linear functional defined on $\mathfrak{A}$ such that $\varphi(a^*a) \geq 0$ for any $a \in \mathfrak{A}$, and $\varphi(1_\mathfrak{A}) = 1$. In this case any $a \in \mathfrak{A}$ is called a random variable. Consider a linearly ordered family $(\mathfrak{A}_i)_{i\in I}$ of $*$-subalgebras of $\mathfrak{A}$, where the index set I is linearly ordered by the relation $<$. The family $(\mathfrak{A}_i)_{i\in I}$ is said to be *monotone independent* if

$$\varphi(a_1 \cdots a_i \cdots a_n) = \varphi(a_i)\varphi(a_1 \cdots a_{i-1}a_{i+1} \cdots a_n),$$

when $a_{i-1} < a_i$ and $a_{i+1} < a_i$, with the elimination of one of the inequalities when $i = 1$ or $i = n$. A family of random variables is said to be monotonically independent if the family of subalgebras generated by each random variable is monotone independent. We first recall the following

Theorem 2.1 ([14], Theorem 3.1)**.** *Let $a_1, a_2, \ldots, a_n \in \mathfrak{A}$ be monotonically independent self-adjoint random variables, in the natural order, over a $*$-algebraic probability space $(\mathfrak{A}, \varphi)$. If μ_{a_i} is the probability distribution of a_i under the state φ, then*

$$H_{\mu_{a_1+a_2+\ldots+a_n}}(z) = H_{\mu_{a_1}}(H_{\mu_{a_2}}(\cdots H_{\mu_{a_n}}(z)\cdots)). \tag{2.1}$$

Moreover, Theorem 3.5 in [14] ensures that for any pair of probability measures μ, ν on $\mathbb{R}$, there exists a unique distribution ρ on $\mathbb{R}$ such that

$$H_\rho(z) = H_\mu(H_\nu(z)).$$

ρ is called the monotonic convolution of μ and ν, and denoted by $\mu \triangleright \nu$. Monotone convolution is associative and affine in the first argument, and Theorem 2.1 entails the law for any partial sum of monotone independent random variables is the monotone convolution of the marginal distributions.

3. The vacuum law for sums of position operators

The section is devoted to present the vacuum distribution for partial sums of position operators in discrete monotone Fock space. It is well known that position operators are a family of monotone independent random variables [15]. As a consequence, it appears quite natural to perform our investigation using the monotone convolution [14]. Furthermore, since any gaussian operator is symmetrically Bernoulli distributed, Theorem 3.1 and Corollary 3.3 in [11] give that the vacuum distribution for the sum of m position operators is a discrete measure with exactly 2^m atoms, and a formula for computing the weights. The main result of the section is Proposition 3.1, where we collect the above results, and give a recurrence formula for computing the atoms of the law.

We first recall some useful features on discrete monotone Fock space, the reader being referred to [4, 5, 12, 15] for further details.

For $k \geq 1$, denote $I_k := \{(i_1, i_2, \dots, i_k) \mid i_1 < i_2 < \cdots < i_k, i_j \in \mathbb{N}\}$. The discrete monotone Fock space is the Hilbert space $\mathcal{F}_m := \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, where for any $k \geq 1$, $\mathcal{H}_k := \ell^2(I_k)$, and $\mathcal{H}_0 = \mathbb{C}\Omega$, Ω being the Fock vacuum. Borrowing the terminology from the physical language, we call each $\mathcal{H}_k$ the kth-particle space and denote by $\mathcal{F}_m^o$ the dense linear manifold of finite particle vectors in $\mathcal{F}_m$, that is

$$\mathcal{F}_m^o := \left\{ \sum_{n=0}^{\infty} c_n \xi_n \mid \xi_n \in \mathcal{H}_n,\ c_n \in \mathbb{C} \text{ s.t. } c_n = 0 \text{ but a finite set} \right\}.$$

Let $(i_1, i_2, \dots, i_k)$ be an increasing sequence of natural integers. The generic element of the canonical basis of $\mathcal{F}_m$ is denoted by $e_{(i_1, i_2, \dots, i_k)}$. Very often, we write $e_{(i)}$ as e_i to simplify the notations. The monotone creation and annihilation operators are respectively given, for any $i \in \mathbb{N}$, by $a_i^\dagger \Omega = e_i$, $a_i \Omega = 0$ and

$$a_i^\dagger e_{(i_1, i_2, \dots, i_k)} := \begin{cases} e_{(i, i_1, i_2, \dots, i_k)} & \text{if } i < i_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$a_i e_{(i_1, i_2, \dots, i_k)} := \begin{cases} e_{(i_2, \dots, i_k)} & \text{if } k \geq 1 \quad \text{and} \quad i = i_1 \\ 0 & \text{otherwise.} \end{cases}$$

One can check that both $a_i^\dagger$ and a_i have unital norm (see [1], Proposition 8), they are mutually adjoint, and satisfy the following relations

$$\begin{aligned} a_i^\dagger a_j^\dagger &= a_j a_i = 0 && \text{if } i \geq j\,, \\ a_i a_j^\dagger &= 0 && \text{if } i \neq j\,. \end{aligned} \tag{3.1}$$

In addition, for any i the following identity holds

$$a_i a_i^\dagger + \sum_{k \leq i} a_k^\dagger a_k = I. \tag{3.2}$$

From now on, for a fixed $i \in \mathbb{N}$ we denote by s_i the sum of creation and annihilation operators with the test function e_i, namely

$$s_i := a_i + a_i^\dagger,$$

which is generally called the position field operator. Moreover, for any $m \in \mathbb{N}$ one takes

$$S_m := \sum_{i=1}^{m} s_i.$$

The distribution μ_m of S_m in the vacuum vector state $\omega_\Omega := \langle \cdot \Omega, \Omega \rangle$ can be deduced using some existing results on monotone convolution [14]. First, one notices it is a compactly supported measure on the real line, since S_m is a bounded self-adjoint operator. It is then determined by the moments $u_{m,n} := \omega_\Omega((S_m)^n)$. As for any $i, n \in \mathbb{N}$ it is easy to see that $s_i^{2n} = s_i^2$ and $s_i^{2n+1} = s_i$, one has $\mu_1 = \frac{1}{2}(\delta_1 + \delta_{-1})$, i.e. μ_1 is the Bernoulli symmetric law. Concerning the case μ_n, $n \geq 2$, the following result gives a recursive formula to compute the atoms and weights for the (necessarily discrete) law of any partial sum of monotone gaussian operators in the vacuum state.

Proposition 3.1. *For any $n \in \mathbb{N}$, the vacuum distribution of S_I, for $I := \{i_1 < \cdots < i_n \mid i_j \in \mathbb{N}\}$ is the discrete measure $\mu_n := \sum_{k=1}^{2^n} b_k^{(n)} \delta_{r_k^{(n)}}$, where $r_1^{(1)} = 1$, $r_2^{(1)} = -1$, $b_1^{(1)} = b_2^{(1)} = 1/2$. Furthermore, for any $j = 0, \ldots, 2^{n-1} - 1$ one finds*

$$r_{2j+h}^{(n)} = \frac{r_{j+1}^{(n-1)} + (-1)^h \sqrt{\left(r_{j+1}^{(n-1)}\right)^2 + 4}}{2}, \quad h = 1, 2 \tag{3.3}$$

and for any $k = 1, \ldots, 2^n$ one achieves

$$b_k^{(n)} = \frac{\prod_{h=1}^{2^{n-1}} \left(r_k^{(n)} - r_h^{(n-1)}\right)}{2 \prod_{\substack{h=1 \\ h \neq k}}^{2^n} \left(r_k^{(n)} - r_h^{(n)}\right)}. \tag{3.4}$$

Proof. We first recall that $(s_i)_{i \geq 1}$ is a family of monotone independent and identically distributed self-adjoint random variables in $(\mathfrak{M}_o, \omega_\Omega)$, where $\mathfrak{M}_o$ is the $*$-algebra generated by $\{a_i \mid i \in \mathbb{N}\}$. Therefore, it is enough to prove the statement for S_n, exploiting Theorem 2.1. Indeed, from (2.1), it follows

$$H_{\mu_n}(z) = H_{\mu_{n-1}}(H_{\mu_1}(z)),$$

as monotone convolution is associative, and the reciprocal Cauchy transform of s_i is $H_{s_i}(z) = \frac{z^2-1}{z}$ for any i. Thus, any zero, say for simplicity r, of H_{μ_n} satisfies the following

$$\frac{r^2 - 1}{r} = r_j^{(n-1)},$$

for each $j = 1, \dots, 2^{n-1}$, and one achieves (3.3). Finally, after denoting $\mu_{2,n}$ the vacuum distribution of $s_2 + \cdots + s_n$, (2.1) gives $\mu_n = \mu_1 \rhd \mu_{2,n}$, and (3.4) follows from Theorem 3.1 in [11], as monotone convolution is affine in the first argument. $\square$

Since μ_1 is a symmetric measure, a standard induction procedure with (3.3) and (3.4) give any μ_n is symmetric too. Furthermore, the 2^n points of the support of the vacuum distribution for the sum of n position operators come in inverse pairs, when $n \geq 2$. Indeed, fix r_0 an element in the support of μ_n, for some $n \geq 1$. Then (3.3) entails that both

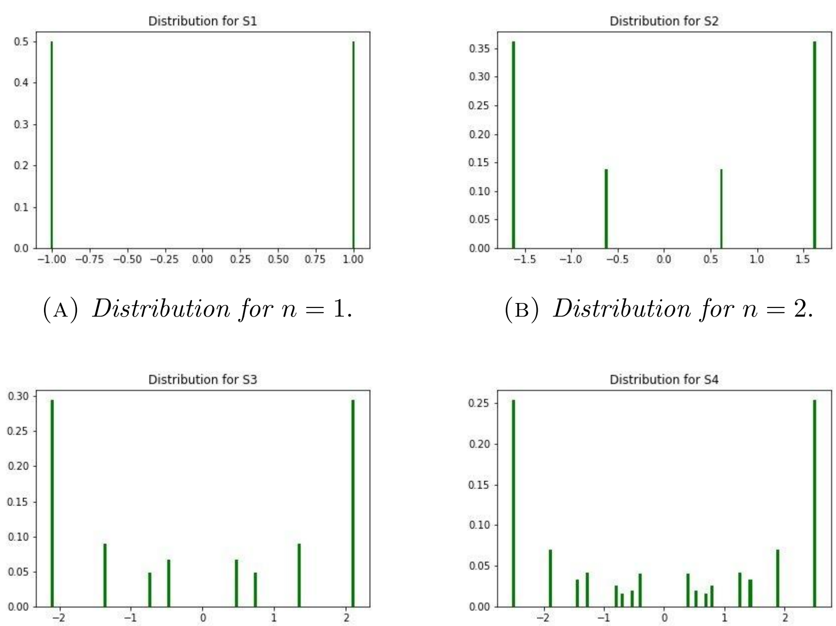
$$r_1 := \frac{r_0 + \sqrt{r_0^2 + 4}}{2}$$

and

$$r_2 := \frac{-r_0 + \sqrt{(-r_0)^2 + 4}}{2}$$

are in the support of μ_{n+1}, and trivially $r_1 r_2 = 1$. As a consequence, one achieves all of the atoms of μ_n just computing half of the positive ones by (3.3), as soon as n is at least 2.

In Figure 1 we report the plots for the vacuum laws of S_n, $n \leq 4$.



(A) *Distribution for* $n = 1$. (B) *Distribution for* $n = 2$.

(C) *Distribution for* $n = 3$. (D) *Distribution for* $n = 4$.

FIGURE 1. Vacuum distribution of S_n, $n = 1, \dots, 4$.

In what follows we give a small error approximation for the extreme values of the support of μ_n, a result appearing useful if compared with the content of Section 4.

Proposition 3.2. *Under the notations introduced above, for any $n \geq 1$ one has*

$$\sqrt{2n-\sqrt{2n}} \leq r^{(n)}_{2^n} < \sqrt{2n} \tag{3.5}$$

Proof. We start by showing the right inequality, which is true for $n=1$ as $r^{(1)}_2 = 1$. Suppose now it holds for any $k \leq n$. Then, since for any n one straightforwardly sees

$$\sqrt{2n}+\sqrt{2n+4} < 2\sqrt{2(n+1)}, \tag{3.6}$$

(3.3) and (3.6) give

$$\begin{aligned} r^{(n+1)}_{2^{n+1}} &= \frac{1}{2}\left(r^{(n)}_{2^n} + \sqrt{\left(r^{(n)}_{2^n}\right)^2+4}\right) \\ &\leq \frac{1}{2}\left(\sqrt{2n}+\sqrt{2n+4}\right) < \sqrt{2(n+1)}, \end{aligned}$$

after recalling the map $\mathbb{R} \ni x \mapsto x+\sqrt{x^2+4}$ is increasing. Proving the inequality $\sqrt{2n-\sqrt{2n}} \leq r^{(n)}_{2^n}$ is longer. It indeed holds for $n=1$. For the remaining cases, after denoting $m := 2n$, we firstly show that

$$\sqrt{m-\sqrt{m}} + \sqrt{\left(m-\sqrt{m}\right)^2+4} \geq 2\sqrt{m+2-\sqrt{m+2}}. \tag{3.7}$$

This is indeed satisfied when $n=1$. Since for any $m \geq 3$, one further has

$$\sqrt{\left(m-\sqrt{m}\right)^2+4} \geq \sqrt{m-\sqrt{m}+4},$$

when $n \geq 2$, the inequality (3.7) holds true if

$$\sqrt{m-\sqrt{m}} + \sqrt{\left(m-\sqrt{m}\right)+4} \geq 2\sqrt{m+2-\sqrt{m+2}}.$$

In fact, after squaring one finds this is equivalent to

$$\sqrt{\left(m-\sqrt{m}\right)\left(m-\sqrt{m}+4\right)} \geq m+2+\sqrt{m}-2\sqrt{m+2},$$

and a further squaring gives

$$\sqrt{m}+\frac{m+3}{m+2}-\frac{m+2+\sqrt{m}}{\sqrt{m+2}} \leq 0.$$

The last inequality is equivalent to

$$4m^4+12m^3-4m^2-24m+1 \geq 0,$$

which is automatically satisfied since the map $f(x) := 4x^4 + 12x^3 - 4x^2 - 24x + 1$ is strictly increasing in $[3, +\infty)$, and $f(3) > 0$.

Suppose now that the left inequality in (3.5) holds for each $k \leq n$. One can extend its validity to $k = n+1$ by means of (3.7) and (3.3). $\square$

As a consequence of (3.5), the condition that the atoms of μ_n come in inverse pair suggests that the littlest positive of them approaches 0 for n going to $+\infty$, and moreover

$$\lim_n \frac{r^{(n)}_{2^n}}{\sqrt{n}} = \sqrt{2}.$$

The last result agrees with the central limit theorem for monotonically independent random variables [15]. Namely, the sum of n position operators, rescaled by a factor $\frac{1}{\sqrt{n}}$, weakly converges to the arcsine law supported in $(-\sqrt{2}, \sqrt{2})$ for $n \to \infty$. In addition, (3.5) suggests $\left(\frac{r^{(n)}_{2^n}}{\sqrt{n}}\right)_n$ approaches $\sqrt{2}$ from the left, and it is an increasing sequence, since by (3.3) it is not difficult to prove that for any n

$$\left(\frac{2}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) r^{(n)}_{2^n} < \sqrt{\frac{(r^{(n)}_{2^n})^2 + 4}{n+1}}.$$

The reader is referred to [6] for similar results in the so-called weakly monotone case.

4. The norm for sums of position operators

As previously pointed out, our approach to compute the norm for partial sums of position operators on monotone Fock space provides an investigation of their spectrum. To this goal, we begin with the definition of the right creators and annihilators on $\mathcal{F}_m$. For any $i \in \mathbb{N}$, take $b^\dagger_i \Omega = e_i$, $b_i \Omega = 0$ and

$$b^\dagger_i e_{(i_1, i_2, \dots, i_k)} := \begin{cases} e_{(i_1, \dots, i_k, i)} & \text{if } i > i_k\,, \\ 0 & \text{otherwise}\,, \end{cases}$$
$$b_i e_{(i_1, i_2, \dots, i_k)} := \begin{cases} e_{(i_1, i_2, \dots, i_{k-1})} & \text{if } k \geq 1 \quad \text{and} \quad i = i_k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Since they are continuous on $\mathcal{F}^o_m$, they can be uniquely extended to the whole $\mathcal{F}_m$ where they are mutually adjoint, and endowed with unital norm. For any i, the right position operator is defined as $r_i := b_i + b^\dagger_i$.

Right creators and annihilators are a powerful tool to study the von Neumann algebra generated by position operators in the q-deformed case, $-1 < q < 1$ [2]. There, the commutant of the von Neumann algebra is generated by right position operators, and the vacuum is a cyclic vector. The latter property entails that the support of the vacuum distribution for sums of position operators covers their spectrum. In the monotone case, the C^*-algebra generated by position operators, say $\mathfrak{S}_m$, is irreducible, as follows (up to replacing $\mathbb{Z}$ with $\mathbb{N}$) from [4], Propositions 5.9 and 5.13. Consequently, Ω is not cyclic for the commutant, and we are forced to reduce to suitable C^*-subalgebras of $\mathfrak{S}_m$.

One preliminary notices that the abelian C^*-algebra generated by s_1 contains the identity operator on $\mathcal{F}_m$, as $s_1^2 = I$. The same happens when one takes the C^*-algebras generated by S_2 and S_3. Indeed, one has $3S_2^2 - S_4^4 = I$, and $7S_3^2 - 13S_3^4 + 7S_3^6 - S_3^8 = I$, respectively. More in general, using the following identities

$$\begin{aligned} &s_i^{2n} = s_i^2, \qquad s_i^{2n+1} = s_i, \\ &s_i^2 s_{i+1} = s_{i+1}, \\ &s_{i+1}^2 + \sum_{j=1}^{i} s_j s_{j+1}^2 s_j = I, \end{aligned}$$

coming from (3.1), (3.2), and Lemma 5.4 and Proposition 5.13 of [4], we conjecture for any n there exists a finite sequence of scalars $(\alpha_k)_k$ such that

$$\sum_{k=1}^{\frac{(n-1)n}{2}+1} \alpha_{2k} S_n^{2k} = I.$$

Its proof is not in the aim of these notes and it does not affect the following results. Thus, from now on we denote by $\mathcal{S}_n$ the abelian C^*-algebra on $\mathcal{F}_m$ generated by S_n for any n, and tacitely suppose it contains I. Next theorem shows that $\mathcal{S}_n$ are indeed the C^*-subalgebras of $\mathfrak{S}_m$ we are looking for, and is crucial to prove that μ_n is supported on the whole spectrum of S_n.

Theorem 4.1. *For any $n \in \mathbb{N}$, the spectral measure $\nu_{\Omega,\Omega}$ is basic on* $\mathrm{sp}(\mathcal{S}_n)$.

Proof. Since [9], Ch. 7 Proposition 2, it is enough to prove that Ω is a cyclic vector for the commutant $\mathcal{S}_n'$ of $\mathcal{S}_n$, $n \in \mathbb{N}$. Namely, by usual approximation arguments, we need to show that for any $k \geq 0$, $i_1 < i - 2 < \cdots i_k$, there exists $T \in \mathcal{S}_n'$ such that $T\Omega = e_{i_1} \otimes \cdots e_{i_k}$, where $k = 0$ gives Ω.
To this aim, we firstly check that for any $k \in \mathbb{N}$

$$[s_k, r_{k+j}] = 0, \qquad j = 0, 1, 2, \ldots$$

Indeed, after fixing $j \geq 0$, one has

$$s_k r_{k+j}\Omega = r_{k+j}s_k\Omega = \begin{cases} e_k \otimes e_{k+j} & \text{if} \quad j \geq 1, \\ \Omega & \text{if} \quad j = 0. \end{cases}$$

Moreover, for any $e_l \in \mathcal{H}$

$$s_k r_{k+j} e_l = r_{k+j} s_k e_l = \begin{cases} e_{k+j} & \text{if} \quad l = k, \\ e_k \otimes e_l \otimes e_{k+j} & \text{if} \quad k < l < k+j, \\ e_k & \text{if} \quad l = k+j, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for $n \geq 2$ and $i_1 < \cdots i_n$,

$$s_k r_{k+j}(e_{i_1} \otimes \cdots \otimes e_{i_n})$$

and

$$r_{k+j} s_k(e_{i_1} \otimes \cdots \otimes e_{i_n})$$

are both equal to

$$\begin{cases} e_{i_2} \otimes \cdots \otimes e_{i_{n-1}} & \text{if} \quad j \geq 1, k+j = i_n, k = i_1, \\ e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} & \text{if} \quad j \geq 1, k+j = i_n, k < i_1, \\ e_{i_2} \otimes \cdots \otimes e_{i_n} \otimes e_{k+j} & \text{if} \quad j \geq 1, k+j > i_n, k = i_1, \\ e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{k+j} & \text{if} \quad j \geq 1, k+j > i_n, k < i_1, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, for each $j \geq 0$

$$[S_n, r_{n+j}] = 0 \tag{4.1}$$

i.e. any r_{n+j} commutes with each element of the $*$-algebra generated by S_n. Thus, a standard approximation argument gives $r_{n+j} \in \mathcal{S}'_n$. Furthermore, one has

$$I\Omega = \Omega \tag{4.2}$$

and

$$r_{i_n} \cdots r_{i_2} r_{i_1}\Omega = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n},$$

for $i_1 < i_2 < \cdots < i_n$ and any $n \geq 1$. Then it then turns out Ω is cyclic for $\mathcal{S}'_1$. Since

$$(S_2 - r_2)\Omega = e_1, \tag{4.3}$$

from (4.1) and (4.3), it follows $e_i = T_i\Omega$, where $T_i \in \mathcal{S}'_2$ and $i \in \mathbb{N}$, whereas (4.2) gives $\Omega = I\Omega$. Moreover, for any $i_1 < i_2 < \cdots < i_n$, it results

$$r_{i_n} \cdots r_{i_2} \overline{r}_{i_1}\Omega = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n},$$

where $\overline{r}_{i_1} = (S_2 - r_2)$ or $\overline{r}_{i_1} = r_{i_1}$, according to whether $i_1 = 1$ or $i_1 > 1$. As a consequence, $\overline{\mathcal{S}'_2\Omega} = \mathcal{F}_m$.

The general case $n \geq 3$ can be performed as follows. As a first step one shows that each e_j, $j = 1, \dots, n-1$ is obtained by the action onto the vacuum of suitable operators belonging to $\mathcal{S}'_n$. Pursuing this requires the replacement of (4.3) with the more general

$$A_n\Omega := (S_n - r_n)\Omega = \sum_{i=1}^{n-1} e_i, \tag{4.4}$$

and the detection of some operators in $\mathcal{S}'_n$ mapping the vacuum into e_j, $j = 2, \dots n-1$. In the latter case e_1 is obtained from (4.4).
More in detail, for $n = 3$, a possible choice for the above mentioned operators goes through

$$B_3 := S_3(S_3 - r_3) - 2I.$$

B_3 indeed gives e_2 by means of $S_3B_3\Omega$. This, together with (4.4), (4.1) and (4.2) allows us to find Ω and the elements of the canonical basis of $\ell^2(\mathbb{N})$. Finally, it results to be cyclic for $\mathcal{S}'_3$, since for $n \geq 2$

$$r_{i_n} \cdots r_{i_3}\overline{r}_{i_2}\overline{r}_{i_1}\Omega = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}, \tag{4.5}$$

where

$$\overline{r}_{i_2}\overline{r}_{i_1} := \begin{cases} B_3 & \text{if } i_1 = 1, i_2 > 2, \\ r_{i_2}(A_3 - S_3B_3) & \text{if } i_1 = 1, i_2 > 2, \\ r_{i_2}S_3B_3 & \text{if } i_1 = 2, \\ r_{i_2}r_{i_2} & \text{if } i_1 > 2. \end{cases}$$

The case $n \geq 4$ appears more complicated. To achieve e_j, $j = 1, \dots, n-1$ it seems natural to start with the natural generalisation of B_3, given by

$$B_n := S_n(S_n - r_n) - (n-1)I.$$

Then we observe that

$$\begin{aligned} S_nB_n\Omega &= S_n\bigg(\sum_{1\leq i<j\leq n-1} e_i \otimes e_j\bigg) \\ &= \sum_{i=2}^{n-1}(i-1)e_i + \sum_{1\leq i<j<k\leq n-1} e_i \otimes e_j \otimes e_k. \end{aligned} \tag{4.6}$$

By means of (4.4) and (4.6), one has

$$\big[(n-2)A_n - S_nB_n\big]\Omega = \sum_{i=1}^{n-2}(n-i-1)e_i \; - \sum_{1\leq i<j<k\leq n-1} e_i \otimes e_j \otimes e_k.$$

After defining

$$\begin{aligned} C_n\Omega :=& \left(S_n[(n-2)A_n - S_nB_n] - \frac{(n-2)(n-3)}{2} r_n^2 \right)\Omega \\ =& \sum_{i=1}^{n-3}\sum_{k=i+1}^{n-2} (n-k-i)e_i \otimes e_k - \sum_{i=2}^{n-2}\sum_{k=i+1}^{n-1} (i-1)e_i \otimes e_k, \end{aligned}$$

one notices it is useful to erase the term $e_{n-2} \otimes e_{n-1}$ from the r.h.s. above to reach our goal. This is achieved by taking

$$D_n\Omega := [(n-3)B_n + C_n]\Omega$$

As a consequence,

$$S_nD_n\Omega = \sum_{i=2}^{n-1} \gamma_i e_i + \sum_{i=1}^{n-4}\sum_{j=i+1}^{n-3}\sum_{k=j+1}^{n-1} \lambda_{ijk} e_i \otimes e_j \otimes e_k,$$

for suitable integers γ_i and λ_{ijk}. Next step consists in erasing e_{n-1} from the first sum above, a result obtained by computing

$$(\gamma_{n-1}A_n - S_nD_n)\Omega.$$

Then one applies again S_n to the quantity above and iterates the procedure. Finally, one recovers the analogue of the r.h.s. of (4.4), i.e.

$$\sum_{i=1}^{n-2} \alpha_i e_i,$$

for some integers α_i. Using similar arguments as above, one erases e_{n-2}, thus reducing the matter to a linear combination of $e_1, \dots, e_{n-3}$. Several iterations of the same procedure lead us to remove, in the following order, $e_{n-3}, \dots, e_3$, and thus to find a suitable $E_n \in \mathcal{S}'_n$ such that

$$E_n\Omega := \beta_1 e_1 + \beta_2 e_2,$$

for $\beta_1, \beta_2 \in \mathbb{Z}$. As a consequence,

$$\big(S_nE_n - (\beta_1 + \beta_2)r_n^2\big)\Omega = \beta_2 e_1 \otimes e_2,$$

and the last equality allows us to get e_2, since

$$\beta_2 e_2 = S_n\big(S_nE_n - (\beta_1 + \beta_2)r_n^2\big)\Omega. \tag{4.7}$$

The remaining e_i, for $i = 3, \dots, n-1$ can be similarly obtained.

The second step consists in finding the remaining elements of the canonical basis of $\mathcal{F}^o_m$. This is obtained, *mutatis mutandis*, as in (4.5). $\square$

To get a flavour of the above exposed procedure, in the case $n = 4$ one finds (4.7) has the following form

$$-e_2 = S_4^2\big(S_4 - r_4 - S_4\big[S_4(S_4 - r_4) - 3I + S_4(B_4 - 3I)\big]\big)\Omega.$$

As a consequence, we have the following

Theorem 4.2. *For any $n \in \mathbb{N}$, one has*

$$\sigma(S_n) = \operatorname{supp}(\mu_n)$$

and

$$\|S_n\| = r^{(n)}_{2^n},$$

where, as usual, σ denotes the spectrum of an operator.

Proof. We first recall that for any n, the map $\Theta_n : \operatorname{sp}(\mathcal{S}_n) \to \sigma(S_n)$ s.t. $\Theta_n(\varphi) = \varphi(S_n)$ is a homeomorphism. As a consequence, for any $f \in C(\operatorname{sp}(\mathcal{S}_n))$

$$\begin{aligned}\int_{\operatorname{sp}(\mathcal{S}_n)} f(\varphi)\mathrm{d}\nu_{\Omega,\Omega}(\varphi) &= \langle (S_n)_f\Omega, \Omega\rangle \\ &= \int_{\sigma(S_n)} (f \circ \Theta_n^{-1})(\varphi(S_n))\mathrm{d}\mu_n(\varphi(S_n)),\end{aligned}$$

where $f \mapsto (S_n)_f$ is the Gelfand isomorphism. Therefore, $\mu_n(B) = \nu_{\Omega,\Omega}(\Theta_n^{-1}(B))$ for any borelian B in $\sigma(S_n)$. Hence, by Theorem 4.1, μ_n results to be supported on the whole compact $\sigma(S_n)$. As $S_n = S_n^*$, the last part of the statement follows from Proposition 3.1. $\square$

The property that Ω is cyclic for any $\mathcal{S}'_n$ is crucial in the proof of Theorem 4.1. Then, one naturally wonders if the vacuum monotone vector is cyclic for the commutant of any C^*-algebra generated by a finite sum of gaussian monotone operators. The following example shows it is not generally true.

Let us take the operator $S_{1,3} := s_1 + s_3$ and denote by $\mathcal{S}_{1,3}$ the unital C^*-algebra generated by it. Here we show there does not exist any $T \in (\mathcal{S}_{1,3})'$ such that $T\Omega = e_2$. To this aim, we preliminary notice that for any $\xi := e_{i_1} \otimes \ldots \otimes e_{i_m}$ of the canonical basis of $\mathcal{F}^o_m$ such that $i_1 \geq 4$, one finds

$$S_{1,3}\xi = e_1 \otimes \xi + e_3 \otimes \xi.$$

As a consequence, we reduce our matter to the action of $S_{1,3}$ on the set

$$\{\Omega, e_1, e_2, e_3, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_3, e_1 \otimes e_2 \otimes e_3\}, \tag{4.8}$$

which is represented by the hermitian matrix $A := (a_{ij})_{i,j=1,\dots,8}$ assuming the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let T be an element in $\mathcal{S}'_{1,3}$, with $B := (b_{ij})_{i,j=1,\dots,8}$ its representing matrix on the vectors (4.8). The condition $[A, B] = 0$ immediately gives $b_{1j} \neq 0$ only when $j = 1, 2, 4, 6$. Moreover, the same condition implies $b_{31} = b_{32} = b_{34} = b_{36} = 0$. Thus e_2 does not belong to $\overline{\mathcal{S}'_{1,3}\Omega}$.

Therefore, it turns out that our approach does not give information on the relation between vacuum law and spectrum for general partial sums of position operators. Nevertheless, in the next lines we will show that, as in the case of vacuum distribution, also the norm depends only on the number of operators in the partial sum.

To this aim, fix an integer m and consider the set I given by a sequence of strictly increasing indices $1 \le i_1 < i_2 < \cdots < i_n = m$ such that there exists i_j, $j = 1, \dots, n$, for which $i_j \neq i_{j-1} + 1$, where $i_0 := 0$. As usual, we denote $S_I := \sum_{h=1}^{n} s_{i_h}$ and we look for the norm of S_I. If $J := \{1, \dots, m\} \setminus I$, one has

$$\mathcal{F}_m = \mathcal{F}_J \oplus \mathcal{F}_J^{\perp} \tag{4.9}$$

where $\mathcal{F}_J$ denotes the closure in $\mathcal{F}_m$ of the subspace $\mathcal{F}_m^o$ generated by the vectors

$$\{e_{j_1} \otimes \cdots \otimes e_{j_k} \mid k \ge 1, j_1 < \cdots < j_k \text{ and } j_l \in J \text{ for some } l\},$$

and $\mathcal{F}_J^{\perp}$ is the orthogonal complement of $\mathcal{F}_J$. The dense subspace of $\mathcal{F}_J^{\perp}$ built similarly as $\mathcal{F}_m^o$ will be denoted by $(\mathcal{F}_J^{\perp})^o$. After noticing that $\mathbb{C}\Omega$ belongs to $\mathcal{F}_J^{\perp}$, it not difficult to check that S_I leaves invariant both the subspaces $\mathcal{F}_J$ and $\mathcal{F}_J^{\perp}$. From now on, we denote by $S_{I,J}^{\perp}$ and $S_{I,J}$ the restrictions of S_I on $\mathcal{F}_J^{\perp}$ and $\mathcal{F}_J$, respectively.

Proposition 4.3. *Under the above notations, one has* $\sigma(S_{I,J}^{\perp}) = \sigma(S_n)$.

Proof. Since the vacuum distributions of $S_{I,J}^{\perp}$ and S_n are equal, as in Theorem 4.1 it is enough to prove that Ω is cyclic for the commutant of the unital C^*-algebra $\mathcal{S}_{I,J}^{\perp}$ generated by $S_{I,J}^{\perp}$. In fact, in this case the thesis will follow arguing as in the proof of Theorem 4.2. To this purpose, we first recall that the right hand side partial shift based on h is the one-to-one map $\theta_h : \mathbb{N} \to \mathbb{N}$ such that

$$\theta_h(k) := \begin{cases} k & \text{if } k < h \\ k+1 & \text{if } k \geq h. \end{cases}$$

If $j_1 < j_2 < \ldots < j_{m-n}$ are the elements of J, we denote by θ_J the composition $\theta_{j_{m-n}} \circ \cdots \circ \theta_{j_1}$. Consider $U_{\theta,J} \in \mathcal{B}(\mathcal{F}_m, \mathcal{F}_J^\perp)$ such that

$$\begin{aligned} &U_{\theta,J}\Omega := \Omega \\ &U_{\theta,J} e_B := e_{\theta_J(B)}\,, \end{aligned}$$

e_B being a generic element of the canonical basis of $\mathcal{F}_m$. As

$$\begin{aligned} &U^*_{\theta,J}\Omega = \Omega \\ &U^*_{\theta,J} e_C = e_{\theta_J^{-1}(C)}, \end{aligned}$$

where e_C is an arbitrary element of the canonical basis of $\mathcal{F}_J^\perp$, one achieves $U_{\theta,J}$ is unitary, i.e.

$$\begin{aligned} &U^*_{\theta,J} U_{\theta,J} = I \\ &U_{\theta,J} U^*_{\theta,J} = I_{\mathcal{F}_J^\perp}. \end{aligned} \tag{4.10}$$

Denote $\mathcal{S}^\perp_{I,J}$ the C^*-algebra generated by $S^\perp_{I,J}$ and $I_{\mathcal{F}_J^\perp}$. After recalling that $S^\perp_{I,J}$ leaves invariant $\mathcal{F}_J^\perp$, one finds for any k

$$S_n^k = U^*_{\theta,J} (S^\perp_{I,J})^k U_{\theta,J}.$$

This gives

$$[(S^\perp_{I,J})^k, U_{\theta,J} T U^*_{\theta,J}] = 0 \tag{4.11}$$

for any $T \in \mathcal{S}'_n$. Then it follows Ω is cyclic for $(\mathcal{S}^\perp_{I,J})'$, since from (4.10) and (4.11)

$$U_{\theta,J} \mathcal{S}'_n U^*_{\theta,J} \Omega = U_{\theta,J} \mathcal{F}^o_m = (\mathcal{F}_J^\perp)^o.$$

□

The next result allows us to compute the norm of any sum of monotone gaussian operators.

Proposition 4.4. *Under the notation introduced above, one has*

$$\|S_I\| = r^{(n)}_{2^n}$$

Proof. Indeed, as S_I leaves both the subspaces $\mathcal{F}_J$ and $\mathcal{F}_J^\perp$ invariant, from (4.9) one finds

$$S_I = S_{I,J} \oplus S^\perp_{I,J}.$$

Fix a generic element ξ in $\mathcal{F}_J^o$, i.e.

$$\xi := \sum_{h=1}^{p} \alpha_h e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}} ,$$

where $p \in \mathbb{N}$, $k_1^{(h)} < \cdots < k_r^{(h)}$ and $k_l^{(h)} \in J$ for some l. After recalling that $S_{I,J}$ acts only on $e_{k_1^{(h)}}$, one finds

$$\begin{aligned} S_{I,J}\xi = & \sum_{1\le h\le p,\ k_1^{(h)}=i_1} \alpha_h e_{k_2^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}} \\ & + \sum_{1\le h\le p,\ i_1<k_1^{(h)}<i_2} \alpha_h e_{i_1} \otimes e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}} \\ & + \sum_{1\le h\le p,\ k_1^{(h)}=i_2} \alpha_h \big(e_{i_1} \otimes e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}} + e_{k_2^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}}\big) \\ & + \sum_{1\le h\le p,\ i_2<k_1^{(h)}<i_3} \alpha_h \big(e_{i_1} + e_{i_2}\big) \otimes e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}} \\ & + \quad \cdots \qquad \cdots \qquad \cdots \\ & + \sum_{1\le h\le p,\ i_n<k_1^{(h)}} \alpha_h \left(\sum_{j=1}^{n} e_{i_j} \right) \otimes e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}} . \end{aligned}$$

Let us take $\eta := U_{\theta,J}\xi \in (\mathcal{F}_J^{\perp})^o$. It results

$$\begin{aligned} S^{\perp}_{I,J}\eta = & \sum_{1\le h\le p,\ \theta_J(k_1)^{(h)}=i_1} \alpha_h U_{\theta,J}\big(e_{k_2^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}}\big) \\ & + \sum_{1\le h\le p,\ i_1<k_1^{(h)}=\theta_J(k_1^{(h)})<i_2} \alpha_h e_{i_1} \otimes U_{\theta,J}\big(e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}}\big) \\ & + \sum_{1\le h\le p,\ i_1<k_1^{(h)}<\theta_J(k_1^{(h)})=i_2} \alpha_h \Big[e_{i_1} \otimes U_{\theta,J}\big(e_{k_1^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}}\big) \\ & + U_{\theta,J}\big(e_{k_2^{(h)}} \otimes \cdots \otimes e_{k_r^{(h)}}\big) \Big] \end{aligned}$$

$$+\sum_{1\le h\le p,\ i_1<k_1^{(h)}<i_2<\theta_J(k_1^{(h)})=i_3}\alpha_h\Big[(e_{i_1}+e_{i_2})\otimes U_{\theta,J}\big(e_{k_1^{(h)}}\otimes\cdots\otimes e_{k_r^{(h)}}\big)$$

$$+U_{\theta,J}\big(e_{k_2^{(h)}}\otimes\cdots\otimes e_{k_r^{(h)}}\big)\Big]$$

$$+\quad\cdots\quad\cdots\quad\cdots$$

$$+\sum_{1\le h\le p,\ i_n<k_1^{(h)}}\alpha_h\bigg(\sum_{j=1}^{n}e_{i_j}\bigg)\otimes e_{k_1^{(h)}}\otimes\cdots\otimes e_{k_r^{(h)}}.$$

As $U_{\theta,J}$ is unitary, it turns out

$$\|S_{I,J}\xi\|\le\|S^{\perp}_{I,J}\eta\|\le\|S^{\perp}_{I,J}\|\|\xi\|,$$

and the density of $\mathcal{F}^o_J$ in $\mathcal{F}_J$ gives $\|S_{I,J}\|\le\|S^{\perp}_{I,J}\|$. Since $S^{\perp}_{I,J}$ is self-adjoint and $\|S_I\|=\sup\{\|S_{I,J}\|,\|S^{\perp}_{I,J}\|\}$, the thesis follows from Proposition 4.3 and Theorem 4.2. □

5. APPENDIX

In the next lines we briefly present how a different approach with respect to the monotone convolution of Section 3 gives us some information about the interlacing structure connecting the atoms of μ_m and those of $\mu_1,\dots,\mu_{m-1}$. Here we point out that a similar achievement can be performed just using monotone independence, as shown in [11], Theorem 3.1. We decided to put here the following results since they refine the latter case, where interlacing relations are established between μ_m and μ_{m-1}, $m\ge 2$.

Let us take

$$u_{m,n}:=\omega_\Omega\big((S_m^2)^n\big).$$

As S_m^2 are bounded self-adjont operators, the sequence $u_{m,n}$ uniquely determines the symmetric probability measure, say ν_m, such that

$$u_{m,n}=\int_{\mathbb{R}}x^n\mathrm{d}\nu_m(x).$$

Using some results contained in [4, 5], one sees the elements of the moment generating functions sequence $(T_m)_m$ are mutually related in the following way

$$T_m(t)=\frac{1}{1-t-t\sum_{k=2}^{m}T_{k-1}(t)}.$$

This gives, for each m

$$T_{m+1}(t)=\frac{T_m(t)}{1-tT_m^2(t)}.\tag{5.1}$$

If $M_m(t)$ denotes the moments generating function of S_m, from (5.1) one achieves

$$M_{m+1}(t) = \frac{M_m(t)}{1 - t^2 M_m^2(t)}. \tag{5.2}$$

If

$$M_1(t) = \frac{Q_1(t)}{P_1(t)},$$

where $P_1(t) = 1 - t^2$ and $Q_1(t) = 1$, and more in general for arbitrary m

$$M_m(t) = \frac{Q_m(t)}{P_m(t)}, \tag{5.3}$$

(5.2) yields the following recursive formulas for $m \geq 2$

$$Q_{m+1}(t) = \prod_{k=1}^{m} P_k(t), \quad P_{m+1}(t) = P_m^2(t) - t^2 Q_m^2(t). \tag{5.4}$$

Thus, properties of the atoms have been reduced to those for roots of the above polynomials. We denote $\mathbb{R}[t]$ the ring of polynomials in the indeterminate t with coefficients in the field $\mathbb{R}$, and $\deg(P)$ is the degree of each $P \in \mathbb{R}[t]$. Furthermore, $\mathcal{Z}_{\mathbb{R}}(P)$ is the (possibly empty) set of the roots of P in $\mathbb{R}$. The following lemma is an application of Bolzano's Theorem.

Lemma 5.1. *Let $P, Q \in \mathbb{R}[t]$ such that $\mathcal{Z}_{\mathbb{R}}(P) = \{p_1, \ldots, p_m\}$ and $\mathcal{Z}_{\mathbb{R}}(Q) = \{q_1, \ldots, q_s\}$, for $m, n \geq 1$. Suppose $\mathcal{Z}_{\mathbb{R}}(P)$ and $\mathcal{Z}_{\mathbb{R}}(Q)$ are disjoint, and $Q(p_j) > 0$ for any $j = 1, \ldots, m$. Let $\{r_1, \ldots, r_{m+s}\}$ be the set of the real roots of P and Q taken in an increasing order. If there exists j such that r_j and r_{j+1} are zeros of different polynomials (*i.e *if $r_j \in \mathcal{Z}_{\mathbb{R}}(P)$, then $r_{j+1} \in \mathcal{Z}_{\mathbb{R}}(Q)$, and viceversa), one has $P^2(t) - Q(t)$ possesses at least a real root on (r_j, r_{j+1}).*

Let $(P_n)_n$ and $(Q_n)_n$ be sequences in $\mathbb{R}[t]$ such that

$$\begin{aligned} &\deg(P_0) > 0, \quad P_0(0) \neq 0, \quad Q_0(t) := 1 \\ &Q_{n+1}(t) := \prod_{k=0}^{n} P_k(t), \quad P_{n+1}(t) := P_n^2(t) - t^2 Q_n^2(t). \end{aligned} \tag{5.5}$$

Lemma 5.2. *Let $(P_n)_n$ and $(Q_n)_n$ be sequences satisfying* (5.5)*. Then, for any $n \geq 0$*

$$\mathcal{Z}_{\mathbb{R}}(P_n) \bigcap \mathcal{Z}_{\mathbb{R}}(Q_n) = \emptyset \tag{5.6}$$

and

$$\mathcal{Z}_{\mathbb{R}}(Q_{n+1}) = \bigcup_{k=0}^{n} \mathcal{Z}_{\mathbb{R}}(P_k), \qquad \mathcal{Z}_{\mathbb{R}}(P_n) \bigcap \mathcal{Z}_{\mathbb{R}}(P_m) = \emptyset, \ \ \text{if } m \neq n.$$

Proof. We preliminary notice that if there exists a common root t_0 for P_n and Q_n for some n, then t_0 is not null, as $P_0(0) \neq 0$ and (5.5). Denote now

$$J := \{n \in \mathbb{N} \mid P_{n+1}(t_0) = Q_{n+1}(t_0), \ \text{ for some } \ t_0 \in \mathbb{C}\}.$$

If we prove J is empty, (5.6) follows. In fact, suppose $J \neq \emptyset$ and take m as its minimum. Since $Q_{m+1}(t_0) = 0$ for some t_0, (5.5) gives that either $P_m(t_0) = 0$ or $Q_m(t_0) = 0$. The assumption $P_{m+1}(t_0) = 0$, together with (5.5) yields that both P_m and Q_m vanish in t_0, since $t_0 \neq 0$. This contradicts the minimum assumption on m.
Finally, one easily obtains $\mathcal{Z}_{\mathbb{R}}(Q_{n+1}) = \bigcup_{k=0}^{n} \mathcal{Z}_{\mathbb{R}}(P_k)$. If in addition $n \neq m$, say $n > m$, the last part of the statement follows from (5.6), as $\mathcal{Z}_{\mathbb{R}}(P_m) \subseteq \mathcal{Z}_{\mathbb{R}}(Q_n)$. □

As a consequence, if $P_n(t)$ and $Q_n(t)$ are as in (5.4), with $P_1(t) = 1 - t^2$, $Q_1(t) = 1$, one has

$$\begin{aligned} &\mathcal{Z}_{\mathbb{R}}(P_n) \bigcap \mathcal{Z}_{\mathbb{R}}(Q_n) = \emptyset, \\ &\mathcal{Z}_{\mathbb{R}}(Q_{n+1}) = \bigcup_{k=1}^{n} \mathcal{Z}_{\mathbb{R}}(P_k), \\ &\mathcal{Z}_{\mathbb{R}}(P_n) \bigcap \mathcal{Z}_{\mathbb{R}}(P_m) = \emptyset, \ \text{ if } \ m \neq n. \end{aligned} \tag{5.7}$$

Finally, we show the interlacing structure connecting the atoms μ_n and those of all μ_i, $i \leq n-1$. It is achieved by means of the roots of $P_n(t)$ and $Q_n(t)$. As these ones are both even functions, we reduce the matter to the positive half-plane.

Proposition 5.3. *Let $P_n(t)$ and $Q_n(t)$ be as in (5.4), with $P_1(t) = 1 - t^2$, $Q_1(t) = 1$. Then, for any $n \geq 1$ the roots of P_n and Q_n are real and simple. Moreover, if $p_1^{(n+1)} < \cdots < p_{2^n}^{(n+1)}$ and $q_1^{(n+1)} < \cdots < q_{2^n-1}^{(n+1)}$ are the positive zeros of P_{n+1} and Q_{n+1} respectively, one has*

$$p_1^{(n+1)} < q_1^{(n+1)} < p_2^{(n+1)} < q_2^{(n+1)} < \cdots < p_{2^n-1}^{(n+1)} < q_{2^n-1}^{(n+1)} < p_{2^n}^{(n+1)}.$$

Proof. Indeed, when $n = 1$ one finds that the positive zeros of P_2 and Q_2 are $\frac{\sqrt{5}\pm 1}{2}$ and 1, respectively.

Now we suppose the statement holds for any $m \leq n$, and consider the case $m = n+1$. As (5.4) gives

$$Q_{n+1}(t) = P_n(t) Q_n(t),$$

any positive root of Q_{n+1} is either a root of P_n or a zero of Q_n, and they do not share any zero by the induction assumption. As a consequence, Q_{n+1} has exactly $2^n - 1$ positive zeros. Let $p_1^{(n)} < \cdots < p_{2^{n-1}}^{(n)}$ be the totality of the positive zeros of P_n. From (5.4)

$$P_{n+1}\big(p_h^{(n)}\big) = -p_h^{(n)} Q_n^2\big(p_h^{(n)}\big) < 0,$$

for each $h = 1, \ldots, 2^{n-1}$, since Q_n and P_n have no common roots, and $Q_n^2(p_h^{(n)}) > 0$ as follows from (5.7). Similarly, for any $h = 1, \ldots, 2^{n-1}-1$

$$P_{n+1}\big(q_h^{(n)}\big) = P_n^2\big(q_h^{(n)}\big) > 0,$$

where $q_1^{(n)} < q_2^{(n)} < \cdots < q_{2^{n-1}-1}^{(n)}$ are the positive roots of Q_n. The induction assumption and Lemma 5.1 give us P_{n+1} has a root in each of the intervals $(p_h^{(n)}, q_h^{(n)})$ and $(q_h^{(n)}, p_{h+1}^{(n)})$, $h = 1, \ldots, 2^{n-1} - 1$. The thesis is then achieved as soon as one proves

$$P_{n+1}(t) = t^{2^{n+1}} + \sum_{k=1}^{2^{n+1}-2} a_k^{(n+1)} t^k + 1, \tag{5.8}$$

for all $n \geq 1$. (5.8) is indeed satisfied for $n = 1$, as $P_2(t) = t^4 - 3t^2 + 1$. Further, we assume it holds for any $P_m(t)$ with $m \leq n$. As a consequence, $P_n^2(t)$ is a monic polynomial with degree 2^{n+1} with a constant unital term.
From (5.4), one has

$$t^2 Q_n^2(t) = t^2\big(1 - t^2\big)^2 \prod_{r=2}^{n-1} \left(t^{2^r} + \sum_{k=1}^{2^r-2} a_k^r t^k + 1 \right)^2,$$

and since $\displaystyle\sum_{k=1}^{n-1} 2^k = 2^n - 2$, it follows that $\deg(t^2 Q_n^2(t)) = 2^{n+1} - 2$. Finally, (5.8) is achieved using again (5.4). □

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