

# THETA BLOCK CONJECTURE FOR PARAMODULAR FORMS OF WEIGHT 2

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ABSTRACT. In this paper we construct an infinite family of paramodular forms of weight 2 which are simultaneously Borcherds products and additive Jacobi lifts. This proves an important part of the theta-block conjecture of Gritsenko–Poor–Yuen (2013) related to the most important infinite series of theta-blocks of weight 2 and  $q$ -order 1. We also consider some applications of this result.

## 1. INTRODUCTION

Paramodular forms are Siegel modular forms of degree two with respect to the symplectic group  $\Gamma_t$  of elementary divisor  $(1, t)$ , the paramodular group. There are two ways to construct paramodular forms from Jacobi forms. The first one is the additive Jacobi lifting due to Gritsenko (see [6] and [5]) which lifts a holomorphic Jacobi form to a paramodular form. The second method is the multiplicative lifting (Borcherds automorphic product, see [1], [2]) in a form, proposed by Gritsenko–Nikulin in [12], which sends a weakly holomorphic Jacobi form of weight 0 to a meromorphic paramodular form. In [14], V. Gritsenko, C. Poor and D. Yuen investigated the paramodular forms which are simultaneously Borcherds products and Gritsenko lifts.

Let  $f : \mathbb{N} \rightarrow \mathbb{Z}$  be a function with a finite support, where  $\mathbb{N}$  is the set of nonnegative integers. We define *theta block* (see [15])

$$(1.1) \quad \Theta_f(\tau, z) = \eta^{f(0)}(\tau) \prod_{a=1}^{\infty} (\vartheta_a(\tau, z) / \eta(\tau))^{f(a)}$$

as a finite product of the Jacobi theta-series  $\vartheta_a(\tau, z) = \vartheta(\tau, az)$  divided by the Dedekind  $\eta$ -function  $\eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$ , where

$$\vartheta(\tau, z) = q^{\frac{1}{8}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n \geq 1} (1 - q^n \zeta)(1 - q^n \zeta^{-1})(1 - q^n)$$

is the odd Jacobi theta-series which is a holomorphic Jacobi form of weight  $\frac{1}{2}$  and index  $\frac{1}{2}$  with a multiplier system of order 8 (see [12]). We call  $\Theta_f$  a pure theta block if  $f$  is nonnegative on  $\mathbb{N}$ . In general, the pure theta block  $\Theta_f$  is a weak Jacobi form of weight  $f(0)/2$  and index  $N = \frac{1}{2} \sum a^2 f(a)$  with a character. It is important that for some functions  $f$  the theta block is holomorphic at infinity and is a holomorphic Jacobi form (see the general theory of theta blocks in [15]). In this way, one gets the most significant holomorphic Jacobi forms of small weights. The following conjecture was proposed in [14].

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**Theta block conjecture.** *Let the pure theta block  $\Theta_f$  be a holomorphic Jacobi form of weight  $k$  and index  $N$  with vanishing order one in  $q = e^{2\pi i\tau}$ . We define a weak Jacobi form  $\Psi_f = -(\Theta_f|T_-(2))/\Theta_f$  of weight 0 and index  $N$ , where  $T_-(2)$  is the index raising Hecke operator. Then*

$$\text{Grit}(\Theta_f) = \text{Borch}(\Psi_f)$$

*is a holomorphic symmetric paramodular form of weight  $k$  with respect to  $\Gamma_N^+$ .*

This conjecture gives a characterization of paramodular forms which are simultaneously Borcherds products and Gritsenko lifts. The conjecture was proved in [14, §8] for all known series of theta blocks of weights  $3 \leq k \leq 11$ . Another known infinite series is given by the theta blocks of weight 2 of type  $\frac{10-\vartheta}{6-\eta}$  found in [15]:

$$\phi_{2,\mathbf{a}} = \frac{\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_1+a_2}\vartheta_{a_2+a_3}\vartheta_{a_3+a_4}\vartheta_{a_1+a_2+a_3}\vartheta_{a_2+a_3+a_4}\vartheta_{a_1+a_2+a_3+a_4}}{\eta^6} \in J_{2,N(\mathbf{a})}$$

where  $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$  and

$$N(\mathbf{a}) = 2a_1^2 + 3a_1a_2 + 2a_1a_3 + a_1a_4 + 3a_2^2 + 4a_2a_3 + 2a_2a_4 + 3a_3^2 + 3a_3a_4 + 2a_4^2.$$

We remark that there are four infinite series of theta blocks of weight 2 in [15]. The series  $\phi_{2,\mathbf{a}}$  is the most important because it gives the first examples of holomorphic Jacobi forms of weight 2: the Jacobi-Eisenstein series of index 25 and the Jacobi cusp form of index 37 (see §4.2).

**Theorem 1.1.** *For any  $\mathbf{a} \in \mathbb{Z}^4$  such that  $\phi_{2,\mathbf{a}}$  is not identically zero, one has*

$$\text{Grit}(\phi_{2,\mathbf{a}}) = \text{Borch}\left(-\frac{\phi_{2,\mathbf{a}}|T_-(2)}{\phi_{2,\mathbf{a}}}\right) \in M_2(\Gamma_{N(\mathbf{a})}^+).$$

This theorem was announced in [16] with the main idea of the proof. In this paper we give its complete proof and present some applications.

## 2. MODULAR FORMS AND LIFT CONSTRUCTIONS

Let  $N$  be a positive integer. The *paramodular group* of level (or polarization)  $N$  is a subgroup of  $\text{Sp}_2(\mathbb{Q})$  defined by

$$(2.1) \quad \Gamma_N = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \text{Sp}_2(\mathbb{Q}), \quad \text{all } * \in \mathbb{Z}.$$

For  $N > 1$ , we shall use the following double normal extension

$$(2.2) \quad \Gamma_N^+ = \Gamma_N \cup \Gamma_N V_N, \quad V_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -N & 0 \end{pmatrix}.$$

This group acts on the Siegel upper half plane of genus 2

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in M(2, \mathbb{C}) : \text{Im } Z > 0 \right\}$$

in the usual way  $M\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$ ,  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_2(\mathbb{R})$ . We put  $(F|_k M)(Z) = \det(CZ + D)^{-k} F(M\langle Z \rangle)$  for any integer  $k$ .

**Definition 2.1.** A holomorphic function  $F : \mathbb{H}_2 \rightarrow \mathbb{C}$  is called a *Siegel paramodular form* of weight  $k$  and level  $N$  with character  $\chi$  if  $F|_k M = \chi(M)F$  for any  $M \in \Gamma_N$ . We denote the space of such modular forms by  $M_k(\Gamma_N, \chi)$ . A paramodular form  $F$  is called a *cuspidal form* if  $\Phi(F|_k g) = 0$  for all  $g \in \mathrm{Sp}_2(\mathbb{Q})$ , where  $\Phi$  is Siegel's operator. The space of paramodular cuspidal forms is denoted by  $S_k(\Gamma_N, \chi)$ .

Let  $\chi_N : \Gamma_N^+ \rightarrow \{\pm 1\}$  be the nontrivial binary character such that  $\chi_N(V_N) = -1$  and  $\chi_N|_{\Gamma_N} = 1$ . Then  $M_k(\Gamma_N)$  is decomposed into the direct sum of plus and minus  $V_N$ -eigenspaces. For  $F \in M_k(\Gamma_N^+, \chi_N^\epsilon)$  ( $\epsilon = 0$  or  $1$ ), we consider its Fourier and Fourier–Jacobi expansions

$$(2.3) \quad F(Z) = \sum_{m \geq 0} \sum_{\substack{n \in \mathbb{N}, r \in \mathbb{Z} \\ 4nmN - r^2 \geq 0}} c(n, r, m) q^n \zeta^r \xi^{mN} = \sum_{m \geq 0} \phi_{mN}(\tau, z) \xi^{mN},$$

where  $q = \exp(2\pi i\tau)$ ,  $\zeta = \exp(2\pi iz)$ ,  $\xi = \exp(2\pi i\omega)$ . Then we have the equality  $(-1)^{k+\epsilon} F(\tau, z, \omega) = F(N\omega, z, \tau/N)$ , which yields  $c(n, r, m) = (-1)^{k+\epsilon} c(m, r, n)$ . When  $k + \epsilon$  is odd (even),  $F$  is called *antisymmetric* (*symmetric*). We remark that  $F$  is a cuspidal form if and only if  $c(n, r, m) = 0$  unless  $4nmN - r^2 > 0$ .

The Fourier–Jacobi coefficient  $\phi_{mN} \in J_{k, mN}$  is a holomorphic Jacobi form (see [4] and the more general Definition 3.1 below).

We introduce the additive lifting which is completely determined by its first Fourier–Jacobi coefficient. Let  $\phi(\tau, z) \in J_{k, N}$ . The index raising Hecke operator  $T_-(m)$  is defined as follows

$$\phi|_k T_-(m) = m^{-1} \sum_{\substack{ad=m \\ b \bmod d}} a^k \phi\left(\frac{a\tau + b}{d}, az\right) \in J_{k, mN}.$$

The Fourier coefficients of  $\phi|_k T_-(m)$  are given by

$$c(n, r; \phi|_k T_-(m)) = \sum_{\substack{d \in \mathbb{N} \\ d|(n, r, m)}} d^{k-1} c\left(\frac{nm}{d^2}, \frac{r}{d}; \phi\right).$$

**Theorem 2.2** ([6]). *For  $\phi \in J_{k, N}$ , we have*

$$\mathrm{Grit}(\phi)(Z) = c(0, 0; \phi) G_k(\tau) + \sum_{m \geq 1} (\phi|_k T_-(m))(\tau, z) e^{2\pi i m N \omega} \in M_k(\Gamma_N^+, \chi_N^k)$$

where  $G_k(\tau) = (2\pi i)^{-k} (k-1)! \zeta(k) + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$  is the Eisenstein series of weight  $k$  on  $\mathrm{SL}_2(\mathbb{Z})$ . Moreover, if  $\phi$  is a Jacobi cuspidal form then  $\mathrm{Grit}(\phi)$  is a paramodular cuspidal form.

The paramodular form  $\mathrm{Grit}(\phi)$  is always symmetric. We now describe the second lifting, the Borcherds automorphic product (see [1] and [2]) in a form proposed by Gritsenko and Nikulin (see [12]). The Borcherds automorphic product is determined by the first two Fourier–Jacobi coefficients.

**Theorem 2.3** ([12]). *Let  $N$  be a positive integer. Assume that  $\Psi \in J_{0, N}^!$  is a weakly holomorphic Jacobi form of weight 0 and index  $N$  with Fourier expansion*

$$\Psi(\tau, z) = \sum_{n, r \in \mathbb{Z}} c(n, r) q^n \zeta^r$$

and  $c(n, r) \in \mathbb{Z}$  for  $4Nn - r^2 \leq 0$ . We set  $C = \frac{1}{4} \sum_{r \in \mathbb{Z}} r^2 c(0, r)$ . Then the function

$$(2.4) \quad \text{Borch}(\Psi) = \left( \eta^{c(0,0)} \prod_{r>0} \left( \frac{\vartheta_r}{\eta} \right)^{c(0,r)} \xi^C \right) \cdot \exp(-\text{Grit}(\Psi))$$

is a meromorphic paramodular form  $\text{Borch}(\Psi) \in M_k^{\text{mero}}(\Gamma_N^+, \chi_\Psi)$  of weight  $k = c(0,0)/2$  whose divisor in  $\Gamma_N^+ \backslash \mathbb{H}_2$  consists of Humbert modular surfaces

$$\text{Hum}(T_0) = \Gamma_N^+ \{Z \in \mathbb{H}_2 : n_0\tau + r_0z + Nm_0\omega = 0\}, \quad T_0 = \begin{pmatrix} n_0 & r_0/2 \\ r_0/2 & Nm_0 \end{pmatrix}$$

with  $\gcd(n_0, r_0, m_0) = 1$ ,  $m_0 \geq 0$  and  $\det(T_0) < 0$ . The multiplicity of  $\text{Borch}(\Psi)$  on  $\text{Hum}(T_0)$  is  $\sum_{n \geq 1} c(n^2 n_0 m_0, nr_0)$ . For  $\lambda \gg 0$ , on  $\{Z \in \mathbb{H}_2 : \text{Im } Z > \lambda I_2\}$  the following product expansion is valid

$$\text{Borch}(\Psi)(Z) = q^A \zeta^B \xi^C \prod_{\substack{n, r, m \in \mathbb{Z} : m \geq 0 \\ \text{if } m = 0 \text{ then } n \geq 0 \\ \text{if } m = n = 0 \text{ then } r < 0}} (1 - q^n \zeta^r \xi^{Nm})^{c(nm, r)}$$

where

$$24A = \sum_{r \in \mathbb{Z}} c(0, r), \quad 2B = \sum_{r \in \mathbb{N}} rc(0, r), \quad D_0 = \sum_{n < 0} \sigma_0(-n)c(n, 0).$$

The character  $\chi_\Psi$  of the paramodular form is generated by the character (or the multiplier system) of the theta block in  $\text{Borch}(\Psi)$  and the character  $\chi_N^{k+D_0}$ .

The automorphic product  $\text{Borch}(\Psi)$  is uniquely determined by the first two Fourier–Jacobi coefficients. The first factor in (2.4) is a theta block  $\Theta$  defined by the  $q^0$ -part of the Fourier expansion of  $\Psi$ . The second one is the product  $-\Theta\Psi$ . If  $\text{Grit}(\phi)$  is a Borcherds product, then  $\phi = \Theta$  and  $\phi|T_-(2) = -\Theta\Psi$ . Thus, we conclude that  $\phi$  is a theta block and  $\Psi = -(\phi|T_-(2))/\phi$ . Furthermore, we can show that  $\phi$  has exactly vanishing order one in  $q = e^{2\pi i\tau}$ , otherwise the constant term  $c(0,0)$  in the Fourier expansion of  $-(\phi|T_-(2))/\phi$  will be greater than  $2k$  (see [19, Proposition 7.2] for a proof). We conjecture that  $\phi$  should be a pure theta block, i.e.  $c(0, r) \geq 0$  in  $\Psi$ . There is no example contrary to the claim at present.

In [14] the conjecture was proved for the quasi-products of theta-functions

$$\eta^{3(8-l)} \vartheta_{d_1} \cdots \vartheta_{d_l} \in J_{12-l, N}, \quad N = (d_1^2 + \cdots + d_l^2)/2 \in \mathbb{N},$$

and for the products of three *theta-quarks*

$$\theta_{a_1, b_1} \theta_{a_2, b_2} \theta_{a_3, b_3} \in J_{3, N}, \quad N = \sum_{i=1}^3 (a_i^2 + a_i b_i + b_i^2),$$

where  $\theta_{a,b} = \vartheta_a \vartheta_b \vartheta_{a+b}/\eta$  is a holomorphic Jacobi form of weight 1 with a character of order 3 (see [3] and [15]).

The main idea of the proof of the conjecture for the mentioned above theta blocks is the following. The odd theta-function  $\vartheta(\tau, z)$  vanishes with order 1 for  $z = \lambda\tau + \mu$  for any  $\lambda, \mu \in \mathbb{Z}$ . Therefore we know the divisor of the theta block  $\Theta_f$  of  $q$ -order one. The Hecke operator  $T_-(m)$  keeps this divisor of the theta block. Therefore we know a part of the divisor of the lift of  $\Theta_f$ . According to Theorem 2.3, the divisor of  $\text{Borch}(\Psi)$  is determined by the singular Fourier coefficients  $c(n, r)$  with  $4nN - r^2 < 0$  of  $\Psi$ , where  $N$  is the index of the Jacobi form  $\Theta_f$ . By the construction,  $\text{Grit}(\Theta_f)$  vanishes on the divisors determined by the  $q^0$ -term of  $\Psi$ . If  $\text{Borch}(\Psi)$  does not

have another divisor then  $\text{Grit}(\Theta_f)/\text{Borch}(\Psi)$  is a holomorphic modular form of weight zero and then equals a constant by Köcher's principle. Unfortunately for this method of proof,  $\text{Borch}(\Psi)$  usually has additional divisors determined by singular Fourier coefficients from the higher  $q^n$ -terms. In order to pass through this difficulty, we represent  $\Theta_f$  as a pull-back of a Jacobi form  $\Theta_L$  in many variables associated to a certain positive definite lattice  $L$ . Since Jacobi forms in many variables have stronger symmetry, the function  $\Psi_L = -\frac{\Theta_L|_{T-(2)}}{\Theta_L}$  may have much simpler singular Fourier coefficients such that the divisor of  $\text{Borch}(\Psi_L)$  is determined only by the  $q^0$ -term of  $\Psi_L$ . In the next section, we give the corresponding definitions and results in the case of many variables.

### 3. ORTHOGONAL MODULAR FORMS AND JACOBI FORMS OF LATTICE INDEX

We start with the general setup (see [1], [3] or [13] for more details). Let  $M$  be an even lattice of signature  $(2, n)$  with  $n \geq 3$ . Let

$$(3.1) \quad \mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}^+$$

be the corresponding Hermitian symmetric domain of type IV (here  $+$  denotes one of its two connected components). By  $O^+(M)$  we denote the index 2 subgroup of the integral orthogonal group  $O(M)$  preserving  $\mathcal{D}(M)$ . By  $\tilde{O}^+(M)$  we denote the subgroup of  $O^+(M)$  acting trivially on the discriminant group of  $M$ . For any  $v \in M \otimes \mathbb{Q}$  satisfying  $(v, v) < 0$ , the rational quadratic divisor associated to  $v$  is defined as

$$\mathcal{D}_v = \{[Z] \in \mathcal{D}(M) : (Z, v) = 0\}.$$

We assume that  $M$  contains two hyperbolic planes and write  $M = U \oplus U_1 \oplus L(-1)$ , where  $U = \mathbb{Z}e \oplus \mathbb{Z}f$  ( $(e, e) = (f, f) = 0, (e, f) = 1$ ),  $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$  are two hyperbolic planes and  $L$  is an even integral positive definite lattice. We choose a basis of  $M$  of the form  $(e, e_1, \dots, f_1, f)$ , where  $\dots$  denotes a basis of  $L(-1)$ . We fix a tube realization of the homogeneous domain  $\mathcal{D}(M)$  related to the 1-dimensional boundary component determined by the isotropic plane  $F = \langle e, e_1 \rangle$ :

$$\mathcal{H}(L) = \{Z = (\tau, \mathfrak{z}, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\text{Im } Z, \text{Im } Z) > 0\},$$

where  $(\text{Im } Z, \text{Im } Z) = 2 \text{Im } \tau \text{Im } \omega - (\text{Im } \mathfrak{z}, \text{Im } \mathfrak{z})_L$ . In this setting, a Jacobi form is a modular form with respect to the Jacobi group  $\Gamma^J(L)$  which is the parabolic subgroup of  $O^+(M)$  preserving the isotropic plane  $F$  and acting trivially on  $L$ . This group is the semidirect product of  $\text{SL}_2(\mathbb{Z})$  with the Heisenberg group  $H(L)$  of  $L$  (see [6] and [3]). Let  $L^\vee$  denote the dual lattice of  $L$  and  $\text{rank}(L)$  denote the rank of  $L$ . We define Jacobi modular forms with respect to  $L$ .

**Definition 3.1.** For  $k \in \mathbb{Z}$ ,  $t \in \mathbb{N}$ , a holomorphic function  $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \rightarrow \mathbb{C}$  is called a *weakly holomorphic* Jacobi form of weight  $k$  and index  $t$  associated to  $L$ , if it satisfies the functional equations

$$(3.2) \quad \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(i\pi t \frac{c(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\right) \varphi(\tau, \mathfrak{z}),$$

$$(3.3) \quad \varphi(\tau, \mathfrak{z} + x\tau + y) = \exp(-i\pi t((x, x)\tau + 2(x, \mathfrak{z}))) \varphi(\tau, \mathfrak{z}),$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ ,  $x, y \in L$  and if it has a Fourier expansion

$$(3.4) \quad \varphi(\tau, \mathfrak{z}) = \sum_{n \geq n_0} \sum_{\ell \in L^\vee} f(n, \ell) q^n \zeta^\ell,$$

where  $n_0 \in \mathbb{Z}$ ,  $q = e^{2\pi i\tau}$  and  $\zeta^\ell = e^{2\pi i(\ell, \mathfrak{z})}$ . If the Fourier expansion of  $\varphi$  satisfies the condition  $(f(n, \ell) \neq 0 \implies n \geq 0)$  then  $\varphi$  is called a *weak* Jacobi form. If  $(f(n, \ell) \neq 0 \implies 2n - (\ell, \ell) \geq 0)$  (respectively  $> 0$ ) then  $\varphi$  is called a *holomorphic* (respectively, *cuspidal*) Jacobi form.

We denote by  $J_{k,L,t}^!$  (respectively,  $J_{k,L,t}^w$ ,  $J_{k,L,t}$ ,  $J_{k,L,t}^{\text{cusp}}$ ) the vector space of weakly holomorphic Jacobi forms (respectively, weak, holomorphic or cusp Jacobi forms) of weight  $k$  and index  $t$ . We note that the Jacobi forms in one variable  $J_{k,N}$  considered in the previous section are identical to the Jacobi forms  $J_{k,A_1,N}$  for the lattice  $A_1 = \langle 2 \rangle$  of rank 1.

The additive Jacobi lifting and Borcherds product of Jacobi forms in many variables are very similar to the case of one variable. Let  $\varphi \in J_{k,L,t}^!$ . For any positive integer  $m$ , we have

$$(3.5) \quad \varphi|_{k,t} T_-(m)(\tau, \mathfrak{z}) = m^{-1} \sum_{\substack{ad=m, a>0 \\ 0 \leq b < d}} a^k \varphi\left(\frac{a\tau + b}{d}, a\mathfrak{z}\right) \in J_{k,L,mt}^!,$$

and the Fourier coefficients of  $\varphi|_{k,t} T_-(m)(\tau, \mathfrak{z})$  are given by the formula

$$f_m(n, \ell) = \sum_{\substack{a \in \mathbb{N} \\ a|(n, \ell, m)}} a^{k-1} f\left(\frac{nm}{a^2}, \frac{\ell}{a}\right),$$

where  $a | (n, \ell, m)$  means that  $a | (n, m)$  and  $a^{-1}\ell \in L^\vee$ .

**Theorem 3.2** (see [6]). *Let  $\varphi \in J_{k,L,1}$ . Then the function*

$$\text{Grit}(\varphi)(Z) = f(0, 0)G_k(\tau) + \sum_{m \geq 1} \varphi|_{k,1} T_-(m)(\tau, \mathfrak{z}) e^{2\pi i m \omega}$$

*is a modular form of weight  $k$  for the stable orthogonal group  $\tilde{\mathcal{O}}^+(2U \oplus L(-1))$ . Moreover, this modular form is symmetric i.e.  $\text{Grit}(\varphi)(\tau, \mathfrak{z}, \omega) = \text{Grit}(\varphi)(\omega, \mathfrak{z}, \tau)$ .*

We fix an ordering  $\ell > 0$  in the dual lattice  $L^\vee$  in a way similar to positive root systems (see the bottom of page 825 in [7]). The long paper [7] contains, in particular, the results of the preprints [arXiv:1005.3753](https://arxiv.org/abs/1005.3753) and [arXiv:1203.6503](https://arxiv.org/abs/1203.6503) of the first author. In the last preprint the following theorem was proved.

**Theorem 3.3** (see Theorem 4.2 in [7] for details). *Let*

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}, \ell \in L^\vee} f(n, \ell) q^n \zeta^\ell \in J_{0,L,1}^!$$

*Assume that  $f(n, \ell) \in \mathbb{Z}$  for all  $2n - (\ell, \ell) \leq 0$ . There is a meromorphic modular form of weight  $f(0, 0)/2$  and character  $\chi$  with respect to  $\tilde{\mathcal{O}}^+(2U \oplus L(-1))$*

$$\text{Borch}(\varphi) = \left( \Theta_{f(0,*)}(\tau, \mathfrak{z}) \exp(2\pi i C\omega) \right) \exp(-\text{Grit}(\varphi)),$$

*where  $C = \frac{1}{2 \text{rank}(L)} \sum_{\ell \in L^\vee} f(0, \ell)(\ell, \ell)$  and*

$$(3.6) \quad \Theta_{f(0,*)}(\tau, \mathfrak{z}) = \eta(\tau)^{f(0,0)} \prod_{\ell > 0} \left( \frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \right)^{f(0,\ell)}$$

is a general theta block. The character  $\chi$  is induced by the character of the theta-product and by the relation  $\chi(V) = (-1)^D$ , where  $V : (\tau, \mathfrak{z}, \omega) \rightarrow (\omega, \mathfrak{z}, \tau)$ , and  $D = \sum_{n < 0} \sigma_0(-n)f(n, 0)$ .

The poles and zeros of  $\text{Borch}(\varphi)$  lie on the rational quadratic divisors  $\mathcal{D}_v$ , where  $v \in 2U \oplus L^\vee(-1)$  is a primitive vector with  $(v, v) < 0$ . The multiplicity of this divisor is given by

$$\text{mult } \mathcal{D}_v = \sum_{d \in \mathbb{Z}, d > 0} f(d^2 n, d\ell),$$

where  $n \in \mathbb{Z}$ ,  $\ell \in L^\vee$  such that  $(v, v) = 2n - (\ell, \ell)$  and  $v - (0, 0, \ell, 0, 0) \in 2U \oplus L(-1)$ .

The same function has the following infinite product expansion

$$\text{Borch}(\varphi)(Z) = q^A \zeta^{\vec{B}} \xi^C \prod_{\substack{n, m \in \mathbb{Z}, \ell \in L^\vee \\ (n, \ell, m) > 0}} (1 - q^n \zeta^\ell \xi^m)^{f(nm, \ell)},$$

where  $Z = (\tau, \mathfrak{z}, \omega) \in \mathcal{H}(L)$ ,  $q = \exp(2\pi i \tau)$ ,  $\zeta^\ell = \exp(2\pi i (\ell, \mathfrak{z}))$ ,  $\xi = \exp(2\pi i \omega)$ , the notation  $(n, \ell, m) > 0$  means that either  $m > 0$ , or  $m = 0$  and  $n > 0$ , or  $m = n = 0$  and  $\ell < 0$ , and

$$A = \frac{1}{24} \sum_{\ell \in L^\vee} f(0, \ell), \quad \vec{B} = \frac{1}{2} \sum_{\ell > 0} f(0, \ell) \ell.$$

**Remark 3.4.** In the case of arbitrary signature we can write the divisors of  $\text{Borch}(\varphi)$  in a way similar to the case of signature  $(2, 3)$  considered in Theorem 2.3. By the Eichler criterion (see [5, proof of Theorem 3.1] and [10, Proposition 3.3]), if  $v_1, v_2 \in 2U \oplus L^\vee(-1)$  are primitive, have the same norm, and the same image in the discriminant group, i.e.  $v_1 - v_2 \in 2U \oplus L(-1)$ , then there exists  $g \in \tilde{\mathcal{O}}^+(2U \oplus L(-1))$  such that  $g(v_1) = v_2$ . Therefore, for a primitive vector  $v \in 2U \oplus L^\vee(-1)$  with  $(v, v) < 0$ , there exists a vector  $(0, n, \ell, 1, 0) \in 2U \oplus L^\vee(-1)$  such that  $(v, v) = 2n - (\ell, \ell)$ ,  $v - (0, n, \ell, 1, 0) \in 2U \oplus L(-1)$  and

$$\tilde{\mathcal{O}}^+(2U \oplus L(-1)) \cdot \mathcal{D}_v = \tilde{\mathcal{O}}^+(2U \oplus L(-1)) \cdot \mathcal{D}_{(0, n, \ell, 1, 0)}.$$

It is known that the Fourier coefficients of  $\varphi \in J_{k, L, 1}^1$  satisfy (see [5, Lemma 2.1])

$$(3.7) \quad f(n, \ell) = f(n + (x, x)/2 + (\ell, x), \ell + x), \text{ for any } x \in L.$$

Thus the Fourier coefficient  $f(n, \ell)$  depends only on the (hyperbolic) norm  $2n - (\ell, \ell)$  of its index and the image of  $\ell$  in the discriminant group of  $L$ . The divisor of the Borcherds product in Theorem 3.3 is defined by the so-called *singular* Fourier coefficients  $f(n, \ell)$  with  $2n - (\ell, \ell) < 0$ . There are only a finite number of orbits of such coefficients that are supported because the norm  $2n - (\ell, \ell)$  of the indices of the nontrivial Fourier coefficients is restricted from below.

We say that the lattice  $L$  satisfies the condition  $\text{Norm}_2$  (see [8] and [13]) if

$$(3.8) \quad \text{Norm}_2 : \forall \bar{c} \in L^\vee/L \quad \exists h_c \in \bar{c} \quad \text{such that} \quad (h_c, h_c) \leq 2.$$

**Lemma 3.5.** *Let  $\varphi \in J_{0, L, 1}^w$  and assume that  $L$  satisfies the condition  $\text{Norm}_2$ . Then the singular Fourier coefficients of  $\varphi$  and the divisors of  $\text{Borch}(\varphi)$  are determined entirely by the  $q^0$ -term of  $\varphi$ , i.e. any divisor of  $\text{Borch}(\varphi)$  is of the form  $\mathcal{D}_{(0, 0, \ell_1, 1, 0)}$ .*

*Proof.* Suppose that  $f(n, \ell) \neq 0$  is singular, i.e.  $2n - (\ell, \ell) < 0$ . There exists a vector  $\ell_1 \in L^\vee$  such that  $\ell - \ell_1 \in L$  and  $(\ell_1, \ell_1) \leq 2$  because  $L$  satisfies  $\text{Norm}_2$  condition. It is clear that  $(\ell, \ell) - (\ell_1, \ell_1)$  is an even integer. If  $-2 \leq 2n - (\ell, \ell) < 0$ , it follows

that  $2n - (\ell, \ell) = -(\ell_1, \ell_1)$  and then  $f(n, \ell) = f(0, \ell_1)$  by (3.7). If  $2n - (\ell, \ell) < -2$ , then there exists a negative integer  $n_1$  satisfying  $2n - (\ell, \ell) = 2n_1 - (\ell_1, \ell_1)$ . Thus, there will be a nonzero Fourier coefficient  $f(n_1, \ell_1)$  with  $n_1 < 0$ , which contradicts the definition of weak Jacobi forms. According to Remark 3.4, we have  $\tilde{\mathcal{O}}^+(2U \oplus L(-1)) \cdot \mathcal{D}_{(0,n,\ell,1,0)} = \tilde{\mathcal{O}}^+(2U \oplus L(-1)) \cdot \mathcal{D}_{(0,0,\ell_1,1,0)}$ .  $\square$

Many examples of the lattices of type  $\text{Norm}_2$  were found in [8] and [13] by the construction of reflective modular forms. We prove below that  $A_4^\vee(5)$  is also a lattice from this class.

By definition,  $A_4 = \{(a_1, \dots, a_5) \in \mathbb{Z}^5 : a_1 + \dots + a_5 = 0\}$ . We fix the set of simple roots in  $A_4$

$$\begin{aligned} \alpha_1 &= (1, -1, 0, 0, 0) & \alpha_2 &= (0, 1, -1, 0, 0) \\ \alpha_3 &= (0, 0, 1, -1, 0) & \alpha_4 &= (0, 0, 0, 1, -1), \end{aligned}$$

and the set of 10 positive roots in  $A_4$

$$(3.9) \quad R^+(A_4) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}.$$

The fundamental weights of  $A_4$  are the vectors

$$\begin{aligned} w_1 &= \left(\frac{4}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}\right), & w_2 &= \left(\frac{3}{5}, \frac{3}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}\right), \\ w_3 &= \left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, -\frac{3}{5}, -\frac{3}{5}\right), & w_4 &= \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{4}{5}\right). \end{aligned}$$

So  $(\alpha_i, w_j) = \delta_{ij}$  and  $A_4^\vee/A_4 = \{0, w_1, w_2, w_3, w_4\}$ . Therefore the renormalization

$$A_4^\vee(5) = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 + \mathbb{Z}w_4, \quad 5(\cdot, \cdot),$$

is an even lattice of determinant 125 and its dual lattice is  $(A_4^\vee(5))^\vee = \frac{1}{5}A_4(5)$ .

**Lemma 3.6.** *The lattice  $A_4^\vee(5)$  satisfies the condition  $\text{Norm}_2$ .*

*Proof.* Consider the discriminant group  $D = (A_4^\vee(5))^\vee/A_4^\vee(5) = \frac{1}{5}A_4(5)/A_4^\vee(5)$ . We see from above that  $A_4^\vee/A_4$  is the cyclic group of order 5. We conclude that  $D \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$  is the 5-group of order 125.

We see that any two vectors in  $\frac{1}{5}A_4(5)$  of norm  $\frac{2}{5}$  (or of norm  $\frac{4}{5}$ ) are not equivalent modulo  $A_4^\vee(5)$ . The vectors of norm  $\frac{6}{5}$  in  $\frac{1}{5}A_4(5)$  have 30 equivalent classes in  $D$  and each class contains 2 elements. The vectors of norm  $\frac{8}{5}$  in  $\frac{1}{5}A_4(5)$  have 20 equivalent classes in  $D$  and each class contains 3 elements. The vectors of norm 2 in  $\frac{1}{5}A_4(5)$  have 24 equivalent classes in  $D$  and each class contains 5 elements. Moreover, the above 124 classes and the zero element form a representative system of  $D$ . We call this representative system the **standard system** of  $D$ .

As a 5-group,  $D$  is the bouquet of 31 cyclic subgroups of order 5. In the standard representative system, ten of them are generated by  $\frac{2}{5}$ -vectors and each subgroup contains two  $\frac{2}{5}$ -vectors and two  $\frac{8}{5}$ -vectors. Fifteen subgroups are generated by  $\frac{4}{5}$ -vectors and each subgroup contains two  $\frac{4}{5}$ -vectors and two  $\frac{6}{5}$ -vectors. Six subgroups are generated by 2-vectors and each subgroup contains four 2-vectors.

For any vector  $\beta$  of norm 2 in  $\frac{1}{5}A_4(5)$ , the lattice  $A_4^\vee(5) + \mathbb{Z}\beta$  is isomorphic to the root lattice  $A_4$ .  $\square$

From the proof of the above lemma we obtain



**Lemma 3.7.** *Let  $O(A_4^\vee(5))$  be the integral orthogonal group of  $A_4^\vee(5)$  and  $O(D)$  be the orthogonal group of the discriminant group  $D$  of  $A_4^\vee(5)$ .*

- (a)  $O(A_4^\vee(5)) \cong O(A_4) = W(A_4) \rtimes C_2$ , where  $W(A_4)$  is the Weyl group of  $A_4$  and the subgroup  $C_2$  is of order 2 and generated by the operator  $\mathfrak{z} \mapsto -\mathfrak{z}$ .
- (b) In the standard system,  $O(A_4^\vee(5))$  acts transitively on the set of classes of the same norm.
- (c) The natural homomorphism  $O(A_4^\vee(5)) \rightarrow O(D)$  is surjective.

We construct a reflective modular form for the lattice  $2U \oplus A_4^\vee(-5)$ . To this end, we consider the Kac–Weyl denominator function of the affine Lie algebra of type  $A_4$  (see [17], [7])

$$(3.10) \quad \Theta_{A_4}(\tau, \mathfrak{z}) = \eta(\tau)^4 \prod_{r \in R^+(A_4)} \frac{\vartheta(\tau, (r, \mathfrak{z}))}{\eta(\tau)}, \quad \mathfrak{z} \in A_4 \otimes \mathbb{C}$$

where  $R^+(A_4)$  is the set of positive roots of  $A_4$  defined in (3.9).

It is easy to check that  $\Theta_{A_4}$  is a weak Jacobi form of weight 2 and index 1 for the lattice  $A_4^\vee(5)$  (see [7, Corollary 2.7] for the case of any root system). Moreover, it is anti-invariant under the action of the Weyl group  $W(A_4)$  and invariant under the action of  $C_2$ . From [17] it follows that  $\Theta_{A_4}(\tau, \mathfrak{z})$  is holomorphic at infinity but we do not need this fact here.

In the dual basis  $\mathfrak{z} = z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 w_4 \in A_4 \otimes \mathbb{C}$ ,  $z_i \in \mathbb{C}$ , we have

$$(3.11) \quad \Theta_{A_4}(\tau, \mathfrak{z}) = \eta^{-6} \vartheta(z_1) \vartheta(z_2) \vartheta(z_3) \vartheta(z_4) \vartheta(z_1 + z_2) \vartheta(z_2 + z_3) \vartheta(z_3 + z_4) \\ \vartheta(z_1 + z_2 + z_3) \vartheta(z_2 + z_3 + z_4) \vartheta(z_1 + z_2 + z_3 + z_4),$$

where  $\vartheta(z) = \vartheta(\tau, z)$ . Next, we define a weak Jacobi form of weight 0

$$(3.12) \quad \Psi_{A_4}(\tau, \mathfrak{z}) = -\frac{(\Theta_{A_4}|T_-(2))(\tau, \mathfrak{z})}{\Theta_{A_4}(\tau, \mathfrak{z})} \\ = \sum_{\substack{r \in A_4 \\ (r, r) = 2}} \exp(2\pi i(r, \mathfrak{z})) + 4 + O(q) \in J_{0, A_4^\vee(5), 1}^w.$$

The Fourier expansion above contains all types of singular Fourier coefficients of the weak Jacobi form of weight 0 according to Lemma 3.5 and Lemma 3.6. Therefore, the corresponding Borcherds product  $\text{Borch}(\Psi_{A_4})$  is a holomorphic modular form of weight 2 with respect to  $\tilde{O}^+(2U \oplus A_4^\vee(-5))$ . Its first Fourier–Jacobi coefficient is equal to  $\Theta_{A_4}$ . Thus,  $\Theta_{A_4}$  is a holomorphic Jacobi form of weight 2. Moreover, the modular group of  $\text{Borch}(\Psi_{A_4})$  is determined by  $\Theta_{A_4}$ . By Lemma 3.7, we have

$$\Phi_{2, A_4^\vee(5)} = \text{Borch}(\Psi_{A_4}) \in M_2(O^+(2U \oplus A_4^\vee(-5)), \chi_2),$$

and by Lemma 3.5 and Lemma 3.7 (b), its divisor is equal to

$$(3.13) \quad \sum_{\substack{v \in 2U \oplus \frac{1}{5}A_4(-5) \\ (v, v) = -\frac{2}{5}}} \mathcal{D}_v = O^+(2U \oplus A_4^\vee(-5)) \cdot \mathcal{D}_{(0,0,\alpha_1/5,1,0)}.$$

The function  $\text{Grit}(\Theta_{A_4})$  is a modular form of weight 2 for  $O^+(2U \oplus A_4^\vee(-5))$  which is anti-invariant under  $W(A_4)$  and invariant under  $C_2$ .

**Lemma 3.8.** *Let  $\ell$  be a nonzero vector in  $L^\vee$ . If  $\varphi \in J_{k,L,1}$  vanishes to at least order  $m \geq 1$  along the divisor  $(\ell, \mathfrak{z}) \in \mathbb{Z}\tau + \mathbb{Z}$ , then  $\text{Grit}(\varphi)$  vanishes to order at least  $m$  on  $\mathcal{D}_{(0,0,\ell,1,0)}$ .*

*Proof.* In the formula of the action of the parabolic Hecke operator  $T_-(m)$  (see (3.5)) the action on  $\mathfrak{z}$  is linear.  $\square$

By the above lemma, we get  $\text{Div}(\text{Grit}(\Theta_{A_4})) \supset \text{Div}(\text{Borch}(\Psi_{A_4}))$ . Then we have  $\text{Grit}(\Theta_{A_4}) = \text{Borch}(\Psi_{A_4})$  by Köcher's principle. Thus we have proved the following.

**Theorem 3.9.** *The following identity is true for the modular form of weight 2*

$$(3.14) \quad \Phi_{2, A_4^\vee(5)} = \text{Grit}(\Theta_{A_4}) = \text{Borch}(\Psi_{A_4}) \in M_2(\mathcal{O}^+(2U \oplus A_4^\vee(-5)), \chi_2),$$

and it has reflective divisor (3.13). The character  $\chi_2$  is of order 2 and defined by the relations  $\chi_2|_{\tilde{\mathcal{O}}^+(2U \oplus A_4^\vee(-5))} = 1$ ,  $\chi_2|_{C_2} = 1$  and  $\chi_2|_{W(A_4)} = \det$ .

**Remark 3.10.** The function  $\Phi_{2, A_4^\vee(5)}$  is a reflective modular form of singular weight (see [7] and [13] for the definitions of reflective divisors and reflective modular forms) and has been constructed by N. Scheithauer in another way (see [21]). Scheithauer constructed this function at the zero-dimensional cusp related to  $U \oplus U(5) \oplus A_4$  using the lifting from scalar-valued modular forms on congruence subgroups to modular forms for the Weil representation of  $\text{SL}_2(\mathbb{Z})$ . By [18, Corollary 1.13.3], we conclude

$$U \oplus U(5) \oplus A_4 \cong 2U \oplus A_4^\vee(5),$$

because the two lattices belong to the same genus. Our construction corresponds to the one-dimensional cusp related to the decomposition  $2U \oplus A_4^\vee(5)$  and it gives the additive Jacobi lifting of this reflective modular form.

**Remark 3.11. A hyperbolization of the affine Lie algebra  $\hat{\mathfrak{g}}(A_4)$ .** The result of Theorem 3.9 has important applications to the theory of Lie algebras. It shows that there exists a hyperbolization of the affine Lie algebra  $\hat{\mathfrak{g}}(A_4)$ , i.e. a Lorentzian Kac–Moody algebra with the following property: the first Fourier–Jacobi coefficient of the automorphic Kac–Weyl–Borcherds denominator function of such generalized hyperbolic Kac–Moody algebra is the Kac–Weyl denominator function of the affine Lie algebra  $\hat{\mathfrak{g}}(A_4)$ . The generators and relations of this new algebra are defined by the Fourier coefficients of the lift  $\text{Grit}(\Theta_{A_4})$ . This is a new example in a rather short series:  $\hat{\mathfrak{g}}(A_1)$  (see [12]),  $\hat{\mathfrak{g}}(4A_1)$  and  $\hat{\mathfrak{g}}(3A_2)$  (see [7]),  $\hat{\mathfrak{g}}(A_2)$  (see [11]). Twenty-three root systems of Niemeier lattices are also this type of examples (see [7]).

#### 4. APPLICATIONS

**4.1. The proof of Theorem 1.1.** Now we can prove the main result of the paper that the theta-block conjecture is true for the theta blocks of type  $\frac{10-\vartheta}{6-\eta}$ . We prove Theorem 1.1 by considering a specialisation of the modular form of Theorem 3.9 (compare with the proof of [14, Theorem 8.2]). It is known that paramodular forms for  $\Gamma_t$  can be viewed as modular forms for  $\tilde{\mathcal{O}}^+(2U \oplus \langle -2t \rangle)$  (see [5], [12]). Let  $z \in \mathbb{C}$ ,  $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ , and  $\mathbf{v} = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 w_4 \in A_4^\vee(5)$ . The specialisation  $\Phi_{2, A_4^\vee(5)}(\tau, z\mathbf{v}, \omega)$  is a modular form with respect to  $\tilde{\mathcal{O}}^+(2U \oplus \langle -2N(\mathbf{a}) \rangle)$ . In fact, the index  $N(\mathbf{a})$  is the half of the (square) norm of the vector  $\mathbf{v}$  in  $A_4^\vee(5)$ . By (3.10) and (3.11) we get  $\Theta_{A_4}(\tau, z\mathbf{v}) = \phi_{2, \mathbf{a}}(\tau, z)$ . We consider only such  $\mathbf{a}$  that  $\phi_{2, \mathbf{a}} \neq 0$ . Due to the linear action of the parabolic Hecke operators  $T_-(m)$  on the coordinate  $\mathfrak{z}$ , the pull-back  $\text{Grit}(\Theta_{A_4})(\tau, z\mathbf{v}, \omega)$  is equal to  $\text{Grit}(\phi_{2, \mathbf{a}}) \neq 0$ . The Borcherds product is also described in terms of the operators  $T_-(m)$  (see Theorem 3.3). Therefore  $\text{Borch}(\Psi_{A_4})(\tau, z\mathbf{v}, \omega)$  is the Borcherds product defined by  $\Psi_{A_4}(\tau, z\mathbf{v})$ . Moreover,  $\Psi_{A_4}(\tau, z\mathbf{v}) = -\frac{\phi_{2, \mathbf{a}}|_{T_-(2)}}{\phi_{2, \mathbf{a}}}$  because the parabolic

Hecke operator  $T_-(2)$  commutes with the specialisation  $\mathfrak{z} = z\mathbf{v}$ . This gives the relation

$$\text{Grit}(\phi_{2,\mathbf{a}}) = \text{Borch} \left( -\frac{\phi_{2,\mathbf{a}}|T_-(2)}{\phi_{2,\mathbf{a}}} \right).$$

**4.2. Explicit divisors of the paramodular forms of weight 2 and linear relations between Fourier coefficients.** Our construction gives explicit formulas for the divisors of the modular forms from Theorem 1.1.

The first modular form corresponds to  $\mathbf{a} = (1, 1, 1, 1)$ . In this case we obtain the first complementary Jacobi–Eisenstein series  $E_{2,25;1} = \eta^{-6}\vartheta^4\vartheta_2^3\vartheta_3^2\vartheta_4 \in J_{2,25}$  (see [4, §2]). The singular Fourier coefficients of  $\psi_{0,25} = -(E_{2,25;1}|T_-(2))/E_{2,25;1}$  are represented by  $\text{Sing}(\psi_{0,25}) = \zeta^4 + 2\zeta^3 + 3\zeta^2 + 4\zeta + 4$ . Thus the divisor of  $\text{Grit}(E_{2,25;1})$  is completely defined by divisors of the theta block, i.e. the corresponding Borcherds product has no additional divisor.

When the index is larger than 25, the corresponding Borcherds products have additional divisors in general and the additional divisors will yield certain relations between the Fourier coefficients of theta blocks. We first recall one example in [14]. Let  $\mathbf{a} = (1, 1, 1, 2)$ , we get  $\phi_{2,37} = \eta^{-6}\vartheta^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5 \in J_{2,37}^{\text{cusp}}$ . Note that  $\dim J_{2,37}^{\text{cusp}} = 1$ . Let  $\psi_{0,37} = -(\phi_{2,37}|T_-(2))/\phi_{2,37}$ . The singular Fourier coefficients of  $\psi_{0,37}$  are represented by

$$\text{Sing}(\psi_{0,37}) = \zeta^5 + \zeta^4 + 2\zeta^3 + 3\zeta^2 + 3\zeta + 4 + q^6\zeta^{30}.$$

The coefficient  $q^6\zeta^{30}$  determines the divisor which does not appear in the theta block  $\phi_{2,37}$ . We call it the additional divisor of  $\text{Grit}(\phi_{2,37})$ . This is the last term in the formula for the full divisor of  $\text{Borch}(\psi_{0,37})$ :

$$\begin{aligned} & 10 \text{Hum} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 37 \end{pmatrix} + 4 \text{Hum} \begin{pmatrix} 0 & 1 \\ 1 & 37 \end{pmatrix} + 2 \text{Hum} \begin{pmatrix} 0 & 3/2 \\ 3/2 & 37 \end{pmatrix} \\ & + \text{Hum} \begin{pmatrix} 0 & 2 \\ 2 & 37 \end{pmatrix} + \text{Hum} \begin{pmatrix} 0 & 5/2 \\ 5/2 & 37 \end{pmatrix} + \text{Hum} \begin{pmatrix} 6 & 15 \\ 15 & 37 \end{pmatrix}. \end{aligned}$$

The fact that  $\text{Grit}(\phi_{2,37})$  vanishes on the additional divisor is equivalent to the fact (see [14, Page 170]) that the Fourier coefficients of  $\phi_{2,37}$  satisfy the linear relation

$$(4.1) \quad \forall n, r \in \mathbb{Z}, \quad \sum_{a \in \mathbb{Z}} c(6a^2 + na, 30a + r; \phi_{2,37}) = 0.$$

Next, we establish similar relations for other Jacobi forms of weight 2 and small indices. We know from [4] that the dimensions of the spaces of Jacobi forms of weight 2 and index 43, 50, 53 are all 1. The generators can be constructed by the theta blocks

$$\begin{aligned} \mathbf{a} = (-1, 5, -1, -2) : & \quad \phi_{2,43} = \eta^{-6}\vartheta^3\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_5 \in J_{2,43}^{\text{cusp}} \\ \mathbf{a} = (2, -1, -3, 6) : & \quad \phi_{2,50} = \eta^{-6}\vartheta^2\vartheta_2^3\vartheta_3^2\vartheta_4^2\vartheta_6 \in J_{2,50} \\ \mathbf{a} = (1, -6, 3, 1) : & \quad \phi_{2,53} = \eta^{-6}\vartheta^3\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6 \in J_{2,53}^{\text{cusp}}. \end{aligned}$$

We put  $\psi_{0,m} = -(\phi_{2,m}|T_-(2))/\phi_{2,m}$  for  $m = 43, 50, 53$ . Their singular Fourier coefficients are represented by

$$\begin{aligned}\text{Sing}(\psi_{0,43}) &= \zeta^5 + 2\zeta^4 + 2\zeta^3 + 2\zeta^2 + 3\zeta + 4 + q^2\zeta^{19} + q^3\zeta^{23}, \\ \text{Sing}(\psi_{0,50}) &= \zeta^6 + 2\zeta^4 + 2\zeta^3 + 3\zeta^2 + 2\zeta + 4 + q^5\zeta^{32} + 2q^{11}\zeta^{47} + 2q^{12}\zeta^{49}, \\ \text{Sing}(\psi_{0,53}) &= \zeta^6 + \zeta^5 + \zeta^4 + 2\zeta^3 + 2\zeta^2 + 3\zeta + 4 + q\zeta^{15} + q^2\zeta^{21} + q^6\zeta^{36}.\end{aligned}$$

Since the functions  $\text{Grit}(\phi_{2,j})$  vanish on the additional divisors, their Fourier coefficients satisfy the similar relations for all additional divisors. For example, in the case of  $\phi_{2,43}$ , we obtain two relations

$$\begin{aligned}\forall n, r \in \mathbb{Z}, \quad & \sum_{a \in \mathbb{Z}} c(2a^2 + na, 19a + r; \phi_{2,43}) = 0 \\ \forall n, r \in \mathbb{Z}, \quad & \sum_{a \in \mathbb{Z}} c(3a^2 + na, 23a + r; \phi_{2,43}) = 0.\end{aligned}$$

**4.3. Reflective modular form of weight 12.** We give one more property of the lattice  $A_4^\vee(5)$ .

**Lemma 4.1.** *There is a primitive embedding of  $A_4^\vee(5)$  into the Leech lattice  $\Lambda_{24}$ .*

*Proof.* In fact, the lattice  $A_4^\vee(5) < \Lambda_{24}$  is a fixed point sublattice with respect to the automorphism of cycle shape  $5^5/1$ . We refer to [20, §9] for details.  $\square$

Any sublattice of  $\Lambda_{24}$  with property  $\text{Norm}_2$  produces a strongly 2-reflective modular form. This is the pull-back of the Borcherds reflective modular form  $\Phi_{12} \in M_{12}(\text{O}^+(II_{2,26}), \det)$  (see [13, Theorem 4.2]). According to Lemma 3.6, we obtain a new example of reflective modular forms.

**Theorem 4.2.** *Consider the primitive sublattice  $2U \oplus A_4^\vee(-5) \hookrightarrow 2U \oplus \Lambda_{24}(-1)$  and the embedding of the homogenous domains  $\mathcal{D}(2U \oplus A_4^\vee(-5)) \hookrightarrow \mathcal{D}(2U \oplus \Lambda_{24}(-1))$ . Then*

$$\Phi_{12, A_4^\vee(-5)} = \Phi_{12}|_{\mathcal{D}(2U \oplus A_4^\vee(-5))} \in M_{12}(\tilde{\text{O}}^+(2U \oplus A_4^\vee(-5)), \det)$$

*is a strongly reflective modular form with complete  $(-2)$ -divisor*

$$\text{Div}(\Phi_{12, A_4^\vee(-5)}) = \sum_{\substack{r \in 2U \oplus A_4^\vee(-5) \\ (r,r) = -2}} \mathcal{D}_r.$$

**Corollary 4.3.** *The modular form  $\Phi_{12, A_4^\vee(-5)}$  determines a Lorentzian Kac–Moody algebra. For this algebra, the 2-reflective Weyl group of  $U \oplus A_4^\vee(-5)$  has a Weyl vector of norm 0, i.e. it has parabolic type.*

*Proof.* See a general construction of Lorentzian Kac–Moody algebras by 2-reflective modular forms in [13].  $\square$

**Corollary 4.4.** *The modular variety  $\tilde{\text{O}}^+(2U \oplus A_4^\vee(-5)) \setminus \mathcal{D}(2U \oplus A_4^\vee(-5))$  is at least uniruled.*

*Proof.* One can use the automorphic criterion proved in [8]. We note that this modular variety can be considered as a moduli space of lattice polarized  $K3$  surfaces.  $\square$

**4.4. Quasi pull-backs.** We proved the main theorem using a pull-back of the reflective modular form  $\Phi_{2,A_4^\vee(5)}$ . In this subsection we construct more pull-backs and quasi pull-backs of  $\Phi_{2,A_4^\vee(5)}$  in order to obtain interesting relations between liftings of multidimensional theta blocks. Using the same arguments about quasi pull-backs as in [9]–[13], we obtain the next proposition.

**Proposition 4.5.** *Assume that  $T = 2U \oplus T_0(-1) \hookrightarrow 2U \oplus A_4^\vee(-5)$  is a primitive sublattice of signature  $(2, n)$  with  $n = 3, 4, 5$ . We consider the corresponding embedding of the homogenous domains  $\mathcal{D}(T) \hookrightarrow \mathcal{D}(2U \oplus A_4^\vee(-5))$  and a finite set*

$$R_{\frac{2}{5}}(T_0^\perp) = \left\{ v \in \frac{1}{5}A_4(-1) : (v, v) = -\frac{2}{5}, (v, T_0) = 0 \right\}.$$

Let  $\mathcal{D}(T)^\bullet$  be the affine cone of  $\mathcal{D}(T)$ . Then the function

$$\Phi_{2,A_4^\vee(5)}|_T = \frac{\Phi_{2,A_4^\vee(5)}(Z)}{\prod_{v \in R_{\frac{2}{5}}(T_0^\perp)/\pm 1} (Z, v)} \Big|_{\mathcal{D}(T)^\bullet}$$

is a nontrivial modular form of weight  $2 + \frac{1}{2}|R_{\frac{2}{5}}(T_0^\perp)|$  with respect to  $\tilde{O}^+(T)$  with a character of order 2. The modular form  $\Phi_{2,A_4^\vee(5)}|_T$  vanishes only on rational quadratic divisors of type  $\mathcal{D}_u(T)$ , where  $u$  is the orthogonal projection of one vector  $v \in 2U \oplus \frac{1}{5}A_4(-1)$  with  $(v, v) = -\frac{2}{5}$  to  $T^\vee$  satisfying  $-\frac{2}{5} \leq (u, u) < 0$ . If the set  $R_{\frac{2}{5}}(T_0^\perp)$  is non-empty then  $F|_T$  is a cusp form.

**Example 4.6.** Let  $T_0 = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$ . Then  $T_0$  is isomorphic to the lattice  $A_3(5)$ . In this case  $T_0^\perp = \mathbb{Z}w_4$  and the set  $R_{\frac{2}{5}}(T_0^\perp)$  is empty. Let  $\mathfrak{z}_3 = z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3 = (2z_1 - z_2)w_1 + (2z_2 - z_1 - z_3)w_2 + (2z_3 - z_2)w_3 - z_3w_4$ .

The pull-back on any sublattice of this type commutes with the additive Jacobi lifting. In the coordinates fixed above we have

$$\begin{aligned} \Theta_{A_4}|_{A_3(5)}(\tau, \mathfrak{z}_3) &= \eta^{-6} \vartheta(2z_1 - z_2) \vartheta(z_1 + z_2 - z_3) \vartheta(z_1 + z_3) \vartheta(z_1) \\ (4.2) \quad & \vartheta(2z_2 - z_1 - z_3) \vartheta(z_2 + z_3 - z_1) \vartheta(z_2 - z_1) \\ & \vartheta(2z_3 - z_2) \vartheta(z_3 - z_2) \vartheta(z_3) \in J_{2,A_3,5}. \end{aligned}$$

Therefore,  $\text{Grit}(\Theta_{A_4}|_{A_3(5)})$  is a modular form of weight 2 for  $\tilde{O}^+(2U \oplus A_3(-5))$  and also a Borcherds product.

**Example 4.7.** The sublattice  $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3$  is isomorphic to  $2A_1(5)$ . By taking  $z_2 = 0$  in (4.2), we get

$$\Theta_{2A_1(5)} = \frac{\vartheta^2(z_1 + z_3) \vartheta^2(z_1 - z_3) \vartheta^2(z_1) \vartheta^2(z_3) \vartheta(2z_1) \vartheta(2z_3)}{\eta^6} \in J_{2,2A_1,5}.$$

Note that  $\dim J_{2,2A_1,5} = 1$  and 5 is the smallest index such that there exists a nontrivial Jacobi form of weight 2 for  $2A_1$ . Therefore,  $\text{Grit}(\Theta_{2A_1(5)})$  is a modular form of weight 2 for  $\tilde{O}^+(2U \oplus 2A_1(-5))$  and a Borcherds product.

We note that the Jacobi form  $\Theta_{A_4}$  of type  $10\text{-}\vartheta/6\text{-}\eta$  generates a tower of the lifts of quasi pull-backs of type  $9\text{-}\vartheta/3\text{-}\eta$  of weight 3 and  $8\text{-}\vartheta$  of weight 4. The theta block conjecture for these theta blocks was proved in [14] based on the reflective modular forms constructed in [7]. The functions considered below have fewer parameters than the modular forms in [7] but they are cusp forms.

**Example 4.8.** Let  $T_0 = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3$ . Its Gram matrix is

$$A_0 = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}, \quad \det(A_0) = 50, \quad A_0^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3/2 \end{pmatrix}.$$

In this case, we have  $T_0^\perp = \mathbb{Z}\alpha_4$  and  $R_{\frac{2}{5}}(T^\perp) = \{\pm\alpha_4/5\}$ . We write  $\mathfrak{z} = \mathfrak{z}_1 + \mathfrak{z}_2$  with  $\mathfrak{z}_1 \in T_0 \otimes \mathbb{C}$ , and  $\mathfrak{z}_2 \in T_0^\perp \otimes \mathbb{C}$ . The quasi pull-back on  $T_0 \subset A_4^\vee(5)$  can be written in the affine coordinate as the derivative at  $\mathfrak{z}_2 = 0$  (see [9] and [11, §8.4]). In this particular case we have the differential operator with respect to  $z_4$  which commutes with the Hecke operators in the Jacobi lifting. As a result we obtain the following Jacobi form of weight 3 of type 9- $\vartheta/3$ - $\eta$

$$(4.3) \quad \Theta_{T_0} = \frac{\vartheta(z_1)\vartheta(z_2)\vartheta(z_1+z_2)\vartheta^2(z_3)\vartheta^2(z_2+z_3)\vartheta^2(z_1+z_2+z_3)}{\eta^3} \in J_{3,A_0,1}^{\text{cusp}}.$$

Therefore,  $\text{Grit}(\Theta_{T_0}) \in S_3(\tilde{\mathcal{O}}^+(2U \oplus A_0(-1)), \chi_2)$  is a **cusp** form of weight 3 with the Borcherds automorphic product constructed by  $\Psi_{A_4}(\tau, \mathfrak{z})|_{z_4=0}$ .

**Example 4.9.** We can continue the construction of quasi pull-back by setting  $z_2 = 0$ . Then we get a Jacobi lifting of *canonical* weight with Borcherds product in four variables

$$\text{Grit}(\vartheta^2(z_1)\vartheta^4(z_3)\vartheta^2(z_1+z_3)) \in S_4(\tilde{\mathcal{O}}^+(2U \oplus B_0(-1)), \chi_2), \quad B_0 = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}.$$

The examples considered above support the following generalization of the theta block conjecture formulated in the introduction.

**Conjecture 4.10.** *Let  $\phi \in J_{k,L,1}$ . The function  $\text{Grit}(\phi)$  has a Borcherds product expansion, i.e.*

$$\text{Grit}(\phi) = \text{Borch} \left( -\frac{\phi|_{T_-}(2)}{\phi} \right),$$

*if and only if  $\phi$  is a pure theta block of type (3.6) with  $f(0, \ell) \geq 0$  for all  $\ell$  and  $\phi$  has vanishing order one in  $q$ .*

**Remark 4.11.** The ‘‘only if’’ part of the above conjecture has an immediate corollary. If there exists a non-constant modular form of weight  $k$  associated to  $\tilde{\mathcal{O}}^+(2U \oplus L(-1))$  which is simultaneously an additive lift and a Borcherds product, then  $\text{rank}(L) \leq 8$  and  $\text{rank}(L)/2 \leq k \leq 12 - \text{rank}(L)$ . In fact, since  $\phi$  has vanishing order one in  $q$ , the number  $A$  in Theorem 3.3 is equal to 1. Equation (3.6) defines a Jacobi form for  $L$ . The Fourier coefficients of any Jacobi form of weight 0 define a generalized 2-design in the dual lattice  $L^\vee$  (see [7, Proposition 2.6])

$$\sum_{\ell \in L^\vee} f(0, \ell)(\ell, \mathfrak{z})^2 = 2C(\mathfrak{z}, \mathfrak{z}) \quad \forall \mathfrak{z} \in L \otimes \mathbb{C}.$$

Therefore, the number of  $\ell > 0$  with non-zero  $f(0, \ell)$  is at least  $\text{rank}(L)$ . In view of the singular weight i.e.  $k \geq \frac{1}{2} \text{rank}(L)$  (see e.g. [1, Corollary 3.2, 3.3]), we prove the above claim.

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