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ON ANDREWS' INTEGER PARTITIONS WITH EVEN PARTS BELOW ODD PARTS

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ABSTRACT. Recently, Andrews defined a partition function $\mathcal{EO}(n)$ which counts the number of partitions of n in which every even part is less than each odd part. He also defined a partition function $\overline{\mathcal{EO}}(n)$ which counts the number of partitions of n enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. Andrews proposed to undertake a more extensive investigation of the properties of $\overline{\mathcal{EO}}(n)$. In this article, we prove infinite families of congruences for $\overline{\mathcal{EO}}(n)$. We next study distribution of $\overline{\mathcal{EO}}(n)$. We prove that there are infinitely many integers N in every arithmetic progression for which $\overline{\mathcal{EO}}(2N)$ is even; and that there are infinitely many integers M in every arithmetic progression for which $\overline{\mathcal{EO}}(2M)$ is odd so long as there is at least one. We further prove that $\overline{\mathcal{EO}}(n)$ is even for almost all n. Very recently, Uncu has treated a different subset of the partitions enumerated by $\mathcal{EO}(n)$. We prove that Uncu's partition function is divisible by 2^k for almost all k. We use arithmetic properties of modular forms and Hecke eigenforms to prove our results.

1. Introduction and statement of results

A partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n. In a recent paper, Andrews [1] studied the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n where every even part is less than each odd part. He denoted by $\overline{\mathcal{EO}}(n)$, the number of partitions counted by $\mathcal{EO}(n)$ in which *only* the largest even part appears an odd number of times. For example, $\mathcal{EO}(8) = 12$ with the relevant partitions being $8, 6+2, 7+1, 4+4, 4+2+2, 5+3, 5+1+1+1, 2+2+2+2, 3+3+2, 3+3+1+1, 3+1+1+1+1+1, 1+1+1+1+1+1+1+1$; and $\overline{\mathcal{EO}}(8) = 5$, with the relevant partitions being $8, 4+2+2, 3+3+2, 3+3+1+1, 1+1+1+1+1+1+1+1$.

Andrews proved that the partition function $\overline{\mathcal{EO}}(n)$ has the following generating function [1, Eqn. (3.2)]:

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^2} = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}, \tag{1.1}$$

where $(a;q)_\infty := \prod_{n\geq 0}(1-aq^n)$. In the same paper, he proposed to undertake a more extensive investigation of the properties of $\overline{\mathcal{EO}}(n)$. The objective of this

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paper is to study divisibility properties of $\overline{\mathcal{EO}}(n)$. To be specific, we use the theory of Hecke eigenforms to establish the following two infinite families of congruences for $\overline{\mathcal{EO}}(n)$ modulo 2 and 8, respectively.

Theorem 1.1. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \geq 5$ is prime such that $p_i \equiv 2 \pmod 3$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$\overline{\mathcal{EO}}\left(p_1^2 \dots p_{k+1}^2 n + \frac{p_1^2 \dots p_k^2 p_{k+1}(3j + p_{k+1}) - 1}{3}\right) \equiv 0 \pmod 2.$$

Let $p \geq 5$ be a prime such that $p \equiv 2 \pmod 3$. By taking all the primes $p_1, p_2, \dots, p_{k+1}$ to be equal to the same prime p in Theorem 1.1, we obtain the following infinite family of congruences for $\overline{\mathcal{EO}}(n)$:

$$\overline{\mathcal{EO}}\left(p^{2(k+1)} n + p^{2k+1} j + \frac{p^{2(k+1)} - 1}{3}\right) \equiv 0 \pmod 2,$$

where $j \not\equiv 0 \pmod p$. In particular, for all $n \geq 0$ and $j \not\equiv 0 \pmod 5$, we have

$$\overline{\mathcal{EO}}\left(25n + 5j + 8\right) \equiv 0 \pmod 2.$$

Theorem 1.2. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \equiv 1 \pmod{24}$ is prime such that $\overline{\mathcal{EO}}\left(\frac{19p_i - 1}{3}\right) \equiv 0 \pmod 8$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$\overline{\mathcal{EO}}\left(8p_1^2 \dots p_{k+1}^2 n + \frac{p_1^2 \dots p_k^2 p_{k+1}(24j + 19p_{k+1}) - 1}{3}\right) \equiv 0 \pmod 8.$$

Let p be a prime such that $p \equiv 1 \pmod{24}$ and $\overline{\mathcal{EO}}\left(\frac{19p-1}{3}\right) \equiv 0 \pmod 8$. By taking all the primes $p_1, p_2, \dots, p_{k+1}$ to be equal to the same prime p in Theorem 1.2, we obtain the following infinite family of congruences for $\overline{\mathcal{EO}}(n)$:

$$\overline{\mathcal{EO}}\left(8p^{2(k+1)} n + 8p^{2k+1} j + \frac{19p^{2(k+1)} - 1}{3}\right) \equiv 0 \pmod 8,$$

where $j \not\equiv 0 \pmod p$. In particular, if we choose $p = 1009$, then $1009 \equiv 1 \pmod{24}$ and $\frac{19 \times 1009 - 1}{3} = 6390$. Using *Mathematica* we verify that $\overline{\mathcal{EO}}\left(6390\right) \equiv 0 \pmod 8$. Thus, for all $n \geq 0$ and $j \not\equiv 0 \pmod{1009}$, we have

$$\overline{\mathcal{EO}}\left(8144648n + 8072j + 6447846\right) \equiv 0 \pmod 8.$$

In [1], Andrews proved that, for all $n \geq 0$

$$\overline{\mathcal{EO}}(10n + 8) \equiv 0 \pmod 5. \tag{1.2}$$

In this article, we prove that the congruence (1.2) is also true modulo 4 if $n \not\equiv 0 \pmod 5$. To be specific, we prove the following result.

Theorem 1.3. *Let $t \in \{1, 2, 3, 4\}$. Then for all $n \geq 0$ we have*

$$\overline{\mathcal{EO}}(10(5n + t) + 8) \equiv 0 \pmod{20}.$$

We note that Theorem 1.3 is not true if $t = 0$. For example, $\overline{\mathcal{EO}}(8)$ is not divisible by 4.

For a nonnegative integer n, let $p(n)$ denote the number of partitions of n. In [12], Ono proved that there are infinitely many integers N in every arithmetic progression for which $p(N)$ is even; and that there are infinitely many integers M in every arithmetic progression for which $p(M)$ is odd so long as there is at least one. Ono's result gave an affirmative answer to a well-known conjecture on parity of $p(n)$ in an arithmetic progression. In the following theorem, we prove the same for the partition function $\overline{\mathcal{EO}}(n)$. We note that $\overline{\mathcal{EO}}(2n+1)=0$ for all $n\geq 0$.

Theorem 1.4. *For any arithmetic progression* $r \pmod t$*, there are infinitely many integers* $N\equiv r \pmod t$ *for which* $\overline{\mathcal{EO}}(2N)$ *is even. Also, for any arithmetic progression* $r \pmod t$*, there are infinitely many integers* $M\equiv r \pmod t$ *for which* $\overline{\mathcal{EO}}(2M)$ *is odd, provided there is one such* M*. Furthermore, if there does exist an* $M\equiv r \pmod t$ *for which* $\overline{\mathcal{EO}}(2M)$ *is odd, then the smallest such* M *is less than*

$$\frac{2^{9+j}3^7t^6}{d^2}\prod_{p|6t}\left(1-\frac{1}{p^2}\right)-2^j,$$

where $d=\gcd(12r-1,t)$ *and* $2^j>\dfrac{t}{12}$.

A well-known conjecture of Parkin and Shanks [13] states that the even and odd values of $p(n)$ are equally distributed, that is,

$$\lim_{X\to\infty}\frac{\#\left\{0\leq n\leq X : p(n)\equiv r \pmod 2\right\}}{X}=\frac{1}{2},$$

where $r\in\{0,1\}$. Little is known regarding this conjecture. In the following theorem we prove that $\overline{\mathcal{EO}}(2n)$ is almost always even.

Theorem 1.5. *Let* $n\geq 0$*. Then* $\overline{\mathcal{EO}}(8n+6)$ *is almost always divisible by* 8*, namely,*

$$\lim_{X\to\infty}\frac{\#\left\{0\leq n\leq X : \overline{\mathcal{EO}}(8n+6)\equiv 0 \pmod 8\right\}}{X}=1.$$

Recently, Uncu [17] has treated a different subset of the partitions enumerated by $\mathcal{EO}(n)$. Also see [1, p. 435]. We denote by $\mathcal{EO}_u(n)$ the partition function defined by Uncu, and the generating function is given by

$$\sum_{n=0}^{\infty}\mathcal{EO}_u(n)q^n=\frac{1}{(q^2;q^4)_\infty^2}. \tag{1.3}$$

For any fixed positive integer k, Gordon and Ono [5] proved that the number of partitions of n into distinct parts is divisible by 2^k for almost all n. Similar studies are done for some other partition functions, for example see [2, 4, 9, 16]. In this article, we study divisibility of the partition function $\mathcal{EO}_u(n)$ by 2^k. To be specific, we prove the following result.

Theorem 1.6. *Let* k *be a positive integer. Then* $\mathcal{EO}_u(2n)$ *is almost always divisible by* 2^k*, namely,*

$$\lim_{X\to\infty}\frac{\#\left\{0\leq n\leq X : \mathcal{EO}_u(2n)\equiv 0 \pmod{2^k}\right\}}{X}=1.$$

2. Preliminaries

In this section, we recall some definitions and basic facts on modular forms. For more details, see for example [11, 7]. We first define the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\},$$
$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\},$$
$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\},$$
$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\}$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, \text{ and } b \equiv c \equiv 0 \pmod N \right\},$$

where N is a positive integer. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N. The smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the level of Γ. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N. The index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1+p^{-1}),$$

where p denotes a prime.

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}$. We identify ∞ with $\frac{1}{0}$ and define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar+bs}{cr+ds}$, where $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2} (cz+d)^{-\ell} f(\gamma z).$$

Definition 2.1. Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:

(1) We have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^\ell f(z)$$
for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

(2) If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_\ell\gamma)(z)$ has a Fourier expansion of the form

$$(f|_\ell\gamma)(z) = \sum_{n\geq 0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$. That is, f is holomorphic at all the cusps of Γ.

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_\ell(\Gamma)$. A modular form $f \in M_\ell(\Gamma)$ is called a cusp form if f vanishes at all the cusps of Γ. The subspace of $M_\ell(\Gamma)$ consisting of cusp forms is denoted by $S_\ell(\Gamma)$.

Definition 2.2. [11, Definition 1.15] If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$. The corresponding space of cusp forms is denoted by $S_\ell(\Gamma_0(N), \chi)$. If χ is the trivial character then we write $M_\ell(\Gamma_0(N))$ and $S_\ell(\Gamma_0(N))$ for short.

Recall that Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

We now recall two theorems from [11, p. 18] which will be used to prove our result.

Theorem 2.3. [11, Theorem 1.64 and Theorem 1.65] *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^\ell \prod_{\delta|N} \delta^{r_\delta}}{d}\right)$. In addition, if c, d, and N are positive integers with $d \mid N$ and $\gcd(c,d) = 1$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is $\frac{N}{24}\sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d}) d\delta}$.

Suppose that $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.3. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$.

Definition 2.4. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of Hecke operator T_m on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty} \left(\sum_{d|\gcd(n,m)} \chi(d) d^{\ell-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is prime, we have

$$f(z)|T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{\ell-1} a\left(\frac{n}{p}\right) \right) q^n. \tag{2.1}$$

We note that $a(n) = 0$ unless n is a nonnegative integer.

Definition 2.5. A modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exists a complex number $\lambda(m)$ for which

$$f(z)|T_m = \lambda(m) f(z). \tag{2.2}$$

3. Proof of Theorems 1.1 and 1.2

We use the theory of Hecke eigenforms to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} \equiv (q;q)_\infty^8 \pmod 2.$$

This gives

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^{3n+1} \equiv \eta^8(3z) \pmod 2.$$

Let $\eta^8(3z) = \sum_{n=1}^{\infty} a(n)q^n$. Then $a(n) = 0$ if $n \not\equiv 1 \pmod 3$ and for all $n \geq 0$,

$$\overline{\mathcal{EO}}(n) \equiv a(3n+1) \pmod 2. \tag{3.1}$$

By Theorem 2.3, we have $\eta^8(3z) \in S_4(\Gamma_0(9))$. Since $\eta^8(3z)$ is a Hecke eigenform (see, for example [10]), (2.1) and (2.2) yield

$$\eta^8(3z)|T_p = \sum_{n=1}^{\infty} \left(a(pn) + p^3 a\left(\frac{n}{p}\right) \right) q^n = \lambda(p) \sum_{n=1}^{\infty} a(n) q^n,$$

which implies

$$a(pn) + p^3 a\left(\frac{n}{p}\right) = \lambda(p) a(n). \tag{3.2}$$

Putting $n = 1$ and noting that $a(1) = 1$, we readily obtain $a(p) = \lambda(p)$. Since $a(p) = 0$ for all $p \not\equiv 1 \pmod 3$, we have $\lambda(p) = 0$. From (3.2), we obtain

$$a(pn) + p^3 a\left(\frac{n}{p}\right) = 0. \tag{3.3}$$

From (3.3), we derive that for all $n \geq 0$ and $p \nmid r$,

$$a(p^2 n + pr) = 0 \tag{3.4}$$

and

$$a(p^2n) = -p^3 a(n) \equiv a(n) \pmod{2}. \tag{3.5}$$

Substituting n by $3n - pr + 1$ in (3.4) and together with (3.1), we find that

$$\overline{\mathcal{EO}}\left(p^2n + \frac{p^2-1}{3} + pr\frac{1-p^2}{3}\right) \equiv 0 \pmod{2}. \tag{3.6}$$

Substituting n by $3n + 1$ in (3.5) and using (3.1), we obtain

$$\overline{\mathcal{EO}}\left(p^2n + \frac{p^2-1}{3}\right) \equiv \overline{\mathcal{EO}}(n) \pmod{2}. \tag{3.7}$$

Since $p \geq 5$ is prime, so $3 \mid (1-p^2)$ and $\gcd\left(\frac{1-p^2}{3}, p\right) = 1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{3}r$. Thus (3.6) can be rewritten as

$$\overline{\mathcal{EO}}\left(p^2n + \frac{p^2-1}{3} + pj\right) \equiv 0 \pmod{2}, \tag{3.8}$$

where $p \nmid j$.

Now, $p_i \geq 5$ are primes such that $p_i \not\equiv 1 \pmod{3}$. Since

$$p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{3} = p_1^2\left(p_2^2 \dots p_k^2 n + \frac{p_2^2 \dots p_k^2 - 1}{3}\right) + \frac{p_1^2 - 1}{3},$$

using (3.7) repeatedly we obtain that

$$\overline{\mathcal{EO}}\left(p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{3}\right) \equiv \overline{\mathcal{EO}}(n) \pmod{2}. \tag{3.9}$$

Let $j \not\equiv 0 \pmod{p_{k+1}}$. Then (3.8) and (3.9) yield

$$\overline{\mathcal{EO}}\left(p_1^2 \dots p_{k+1}^2 n + \frac{p_1^2 \dots p_k^2 p_{k+1}(3j + p_{k+1}) - 1}{3}\right) \equiv 0 \pmod{2}.$$

This completes the proof of the theorem. □

To prove Theorem 1.2, we need that the eta-quotient $\eta^5(96z)/\eta(24z)$ is an eigenform for the Hecke operators T_p, where $p \equiv 1 \pmod{24}$. This has been observed to be true by Scott Ahlgren. We now present below the proof given by Ahlgren which was communicated to us through an email. Let $F_1 = \eta^5(24z)/\eta(96z)$, $F_7 = \eta^3(24z)\eta(96z)$, $F_{13} = \eta(24z)\eta^3(96z)$, and $F_{19} = \eta^5(96z)/\eta(24z)$. Then F_j is supported on exponents congruent to $j \pmod{24}$. The Hecke operators T_p for $p \equiv 5, 11, 17, 23 \pmod{24}$ annihilate each of these forms. The Hecke operators T_p for $p \equiv 1, 5, 13, 19 \pmod{24}$ map F_j to a multiple of $F_{j'}$, where $j' \equiv pj \pmod{24}$. It turns out that a linear combination of the forms F_j is an eigenform of all of the Hecke operators. In [8, p. 209], equation (13.84) expresses the linear combination as an eigenform. Since the F_j are supported on distinct classes of coefficients, it follows that F_j are eigenforms of all the Hecke operators.

Proof of Theorem 1.2. We first recall the following 2-dissection formula from [3, Entry 25, p. 40]:

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5 (q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty}. \tag{3.10}$$

From (1.1), we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n=\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{3.11}$$

Combining (5.1) and (5.2), and then extracting the terms with odd powers of q, we deduce that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n=2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{3.12}$$

We again combine (5.1) and (5.3), and then extract the terms with odd powers of q to obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^n=4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3}.$$

Since $(q;q)_\infty^2\equiv(q^2;q^2)_\infty \pmod 2$, we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^n\equiv 4\frac{(q^4;q^4)_\infty^5}{(q;q)_\infty} \pmod 8.$$

This gives

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^{24n+19}\equiv 4\frac{\eta(96z)^5}{\eta(24z)} \pmod 8.$$

Let $\frac{\eta(96z)^5}{\eta(24z)}=\sum_{n=1}^{\infty}a(n)q^n$. It is clear that $a(n)=0$ if $n\not\equiv 19 \pmod{24}$. Also, for all $n\geq 0$,

$$\overline{\mathcal{EO}}(8n+6)\equiv 4a(24n+19) \pmod 8. \tag{3.13}$$

By Theorem 2.3, we have $\frac{\eta(96z)^5}{\eta(24z)}\in S_2(\Gamma_0(2304))$. Since $\frac{\eta(96z)^5}{\eta(24z)}$ is a Hecke eigenform for the Hecke operator T_p, where $p\equiv 1 \pmod{24}$, (2.1) and (2.2) yield

$$a(pn)+p\left(\frac{2}{p}\right)a\left(\frac{n}{p}\right)=\lambda(p)a(n). \tag{3.14}$$

Putting $n=19$ in (3.14) and noting that $p\not\equiv 19 \pmod{24}$, we obtain $a(19p)=\lambda(p)a(19)$. Also, $a(19)=1$, and hence $a(19p)=\lambda(p)$. Thus (3.14) gives

$$a(pn)+p\left(\frac{2}{p}\right)a\left(\frac{n}{p}\right)=a(19p)a(n). \tag{3.15}$$

From (3.15), we obtain that for all $n\geq 0$ and $p\nmid r$,

$$a(p^2n)+a(n)\equiv a(19p)a(pn) \pmod 2 \tag{3.16}$$

and

$$a(p^2n+pr)=a(19p)a(pn+r). \tag{3.17}$$

Let $A(n)=a(24n+19)$. Let p be a prime such that $p\equiv 1 \pmod{24}$. Now, replacing n by $24n-pr+19$ in (3.17), we obtain

(3.18)

$$A\left(p^2n+19\frac{p^2-1}{24}+pr\frac{1-p^2}{24}\right)=A\left(19\frac{p-1}{24}\right)A\left(pn+19\frac{p-1}{24}+r\frac{1-p^2}{24}\right).$$

We note that $\gcd\left(\frac{1-p^2}{24}, p\right) = 1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{24}r$. Thus, (3.18) can be rewritten as

$$A\left(p^2n + 19\frac{p^2-1}{24} + pj\right) = A\left(19\frac{p-1}{24}\right) A\left(pn + 19\frac{p-1}{24} + j\right), \tag{3.19}$$

where $p \nmid j$. Similarly, replacing n by $24n+19$ in (3.16), we have, modulo 2

$$A\left(p^2n + 19\frac{p^2-1}{24}\right) + A(n) \equiv A\left(19\frac{p-1}{24}\right) A\left(pn + 19\frac{p-1}{24}\right). \tag{3.20}$$

Let p be such that $\overline{\mathcal{EO}}\left(\frac{19p-1}{3}\right) \equiv 0 \pmod 8$. Then, using the relation $\overline{\mathcal{EO}}(8n+6) \equiv 4A(n) \pmod 8$, we have $A\left(19\frac{p-1}{24}\right) \equiv 0 \pmod 2$. Hence, (3.19) and (3.20) imply

$$A\left(p^2n + 19\frac{p^2-1}{24} + pj\right) \equiv 0 \pmod 2 \tag{3.21}$$

and

$$A\left(p^2n + 19\frac{p^2-1}{24}\right) \equiv A(n) \pmod 2. \tag{3.22}$$

From our hypothesis, we have $p_i \geq 5$ are primes such that $p_i \equiv 1 \pmod{24}$ and $A\left(19\frac{p_i-1}{24}\right) \equiv 0 \pmod 2$. Now, using (3.22) we deduce that

$$A\left(p_1^2 \dots p_k^2 n + 19\frac{p_1^2 \dots p_k^2 - 1}{24}\right) \equiv A(n) \pmod 2.$$

Replacing n by $p_{k+1}^2 n + 19\frac{p_{k+1}^2-1}{24} + p_{k+1}j$, and then using (3.21) we obtain

$$A\left(p_1^2 \dots p_k^2 p_{k+1}^2 n + 19\frac{p_1^2 \dots p_k^2 p_{k+1}^2 - 1}{24} + p_1^2 \dots p_k^2 p_{k+1} j\right) \equiv 0 \pmod 2.$$

We complete the proof by using the fact that $\overline{\mathcal{EO}}(8n+6) \equiv 4A(n) \pmod 8$. $\square$

4. Proof of Theorem 1.3

We prove Theorem 1.3 using the approach developed in [14, 15]. To this end, we first recall some definitions and results from [14, 15]. For a positive integer M, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. If $r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of M, we write $r = (r_{\delta_1}, \dots, r_{\delta_k})$. Define $c_r(n)$ by

$$\sum_{n=0}^{\infty} c_r(n)q^n := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_\delta}. \tag{4.1}$$

The approach to proving congruences for $c_r(n)$ developed by Radu [14, 15] reduces the number of cases that one must check as compared with the classical method which uses Sturm's bound alone.

Let m be a positive integer. For any integer s, let $[s]_m$ denote the residue class of s in $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathbb{S}_m \subseteq \mathbb{Z}_m$

be the set of all squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : \exists [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta \mid M} \delta r_\delta \pmod{m} \right\}.$$

Definition 4.1. Suppose m, M and N are positive integers, $r = (r_\delta) \in R(M)$ and $t \in \{0, 1, \ldots, m-1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write

$$\prod_{\delta \mid M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are nonnegative integers with j odd. The set Δ^* consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

(1) Each prime divisor of m is also a divisor of N.
(2) $\delta \mid M$ implies $\delta \mid mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.
(3) $kN \sum_{\delta \mid M} r_\delta mN/\delta \equiv 0 \pmod{24}$.
(4) $kN \sum_{\delta \mid M} r_\delta \equiv 0 \pmod{8}$.
(5) $\frac{24m}{\gcd(-24kt - k\sum_{\delta \mid M} \delta r_\delta, 24m)}$ divides N.
(6) If $2 \mid m$, then either $4 \mid kN$ and $8 \mid sN$ or $2 \mid s$ and $8 \mid (1-j)N$.

Throughout this section we take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let m, M, N be positive integers. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta \mid M} r_\delta \frac{\gcd^2(\delta a + \delta k \lambda c, mc)}{\delta m}$$

and

$$p_{r'}^*(\gamma) := \frac{1}{24} \sum_{\delta \mid N} r'_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

Lemma 4.2. [14, Lemma 4.5] *Let u be a positive integer, $(m, M, N, r = (r_\delta), t) \in \Delta^*$ and $r' = (r'_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_{r'}^*(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{min} = \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24} \left\{ \left(\sum_{\delta \mid M} r_\delta + \sum_{\delta \mid N} r'_\delta \right) [\Gamma : \Gamma_0(N)] - \sum_{\delta \mid N} \delta r'_\delta \right\} - \frac{1}{24m} \sum_{\delta \mid M} \delta r_\delta - \frac{t_{min}}{m}.$$

If the congruence $c_r(mn + t') \equiv 0 \pmod{u}$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then it holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

To apply Lemma 4.2 we utilize the following result, which gives a complete set of representatives of the double cosets in $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$.

Lemma 4.3. [18, Lemma 4.3] *If N or $\frac{1}{2}N$ is a square-free integer, then*

$$\bigcup_{\delta \mid N} \Gamma_0(N) \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \Gamma_\infty = \Gamma.$$

Proof of Theorem 1.3. Due to (1.2) we need to prove our congruences modulo 4 only. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(n)q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} &= \frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^4}\\ &\equiv \frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^3}{(q^4;q^4)_\infty^2} \pmod 4\\ &= (q^2;q^2)_\infty^2(q^4;q^4)_\infty \pmod 4.\end{aligned}$$

Let $(m, M, N, r, t) = (50, 8, 10, (0, 2, 1, 0), 18)$. It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{18, 28, 38, 48\}$. From Lemma 4.3 we know that $\left\{\begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix} : \delta|10\right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $r' = (0, 0, 0, 0, 0, 0) \in R(10)$. We have used *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta \mid N$, where $\gamma_\delta = \begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix}$. We compute that the upper bound in Lemma 4.2 is $\lfloor\nu\rfloor = 1$. Using *Mathematica* we verify that $\overline{\mathcal{EO}}(50n + t') \equiv 0 \pmod 4$ for $n \leq 1$ and $t' \in P_{m,r}(t)$. Thus, by Lemma 4.2, we conclude that $\overline{\mathcal{EO}}(50n + t') \equiv 0 \pmod 4$ for any $n \geq 0$, where $t' \in \{18, 28, 38, 48\}$. This completes the proof of the theorem. □

5. Proof of Theorems 1.4, 1.5 and 1.6

We prove Theorem 1.4 by using the approach developed in [12]. Recently, Jameson and Wieczorek [6] have done a similar study for the generalized Frobenius partitions. To make this paper self-contained, we recall two results from [6]. Also see [12]. Let $M_k^!(\Gamma_0(N_0), \chi)$ denote the space of weakly holomorphic modular forms.

Theorem 5.1. [6, Theorem 5] *Let N_0, α, β, t be integers with N_0, α, t positive, and let*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in M_k^!\left(\Gamma_0(N_0), \chi\right),$$

where $c(n)$ are algebraic integers in some number field. For any arithmetic progression $r \pmod t$, there are infinitely many integers $N \equiv r \pmod t$ for which $c(N)$ is even.

Theorem 5.2. [6, Theorem 6] *Let N_0, α, β, t be integers with N_0, α positive, and $t > 1$, and let*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in M_k^!\left(\Gamma_0(N_0), \chi\right),$$

where $c(n)$ are algebraic integers in some number field. For any arithmetic progression $r \pmod t$, there are infinitely many integers $M \equiv r \pmod t$ for which $c(M)$ is odd, provided there is one such M.

Furthermore, if there does exist an $M \equiv r \pmod t$ for which $c(M)$ is odd, then the smallest such M is less than $C_{r,t}$ for

$$C_{r,t} := \frac{2^j\cdot 12 + k}{12\alpha}\left[\frac{N\alpha^2t^2}{d}\right]^2 \prod_{p|N\alpha t}\left(1 - \frac{1}{p^2}\right) - 2^j,$$

where $N := \mathrm{lcm}(\alpha t, N_0)$, $d := \gcd(\alpha r + \beta, t)$, *and* j *is a sufficiently large integer.*

Proof of Theorem 1.4. We have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}.$$

We rewrite the above identity in terms of η-quotients, and then use the binomial theorem to obtain

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^{6n+1} = \frac{\eta(12z)^3}{\eta(6z)^2} \equiv \frac{\eta(12z)^4}{\eta(6z)^4} \pmod 2.$$

By Theorem 2.3, we have

$$\frac{\eta(12z)^4}{\eta(6z)^4} \in M_0^!\left(\Gamma_0(72)\right).$$

Let $f_t(z) := \dfrac{\eta(12z)^4}{\eta(6z)^4}\Delta^{2^j}(6tz)$, where $\Delta(z) := \eta^{24}(z)$. The cusps of $\Gamma_0(72t)$ are represented by fractions $\frac{c}{d}$ where $d \mid 72t$ and $\gcd(c,d) = 1$. Now, $f_t(z)$ vanishes at the cusp $\frac{c}{d}$ if and only if

$$4\frac{\gcd(d,12)^2}{12} - 4\frac{\gcd(d,6)^2}{6} + 24 \cdot 2^j \frac{\gcd(d,6t)^2}{6t} > 0.$$

We have

$$4\frac{\gcd(d,12)^2}{12} - 4\frac{\gcd(d,6)^2}{6} + 24 \cdot 2^j \frac{\gcd(d,6t)^2}{6t} \geq 2^j \frac{6}{t} - \frac{1}{2}.$$

Hence, if j is an integer such that $2^j > \frac{t}{12}$, then $f_t(z) \in S_{12\cdot 2^j}\left(\Gamma_0(72t)\right)$. Finally, our desired result follows immediately by applying Theorems 5.1 and 5.2 to $\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^{6n+1}$. □

Proof of Theorem 1.5. We first recall the following 2-dissection formula from [3, Entry 25, p. 40]:

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}. \tag{5.1}$$

From (1.1), we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{5.2}$$

Combining (5.1) and (5.2), and then extracting the terms with odd powers of q, we deduce that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{5.3}$$

We again combine (5.1) and (5.3), and then extract the terms with odd powers of q to obtain

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^n = 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3}.$$

Since $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2$, we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^n \equiv 4\frac{(q^4;q^4)_\infty^5}{(q;q)_\infty} \pmod 8.$$

We rewrite the above equation in terms of η-quotients and obtain

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^{24n+19} \equiv 4\frac{\eta^5(96z)}{\eta(24z)} \pmod 8. \tag{5.4}$$

Let $A(z) = \dfrac{\eta^2(24z)}{\eta(48z)}$. Then, $A^2(z) \equiv 1 \pmod 4$. Also, let $B(z) = \dfrac{\eta^5(96z)\eta^3(24z)}{\eta^2(48z)}$. Then we have

$$B(z) = \frac{\eta^5(96z)}{\eta(24z)}A^2(z) \equiv \frac{\eta^5(96z)}{\eta(24z)} \pmod 4. \tag{5.5}$$

The cusps of $\Gamma_0(2304)$ are represented by fractions $\frac{c}{d}$ where $d \mid 2304$ and $\gcd(c,d) = 1$. By Theorem 2.3, $B(z)$ is holomorphic at the cusp $\frac{c}{d}$ if and only if

$$5\frac{\gcd(d,96)^2}{96} + 3\frac{\gcd(d,24)^2}{24} - 2\frac{\gcd(d,48)^2}{48} \geq 0.$$

Now,

$$\begin{aligned} &5\frac{\gcd(d,96)^2}{96} + 3\frac{\gcd(d,24)^2}{24} - 2\frac{\gcd(d,48)^2}{48} \\ &= \frac{\gcd(d,48)^2}{24}\left(\frac{5}{4}\frac{\gcd(d,96)^2}{\gcd(d,48)^2} + 3\frac{\gcd(d,24)^2}{\gcd(d,48)^2} - 1\right) \\ &> 0. \end{aligned}$$

Hence, by Theorem 2.3, $B(z) \in S_3(\Gamma_0(2304), \left(\frac{-4}{\bullet}\right))$.

Let m be a positive integer. By a deep theorem of Serre [11, p. 43], if $f(z) \in M_\ell(\Gamma_0(N), \chi)$ has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]],$$

then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod m\} = \mathcal{O}\left(\frac{X}{(\log X)^\alpha}\right).$$

Since $B(z) \in S_3(\Gamma_0(2304), \left(\frac{-4}{\bullet}\right))$, the Fourier coefficients of $B(z)$ are almost always divisible by m. Hence, using (5.5) and (5.4) we complete the proof of the theorem. □

Proof of Theorem 1.6. The generating function of $\mathcal{EO}_u(2n)$ is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}_u(2n)q^n = \frac{1}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}. \tag{5.6}$$

We note that $\eta(24z) = q\prod_{n=1}^{\infty}(1-q^{24n})$ is a power series of q. As in the proof of Theorem 1.5, let

$$A(z) = \prod_{n=1}^{\infty} \frac{(1-q^{24n})^2}{(1-q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}.$$

Then using binomial theorem we have

$$A^{2^k}(z) = \frac{\eta^{2^{k+1}}(24z)}{\eta^{2^k}(48z)} \equiv 1 \pmod{2^{k+1}}. \tag{5.7}$$

Define $B_k(z)$ by

$$B_k(z) = \left(\frac{\eta(48z)}{\eta(24z)}\right)^2 A^{2^k}(z). \tag{5.8}$$

Modulo 2^{k+1}, we have

$$B_k(z) = \frac{\eta^2(48z)}{\eta^2(24z)} A^{2^k}(z) \equiv \frac{\eta^2(48z)}{\eta^2(24z)} = q^2 \frac{(q^{48};q^{48})_\infty^2}{(q^{24};q^{24})_\infty^2}. \tag{5.9}$$

Combining (5.6) and (5.9), we obtain

$$B_k(z) \equiv \sum_{n=0}^{\infty} \mathcal{EO}_u(2n) q^{24n+2} \pmod{2^{k+1}}. \tag{5.10}$$

The cusps of $\Gamma_0(576)$ are represented by fractions $\frac{c}{d}$ where $d \mid 576$ and $\gcd(c,d)=1$. By Theorem 2.3, it is easily seen that $B_k(z)$ is a form of weight 2^{k-1} on $\Gamma_0(576)$. Therefore, $B_k(z) \in M_{2^{k-1}}(\Gamma_0(576))$ if and only if $B_k(z)$ is holomorphic at the cusp $\frac{c}{d}$. We know that $B_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24}\left(2^{k+1}-2\right) + \frac{\gcd(d,48)^2}{24}\left(1-2^{k-1}\right) \geq 0.$$

Now,

$$\begin{aligned}
&\gcd(d,24)^2\left(2^{k+1}-2\right) + \gcd(d,48)^2\left(1-2^{k-1}\right)\\
&= \gcd(d,48)^2\left(\frac{\gcd(d,24)^2}{\gcd(d,48)^2}(2^{k+1}-2) + (1-2^{k-1})\right)\\
&\geq \frac{1}{4}(2^{k+1}-2) + (1-2^{k-1})\\
&> 0.
\end{aligned}$$

Hence, $B_k(z) \in M_{2^{k-1}}(\Gamma_0(576))$. Now, using Serre's theorem [11, p. 43] as shown in the proof of Theorem 1.5, we arrive at the desired result due to (5.10). □

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