

## A CERTAIN RECIPROCAL POWER SUM IS NEVER AN INTEGER

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ABSTRACT. By  $(\mathbb{Z}^+)^{\infty}$  we denote the set of all the infinite sequences  $\mathcal{S} = \{s_i\}_{i=1}^{\infty}$  of positive integers (note that all the  $s_i$  are not necessarily distinct and not necessarily monotonic). Let  $f(x)$  be a polynomial of nonnegative integer coefficients. For any integer  $n \geq 1$ , one lets  $\mathcal{S}_n := \{s_1, \dots, s_n\}$  and  $H_f(\mathcal{S}_n) := \sum_{k=1}^n \frac{1}{f(k)^{s_k}}$ . When  $f(x)$  is linear, it is proved in [Y.L. Feng, S.F. Hong, X. Jiang and Q.Y. Yin, A generalization of a theorem of Nagell, Acta Math. Hungari, to appear] that for any infinite sequence  $\mathcal{S}$  of positive integers,  $H_f(\mathcal{S}_n)$  is never an integer if  $n \geq 2$ . Now let  $\deg f(x) \geq 2$ . Clearly,  $0 < H_f(\mathcal{S}_n) < \zeta(2) < 2$ . But it is not clear whether the reciprocal power sum  $H_f(\mathcal{S}_n)$  can take 1 as its value. In this paper, with the help of a result of Erdős, we use the analytic and  $p$ -adic method to show that for any infinite sequence  $\mathcal{S}$  of positive integers and any positive integer  $n \geq 2$ ,  $H_f(\mathcal{S}_n)$  is never equal to 1. Furthermore, we use a result of Kakeya to show that if  $\frac{1}{f(k)} \leq \sum_{i=1}^{\infty} \frac{1}{f(k+i)}$  holds for all positive integers  $k$ , then the union set  $\bigcup_{\mathcal{S} \in (\mathbb{Z}^+)^{\infty}} \{H_f(\mathcal{S}_n) | n \in \mathbb{Z}^+\}$  is dense in the

interval  $(0, \alpha_f)$  with  $\alpha_f := \sum_{k=1}^{\infty} \frac{1}{f(k)}$ . It is well known that  $\alpha_f = \frac{1}{2}(\pi \frac{e^{2\pi} + 1}{e^{2\pi} - 1} - 1) \approx 1.076674$  when  $f(x) = x^2 + 1$ . Our dense result infers that when  $f(x) = x^2 + 1$ , for any sufficiently small  $\varepsilon > 0$ , there are positive integers  $n_1$  and  $n_2$  and infinite sequences  $\mathcal{S}^{(1)}$  and  $\mathcal{S}^{(2)}$  of positive integers such that  $1 - \varepsilon < H_f(\mathcal{S}_{n_1}^{(1)}) < 1$  and  $1 < H_f(\mathcal{S}_{n_2}^{(2)}) < 1 + \varepsilon$ .

## 1. INTRODUCTION

Let  $\mathbb{Z}$ ,  $\mathbb{Z}^+$  and  $\mathbb{Q}$  be the set of integers, the set of positive integers and the set of rational numbers, respectively. Let  $n \in \mathbb{Z}^+$ . In 1915, Theisinger [11] showed that the  $n$ -th harmonic sum  $1 + \frac{1}{2} + \dots + \frac{1}{n}$  is never an integer if  $n > 1$ . In 1923, Nagell [10] extended Theisinger's result by showing that if  $a$  and  $b$  are positive integers and  $n \geq 2$ , then the reciprocal sum  $\sum_{i=0}^{n-1} \frac{1}{a+bi}$  is never an integer. Erdős and Niven [3] generalized Nagell's result by considering the integrality of the elementary symmetric functions of  $\frac{1}{a}, \frac{1}{a+b}, \dots, \frac{1}{a+(n-1)b}$ . In the recent years, Erdős and Niven's result [3] was extended to the general polynomial sequence, see [1], [5], [9] and [12]. Another interesting and related topic is presented in [14].

By  $(\mathbb{Z}^+)^{\infty}$  we denote the set of all the infinite sequence  $\{s_i\}_{i=1}^{\infty}$  of positive integers (note that all the  $s_i$  are not necessarily distinct and not necessarily monotonic). For any given  $\mathcal{S} = \{s_i\}_{i=1}^{\infty} \in (\mathbb{Z}^+)^{\infty}$ , we let  $\mathcal{S}_n := \{s_1, \dots, s_n\}$ . Associated to the infinite sequence  $\mathcal{S}$  of positive integers and a polynomial  $f(x)$  of nonnegative integer coefficients, one can form an infinite sequence  $\{H_f(\mathcal{S}_n)\}_{n=1}^{\infty}$  of positive rational fractions with  $H_f(\mathcal{S}_n)$  being

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defined as follows:

$$H_f(\mathcal{S}_n) := \sum_{k=1}^n \frac{1}{f(k)^{s_k}}.$$

Very recently, Feng, Hong, Jiang and Yin [4] showed that when  $f(x)$  is linear, the reciprocal power sum  $H_f(\mathcal{S}_n)$  is never an integer if  $n \geq 2$ . It is natural to ask the following interesting question: Is the similar result still true when  $f(x)$  is of degree at least two and nonnegative integer coefficients?

In this paper, our main goal is to study this question. In fact, by using the analytic and  $p$ -adic method and with the help of Erdős theorem [2] on the distribution in the arithmetic progression  $\{4n+1\}_{n=1}^{\infty}$ , we will show the following result that is the first main result of this paper.

**Theorem 1.1.** *Let  $f(x)$  be a polynomial of nonnegative integer coefficients and of degree at least two. Then for any infinite sequence  $\mathcal{S}$  of positive integers and for any positive integer  $n \geq 2$ , the reciprocal power sum  $H_f(\mathcal{S}_n)$  is never an integer.*

Clearly, Theorem 1.1 gives an affirmative answer to the above mentioned question.

Associated to any given infinite sequence  $\mathcal{S}$  of positive integers, we let

$$H_f(\mathcal{S}) := \{H_f(\mathcal{S}_n) | n \in \mathbb{Z}^+\}$$

and

$$\alpha_f(\mathcal{S}) := \sum_{k=1}^{\infty} \frac{1}{f(k)^{s_k}}.$$

Put

$$\alpha_f := \sum_{k=1}^{\infty} \frac{1}{f(k)}. \quad (1)$$

Note that  $\alpha_f$  may be  $+\infty$ . Then  $\alpha_f(\mathcal{S}) \leq \alpha_f$  and  $H_f(\mathcal{S}) \subseteq (\inf H_f(\mathcal{S}), \alpha_f(\mathcal{S}))$ . It is clear that  $H_f(\mathcal{S})$  is not dense (nowhere dense) in the interval  $(\inf H_f(\mathcal{S}), \alpha_f(\mathcal{S}))$ . However, if we put all the sets  $H_f(\mathcal{S})$  together, then one arrives at the following interesting dense result that is the second main result of this paper.

**Theorem 1.2.** *Let  $f(x)$  be a polynomial of nonnegative integer coefficients and let  $U_f$  be the union set defined by*

$$U_f := \bigcup_{\mathcal{S} \in (\mathbb{Z}^+)^{\infty}} H_f(\mathcal{S}).$$

(i). *If  $\deg f(x) = 1$ , then  $U_f$  is dense in the interval  $(\delta, +\infty)$  with  $\delta = 1$  if  $f(x) = x$ , and  $\delta = 0$  otherwise.*

(ii). *If  $\deg f(x) \geq 2$  and*

$$\frac{1}{f(k)} \leq \sum_{i=1}^{\infty} \frac{1}{f(k+i)} \quad (2)$$

*holds for all positive integers  $k$ , then  $U_f$  is dense in the interval  $(0, \alpha_f)$  with  $\alpha_f$  being given in (1).*

Let  $H(\mathcal{S}_n) := H_f(\mathcal{S}_n)$  and  $H(\mathcal{S}) := H_f(\mathcal{S})$  if  $f(x) = x^2 + 1$ . It is well known that (see, for instance, [7])

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 1} = \frac{1}{2} \left( \pi \frac{e^{2\pi} + 1}{e^{2\pi} - 1} - 1 \right) := \alpha. \quad (3)$$

Furthermore,  $\alpha \approx 1.076674$ . Evidently, for any positive integer  $n$ , we have

$$0 < H(\mathcal{S}_n) \leq \sum_{k=1}^n \frac{1}{k^2 + 1} < \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} < 2.$$

Theorem 1.1 tells us that  $H(\mathcal{S}_n)$  is never equal to 1 for any infinite sequence  $\mathcal{S}$  of positive integers and any positive integer  $n$ . This extends the corresponding result in [9] and [14] which states that for the infinite sequence  $\mathcal{S}$  with  $s_i = s_1$  for all integers  $i \geq 1$ ,  $H(\mathcal{S}_n)$  is not equal to 1. On the other hand, one can easily check that (2) is true when  $f(x) = x^2 + 1$ . So Theorem 1.2 infers that for any sufficiently small  $\varepsilon > 0$ , there are positive integers  $n_1$  and  $n_2$  and infinite sequences  $\mathcal{S}^{(1)}$  and  $\mathcal{S}^{(2)}$  of positive integers such that  $1 - \varepsilon < H(\mathcal{S}_{n_1}^{(1)}) < 1$  and  $1 < H(\mathcal{S}_{n_2}^{(2)}) < 1 + \varepsilon$ .

This paper is organized as follows. First, in Section 2, we recall the early results due to Erdős [2] and Takeya [6], respectively, and then show some preliminary lemmas which are needed in the proofs of Theorems 1.1 and 1.2. Then in Sections 3 and 4, we supply the proofs of Theorems 1.1 and 1.2, respectively. The final section is devoted to some remarks. Actually, a conjecture on the case of integer coefficients polynomial is proposed there.

## 2. AUXILIARY LEMMAS

In this section, we present several auxiliary lemmas that are needed in the proofs of Theorems 1.1 and 1.2. We begin with a well-known result due to Erdős.

**Lemma 2.1.** [2] *For any real number  $\xi \geq 7$ , there exists a prime  $p \in (\xi, 2\xi]$  such that  $p \equiv 1 \pmod{4}$ .*

For any given prime  $p$  with  $p \equiv 1 \pmod{4}$ , the congruence  $x^2 + 1 \equiv 0 \pmod{p}$  is solvable, and in the remaining part of this paper, we use  $r_p$  to stand for the smallest positive root of  $x^2 + 1 \equiv 0 \pmod{p}$ . In the conclusion of this section, we use Lemma 2.1 to show the following result that is vital in the proof of Theorem 1.1.

**Lemma 2.2.** *For any integer  $n \geq 2$ , there is a prime  $p$  with  $p \equiv 1 \pmod{4}$  such that  $r_p \leq n < p$ .*

*Proof.* If  $n = 2, 3$  or  $4$ , then letting  $p = 5$  gives us that  $r_p = 2$ . So Lemma 2.2 is true in this case.

If  $n = 5$  or  $6$ , then picking  $p = 13$  gives us that  $r_p = 5$ . Lemma 2.2 holds in this case.

Now let  $n \geq 7$ . At this moment, Lemma 2.1 guarantees the existence of a prime  $p$  such that  $p \equiv 1 \pmod{4}$  and  $n < p < 2n$ . Since  $p - r_p$  is another positive root of  $x^2 + 1 \equiv 0 \pmod{p}$  and  $r_p < p - r_p$ , it follows that

$$r_p \leq \frac{p-1}{2} < \frac{p}{2} < n < p.$$

as required. Hence Lemma 2.2 is proved.  $\square$

Now let us state a result obtained by Takeya in 1914.

**Lemma 2.3.** [6] *Let  $\sum_{k=1}^{\infty} a_k$  be an absolutely convergent infinite series of real numbers and let the set, denoted by  $SPS$ , of all the partial sums of the series  $\sum_{k=1}^{\infty} a_k$  be defined by*

$$SPS := \left\{ \sum_{i=1}^m a_{k_i} \mid m \in \mathbb{Z}^+ \cup \{\infty\}, 1 \leq k_1 < \dots < k_m \right\}.$$

Let  $u := \inf SPS$  and  $v := \sup SPS$  (note that  $u$  may be  $-\infty$  and  $v$  may be  $+\infty$ ). Then the set  $U$  consists of all the values in the interval  $(u, v)$  if and only if

$$|a_k| \leq \sum_{i=1}^{\infty} |a_{k+i}|$$

holds for all  $k \in \mathbb{Z}^+$ .

Using Lemma 2.3, we can prove the following two useful results that play key roles in the proof of Theorem 1.2.

**Lemma 2.4.** Let  $\sum_{k=1}^{\infty} a_k$  be a convergent infinite series of positive real numbers and

$$V := \left\{ \sum_{i=1}^m a_{k_i} \mid m \in \mathbb{Z}^+, 1 \leq k_1 < \dots < k_m \right\}.$$

If

$$a_k \leq \sum_{i=1}^{\infty} a_{k+i} \tag{4}$$

holds for all  $k \in \mathbb{Z}^+$ , then the set  $V$  is dense in the interval  $(0, v)$  with  $v := \sum_{k=1}^{\infty} a_k$ .

*Proof.* From the condition (4) and Lemma 2.3, we know that the set

$$SPS = \left\{ \sum_{i=1}^m a_{k_i} \mid m \in \mathbb{Z}^+ \cup \{\infty\}, 1 \leq k_1 < \dots < k_m \right\}$$

consists of all the values in the interval  $(0, v)$  since here  $\inf SPS = 0$ . Let  $r$  be any given real number in  $(0, v)$  and  $\varepsilon$  be any sufficiently small positive number (one may let  $\varepsilon < \min(r, v - r)$ ). Then  $r \in SPS$  which implies that there is an integer  $m \in \mathbb{Z}^+ \cup \{\infty\}$  and there are  $m$  integers  $k_1, \dots, k_m$  with  $1 \leq k_1 < \dots < k_m$  such that  $r = \sum_{i=1}^m a_{k_i}$ .

If  $m \in \mathbb{Z}^+$ , then  $r \in V$ . Lemma 2.4 is true in this case.

If  $m = \infty$ , then  $r = \sum_{i=1}^{\infty} a_{k_i}$ . That is,  $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_{k_i} = r$ . Thus there is a positive integer  $m'$  such that  $|r - \sum_{i=1}^{m'} a_{k_i}| < \varepsilon$ . Noticing that all  $a_{k_i}$  are positive, we deduce that  $r - \varepsilon < \sum_{i=1}^{m'} a_{k_i} < r$  as desired.

This completes the proof of Lemma 2.4.  $\square$

**Lemma 2.5.** Let  $\sum_{k=1}^{\infty} a_k$  be a divergent infinite series of positive real numbers with  $a_k$  decreasing as  $k$  increasing and  $a_k \rightarrow 0$  as  $k \rightarrow \infty$ . Define

$$V := \left\{ \sum_{i=1}^m a_{k_i} \mid m \in \mathbb{Z}^+, 1 \leq k_1 < \dots < k_m \right\}.$$

Then the set  $V$  is dense in the interval  $(0, +\infty)$ .

*Proof.* Let  $r$  be any given real number in  $(0, +\infty)$  and  $\varepsilon$  be any sufficiently small positive number (one may let  $\varepsilon < r$ ). Let  $a_0 := 0$  and  $m_0 = 0$ . Since the series  $\sum_{k=0}^{\infty} a_k$  is divergent, there exists a unique integer  $m_1 \geq 0$  such that

$$\sum_{k=m_0}^{m_1} a_k < r$$

and

$$\sum_{k=m_0}^{m_1} a_k + a_{m_1+1} \geq r.$$

On the one hand, since  $a_k$  decreases as  $k$  increases and  $a_k \rightarrow 0$  as  $k \rightarrow \infty$ , there is an integer  $m_2$  with  $m_2 > m_1 + 1$  and

$$a_{m_2} < r - \sum_{k=m_0}^{m_1} a_k \leq a_{m_1+1}.$$

Moreover, there exists an integer  $m_3$  with  $m_3 \geq m_2$  and

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k < r$$

and

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + a_{m_3+1} \geq r$$

since  $\sum_{k=m_2}^{\infty} a_k$  also diverges.

Continuing in this way, we can form an increasing sequence  $\{m_k\}_{k=0}^{\infty}$  such that

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k < r$$

but

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k + a_{m_{2t+1}+1} \geq r$$

for any nonnegative integer  $t$ . Obviously, one has

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k \in V.$$

On the other hand, since  $\lim_{k \rightarrow +\infty} a_k = 0$ , it follows that there exists a nonnegative integer  $t_0$  such that  $a_{m_{2t_0+1}+1} < \varepsilon$ . That is, we have

$$r - \varepsilon < r - a_{m_{2t_0+1}+1} \leq \sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t_0}}^{m_{2t_0+1}} a_k < r.$$

Hence  $V$  is dense in the interval  $(0, +\infty)$ .

This concludes the proof of Lemma 2.5.  $\square$

### 3. PROOF OF THEOREM 1.1

As usual, for any prime  $p$  and for any integer  $x$ , we let  $v_p(x)$  stand for the  $p$ -adic valuation of  $x$ , i.e.,  $v_p(x)$  is the biggest nonnegative integer  $r$  with  $p^r$  dividing  $x$ . If  $x = \frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b \neq 0$ , then define  $v_p(x) := v_p(a) - v_p(b)$ .

We can now prove Theorem 1.1 as follows.

*Proof of Theorem 1.1.* We just need to prove that  $H_f(\mathcal{S}_n)$  is between two adjacent integers or  $v_p(H_f(\mathcal{S}_n)) < 0$  for some prime  $p$ . Let  $f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$  with  $m \geq 2$  and  $a_m \geq 1$ .

If there is some  $a_i \neq 0$  with  $1 \leq i \leq m-1$ , then  $f(k) \geq a_m k^m + a_i k^i \geq k^2 + k$  for all  $k \geq 1$ . Therefore,

$$H_f(\mathcal{S}_n) = \sum_{k=1}^n \frac{1}{f(k)^{s_k}} \leq \sum_{k=1}^n \frac{1}{k^2 + k} = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} < 1.$$

If  $a_i = 0$  for all  $0 \leq i \leq m-1$ , then  $f(x) = a_m x^m$ . Furthermore,

$$1 < H_f(\mathcal{S}_n) = \sum_{k=1}^n \frac{1}{(k^m)^{s_k}} \leq \sum_{k=1}^n \frac{1}{k^2} < \zeta(2) = \frac{\pi^2}{6} < 2$$

when  $a_m = 1$ , and

$$0 < H_f(\mathcal{S}_n) = \sum_{k=1}^n \frac{1}{(a_m k^m)^{s_k}} \leq \frac{1}{2} \sum_{k=1}^n \frac{1}{k^2} < 1.$$

when  $a_m \geq 2$ .

If  $a_i = 0$  for all  $1 \leq i \leq m-1$  and  $a_0 \neq 0$ , then  $f(x) = a_m x^m + a_0$ . Moreover, if  $a_m \geq 2$  or  $a_0 \geq 2$ , then  $f(k) = k^m + (a_m - 1)k^m + a_0 \geq k^2 + 2$  for all  $k \geq 1$ . So

$$0 < H_f(\mathcal{S}_n) \leq \sum_{k=1}^n \frac{1}{k^2 + 2} \leq \sum_{k=1}^n \frac{1}{k^2 + 1} - \frac{1}{2} - \frac{1}{5} + \frac{1}{3} + \frac{1}{6} < \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} - \frac{1}{5} < 1.$$

If  $m \geq 3$ , then  $f(k) \geq k^3 + 1$  for all  $k \geq 1$ . So

$$0 < H_f(\mathcal{S}_n) \leq \sum_{k=1}^n \frac{1}{k^3 + 1} \leq \sum_{k=1}^n \frac{1}{k^3} - 1 - \frac{1}{8} + \frac{1}{2} + \frac{1}{9} < \zeta(3) - \frac{37}{72} < 1.$$

In what follows, we let  $f(x) = x^2 + 1$ .

By Lemma 2.2, there is a prime  $p$  such that  $p \equiv 1 \pmod{4}$  and  $r_p \leq n < p$  where  $r_p$  is the smallest positive root of  $x^2 + 1 \equiv 0 \pmod{p}$ . Since  $p \mid (r_p^2 + 1)$ , one has  $r_p^2 \geq p - 1$ . Noticing that  $p \geq 5$ , it follows that

$$2 \leq \sqrt{p-1} \leq r_p \leq \frac{p-1}{2} < p - r_p.$$

Therefore

$$0 < r_p^2 + 1 < (p - r_p)^2 + 1 \leq (p - 2)^2 + 1 < p^2.$$

This infers that  $v_p(r_p^2 + 1) = 1$  and  $v_p((p - r_p)^2 + 1) = 1$ . So we have

$$v_p\left(\frac{1}{(r_p^2 + 1)^{s_{r_p}}}\right) = -s_{r_p} < 0,$$

$$v_p\left(\frac{1}{((p - r_p)^2 + 1)^{s_{p-r_p}}}\right) = -s_{p-r_p} < 0$$

and

$$v_p\left(\frac{1}{(k^2 + 1)^{s_k}}\right) = 0$$

for any integer  $k$  with  $1 \leq k \leq p$  and  $k \notin \{r_p, p - r_p\}$ .

Now we divide the proof into the following two cases.

CASE 1.  $r_p \leq n < p - r_p$ . Since

$$v_p\left(\sum_{\substack{k=1 \\ k \neq r_p}}^n \frac{1}{(k^2 + 1)^{s_k}}\right) \geq 0,$$

it follows from the isosceles triangle principle (see, for example, [8]) that

$$v_p(H(\mathcal{S}_n)) = v_p\left(\frac{1}{(r_p^2 + 1)^{s_{r_p}}} + \sum_{\substack{k=1 \\ k \neq r_p}}^n \frac{1}{(k^2 + 1)^{s_k}}\right) = v_p\left(\frac{1}{(r_p^2 + 1)^{s_{r_p}}}\right) = -s_{r_p} < 0.$$

Namely,  $H(\mathcal{S}_n) \notin \mathbb{Z}$ . So Theorem 1.1 is proved in this case.

CASE 2.  $p - r_p \leq n < p$ . Let  $H(\mathcal{S}_n) = A + B$ , where

$$A := \frac{1}{(r_p^2 + 1)^{s_{r_p}}} + \frac{1}{((p - r_p)^2 + 1)^{s_{p-r_p}}}$$

and

$$B := \sum_{\substack{k=1 \\ k \neq r_p, k \neq p-r_p}}^n \frac{1}{(k^2 + 1)^{s_k}}.$$

Evidently, one has  $v_p(B) \geq 0$ . We claim that  $v_p(A) < 0$ . Then by the claim and the isosceles triangle principle again, we obtain that

$$v_p(H(\mathcal{S}_n)) = v_p(A + B) = v_p(A) < 0,$$

which implies that  $H(\mathcal{S}_n) \notin \mathbb{Z}$  as desired. It remains to show the truth of the claim.

If  $s_{r_p} \neq s_{p-r_p}$ , then it is obvious that  $v_p(A) = \min(-s_{r_p}, -s_{p-r_p}) < 0$ . So the claim is true if  $s_{r_p} \neq s_{p-r_p}$ .

Now let  $s_{r_p} = s_{p-r_p} := s$ . Then

$$A = \frac{((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s}{(r_p^2 + 1)^s((p - r_p)^2 + 1)^s}.$$

We introduce an auxiliary function  $g(x)$  as follows:

$$g(x) := \left( \left( \frac{p}{2} + x \right)^2 + 1 \right)^s + \left( \left( \frac{p}{2} - x \right)^2 + 1 \right)^s.$$

Then the derivative of  $f(x)$  is

$$g'(x) = s \left( \left( \frac{p}{2} + x \right)^2 + 1 \right)^{s-1} (p + 2x) - s \left( \left( \frac{p}{2} - x \right)^2 + 1 \right)^{s-1} (p - 2x).$$

So  $g'(x) > 0$  if  $x \in (0, \frac{p}{2}]$ . This implies that  $g(x)$  is increasing if  $x \in (0, \frac{p}{2}]$ .

Since  $2 \leq r_p < \frac{p}{2}$ , one derives that  $0 < \frac{p}{2} - r_p \leq \frac{p}{2} - 2 < \frac{p}{2}$ . Hence

$$\begin{aligned} ((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s &= g\left(\frac{p}{2} - r_p\right) \\ &\leq g\left(\frac{p}{2} - 2\right) \\ &= (2^2 + 1)^s + ((p - 2)^2 + 1)^s \\ &< 5^s + (p - 1)^{2s} \\ &< p^{2s}, \end{aligned}$$

where the last inequality follows from the fact that  $p \geq 5$  implies that

$$p^{2s} - (p - 1)^{2s} = (p^s + (p - 1)^s)(p^s - (p - 1)^s) > 5^s.$$

Thus

$$v_p(((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s) < 2s.$$

Therefore

$$\begin{aligned} v_p(A) &= v_p(((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s) - v_p((r_p^2 + 1)^s((p - r_p)^2 + 1)^s) \\ &< 2s - 2s = 0. \end{aligned}$$

The claim holds if  $s_{r_p} = s_{p-r_p}$ . The claim is proved.

This finishes the proof of Theorem 1.1.  $\square$

## 4. PROOF OF THEOREM 1.2

In the section, we present the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Let

$$V_f := \left\{ \sum_{i=1}^m \frac{1}{f(k_i)} \mid m \in \mathbb{Z}^+, 1 \leq k_1 < \dots < k_m \right\}$$

and

$$\bar{V}_f := \left\{ \sum_{i=1}^m \frac{1}{f(k_i)} \mid m \in \mathbb{Z}^+, 2 \leq k_1 < \dots < k_m \right\}.$$

Pick any given real number  $r$  in  $(\inf U_f, \sup U_f)$  and let  $\varepsilon$  be any sufficiently small positive number (one may let  $\varepsilon < \min(r - \inf U_f, \sup U_f - r)$ ).

(i). Since  $f(x)$  is a polynomial of nonnegative integer and degree one, it follows that  $\sum_{k=1}^{\infty} \frac{1}{f(k)}$  (resp.  $\sum_{k=2}^{\infty} \frac{1}{f(k)}$ ) is a divergent infinite series of positive real numbers with  $\{\frac{1}{f(k)}\}_{k=1}^{\infty}$  (resp.  $\{\frac{1}{f(k)}\}_{k=2}^{\infty}$ ) directly decreasing to 0 as  $k$  increases. By Lemma 2.5, we know that  $V_f$  (resp.  $\bar{V}_f$ ) is dense in the interval  $(0, +\infty)$ . Clearly, we have  $\sup U_f = \sup V_f = +\infty$ .

If  $f(1) = 1$ , then  $f(x) = x$  which implies that  $f(2) > 1$ ,  $\inf U_f = 1$  and  $r \in (\inf U_f, \sup U_f) = (1, +\infty)$ . Since  $\bar{V}_f$  is dense in the interval  $(0, +\infty)$ , there is an element

$$\sum_{i=1}^m \frac{1}{f(k_i)} \in \left( r - 1 - \varepsilon, r - 1 - \frac{\varepsilon}{2} \right) \quad (5)$$

with  $2 \leq k_1 < \dots < k_m$ . Now let  $s_k = 1$  for  $k \in \{k_1, \dots, k_m\}$  and  $s_k > \frac{\log \frac{2k_m}{\varepsilon}}{\log f(2)}$  for  $k \in \{2, 3, \dots, k_m\} \setminus \{k_1, \dots, k_m\}$ . Then

$$0 \leq \sum_{\substack{k=2 \\ k \notin \{k_1, \dots, k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{f(2)^{\frac{\log \frac{2k_m}{\varepsilon}}{\log f(2)}}} = \frac{\varepsilon}{2}. \quad (6)$$

It follows from (5) and (6) that

$$\sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} = 1 + \sum_{\substack{k=2 \\ k \notin \{k_1, \dots, k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^m \frac{1}{f(k_i)^{s_{k_i}}} \in (r - \varepsilon, r).$$

That is,  $U_f$  is dense in the interval  $(\inf U_f, \sup U_f) = (1, +\infty)$  in this case.

If  $f(1) > 1$ , then  $\inf U_f = 0$  and  $r \in (\inf U_f, \sup U_f) = (0, +\infty)$ . Since  $V_f$  is dense in the interval  $(0, +\infty)$ , there is an element

$$\sum_{i=1}^m \frac{1}{f(k_i)} \in \left( r - \varepsilon, r - \frac{\varepsilon}{2} \right) \quad (7)$$

with  $1 \leq k_1 < \dots < k_m$ . Now, let  $s_k = 1$  for  $k \in \{k_1, \dots, k_m\}$  and  $s_k > \frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}$  for  $k \in \{1, 2, \dots, k_m\} \setminus \{k_1, \dots, k_m\}$ . One has

$$0 \leq \sum_{\substack{k=1 \\ k \notin \{k_1, \dots, k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{f(1)^{\frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}}} = \frac{\varepsilon}{2}, \quad (8)$$



and so by (7) and (8),

$$\sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} = \sum_{\substack{k=1 \\ k \notin \{k_1, \dots, k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^m \frac{1}{f(k_i)^{s_{k_i}}} \in (r - \varepsilon, r).$$

Namely,  $U_f$  is dense in the interval  $(\inf U_f, \sup U_f) = (0, +\infty)$  in this case.

(ii). First of all, since  $f(x)$  is a polynomial of nonnegative integer and  $\deg f(x) \geq 2$ , we know that  $\sum_{k=1}^{\infty} \frac{1}{f(k)}$  is a convergent infinite series of positive real numbers. With the hypothesis  $\frac{1}{f(k)} \leq \sum_{i=1}^{\infty} \frac{1}{f(k+i)}$  for any positive integer  $k$ , Lemma 2.4 yields that  $V_f$  is dense in the interval  $(0, \sup V_f)$ .

We claim that  $f(1) > 1$ . Otherwise,  $f(1) = 1$ . Then  $f(x) = x^m$  with  $m \geq 2$ . However,

$$\frac{1}{f(1)} = 1 > \frac{\pi^2}{6} - 1 = \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \geq \sum_{i=1}^{\infty} \frac{1}{f(1+i)},$$

which contradicts with our hypothesis. So we must have  $f(1) > 1$ . The claim is proved.

In the following, we let  $f(1) > 1$ . Then  $\inf U_f = 0$ ,  $\sup U_f = \sup V_f = \alpha_f$  and  $r \in (\inf U_f, \sup U_f) = (0, \alpha_f)$ . Since  $V_f$  is dense in the interval  $(0, \sup V_f) = (0, \alpha_f)$ , there is an element

$$\sum_{i=1}^m \frac{1}{f(k_i)} \in \left(r - \varepsilon, r - \frac{\varepsilon}{2}\right)$$

with  $1 \leq k_1 < \dots < k_m$ . Then letting  $s_k = 1$  for  $k \in \{k_1, \dots, k_m\}$  and  $s_k > \frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}$  for  $k \in \{1, 2, \dots, k_m\} \setminus \{k_1, \dots, k_m\}$  gives us that

$$0 \leq \sum_{\substack{k=1 \\ k \notin \{k_1, \dots, k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{f(1)^{\frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}}} = \frac{\varepsilon}{2}.$$

It infers that

$$\sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} = \sum_{\substack{k=1 \\ k \notin \{k_1, \dots, k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^m \frac{1}{f(k_i)^{s_{k_i}}} \in (r - \varepsilon, r).$$

In other words,  $U_f$  is dense in the interval  $(0, \alpha_f)$ . So part (ii) is proved.

The proof of Theorem 1.2 is complete.  $\square$

## 5. CONCLUDING REMARKS

1. Let  $\mathcal{T} := ((0, \alpha) \cap \mathbb{Q}) \setminus \bigcup_{\mathcal{S} \in (\mathbb{Z}^+)^{\infty}} H(\mathcal{S})$ . Then Theorem 1.1 tells us that  $1 \in \mathcal{T}$ .

But it is well known that if  $p$  is a prime, then the congruence  $x^2 + 1 \equiv 0 \pmod{p}$  is solvable if and only if either  $p = 2$ , or  $p \equiv 1 \pmod{4}$ . Thus for any infinite sequence  $\mathcal{S}$  of positive integers and for any positive integer  $n$ , if one writes  $H(\mathcal{S}_n) = \frac{H_1(\mathcal{S}_n)}{H_2(\mathcal{S}_n)}$ , where  $H_1(\mathcal{S}_n), H_2(\mathcal{S}_n) \in \mathbb{Z}^+$  and  $\gcd(H_1(\mathcal{S}_n), H_2(\mathcal{S}_n)) = 1$ , then  $H_2(\mathcal{S}_n)$  is not divisible by any prime  $p$  with  $p \equiv 3 \pmod{4}$ . It follows that  $\mathcal{L} := \{\frac{a}{b} \in (0, \alpha) \mid a, b \in \mathbb{Z}, (a, b) = 1, b \text{ is divisible by at least a prime } p \text{ with } p \equiv 3 \pmod{4}\} \subset \mathcal{T}$ . An interesting question naturally arises: Are there other elements in the set  $\mathcal{T}$  except for the elements in  $\{1\} \cup \mathcal{L}$ ? Further, one would like to determine the set  $\mathcal{T}$ . This problem is kept open so far.

2. We let  $f(x)$  be a polynomial of nonnegative integer coefficients and of degree at least two, and let  $U_f$  be the union set given in Theorem 1.2. Then part (ii) of Theorem

1.2 says that the condition (2) is a sufficient condition such that the union set  $U_f$  is dense in the interval  $(0, \alpha_f)$ . One may ask the following interesting question: What is the sufficient and necessary condition for the union set  $U_f$  to be dense in the interval  $(0, \alpha_f)$ ?

3. Now let  $f(x)$  be a nonzero polynomial of integer coefficients. Let  $Z_f := \{x \in \mathbb{Z} : f(x) = 0\}$  be the set of integer roots of  $f(x)$  and  $\{a_i\}_{i=1}^{\infty} := \mathbb{Z}^+ \setminus Z_f$  be arranged in the increasing order. Then  $f(a_i) \neq 0$  for all integers  $i \geq 1$ . Let  $n$  and  $k$  be integers such that  $1 \leq k \leq n$  and let  $H_f^{(k)}(\mathcal{S}_n)$  stand for the  $k$ -th elementary symmetric functions of

$$\frac{1}{f(a_1)^{s_1}}, \frac{1}{f(a_2)^{s_2}}, \dots, \frac{1}{f(a_n)^{s_n}}.$$

That is,

$$H_f^{(k)}(\mathcal{S}_n) := \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \frac{1}{f(a_{i_j})^{s_{i_j}}}.$$

Then  $H_f^{(1)}(\mathcal{S}_n) = H_f(\mathcal{S}_n)$ . Let

$$\bar{H}_f^{(k)}(\mathcal{S}_n) := \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \prod_{j=1}^k \frac{1}{f(a_{i_j})^{s_{i_j}}}.$$

When  $f(x)$  is of nonnegative integer coefficients and  $s_i = 1$  for all integers  $i \geq 1$ , the integrality of  $H_f^{(k)}(\mathcal{S}_n)$  ( $1 \leq k \leq n$ ) was previously investigated in [1], [3], [5], [9] and [12]. But such integrality problem has not been studied when  $f(x)$  contains negative coefficients. On the one hand, for any given integer  $N_0 \geq 1$ , one can easily find a polynomial  $f_0(x)$  of integer coefficients such that for all integers  $n$  and  $k$  with  $1 \leq k \leq n \leq N_0$ , both of  $H_{f_0}^{(k)}(\mathcal{S}_n)$  and  $\bar{H}_{f_0}^{(k)}(\mathcal{S}_n)$  are integers. Actually, letting

$$f_0(x) = \prod_{i=1}^{N_0} (x - i) \pm 1$$

gives us the expected result. On the other hand, for any given nonzero polynomial  $f(x)$  of integer coefficients, we believe that the similar integrality result is still true. So in concluding this paper, we suggest the following more general conjecture that generalizes Conjecture 4.1 of [4] and Conjecture 3.1 of [9].

*Conjecture 5.1.* Let  $f(x)$  be a nonzero polynomial of integer coefficients and  $\mathcal{S} = \{s_i\}_{i=1}^{\infty}$  be an infinite sequence of positive integers (not necessarily increasing and not necessarily distinct). Then there is a positive integer  $N$  such that for any integer  $n \geq N$  and for all integers  $k$  with  $1 \leq k \leq n$ , both of  $H_f^{(k)}(\mathcal{S}_n)$  and  $\bar{H}_f^{(k)}(\mathcal{S}_n)$  are not integers.

By Theorem 1.1, one knows that Conjecture 5.1 holds when  $k = 1$ . It is clear that Conjecture 5.1 is true when  $k = n$ . Thus we need just to look at the case  $2 \leq k \leq n - 1$ . Obviously, the results presented in [1], [3]-[5], [9]-[13] and Theorem 1.1 of this paper supply evidences to Conjecture 5.1.

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