A Bezout ring of stable range 2 which has square stable range 1

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Abstract: In this paper we introduced the concept of a ring of stable range 2 which has square stable range 1. We proved that a Hermitian ring R which has (right) square stable range 1 is an elementary divisor ring if and only if R is a duo ring of neat range 1. And we proved that a commutative Hermitian ring R is a Toeplitz ring if and only if R is a ring of (right) square range 1. We proved that if R be a commutative elementary divisor ring of (right) square stable range 1, then for any matrix $A \in M_2(R)$ one can find invertible Toeplitz matrices P and Q such that $PAQ = \begin{pmatrix} e_1 & e_2 \\ e_1 & e_2 \end{pmatrix}$, where e_i is a divisor of e_2 .

Key words and phrases: Hermitian ring, elementary divisor ring, stable range 1, stable range 2, square stable range 1, Toeplitz matrix, duo ring, quasi-duo ring.

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1 Introduction

The notion of a stable range of a ring was introduced by H. Bass, and became especially popular because of its various applications to the problem of cancellation and substitution of modules. Let us say that a module A satisfies the power-cancellation property if for all modules B and C, $A \oplus B \cong A \oplus C$ implies that $B^n \cong C^n$ for some positive integer n (here B^n denotes the direct sum of n copies of B). Let us say that a right R-module A has the power-substitution property if given any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ which each $A_i \cong A$, there exist a positive integer nand a submodule $C \subseteq M^n$ such that $M^n = C \oplus B_1^n = C \oplus B_2^n$.

Prof. K. Goodearl pointed out that a commutative rind R has the powersubstitution property if and only if R is of (right) power stable range 1, i.e. if aR + bR = R than $(a^n + bx)R = R$ for some $x \in R$ and some integer $n \ge 2$ depending on $a, b \in R$ [1]. Recall that a ring R is said to have 1 in the stable range provided that whenever ax + b = 1 in R, there exists $y \in R$ such that a + by is a unit in R. The following Warfield's theorem shows that 1 in the stable range is equivalent to a substitution property.

Theorem 1. [1] Let A be a right R-module, and set $E = \text{End}_R(A)$. Then E has 1 in the stable range if and only if for any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with each $A_i \cong A$, there exists a submodule $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$.

A ring R is said to have 2 in the stable range if for any $a_1, \ldots, a_r \in R$ where $r \geq 3$ such that $a_1R + \cdots + a_rR = R$, there exist elements $b_1, \ldots, b_{r-1} \in R$ such that $(a_1 + a_rb_1)R + (a_2 + a_rb_2)R + \cdots + (a_{r-1} + a_rb_{r-1})R = R$.

K. Goodearl pointed out to us the following result.

Proposition 1. [1] Let R be a commutative ring which has 2 in the stable range. If R satisfies right power-substitution, then so does $M_n(R)$, for all n.

Our goal this paper is to study certain algebraic versions of the notion of stable range 1. In this paper we study a Bezout ring which has 2 in the stable range and which is a ring square stable range 1.

A ring R is said to have (right) square stable range 1 (written ssr(R) = 1) if aR + bR = R for any $a, b \in R$ implies that $a^2 + bx$ is an invertible element of R for some $x \in R$. Considering the problem of factorizing the matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ into a product of two Toeplitz matrices. D. Khurana, T.Y. Lam and Zhou Wang were led to ask go units of the form $a^2 + bx$ given that aR + bR = R.

Obviously, a commutative ring which has 1 in the stable range is a ring which has (right) square stable range 1, but not vice versa in general. Examples of rings which have (right) square stable range 1 are rings of continuous real-valued functions on topological spaces and real holomorphy rings in formally real fields [2].

Proposition 2. [2] For any ring R with ssr(R) = 1, we have that R is right quasi-duo (i.e. R is a ring in which every maximal right ideal is an ideal).

We say that matrices A and B over a ring R are equivalent if there exist invertible matrices P and Q of appropriate sizes such that B = PAQ. If for a matrix A there exists a diagonal matrix $D = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that A and D are equivalent and $R\varepsilon_{i+1}R \subseteq \varepsilon_iR \cap R\varepsilon_i$ for every i then we say that the matrix A has a canonical diagonal reduction. A ring R is called an elementary divisor ring if every matrix over R has a canonical diagonal reduction. If every (1×2) -matrix $((2 \times 1)$ -matrix) over a ring R has a canonical diagonal reduction then R is called a right (left) Hermitian ring. A ring which is both right and left Hermitian is called an Hermitian ring. Obviously, a commutative right (left) Hermitian ring is an Hermitian ring. We note that a right Hermitian ring is a ring in which every finitely generated right ideal is principal.

Theorem 2. [3] Let R be a right quasi-duo elementary divisor ring. Then for any $a \in R$ there exists an element $b \in R$ such that RaR = bR = Rb. If in addition all zero-divisors of R lie in the Jacobson radical, then R is a duo ring.

Recall that a right (left) duo ring is a ring in which every right (left) ideal is two-sided. A duo ring is a ring which is both left and right duo ring.

We have proved the next result.

Theorem 3. Let R be an elementary divisor ring which has (right) square stable range 1 and which all zero-divisors of R lie in Jacobson radical of R, then R is a duo ring.

Proof. By Proposition 2 we have that R is a right quasi-duo ring. By Theorem 2 we have that R is a duo ring.

Proposition 3. Let R be a Hermitian duo ring. For every $a, b, c \in R$ such that aR + bR + cR = R the following conditions are equivalent:

- 1) there exist elements $p, q \in R$ such that paR + (pb + qc)R = R;
- 2) there exist elements $\lambda, u, v \in R$ such that $b + \lambda c = vu$, where uR + aR = R and vR + cR = R.

Proof. 1) \Rightarrow 2) Since paR + (pb + qc)R = R we have pR + qcR = R and since R is a duo ring we have pR + cR = R. Than Rp + Rc = R, i.e. vp + jc = 1 for some elements $v, j \in R$. Then vpb + jcb = b and b - vpb = jcb = cj'b = ct where t = j'b and jc = cj'. Element j' exist, since R is a duo ring. Then v(pb + qc) = vpb + vqc = b + ct + vqc = b + ct + ck, where vqc = ck for some element $k \in R$. That is, we have $v(pb + qc) - b = c\lambda$ for some element $\lambda \in R$. We have $b + c\lambda = v(pb + qc)$. Let u = pb + qc. We have $b + c\lambda = vu$, where vR + cR = R, since vp + cj' = 1 and uR + aR = R, since paR + (pb + qc)R = R.

 $2)\Rightarrow 1)$ Since vR + cR = R then Rv + Rc = R. Let pv + jc = 1 for some elements $p, j \in R$. Then pR + cR = R. Since $b + \lambda c = vu$, we have $pb = p(vu - \lambda c) = (pv)u - p\lambda c = (1 - jc)u + p\lambda c = u - ju'c + p\lambda c = u + qc$ for some element $q = p\lambda - ju'$, where cu = u'c for some element $u' \in R$. Since u = pb + qc, therefore (pb + qc)R + aR = R. Since R is an Hermitian duo ring then we have pR + qR = dR where $p = dp_1, q = dq_1$ and $p_1R + q_1R = R$. Then $p_1R + (p_1b + q_1c)R = R$ since $pR \subset p_1R$ and pR + cR = R, $p_1R + q_1R = R$, i.e. we have $p_1R + (p_1b + q_1c)R = R$. Hence, $aR + (p_1b + q_1c)R$ we have $p_1aR + (p_1b + q_1c)R = R$.

Remark 1. In Proposition 3 we can choose the elements u and v such that uR + vR = R.

Proposition 4. Let R be an Hermitian duo ring. Then the following conditions are equivalent:

- 1) R is an elementary divisor duo ring;
- 2) for every $x, y, z, t \in R$ such that xR + yR = R and zR + tR = R there exists an element $\lambda \in R$ such that $x + \lambda y = vu$, where vR + zR = R and uR + tR = R.

Proof. 1) \Rightarrow 2) Let R be an elementary divisor ring. By [4] for any a, b, c such that aR + bR + cR = R there exist elements $p, q \in R$ such that paR + (pb + qc)R = R.

Since xR + yR = R, zR + tR = R and the fact that R is a Hermitian duo ring we have zR + xR + ytR = R. By Proposition 3 we have $x + \lambda yt = uv$ where uR + zR = R, vR + ytR = R. Since $x + (\lambda t)y = x + \mu y = uv$ where $\mu = \lambda t$, we have uR + zR = R, vR + yR = R.

2) \Rightarrow 1) Let aR + bR + cR = R and Rb + Rc = Rd and $b = b_1d$, $c = c_1d$, where $Rb_1 = Rc_1 = R$. Since R is a duo ring then $b_1R + c_1R = R$. So now dR = Rd and aR + bR + cR = R, Rb + Rc = Rd we have aR + dR = R, i.e. $dd_1 + ax = 1$ for some elements $d_1, x \in R$. Then $1 - dd_1 \in aR$.

Since $b_1R + c_1R = R$, by Conditions 2 of Proposition 3 there exists an element $\lambda_1 \in R$ such that $b_1 + c_1\lambda = vu_1$ where $u_1R + (1 - dd_1)R = R$ and $vR + dd_1R = R$. Since $(1 - dd_1) \in aR$ and $u_1R + (1 - dd_1)R = R$. We have uR + aR = R. Let $u = u_1d$. Since $u_1R + aR = R$ and dR + aR = R we have uR + aR = R. Since $b_1 + c_1\lambda = vu_1$, we have $b + c\mu + vu$, where $\lambda d = d\mu$.

Recall that $vR + dd_1R = R$ then vR + dR = R. Since $vR + cR = vR + c_1dR = vR + c_1R$. So $b_1 + c_1\lambda = vu_1$ and $b_1R + c_1R = R$ then $vR + c_1R = R$.

Therefore, vR+cR = R. This means that the Condition 2 of Proposition 3 is true. By Proposition 3 we conclude that for every $a, b, c \in R$ with aR + bR + cR = R there exist elements $p, q \in R$ such that paR + (pb + qc)R = R, i.e. according to [4], R is an elementary divisor ring.

Definition 1. Let R be a duo ring. We say that an element $a \in R \setminus \{0\}$ is neat if for any elements $b, c \in R$ such that bR + cR = R there exist elements $r, s \in R$ such that a = rs, where rR + bR = R, sR + cR = R, rR + sR = R.

Definition 2. We say that a duo ring R has neat range 1 if for every $a, b \in R$ such that aR + bR = R there exists an element $t \in R$ such that a + bt is a neat element.

According to Propositions 3, 4 and Remark 1 we have the following result.

Theorem 4. A Hermitian duo ring R is an elementary divisor ring if and only if R has neat range 1.

The term "neat range 1" substantiates the following theorem.

Theorem 5. Let R be a Hermitian duo ring. If c is a neat element of R then R/cR is a clean ring.

Proof. Let c = rs, where rR + aR = R, sR + (1 - a)R = R for any element $a \in R$. Let $\bar{r} = r + cR$, $\bar{s} = s + aR$. From the equality rR + sR = R we have ru + sv = 1 for some elements $u, v \in R$. Hence $r^2u + srv = r$ and $rsu + s^2v = s$ we have $\bar{r}^2\bar{u} = \bar{r}$, $\bar{s}^2\bar{v} = \bar{s}$. Let $\bar{s}\bar{v} = \bar{e}$. It is obvious that $\bar{e}^2 = \bar{e}$ and $\bar{1} - \bar{e} = \bar{u}\bar{r}$. Since rR + aR = R, we have rx + ay = 1 for elements $x, y \in R$. Hence rxsv + aysv = sv we have rsx'v + aysv = sv where xs = sx' for some element $x' \in R$. Then $\bar{a}\bar{y}\bar{e} = \bar{e}$, i.e. $\bar{e} \in \bar{a}R$. Similarly from the equality sR + (1 - a)R = R, it follows $\bar{1} - \bar{e} \in (\bar{1} - \bar{a})R$. According to [5] R/cR is an exchange ring. Since R is a duo ring, R/cR is a clean ring.

Taking into account the Theorem 3 and Theorem 4 we have the following result.

Theorem 6. A Hermitian ring R which has (right) square stable range 1 is an elementary divisor ring if and only if R is a duo ring of neat range 1. Let R be a commutative Bezout ring. The matrix A of order 2 over R is said to be a Toeplitz matrix if it is of the form

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

where $a, b, c \in R$.

Notice that if A is an invertible Toeplitz matrix, then A^{-1} is also an invertible Toeplitz matrix.

Definition 3. A commutative Hermitian ring R is called a Toeplitz ring if for any $a, b \in R$ there exist an invertible Toeplitz matrix T such that (a, b)T = (d, 0) for some element $d \in R$.

Theorem 7. A commutative Hermitian ring R is a Toeplitz ring if and only if R is a ring of (right) square range 1.

Proof. Let R be a commutative Hermitian ring of (right) square stable range 1 and aR + bR = R for some elements $a, b \in R$. Then $a^2 + bt = u$, where u is an invertible element of R.

Let

$$S = \begin{pmatrix} a & -b \\ t & a \end{pmatrix}, \quad K = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Then

$$(a,b)S = (u,0), \quad (u,0)K = (1,0),$$

i.e. we have

$$(a,b)SK = (1,0).$$

Since

$$\begin{pmatrix} a & -b \\ t & a \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} au^{-1} & -bu^{-1} \\ -tu^{-1} & au^{-1} \end{pmatrix} = T$$

we have that T = SK is a Toeplitz matrix. So (a, b)T = (1, 0). If $a, b \in R$ and aR + bR = dR then by $a = da_0$, $b = db_0$ and $a_0R + b_0R = R$ [4]. Then there exists an element $t \in R$ such that $a_0 + b_0t = u$, where a is an invertible element of R.

Let

$$\begin{pmatrix} a_0 & -b_0 \\ t & a_0 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix}$$

Note that T is an invertible Toeplitz matrix. Then (a, b)T = (d, 0), i.e. R is a Toeplitz ring.

Let R be a Toeplitz ring and aR + bR = R. The exists an invertible Toeplitz matrix T such that (a, b)T = (1, 0). Let $S = T^{-1} = \begin{pmatrix} x & t \\ y & x \end{pmatrix}$, where $x, y, t \in R$. So det $T^{-1} = z^2 + ty = u$ is an invertible element of R. Since $(a, b) = (1, 0)T^{-1}$, we have a = x, b = t. By equality $x^2 + ty = u$ we have $a^2 + by = u$, i.e. R is a ring of (right) square stable range 1.

Theorem 8. Let R be a commutative ring of square stable range 1. Then for any row (a,b), where aR + bR = R, there exists an invertible Toeplitz matrix

$$T = \begin{pmatrix} a & b \\ x & a \end{pmatrix},$$

where $x \in R$.

Proof. By Theorem 7 we have (a, b) = (1, 0)T for some invertible Toeplitz matrix T. Let $T = \begin{pmatrix} x & t \\ y & x \end{pmatrix}$. Then a = x, b = t and $T = \begin{pmatrix} a & b \\ y & a \end{pmatrix}$ is an invertible Toeplitz matrix.

Recall that $GE_n(R)$ denotes a group of $n \times n$ elementary matrices over ring R. The following theorem demonstrated that it is sufficient to consider only the case of matrices of order 2 in Theorem 7.

Theorem 9. [4] Let R be a commutative elementary divisor ring. Then for any $n \times m$ matrix A (n > 2, m > 2) one can find matrices $P \in GE_n(R)$ and $Q \in GE_m(R)$ such that

$$PAQ = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0\\ 0 & e_2 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & e_s & 0\\ 0 & 0 & \dots & 0 & A_0 \end{pmatrix}$$

where e_i is a divisor of e_{i+1} , $1 \le i \le s-1$, and A_0 is a $2 \times k$ or $k \times 2$ matrix for some $k \in \mathbb{N}$.

Theorem 10. Let R be a commutative elementary divisor ring of (right) square stable range 1. Then for any 2×2 matrix A one can find invertible Toeplitz matrices P and Q such that

$$PAQ = \begin{pmatrix} e_1 & 0\\ 0 & e_2 \end{pmatrix},$$

where e_i is a divisor of e_2 .

Proof. Since R is a Toeplitz ring it is enough to consider matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where aR + bR + cR = R. Since R is an elementary divisor ring by [4] there exist elements $p, q \in R$ such that paR + (pb+qc)R = R, i.e. par + (pb+qc)s = 1 for some elements $r, s \in R$. Since pR + qR = R and rR + sR = R, by Theorem 8 we have the invertible Toeplitz matrices $P = \begin{pmatrix} p & q \\ * & * \end{pmatrix}$, $Q = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ such that

$$PAQ = \begin{pmatrix} 1 & x \\ y & z \end{pmatrix} = A_1.$$

Then

$$\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} A_1 \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ac \end{pmatrix},$$

where $S = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ are invertible Toeplitz matrices. So

$$SPAQT = \begin{pmatrix} 1 & 0 \\ 0 & ac \end{pmatrix}.$$

Theorem is proved.

Open Question. Is it true that every commutative Bezout domain of stable range 2 which has (right) square stable range 1 is an elementary divisor ring?

References

K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math.* 64 (1976) 487–411.

- [2] D. Khurana, T. Y. Lam and Zh. Wang, Rings of square stable range one, J. Algebra 338 (2011) 122–143.
- [3] B. V. Zabavsky and M. Ya. Komarnytskii, Distributive elementary divisor domains, Ukr. Math. J. 42 (1990) 890–892.
- [4] B. V. Zabavsky Diagonal reduction of matrices over rings (Mathematical Studies, Monograph Series, volume XVI, VNTL Publishers, Lviv, 2012).
- [5] W. K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc. 229 (1977) 269–278.