A Bezout ring of stable range 2 which has square stable range 1

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Abstract: In this paper we introduced the concept of a ring of stable range 2 which has square stable range 1. We proved that a Hermitian ring R which has (right) square stable range 1 is an elementary divisor ring if and only if R is a duo ring of neat range 1. And we proved that a commutative Hermitian ring R is a Toeplitz ring if and only if R is a ring of (right) square range 1. We proved that if R be a commutative elementary divisor ring of (right) square stable range 1, then for any matrix $A \in M_2(R)$ one can find invertible Toeplitz matrices P and Q such that $PAQ = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$, where e_i is a divisor of e_2 .

Key words and phrases: Hermitian ring, elementary divisor ring, stable range 1, stable range 2, square stable range 1, Toeplitz matrix, duo ring, quasi-duo ring.

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1 Introduction

The notion of a stable range of a ring was introduced by H. Bass, and became especially popular because of its various applications to the problem of cancellation and substitution of modules. Let us say that a module A satisfies the power-cancellation property if for all modules B and C, $A \oplus B \cong A \oplus C$ implies that $B^n \cong C^n$ for some positive integer n (here B^n denotes the direct sum of n copies of B). Let us say that a right R-module A has the power-substitution property if given any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ which each $A_i \cong A$, there exist a positive integer n and a submodule $C \subseteq M^n$ such that $M^n = C \oplus B_1^n = C \oplus B_2^n$.

Prof. K. Goodearl pointed out that a commutative rind R has the powersubstitution property if and only if R is of (right) power stable range 1, i.e. if $aR + bR = R$ than $(aⁿ + bx)R = R$ for some $x \in R$ and some integer $n \ge 2$ depending on $a, b \in R$ [\[1\]](#page-7-0).

Recall that a ring R is said to have 1 in the stable range provided that whenever $ax + b = 1$ in R, there exists $y \in R$ such that $a + by$ is a unit in R. The following Warfield's theorem shows that 1 in the stable range is equivalent to a substitution property.

Theorem 1. *[\[1\]](#page-7-0)* Let A be a right R-module, and set $E = \text{End}_R(A)$. Then E *has 1 in the stable range if and only if for any right* R*-module decomposition* $M = A_1 \oplus B_1 = A_2 \oplus B_2$ *with each* $A_i \cong A$ *, there exists a submodule* $C \subseteq M$ *such that* $M = C \oplus B_1 = C \oplus B_2$.

A ring R is said to have 2 in the stable range if for any $a_1, \ldots, a_r \in R$ where $r \geq 3$ such that $a_1R+\cdots+a_rR=R$, there exist elements $b_1,\ldots,b_{r-1} \in$ R such that $(a_1 + a_r b_1)R + (a_2 + a_r b_2)R + \cdots + (a_{r-1} + a_r b_{r-1})R = R$.

K. Goodearl pointed out to us the following result.

Proposition 1. *[\[1\]](#page-7-0) Let* R *be a commutative ring which has 2 in the stable range. If* R *satisfies right power-substitution, then so does* $M_n(R)$ *, for all n.*

Our goal this paper is to study certain algebraic versions of the notion of stable range 1. In this paper we study a Bezout ring which has 2 in the stable range and which is a ring square stable range 1.

A ring R is said to have (right) square stable range 1 (written $ssr(R) = 1$) if $aR + bR = R$ for any $a, b \in R$ implies that $a^2 + bx$ is an invertible element of R for some $x \in R$. Considering the problem of factorizing the matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ into a product of two Toeplitz matrices. D. Khurana, T.Y. Lam and Zhou Wang were led to ask go units of the form $a^2 + bx$ given that $aR + bR = R$.

Obviously, a commutative ring which has 1 in the stable range is a ring which has (right) square stable range 1, but not vice versa in general. Examples of rings which have (right) square stable range 1 are rings of continuous real-valued functions on topological spaces and real holomorphy rings in formally real fields [\[2\]](#page-8-0).

Proposition 2. [\[2\]](#page-8-0) For any ring R with $ssr(R) = 1$, we have that R is right *quasi-duo (i.e.* R *is a ring in which every maximal right ideal is an ideal).*

We say that matrices A and B over a ring R are equivalent if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$. If for a matrix A there exists a diagonal matrix $D = diag(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r, 0, \dots, 0)$ such that A and D are equivalent and $R\varepsilon_{i+1}R \subseteq \varepsilon_i R \cap R\varepsilon_i$ for every i then we say that the matrix A has a canonical diagonal reduction. A ring R is called an elementary divisor ring if every matrix over R has a canonical diagonal reduction. If every (1×2) -matrix $((2 \times 1)$ -matrix) over a ring R has a canonical diagonal reduction then R is called a right (left) Hermitian ring. A ring which is both right and left Hermitian is called an Hermitian ring. Obviously, a commutative right (left) Hermitian ring is an Hermitian ring. We note that a right Hermitian ring is a ring in which every finitely generated right ideal is principal.

Theorem 2. *[\[3\]](#page-8-1) Let* R *be a right quasi-duo elementary divisor ring. Then for any* $a \in R$ *there exists an element* $b \in R$ *such that* $RaR = bR = Rb$ *. If in addition all zero-divisors of* R *lie in the Jacobson radical, then* R *is a duo ring.*

Recall that a right (left) duo ring is a ring in which every right (left) ideal is two-sided. A duo ring is a ring which is both left and right duo ring.

We have proved the next result.

Theorem 3. *Let* R *be an elementary divisor ring which has (right) square stable range 1 and which all zero-divisors of* R *lie in Jacobson radical of* R*, then* R *is a duo ring.*

Proof. By Proposition [2](#page-1-0) we have that R is a right quasi-duo ring. By Theo-rem [2](#page-2-0) we have that R is a duo ring. \Box

Proposition 3. Let R be a Hermitian duo ring. For every $a, b, c \in R$ such *that* $aR + bR + cR = R$ *the following conditions are equivalent:*

- *1)* there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$;
- *2)* there exist elements $\lambda, u, v \in R$ such that $b + \lambda c = vu$, where $uR + aR =$ R and $vR + cR = R$.

Proof. 1) \Rightarrow 2) Since $paR + (pb + qc)R = R$ we have $pR + qcR = R$ and since R is a duo ring we have $pR + cR = R$. Than $Rp + Rc = R$, i.e. $vp + jc = 1$ for some elements $v, j \in R$. Then $vpb + jcb = b$ and $b - vpb = jcb = cj'b = ct$ where $t = j'b$ and $jc = cj'$. Element j' exist, since R is a duo ring. Then $v(pb + qc) = vpb + vqc = b + ct + vqc = b + ct + ck$, where $vqc = ck$ for some element $k \in R$. That is, we have $v(pb+qc) - b = c\lambda$ for some element $\lambda \in R$. We have $b + c\lambda = v(pb + qc)$. Let $u = pb + qc$. We have $b + c\lambda = vu$, where $vR+cR = R$, since $vp+cj' = 1$ and $uR+aR = R$, since $paR+(pb+qc)R = R$.

2)⇒1) Since $vR + cR = R$ then $Rv + Rc = R$. Let $pv + jc = 1$ for some elements $p, j \in R$. Then $pR + cR = R$. Since $b + \lambda c = vu$, we have $pb = p(vu - \lambda c) = (pv)u - p\lambda c = (1 - jc)u + p\lambda c = u - ju'c + p\lambda c = u + qc$ for some element $q = p\lambda - ju'$, where $cu = u'c$ for some element $u' \in R$. Since $u = pb + qc$, therefore $(ph+qc)R+aR=R$. Since R is an Hermitian duo ring then we have $pR+qR = dR$ where $p = dp_1$, $q = dq_1$ and $p_1R+q_1R = R$. Then $p_1R + (p_1b + q_1c)R = R$ since $pR \subset p_1R$ and $pR + cR = R$, $p_1R + q_1R = R$, i.e. we have $p_1R + (p_1b + q_1c)R = R$. Hence, $aR + (p_1b + q_1c)R$ we have $p_1aR + (p_1b + q_1c)R = R.$ □

Remark 1. *In Proposition [3](#page-2-1) we can choose the elements* u *and* v *such that* $uR + vR = R$.

Proposition 4. *Let* R *be an Hermitian duo ring. Then the following conditions are equivalent:*

- *1)* R *is an elementary divisor duo ring;*
- *2)* for every $x, y, z, t \in R$ such that $xR + yR = R$ and $zR + tR = R$ there *exists an element* $\lambda \in R$ *such that* $x + \lambda y = vu$ *, where* $vR + zR = R$ $and uR + tR = R.$

Proof. 1)⇒2) Let R be an elementary divisor ring. By [\[4\]](#page-8-2) for any a, b, c such that $aR + bR + cR = R$ there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R.$

Since $xR + yR = R$, $zR + tR = R$ and the fact that R is a Hermitian duo ring we have $zR + xR + ytR = R$. By Proposition [3](#page-2-1) we have $x + \lambda yt = uv$ where $uR + zR = R$, $vR + ytR = R$. Since $x + (\lambda t)y = x + \mu y = uv$ where $\mu = \lambda t$, we have $uR + zR = R$, $vR + yR = R$.

2)⇒1) Let $aR + bR + cR = R$ and $Rb + Rc = Rd$ and $b = b_1d$, $c = c_1d$, where $Rb_1 = Rc_1 = R$. Since R is a duo ring then $b_1R + c_1R = R$. So now $dR = Rd$ and $aR + bR + cR = R$, $Rb + Rc = Rd$ we have $aR + dR = R$, i.e. $dd_1 + ax = 1$ for some elements $d_1, x \in R$. Then $1 - dd_1 \in aR$.

Since $b_1R + c_1R = R$, by Conditions 2 of Proposition [3](#page-2-1) there exists an element $\lambda_1 \in R$ such that $b_1 + c_1 \lambda = vu_1$ where $u_1R + (1 - dd_1)R = R$ and $vR + dd_1R = R$. Since $(1 - dd_1) \in aR$ and $u_1R + (1 - dd_1)R = R$. We have $uR + aR = R$. Let $u = u_1d$. Since $u_1R + aR = R$ and $dR + aR = R$ we have $uR + aR = R$. Since $b_1 + c_1\lambda = vu_1$, we have $b + c\mu + vu$, where $\lambda d = d\mu$.

Recall that $vR + dd_1R = R$ then $vR + dR = R$. Since $vR + cR =$ $vR + c_1dR = vR + c_1R$. So $b_1 + c_1\lambda = vu_1$ and $b_1R + c_1R = R$ then $vR + c_1R = R.$

Therefore, $vR+cR = R$. This means that the Condition 2 of Proposition [3](#page-2-1) is true. By Proposition [3](#page-2-1) we conclude that for every $a, b, c \in R$ with aR + $bR + cR = R$ there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$, i.e. according to [\[4\]](#page-8-2), R is an elementary divisor ring. \Box

Definition 1. Let R be a duo ring. We say that an element $a \in R \setminus \{0\}$ is neat if for any elements $b, c \in R$ such that $bR + cR = R$ there exist elements $r, s \in R$ such that $a = rs$, where $rR + bR = R$, $sR + cR = R$, $rR + sR = R$.

Definition 2. We say that a duo ring R has neat range 1 if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $a + bt$ is a neat element.

According to Propositions [3,](#page-2-1) [4](#page-3-0) and Remark [1](#page-3-1) we have the following result.

Theorem 4. *A Hermitian duo ring* R *is an elementary divisor ring if and only if* R *has neat range 1.*

The term "neat range 1" substantiates the following theorem.

Theorem 5. *Let* R *be a Hermitian duo ring. If* c *is a neat element of* R *then* R/cR *is a clean ring.*

Proof. Let $c = rs$, where $rR + aR = R$, $sR + (1 - a)R = R$ for any element $a \in R$. Let $\bar{r} = r + cR$, $\bar{s} = s + aR$. From the equality $rR + sR = R$ we have $ru + sv = 1$ for some elements $u, v \in R$. Hence $r^2u + sv = r$ and $rsu + s^2v = s$ we have $\bar{r}^2\bar{u} = \bar{r}, \bar{s}^2\bar{v} = \bar{s}$. Let $s\bar{v} = \bar{e}$. It is obvious that $\bar{e}^2 = \bar{e}$ and $\bar{1} - \bar{e} = \bar{u}\bar{r}$. Since $rR + aR = R$, we have $rx + ay = 1$ for elements $x, y \in R$. Hence $rxsv + aysv = sv$ we have $rsx'v + aysv = sv$ where $xs = sx'$ for some element $x' \in R$. Then $\overline{a}\overline{y}\overline{e} = \overline{e}$, i.e. $\overline{e} \in \overline{a}\overline{R}$. Similarly from the equality $sR + (1 - a)R = R$, it follows $\overline{1} - \overline{e} \in (\overline{1} - \overline{a})\overline{R}$. According to [\[5\]](#page-8-3) R/cR is an exchange ring. Since R is a duo ring, R/cR is a clean ring. \Box

Taking into account the Theorem [3](#page-2-2) and Theorem [4](#page-4-0) we have the following result.

Theorem 6. *A Hermitian ring* R *which has (right) square stable range 1 is an elementary divisor ring if and only if* R *is a duo ring of neat range 1.*

Let R be a commutative Bezout ring. The matrix A of order 2 over R is said to be a Toeplitz matrix if it is of the form

$$
\begin{pmatrix} a & b \\ c & a \end{pmatrix}
$$

where $a, b, c \in R$.

Notice that if A is an invertible Toeplitz matrix, then A^{-1} is also an invertible Toeplitz matrix.

Definition 3. A commutative Hermitian ring R is called a Toeplitz ring if for any $a, b \in R$ there exist an invertible Toeplitz matrix T such that $(a, b)T = (d, 0)$ for some element $d \in R$.

Theorem 7. *A commutative Hermitian ring* R *is a Toeplitz ring if and only if* R *is a ring of (right) square range 1.*

Proof. Let R be a commutative Hermitian ring of (right) square stable range 1 and $aR + bR = R$ for some elements $a, b \in R$. Then $a^2 + bt = u$, where u is an invertible element of R.

Let

$$
S = \begin{pmatrix} a & -b \\ t & a \end{pmatrix}, \quad K = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix}.
$$

Then

$$
(a, b)S = (u, 0), \quad (u, 0)K = (1, 0),
$$

i.e. we have

$$
(a, b)SK = (1, 0).
$$

Since

$$
\begin{pmatrix} a & -b \ t & a \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} au^{-1} & -bu^{-1} \ -tu^{-1} & au^{-1} \end{pmatrix} = T
$$

we have that $T = SK$ is a Toeplitz matrix. So $(a, b)T = (1, 0)$. If $a, b \in R$ and $aR + bR = dR$ then by $a = da_0$, $b = db_0$ and $a_0R + b_0R = R$ [\[4\]](#page-8-2). Then there exists an element $t \in R$ such that $a_0 + b_0 t = u$, where a is an invertible element of R.

Let

$$
\begin{pmatrix} a_0 & -b_0 \ t & a_0 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \ 0 & u^{-1} \end{pmatrix}.
$$

Note that T is an invertible Toeplitz matrix. Then $(a, b)T = (d, 0)$, i.e. R is a Toeplitz ring.

Let R be a Toeplitz ring and $aR + bR = R$. The exists an invertible Toeplitz matrix T such that $(a, b)T = (1, 0)$. Let $S = T^{-1} = \begin{pmatrix} x & t \\ y & x \end{pmatrix}$, where $x, y, t \in R$. So det $T^{-1} = z^2 + ty = u$ is an invertible element of R. Since $(a, b) = (1, 0)T^{-1}$, we have $a = x$, $b = t$. By equality $x^2 + ty = u$ we have $a^2 + by = u$, i.e. R is a ring of (right) square stable range 1. □

Theorem 8. *Let* R *be a commutative ring of square stable range 1. Then for any row* (a, b) *, where* $aR + bR = R$ *, there exists an invertible Toeplitz matrix*

$$
T = \begin{pmatrix} a & b \\ x & a \end{pmatrix},
$$

where $x \in R$ *.*

Proof. By Theorem [7](#page-5-0) we have $(a, b) = (1, 0)T$ for some invertible Toeplitz $\begin{pmatrix} x & t \\ y & x \end{pmatrix}$. Then $a = x$, $b = t$ and $T =$ $\begin{pmatrix} a & b \\ y & a \end{pmatrix}$ is an matrix T . Let $T =$ invertible Toeplitz matrix. \Box

Recall that $GE_n(R)$ denotes a group of $n \times n$ elementary matrices over ring R. The following theorem demonstrated that it is sufficient to consider only the case of matrices of order 2 in Theorem [7.](#page-5-0)

Theorem 9. *[\[4\]](#page-8-2) Let* R *be a commutative elementary divisor ring. Then for any* $n \times m$ *matrix* A $(n > 2, m > 2)$ *one can find matrices* $P \in \mathbb{G}E_n(R)$ *and* $Q \in GE_m(R)$ *such that*

$$
PAQ = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0 \\ 0 & e_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e_s & 0 \\ 0 & 0 & \dots & 0 & A_0 \end{pmatrix}
$$

where e_i *is a divisor of* e_{i+1} , $1 \leq i \leq s-1$, and A_0 *is a* $2 \times k$ *or* $k \times 2$ *matrix for some* $k \in \mathbb{N}$ *.*

Theorem 10. *Let* R *be a commutative elementary divisor ring of (right) square stable range 1. Then for any* 2×2 *matrix* A *one can find invertible Toeplitz matrices* P *and* Q *such that*

$$
PAQ = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},
$$

where e_i *is a divisor of* e_2 *.*

Proof. Since R is a Toeplitz ring it is enough to consider matrices of the form

$$
A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},
$$

where $aR + bR + cR = R$. Since R is an elementary divisor ring by [\[4\]](#page-8-2) there exist elements $p, q \in R$ such that $paR+(pb+qc)R = R$, i.e. $par+(pb+qc)s = 1$ for some elements $r, s \in R$. Since $pR + qR = R$ and $rR + sR = R$, by Theorem [8](#page-6-0) we have the invertible Toeplitz matrices $P =$ $\begin{pmatrix} p & q \\ * & * \end{pmatrix}$, $Q =$ $\int r \cdot *$ \setminus such that

$$
PAQ = \begin{pmatrix} 1 & x \\ y & z \end{pmatrix} = A_1.
$$

Then

s ∗

$$
\begin{pmatrix} 1 & 0 \ -y & 1 \end{pmatrix} A_1 \begin{pmatrix} 1 & -x \ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \ 0 & ac \end{pmatrix},
$$

where $S =$ $\begin{pmatrix} 1 & 0 \end{pmatrix}$ $-y$ 1 \setminus and $T =$ $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ are invertible Toeplitz matrices. So

$$
SPAQT = \begin{pmatrix} 1 & 0 \\ 0 & ac \end{pmatrix}.
$$

Theorem is proved.

Open Question. Is it true that every commutative Bezout domain of stable range 2 which has (right) square stable range 1 is an elementary divisor ring?

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 \Box

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