

# TOPOLOGICAL DEGREE FOR EQUIVARIANT GRADIENT PERTURBATIONS OF AN UNBOUNDED SELF-ADJOINT OPERATOR IN HILBERT SPACE

PIOTR BARTŁOMIEJCZYK, BARTOSZ KAMEDULSKI,  
AND PIOTR NOWAK-PRZYGODZKI

**ABSTRACT.** We present a version of the equivariant gradient degree defined for equivariant gradient perturbations of an equivariant unbounded self-adjoint operator with purely discrete spectrum in Hilbert space. Two possible applications are discussed.

## INTRODUCTION

To obtain new bifurcation results, N. Dancer [5] introduced in 1985 a new topological invariant for  $S^1$ -equivariant gradient maps, which provides more information than the usual equivariant one. In 1994 S. Rybicki [14, 16] developed the complete degree theory for  $S^1$ -equivariant gradient maps and 3 years later K. Gęba extended this theory to an arbitrary compact Lie group. In 2001 S. Rybicki [15] defined the degree for  $S^1$ -equivariant strongly indefinite functionals in Hilbert space. 10 years later A. Gołębiewska and S. Rybicki [8] generalized this degree to compact Lie groups. The relation between equivariant and equivariant gradient degree theories were studied in [1, 2, 7].

The main goal of this paper is to present a construction and properties of a new degree-type topological invariant  $\text{Deg}_G^\nabla$ , which is defined for equivariant gradient perturbations of a equivariant unbounded self-adjoint Hilbert operator with a purely discrete spectrum (in the general case a compact Lie group). As far as we know, the idea of the construction of such an invariant should be attributed to K. Gęba.

It is worth pointing out that equivariant gradient perturbations of an equivariant unbounded self-adjoint operator with a purely discrete spectrum appear naturally in a variety of problems in nonlinear analysis, such as the search for periodic solutions of Hamiltonian systems

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or the study of Seiberg-Witten equations for three dimensional manifolds. The purpose of our work is to provide a topological tool that allows us to solve problems similar to the above mentioned ones.

The paper is organized as follows. Section 1 contains some preliminaries. In Section 2 we present the construction that leads to the definition of the degree  $\text{Deg}_G^\nabla$ . The correctness of this definition is proved in Section 3. The properties of the degree  $\text{Deg}_G^\nabla$  are examined in Section 4. Finally, in Section 5 we discuss two examples of possible applications.

## 1. PRELIMINARIES

The preliminaries are divided into five brief subsections.

**1.1. Unbounded self-adjoint operators in Hilbert space.** This subsection is based on [17]. Let  $E$  be a real separable Hilbert space with inner product  $\langle \cdot | \cdot \rangle$  and  $A: D(A) \subset E \rightarrow E$  be a linear operator (not necessarily bounded) such that its domain  $D(A)$  is dense in  $E$ . Set

$$D(A^*) = \{y \in E \mid \exists u \in E \forall x \in D(A) \langle Ax \mid y \rangle = \langle x \mid u \rangle\}.$$

Since  $D(A)$  is dense in  $E$ , the vector  $u \in E$  is uniquely determined by  $y$ . Therefore by setting  $A^*y = u$  we obtain a well-defined linear operator from  $D(A^*)$  to  $E$ . The operator  $A^*$  is called the *adjoint* operator of  $A$ . We say that  $A$  is *self-adjoint* if  $A = A^*$ . By the Hellinger-Toeplitz theorem, if  $A$  is self-adjoint and  $D(A) = E$  then  $A$  is bounded.

It is easy to see that

$$\langle x \mid y \rangle_1 = \langle x \mid y \rangle + \langle Ax \mid Ay \rangle$$

defines an inner product on the domain  $D(A)$ . Under this product  $D(A)$  becomes a Hilbert space, which will be denoted by  $E_1$ . Thus  $D(A)$  and  $E_1$  are equal as sets but equipped with different inner products. Note that  $A$  treated as an operator from  $E_1$  to  $E$  is bounded.

We say that a self-adjoint operator  $A$  has a *purely discrete spectrum* if its spectrum consists only of isolated eigenvalues of finite multiplicity. If  $E$  is an infinite dimensional Hilbert space then following conditions are equivalent:

- (1)  $A$  has a purely discrete spectrum.
- (2) There is a real sequence  $\{\lambda_n\}$  and an orthonormal basis  $\{e_n\}$  such that  $\lim |\lambda_n| = \infty$  and  $Ae_n = \lambda_n e_n$  for  $n \in \mathbb{N}$ .
- (3) The embedding  $\iota: E_1 \rightarrow E$  is compact.

### 1.2. Local maps in Hilbert space.

- Let
- $E$  be a real Hilbert orthogonal representation of a compact Lie group  $G$ ,
  - $A: D(A) \subset E \rightarrow E$  be an unbounded self-adjoint operator with a purely discrete spectrum,
  - $D(A)$  be invariant and  $A$  equivariant.

**Definition 1.1.** We write  $f \in \mathcal{G}_G(E)$  if

- $f: D_f \subset E_1 \rightarrow E$ , where  $D_f$  is an open invariant subset of  $E_1$ ,
- $f(x) = Ax - \nabla\varphi(x)$ , where  $\varphi: E \rightarrow \mathbb{R}$  is  $C^1$  and invariant,
- $f^{-1}(0)$  is compact.

Elements of  $\mathcal{G}_G(E)$  will be called *local maps*.

**1.3. Otopies in Hilbert space.** Let  $I = [0, 1]$ . Assume that  $G$  acts trivially on  $I$ . A map  $h: \Lambda \subset I \times E_1 \rightarrow E$  is called an *otopy* if

- $\Lambda$  is an open invariant subset of  $I \times E_1$ ,
- $h(t, \cdot) \in \mathcal{G}_G(E)$  for each  $t \in I$ ,
- $h^{-1}(0)$  is compact.

Given an otopy  $h: \Lambda \subset I \times E_1 \rightarrow E$  we can define for each  $t \in I$ :

- sets  $\Lambda_t = \{x \in E_1 \mid (t, x) \in \Lambda\}$ ,
- maps  $h_t: \Lambda_t \rightarrow E$  with  $h_t(x) = h(t, x)$ .

If  $h$  is an otopy, we say that  $h_0$  and  $h_1$  are *otopic*. The relation of being otopic is an equivalence relation in  $\mathcal{G}_G(E)$ .

Observe that if  $f$  is a local map and  $U$  is an open subset of  $D_f$  such that  $f^{-1}(0) \subset U$ , then  $f$  and  $f|_U$  are otopic. This property of local maps is called the *restriction property*. In particular, if  $f^{-1}(0) = \emptyset$  then  $f$  is otopic to the empty map.

**1.4. Euler-tom Dieck ring.** Recall the notion of the *Euler-tom Dieck ring* following [19]. For a compact Lie group  $G$  let  $\mathfrak{U}(G)$  denote the set of equivalence classes of finite  $G$ -CW-complexes. Two complexes  $X$  and  $Y$  are identified if the quotients  $X^H/WH$  and  $Y^H/WH$  have the same Euler characteristic for all closed subgroups  $H$  of  $G$ . Recall that  $X^H$  stands here for the  $H$ -fixed point set of  $X$ , i.e.  $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$  and  $WH$  for the Weyl group of  $H$ , i.e.  $WH = NH/H$ . Addition and multiplication in  $\mathfrak{U}(G)$  are induced by disjoint union and cartesian product with diagonal  $G$ -action, i.e.

$$[X] + [Y] = [X \sqcup Y], \quad [X] \cdot [Y] = [X \times Y],$$

where the square brackets stand for an equivalence class of finite  $G$ -CW-complexes. In this way  $\mathfrak{U}(G)$  becomes a commutative ring with unit and is called the *Euler-tom Dieck ring* of  $G$ .

Additively,  $\mathfrak{U}(G)$  is a free abelian group with basis elements  $[G/H]$ , where  $H$  is a closed subgroup of  $G$ . In consequence, each element of  $\mathfrak{U}(G)$  can be uniquely written as a finite sum  $\sum d_{(H)} [G/H]$ , where  $d_{(H)}$  is an integer, which depends only on the conjugacy class of  $H$ . The ring unit is  $[G/G]$ .

**1.5. Finite dimensional equivariant gradient degree  $\deg_G^\nabla$ .** Assume that  $V$  is a real finite dimensional orthogonal representation of a compact Lie group  $G$ . We write  $f \in \mathcal{G}_G(V)$  if  $f$  is an equivariant gradient map from an open invariant subset of  $V$  to  $V$  and  $f^{-1}(0)$  is compact. In the papers [1, 2, 6, 16] the authors defined the equivariant gradient degree

$$\deg_G^\nabla: \mathcal{G}_G(V) \rightarrow \mathfrak{U}(G)$$

and proved that the degree has the following properties: additivity, otopy invariance, existence and normalization. The product property formulated below was proved in [6] and [9].

**Theorem 1.2** (Product property). *Let  $V$  and  $W$  be real finite dimensional orthogonal representations of a compact Lie group  $G$ . If  $f \in \mathcal{G}_G(V)$  and  $f' \in \mathcal{G}_G(W)$ , then  $f \times f' \in \mathcal{G}_G(V \oplus W)$  and*

$$\deg_G^\nabla(f \times f') = \deg_G^\nabla(f) \cdot \deg_G^\nabla(f') \text{ in } \mathfrak{U}(G).$$

In the next section we will make use of the following result, which can be found in [8, Cor. 2.1].

**Theorem 1.3.** *Let  $V$  be a real finite dimensional orthogonal representation of a compact Lie group  $G$ . If  $B$  is an equivariant self-adjoint isomorphism of  $V$  then  $\deg_G^\nabla(B)$  is invertible in  $\mathfrak{U}(G)$ .*

*Remark 1.4.* Note that Theorem 1.3 holds even if  $V$  is trivial. In this case  $\deg_G^\nabla(B)$  is equal to the unit of  $\mathfrak{U}(G)$ .

## 2. DEFINITION OF DEGREE

In this section we present the construction of the degree  $\text{Deg}_G^\nabla$  using finite dimensional approximations.

**2.1. Finite dimensional approximations.** Let us start with some notations:

- for  $\lambda \in \sigma(A)$  denote by  $V(\lambda)$  the corresponding eigenspace;
- for  $n \in \mathbb{N}$  write  $V_n = \bigoplus_{|\lambda| \leq n} V(\lambda)$ ,  $V^n = \bigoplus_{n-1 < |\lambda| \leq n} V(\lambda)$  and  $A_n = A|_{V^n}$ ; hence  $V_n = V_{n-1} \oplus V^n$ ;
- let  $P_n: E \rightarrow V_n$  denote the orthogonal projection.

Assume that  $U$  is an open bounded invariant subset of  $D_f$  such that

$$f^{-1}(0) \subset U \subset \text{cl } U \subset D_f.$$

Set  $U_n = U \cap V_n$ . Finally, let  $f_n: U_n \rightarrow V_n$  be given by

$$f_n(x) = Ax - P_n F(x),$$

where  $F(x) = \nabla \varphi(x)$ .

The following two lemmas are needed to prove Lemma 2.3, which is crucial for the definition of  $\text{Deg}_G^\nabla$ .

**Lemma 2.1.** *There is  $\epsilon > 0$  such that  $|f(x)| \geq 2\epsilon$  for all  $x \in \partial U$ .*

*Proof.* The fact  $F$  is compact and  $\partial U$  is closed and bounded implies our claim.  $\square$

Let us introduce an auxiliary map  $\tilde{f}_n: D_f \rightarrow E$  given by  $\tilde{f}_n(x) = Ax - P_n F(x)$ . By definition,  $\tilde{f}_n|_{U_n} = f_n$ .

**Lemma 2.2.** *There is  $N$  such that for  $n \geq N$  we have*

- (1)  $|f(x) - \tilde{f}_n(x)| < \epsilon$  for  $x \in \text{cl } U$ ,
- (2)  $|\tilde{f}_n(x)| > \epsilon$  for  $x \in \partial U$ .

*Proof.* Since  $F$  is compact,  $F$  is close to  $P_n F$ , which gives (1). In turn (2) follows from (1) and Lemma 2.1.  $\square$

**Lemma 2.3.** *For  $n \geq N$  we have  $f_n \in \mathcal{G}_G(V_n)$  and, in consequence,  $\text{deg}_G^\nabla(f_n) \in \mathfrak{U}(G)$  is well-defined.*

*Proof.* Since  $f_n$  is obviously gradient, it is enough to check that  $f_n^{-1}(0)$  is compact. Note that  $\tilde{f}_n$  can be considered as an extension of  $f_n$  on  $\text{cl } U_n$ . By (2) from Lemma 2.2,  $\tilde{f}_n$  does not have zeroes in  $\partial U_n \subset \partial U$ , which implies that  $f_n^{-1}(0) = \tilde{f}_n^{-1}(0) \cap U_n$  is compact.  $\square$

**2.2. Degree definition.** Observe that  $A_n$  is an equivariant self-adjoint isomorphism for  $n \geq 1$ . By Theorem 1.3, elements  $a_n := \text{deg}_G^\nabla(A_n)$  are invertible in  $\mathfrak{U}(G)$ . Set  $m_n := a_1^{-1} \cdot a_2^{-1} \cdot \dots \cdot a_n^{-1}$ .

**Definition 2.4.** Let  $\text{Deg}_G^\nabla: \mathcal{G}_G(E) \rightarrow \mathfrak{U}(G)$  be defined by

$$\text{Deg}_G^\nabla(f) := m_n \cdot \text{deg}_G^\nabla(f_n)$$

for  $n \geq N$ .

An alternative definition of  $\text{Deg}_G^\nabla$  in terms of the direct limit is given in Appendix A.

### 3. CORRECTNESS OF THE DEFINITION

We have to prove that our definition does not depend on the choice of  $n$  and the neighbourhood  $U$ .

**3.1. Independence from the choice of  $n$ .** To show this we will need the following lemma.

**Lemma 3.1.** *For  $n$  large enough  $f_{n+1}$  is otopic to  $f_n \times A_{n+1}$  in  $\mathcal{G}_G(V_{n+1})$  and hence*

$$\deg_G^\nabla(f_{n+1}) = \deg_G^\nabla(f_n \times A_{n+1}).$$

*Proof.* First observe there is an open  $W \subset U$  and natural number  $N$  such that

- $f^{-1}(0) \subset W \subset U$ ,
- $P_n(\text{cl } W) \subset U_n$  for all  $n \geq N$ .

Define  $h_{n+1}: I \times \text{cl } W_{n+1} \rightarrow V_{n+1}$  by

$$h_{n+1}(t, x) = (1 - t)f_{n+1}(x) + t(f_n \times A_{n+1})(x).$$

We set  $n$  sufficiently large. One can show that  $h_{n+1}(t, x) \neq 0$  for  $t \in I$  and  $x \in \partial W_{n+1}$ . In consequence,  $h_{n+1}|_{I \times W_{n+1}}$  is a finite dimensional equivariant gradient otopy between  $f_{n+1}|_{W_{n+1}}$  and  $f_n \times A_{n+1}|_{W_{n+1}}$  (otherwise there would be a point  $x_0 \in \partial W$  such that  $f(x_0) = 0$ , a contradiction). On the other hand, by the restriction property,  $f_{n+1}$  and  $f_n \times A_{n+1}$  are otopic to their restrictions to  $W_{n+1}$ , which completes the proof.  $\square$

From Lemma 3.1 and Theorem 1.2 we can easily conclude that

$$\begin{aligned} \deg_G^\nabla(f_{n+1}) &\stackrel{3.1}{=} \deg_G^\nabla(f_n \times A_{n+1}) \stackrel{1.2}{=} \\ &\deg_G^\nabla(f_n) \cdot \deg_G^\nabla(A_{n+1}) = a_{n+1} \cdot \deg_G^\nabla(f_n). \end{aligned}$$

This gives

$$m_{n+1} \cdot \deg_G^\nabla(f_{n+1}) = m_{n+1} \cdot a_{n+1} \cdot \deg_G^\nabla(f_n) = m_n \cdot \deg_G^\nabla(f_n),$$

which shows that  $\text{Deg}_G^\nabla(f)$  does not depend on the choice of  $n$  large enough.

**3.2. Independence from the choice of  $U$ .** According to our definition  $\text{Deg}_G^\nabla(f) = \text{Deg}_G^\nabla(f|_U)$ . Now we will prove that in fact  $\text{Deg}_G^\nabla(f)$  is independent from the choice of the neighbourhood  $U$ .

**Lemma 3.2.** *Let  $W$  and  $U$  be open bounded sets such that*

$$f^{-1}(0) \subset W \subset U \subset \text{cl } U \subset D_f.$$

*Then  $\text{Deg}_G^\nabla(f|_W) = \text{Deg}_G^\nabla(f|_U)$ .*

*Proof.* By the analogue of Lemma 2.1 (with  $\partial U$  replaced by  $\text{cl } U \setminus W$ ),  $|f(x)| \geq 2\epsilon$  for  $x \in \text{cl } U \setminus W$  and by Lemma 2.2,  $|f(x) - \tilde{f}_n(x)| < \epsilon$  for  $x \in \text{cl } U$ . Hence  $\tilde{f}_n(x) \neq 0$  for  $x \in \text{cl } U \setminus W$ . In consequence,  $f_n(x) \neq 0$  for  $x \in \text{cl } U_n \setminus W_n$ . Therefore

$$\text{Deg}_G^\nabla(f \upharpoonright U) = m_n \cdot \text{deg}_G^\nabla(f_n \upharpoonright U_n) = m_n \cdot \text{deg}_G^\nabla(f_n \upharpoonright W_n) = \text{Deg}_G^\nabla(f \upharpoonright W). \quad \square$$

**Corollary 3.3.** *Let  $U$  and  $U'$  be open bounded subsets of  $D_f$  such that*

$$f^{-1}(0) \subset U \cap U' \subset \text{cl}(U \cup U') \subset D_f.$$

*Then  $\text{Deg}_G^\nabla(f \upharpoonright U) = \text{Deg}_G^\nabla(f \upharpoonright U \cap U') = \text{Deg}_G^\nabla(f \upharpoonright U')$ .*

In this way we have proved that  $\text{Deg}_G^\nabla(f)$  does not depend on the choice of admissible  $U$ .

#### 4. DEGREE PROPERTIES

In this section we prove that our degree  $\text{Deg}_G^\nabla: \mathcal{G}_G(E) \rightarrow \mathfrak{U}(G)$  has all properties analogous to the well-known properties of the finite dimensional equivariant gradient degree  $\text{deg}_G^\nabla$ .

**Additivity property.** *If  $f, f' \in \mathcal{G}_G(E)$  and  $D_f \cap D_{f'} = \emptyset$  then*

$$\text{Deg}_G^\nabla(f \sqcup f') = \text{Deg}_G^\nabla(f) + \text{Deg}_G^\nabla(f').$$

**Otopy invariance property.** *Let  $f, f' \in \mathcal{G}_G(E)$ . If  $f$  is otopic to  $f'$  then*

$$\text{Deg}_G^\nabla(f) = \text{Deg}_G^\nabla(f').$$

**Existence property.** *If  $\text{Deg}_G^\nabla(f) \neq 0$  then  $f(x) = 0$  for some  $x \in D_f$ .*

**Normalization property.**

$$\text{Deg}_G^\nabla(A + P_0) = [G/G] = 1_{\mathfrak{U}(G)},$$

where  $P_0: E_1 \rightarrow V_0 = \ker A$  is the orthogonal projection.

**Product property.** *Let  $E$  and  $E'$  be real Hilbert orthogonal representations of a compact Lie group  $G$ . If  $f \in \mathcal{G}_G(E)$  and  $f' \in \mathcal{G}_G(E')$ , then  $f \times f' \in \mathcal{G}_G(E \oplus E')$  and*

$$\text{Deg}_G^\nabla(f \times f') = \text{Deg}_G^\nabla(f) \cdot \text{Deg}_G^\nabla(f'),$$

where the dot here denotes the multiplication in  $\mathfrak{U}(G)$ .

*Proof.*

**Additivity.** Immediately from the additivity of  $\text{deg}_G^\nabla$  we obtain

$$\begin{aligned} \text{Deg}_G^\nabla(f \sqcup f') &= m_n \cdot \text{deg}_G^\nabla(f_n \sqcup f'_n) = \\ &= m_n \cdot (\text{deg}_G^\nabla(f_n) + \text{deg}_G^\nabla(f'_n)) = \text{Deg}_G^\nabla(f) + \text{Deg}_G^\nabla(f'). \end{aligned}$$

**Otopy invariance.** Let the map  $h: \Lambda \subset I \times E_1 \rightarrow E$  given by  $h(t, x) = Ax - F(t, x)$  be an otopy. We introduce the following notation:

$$\begin{aligned} \Lambda^t &= \{x \in E_1 \mid (t, x) \in \Lambda\}, & h^t: \Lambda^t &\rightarrow E, & h^t(x) &= h(t, x), \\ \Lambda_n &= \Lambda \cap (I \times V_n), & h_n: \Lambda_n &\rightarrow V_n, & h_n(t, x) &= Ax - P_n F(t, x), \\ \Lambda_n^t &= \Lambda^t \cap V_n, & h_n^t: \Lambda_n^t &\rightarrow V_n, & h_n^t(x) &= h_n(t, x). \end{aligned}$$

Note that for the needs of this subsection the time parameter  $t$  of the otopy is a superscript, not a subscript. According to the above notation we have to show that  $\text{Deg}_G^\nabla(h^0) = \text{Deg}_G^\nabla(h^1)$ . Since  $h^{-1}(0)$  is compact, there is an open bounded set  $W \subset I \times E_1$  such that

$$h^{-1}(0) \subset W \subset \text{cl } W \subset \Lambda.$$

Hence for  $i = 0, 1$  we have

$$(h^i)^{-1}(0) \subset W^i \subset \text{cl } W^i \subset \Lambda^i,$$

where  $W^i = \{x \in E_1 \mid (i, x) \in W\}$ . Similarly as in Lemma 2.1, there is  $\epsilon > 0$  such that  $|h(z)| \geq 2\epsilon$  for  $z \in \partial W$ . On the other hand, similarly as in Lemma 2.2, there is  $N$  such that  $|h(z) - \tilde{h}_n(z)| < \epsilon$  for  $z \in \text{cl } W$  and  $n \geq N$ , where  $\tilde{h}_n: \Lambda \rightarrow E$  is given by  $\tilde{h}_n(t, x) = Ax - P_n F(t, x)$ . Therefore  $|h_n(z)| \geq \epsilon$  for  $z \in \partial W_n \subset \partial W$ . From the above:

- $h_n \upharpoonright_{W_n}$  is a finite dimensional equivariant gradient otopy,
- $\text{Deg}_G^\nabla(h^i) = m_n \cdot \text{deg}_G^\nabla(h_n^i \upharpoonright_{W_n^i})$ ,

which, by the otopy invariance of  $\text{deg}_G^\nabla$ , gives

$$\text{Deg}_G^\nabla(h^0) = m_n \cdot \text{deg}_G^\nabla(h_n^0 \upharpoonright_{W_n^0}) = m_n \cdot \text{deg}_G^\nabla(h_n^1 \upharpoonright_{W_n^1}) = \text{Deg}_G^\nabla(h^1).$$

**Existence.** If  $f^{-1}(0) = \emptyset$  then  $f$  is otopic with the empty map. Hence

$$\text{Deg}_G^\nabla(f) = \text{Deg}_G^\nabla(\emptyset) = 0.$$

**Normalization.** Observe that  $A + P_0$  is an injection and

$$\text{deg}_G^\nabla((A + P_0)_n) = \text{deg}_G^\nabla(\text{Id} \upharpoonright_{V_0}) \cdot \text{deg}_G^\nabla(A_1) \cdot \dots \cdot \text{deg}_G^\nabla(A_n) = m_n^{-1}$$

for any  $n \geq 1$ . Hence

$$\text{Deg}_G^\nabla(A + P_0) = m_n \cdot \text{deg}_G^\nabla((A + P_0)_n) = [G/G].$$



**Product formula.** Let  $f(x) = Ax - F(x)$  and  $f'(x) = A'x - F'(x)$ . Observe that, by Theorem 1.2, if  $f_n \in \mathcal{G}_G(V_n)$  and  $f'_n \in \mathcal{G}_G(V'_n)$  then  $f_n \times f'_n \in \mathcal{G}_G(V_n \oplus V'_n)$  and

$$\deg_G^\nabla(f_n \times f'_n) = \deg_G^\nabla(f_n) \cdot \deg_G^\nabla(f'_n).$$

Moreover, for  $n$  large enough

$$\text{Deg}_G^\nabla(f) = m_n \cdot \deg_G^\nabla(f_n),$$

$$\text{Deg}_G^\nabla(f') = m'_n \cdot \deg_G^\nabla(f'_n).$$

Since for any  $i \geq 1$

$$\deg_G^\nabla((A \times A')_i) = \deg_G^\nabla(A_i \times A'_i) = \deg_G^\nabla(A_i) \cdot \deg_G^\nabla(A'_i),$$

we have

$$\begin{aligned} \text{Deg}_G^\nabla(f \times f') &= m_n \cdot m'_n \cdot \deg_G^\nabla(f_n \times f'_n) = \\ &= m_n \cdot m'_n \cdot \deg_G^\nabla(f_n) \cdot \deg_G^\nabla(f'_n) = \text{Deg}_G^\nabla(f) \cdot \text{Deg}_G^\nabla(f'). \end{aligned}$$

□

*Remark 4.1.* The normalization property can be formulated more generally, but the proof of this fact will appear elsewhere. Namely, let  $x_0 \in V_n$  and, in consequence,  $Gx_0 \subset V_n$ . Define

$$\begin{aligned} U &= \{x + y + z \mid x \in Gx_0, y \in (T_{x_0}(Gx_0))^\perp \subset V_n, \\ &\quad |y| < \epsilon, z \in (V_n)^\perp \subset E_1\} \end{aligned}$$

and  $f: U \rightarrow E$  by

$$f(x + y + z) = (A + P_0)(y + z).$$

Then  $\text{Deg}_G^\nabla(f) = [G/G_{x_0}]$ .

## 5. POSSIBLE APPLICATIONS

We should emphasize that this section contains not real applications of the theory but only two exemplary situations illustrating potential applications.

**5.1. Applications to Hamiltonian systems.** The search for periodic solutions in Hamiltonian systems is one of the fundamental problems in nonlinear analysis (see for instance [3, 12, 13, 20]). Consider the Hamiltonian system of ODE

$$\frac{dp}{dt} = -H_q, \quad \frac{dq}{dt} = H_p,$$

where  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and  $p, q \in \mathbb{R}^n$  or equivalently

$$\frac{dz}{dt} = \mathcal{J}H_z,$$

where  $z = (p, q)$  and

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The function  $H$  is called the hamiltonian or energy.

Rewrite the Hamiltonian system as

$$(*) \quad \dot{z} = \mathcal{J}\nabla H(z), \quad z \in \mathbb{R}^{2n}$$

or equivalently  $-\mathcal{J}\dot{z} - \nabla H(z) = 0$ .

We are searching for solutions  $z \in H_T^1$  of the equation  $(*)$ , where  $H_T^1$  ( $T > 0$ ) denotes the completion of the set of smooth  $T$ -periodic functions from  $\mathbb{R}$  to  $\mathbb{R}^{2n}$  in the norm associated to the inner product  $(u|v)_{H_T^1} = \int_0^T uv \, dt + \int_0^T \dot{u}\dot{v} \, dt$ . For this purpose we apply the method of the topological degree  $\text{Deg}_{S^1}^\nabla$ . Namely, let  $E = L^2(S^1, \mathbb{R}^{2n})$  and  $E_1 = H^1(S^1, \mathbb{R}^{2n})$ . Moreover, denote by  $D$  the set  $E_1$  equipped with the inner product from  $E$ .

Observe that

- $E$  and  $E_1$  are Hilbert spaces and orthogonal representations of the group  $SO(2) = S^1$  with the  $S^1$ -action given by the shift in time,
- $A: D \rightarrow E$  given by  $Az = -\mathcal{J}\dot{z}$  is an equivariant unbounded self-adjoint operator with a purely discrete spectrum,
- $\nabla H(z)$  is a gradient of the invariant functional  $\varphi: E \rightarrow \mathbb{R}$  defined by  $\varphi(z) = \int_0^{2\pi} H(z(t)) \, dt$ ,
- $\nabla H \circ \iota: E_1 \rightarrow E$  is a compact map by the compactness of the inclusion  $\iota: E_1 \rightarrow E$ .

We can now formulate the main result of this subsection.

**Theorem 5.1.** *Assume that  $\lambda > 0$  and the set of zeros of the map  $f_\lambda(z) = -\mathcal{J}\dot{z} - \lambda\nabla H(z)$  is compact. If  $\text{Deg}_{S^1}^\nabla(f_\lambda) \neq 0$  then the equation  $(*)$  has a solution in  $H_{2\pi\lambda}^1$ .*

*Proof.* First note that if  $f_\lambda^{-1}(0)$  is compact then  $f_\lambda$  is an element of  $\mathcal{G}_{S^1}(E)$ . By the existence property,  $\text{Deg}_{S^1}^\nabla(f_\lambda) \neq 0$  implies that  $f_\lambda(z) = 0$  for some  $z \in E_1$ . Hence a lift  $\tilde{z} \in H_{2\pi\lambda}^1$  of  $z$  given by  $\tilde{z}(t) = z(\rho(t))$ , where  $\rho: \mathbb{R} \rightarrow S^1$  is the standard covering projection, is a solution of  $(*)$ , which is our claim.  $\square$

**5.2. Applications to the Seiberg-Witten equations.** The description of the Seiberg-Witten equations presented here is necessarily sketchy (for more details we refer the reader to [4, 10, 11, 18]). Let  $M$  be a closed oriented Riemannian 3-manifold. A  $\text{Spin}^c$ -structure on  $M$  consists of rank two Hermitian vector bundle  $S \rightarrow M$  called the *spinor bundle*. We write  $\Omega^1(M, i\mathbb{R})$  for the space of smooth imaginary-valued 1-forms on  $M$  and  $\Gamma(S)$  for the space of smooth cross-sections of the spinor bundle  $S \rightarrow M$ . For each  $\alpha \in \Omega^1(M, i\mathbb{R})$  there is an associated *Dirac operator*  $D_\alpha: \Gamma(S) \rightarrow \Gamma(S)$ .

Recall that, in what follows,  $d$  stands for the exterior derivative and  $*$  denotes the Hodge star. For a pair  $(\alpha, \varphi) \in \Omega^1(M, i\mathbb{R}) \oplus \Gamma(S)$  the *Seiberg-Witten equations* are

$$\begin{cases} D_\alpha \varphi = 0 \\ *d\alpha = Q(\varphi), \end{cases}$$

where  $Q(\varphi) \in \Omega^1(M, i\mathbb{R})$  is a certain quadratic form (nonlinear part of the equations). The solutions of Seiberg-Witten equations are zeros of the *Seiberg-Witten map*

$$\text{SW}: \Omega^1(M, i\mathbb{R}) \oplus \Gamma(S) \rightarrow \Omega^1(M, i\mathbb{R}) \oplus \Gamma(S)$$

given by

$$\text{SW}(\alpha, \varphi) = (*d\alpha - Q(\varphi), -D_\alpha \varphi).$$

After suitable Sobolev completion the Seiberg-Witten map  $\text{SW}$  can be written in the form  $A - F$ , where  $A = (*d\alpha, -D_\alpha \varphi)$  is an unbounded self-adjoint operator with a purely discrete spectrum and  $F$  is a gradient map. Moreover, the Seiberg-Witten map is equivariant for the action of the group  $S^1$ , which acts trivially on the component arising from the differential forms and as complex multiplication on the spinor component. It suggests that the SW map should fit to our abstract setting of the degree  $\text{Deg}_{S^1}^\nabla$ . Unfortunately, the set of zeros of the SW map is not compact. However, we hope that it is possible to reduce our problem to some subspace of  $\Omega^1(M, i\mathbb{R})$  in such a way that the reduced SW map will have a compact set of zeros, which will be contained in the set of zeros of the original SW map. Verifying this claim is, however, still in progress.

#### APPENDIX A.

Definition 2.4 may be seen as a simple particular case of a more general construction called the *direct limit of a direct system of groups*.

Namely, for  $i = 0, 1, \dots$  let  $G_i$  denote an abelian group and  $\alpha_i: G_i \rightarrow G_{i+1}$  a group homomorphism. With this notation we get the sequence

$$G_0 \xrightarrow{\alpha_0} G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \rightarrow \dots$$

Let  $\tilde{G} := \coprod_{i=0}^{\infty} G_i$  denote a disjoint union, i.e.

$$\tilde{G} = \{(i, m) \mid i \in \mathbb{N}, m \in G_i\}.$$

We introduce in  $\tilde{G}$  an equivalence relation. For  $i > j$  we write  $(i, m) \sim (j, l)$  if

$$\alpha_{i-1} \circ \dots \circ \alpha_{j+1} \circ \alpha_j(l) = m.$$

The *direct limit of groups* is the set of equivalence classes of the above relation, denoted by

$$\varinjlim G_i = \tilde{G} / \sim.$$

Let  $\varinjlim \mathfrak{U}(G)$  denote a direct limit of groups, where

- $G_i = \mathfrak{U}(G)$  for all  $i$ ,
- $\alpha_i$  is multiplication by an element  $a_i = \deg_G^\nabla(A_i, V^i) \in \mathfrak{U}(G)$ .

With this notation we can alternatively define our degree as a function  $\text{Deg}_G^\nabla: \mathfrak{S}_G(E) \rightarrow \varinjlim \mathfrak{U}(G) \approx \mathfrak{U}(G)$  given by

$$\text{Deg}_G^\nabla(f) := [(n, \deg_G^\nabla(f_n, \mathcal{U}_n))]$$

for  $n$  large enough.

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FACULTY OF APPLIED PHYSICS AND MATHEMATICS, GDAŃSK UNIVERSITY OF TECHNOLOGY, GABRIELA NARUTOWICZA 11/12, 80-233 GDAŃSK, POLAND  
*E-mail address:* piobart1@pg.edu.pl

FACULTY OF NAVIGATION, GDYNIA MARITIME UNIVERSITY, JANA PAWŁA II 3, 81-345 GDYNIA, POLAND  
*E-mail address:* b.kamedulski@wn.umg.edu.pl

FACULTY OF APPLIED PHYSICS AND MATHEMATICS, GDAŃSK UNIVERSITY OF TECHNOLOGY, GABRIELA NARUTOWICZA 11/12, 80-233 GDAŃSK, POLAND  
*E-mail address:* piotrnp@wp.pl