

# The Beltrami equation with parameters and uniformization of holomorphic foliations with hyperbolic leaves\*

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## Introduction

Let  $X$  be a complex manifold. We say that a foliation with singularities is defined on  $X$  if there exists an analytic subset  $\Sigma$  of codimension at least two and a foliation of its complement by analytic curves that cannot be extended to a neighborhood of any point of  $\Sigma$ . A foliation can be locally defined by polynomial vector fields. In generic case the singular set  $\Sigma$  consists of isolated points.

A covering manifold of leaves of a foliation was defined in [Il1], [Il2]. Let  $\mathcal{F}$  be a foliation with singularities on a complex manifold  $X$  and let  $B$  be transversal cross-section. Let  $\varphi_p$  be a leaf passing through a point  $p \in B$  and let  $\hat{\varphi}_p$  be the universal covering over this leaf with the marked point  $p$ . Define  $M = \bigcup_{p \in B} \hat{\varphi}_p$ . It is shown in [Il1], [Il2] that at least in affine case or, in more general Stein case, a topology and a complex structure on this union can be defined so that it is a complex manifold with locally biholomorphic projection  $\tilde{\pi} : M \rightarrow X$  and a holomorphic section  $B \rightarrow M$  right inverse to the

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holomorphic retraction  $\pi : M \rightarrow B$ . For any leaf  $\varphi_p$  the restriction of  $\tilde{\pi}$  to  $\hat{\varphi}_p$  is the universal covering map over  $\varphi_p$ . For a foliation of a compact manifold the manifold of universal coverings can be non-Hausdorff but in a generic case it is Hausdorff (see [Br1], [Br2]). It is possible to define a Hausdorff universal covering for general foliations of compact Kahler manifolds if we include the singular points in the leaves in some not generic cases ([Br1], [Br3]) but here we don't consider such situations.

Let  $T_{\mathcal{F}}$  be the linear bundle tangent to the leaves. If this bundle is negative, then there exists an hermitean metric on  $X \setminus \Sigma$  and restriction of this metric on each leaf has a negative curvature. For generic such foliation each leaf is hyperbolic ([G11] or [LN]). In particular, it's true for a generic foliation of  $\mathbb{C}\mathbb{P}^n$ . The uniformizing map of every leaf is unique modulo automorphisms of the disk, and after some normalization (to get uniqueness) we may ask: how the uniformizing map of  $\varphi_p$  depends on the point  $p$ ? Equivalently, we may put on every leaf its Poincare metric, i. e., the unique complete hermitian metric of curvature -1 and ask about dependence of this metric on the point  $p$ . It is known that the Poincare metric is continuous [V] and even Holder-continuous [DNS]. The simultaneous uniformization conjecture states that there exists an analytic in  $p$  biholomorphism of  $\varphi_p$  onto an appropriate  $p$ -depending domain on the Riemann sphere. It is known that this conjecture is wrong for general foliation in dimension of more than two or even for foliations of general two-dimensional manifolds [G12]. It is not known is this conjecture true or not for generic foliations of  $\mathbb{C}^2$  or  $\mathbb{C}\mathbb{P}^2$ .

One of the main problems of the theory of holomorphic foliations is the problem of analytic continuation of the Poincare map defined on a transversal to the leaves and the related problem of the persistence of cycles. It was shown in [II3] that these problems have the positive solution if there exists an analytic simultaneous uniformization and the image domains continuously depend on the initial conditions. In the absence of an analytic simultaneous uniformization we can consider the results below as its more feeble version. We hope that these results can be useful in following attempts to clear the situation with the persistence problems. Though the situation can not be simple. There exists examples of the non-extendability of the Poincare map though for rather special cases [CDFG].

There were shown in [Sh1], [Sh2] that for generic foliation with negative  $T_{\mathcal{F}}$  we can define the complex structure on  $M$  as an almost complex structure on the product  $B \times D$  ( $D$  is the unit disk) and this almost complex structure

can be defined by forms of type (1,0)

$$dz_i, \quad i = 1, \dots, n, \quad (0.1)$$

$$dw + \mu d\bar{w} + \langle c, d\bar{z} \rangle, \quad (0.2)$$

where  $n+1$  is the dimension of  $X$ ,  $z, w$  are charts on  $B$  and  $D$  correspondingly,  $c$  is a smooth vector-function,  $\langle c, d\bar{z} \rangle = c_1 d\bar{z}_1 + \dots + c_n d\bar{z}_n$ ,  $\mu$  is a smooth function satisfying the estimate  $|\mu| \leq d < 1$  for some non-negative  $d$ . I.e., we can say that this almost complex structure is quasiconformal on each fiber.

The conclusion made in [Sh1] that the Poincare metric smoothly depends on a base point isn't correct because there isn't satisfied the sufficient condition: uniform boundedness along the fibers of derivatives with respect to the parameters, i.e., to the coordinates on the base (see [AhB]). It was shown by B. Deroin that the Poincare metric isn't smooth for the foliation of a neighborhood of a hyperbolic singular point.

However in a generic case there exists a finitely smooth map holomorphic on the fibers and mapping each fiber on a bounded domain in  $\mathbb{C}$  continuously depending on a base point. Moreover, there exist estimates for derivatives of this map similar to the estimates of  $\mu$  and  $c$  obtained in [Sh2]. Now we formulate our main result. We denote by  $\partial^{(k)}f$  any derivative with respect to the variable  $w, \bar{w}, z_i, \bar{z}_i$  of the total order  $|k|$ .

**Theorem 1** *Suppose  $X$  is a compact complex manifold of dimension  $n + 1$  and  $\mathcal{F}$  is a holomorphic foliation of  $X$  with negative with  $T_{\mathcal{F}}$ . Suppose that the singular set  $\Sigma$  is finite and in some neighborhood of each singular point the vector field locally defining the foliation is analytically linearizable and the linear part is diagonalizable. Let  $M$  be a manifold of universal covering with simply connected base  $B$  and let the complex structure on  $M$  be defined by forms (0.1), (0.2). Then for every integer  $p \geq 0$  there exists a fiberwise map  $f : M \rightarrow B \times \mathbb{C}$  differentiable up to the order  $p$ , holomorphic on the fibers, continuously depending on a base point in  $C^0(\mathbb{C})$  and satisfying the estimates*

$$|f_w(w, z)| \leq C(1 - |w|)^{-\alpha}, \quad |f_{\bar{w}}(w, z)| \leq C(1 - |w|)^{-\alpha}, \quad 0 \leq \alpha < 1, \quad (0.3)$$

$$|f_{w^m \bar{w}^l}(w, z)| / |f_w(w, z)| \leq C(1 - |w|)^{-(m+l-1)}, \quad m + l \leq p, \quad (0.4)$$

$$|\partial^{(k)}f(w, z)| \leq C(1 - |w|)^{-N(p)}, \quad |k| \leq p. \quad (0.5)$$

Here  $C$  is some uniform constant and  $N(p)$  is a constant depending on  $p$ .

Since our complex structure is defined by forms (0.1), (0.2) the assertion that  $f$  is holomorphic on the fibers means that  $f$  satisfies the Beltrami equation  $f_{\bar{z}} = \mu f_z$ . The proof of the theorem is based on the estimates obtained in [Sh2] and the present article can be considered as a continuation of that work. In fact, all that follows is a study of the Beltrami equation with a coefficient depending on parameters and satisfying the estimates of [Sh2].

## 1 Preliminary notes and the sketch of the proof

In what follows  $D$  is the unit disk,  $D_r$  is the disk of radii  $r$  centered at zero,  $D_{a,r}$  is the disk of radii  $r$  centered at  $a$ . Suppose  $f$  is a function of a vector variable  $z$  of dimension  $n$  and of a scalar variable  $w$ , and  $(k)$  is a multi-index  $(k) = \{k_0, k_{\bar{0}}, k_1, \dots, k_n, k_{\bar{1}}, \dots, k_{\bar{n}}\}$ . We denote by  $\partial^{(k)}f$  or  $f_{(k)}$  the derivative  $f_{w^{k_0} \bar{w}^{k_{\bar{0}}} z_1^{k_1} \dots z_n^{k_n} \bar{z}_1^{k_{\bar{1}}} \dots \bar{z}_n^{k_{\bar{n}}}}$  and define  $|(k)| = k_0 + k_{\bar{0}} + k_1 + \dots + k_n + k_{\bar{1}} + \dots + k_{\bar{n}}$ . Also, sometimes we shall use double multi-indexes  $(k, l) = \{\{k_1, k_{\bar{1}}\}, \{l_1, \dots, l_n, l_{\bar{1}}, \dots, l_{\bar{n}}\}\}$  and denote by  $\partial^{(k,l)}f$  or  $f_{(k,l)}$  the derivatives  $f_{w^{k_1} \bar{w}^{k_{\bar{1}}} z_1^{l_1} \dots z_n^{l_n} \bar{z}_1^{l_{\bar{1}}} \dots \bar{z}_n^{l_{\bar{n}}}}$ . In this case we define  $|(k)| = k_1 + k_{\bar{1}}, |(l)| = l_1 + \dots + l_n + l_{\bar{1}} + \dots + l_{\bar{n}}$ . The main result of [Sh2] is the theorem about almost complex structures on manifolds of universal coverings:

**Theorem ACS.** *Suppose  $X$  is a compact complex manifold of dimension  $n + 1$  and  $\mathcal{F}$  is a holomorphic foliation of  $X$  with negative  $T_{\mathcal{F}}$ . Suppose that the singular set  $\Sigma$  is finite and in some neighborhood of each singular point the vector field locally defining the foliation is analytically linearizable and the linear part is diagonalizable. Let  $M$  be a manifold of universal covering with a simply connected base  $B$ . Then the complex structure on  $M$  can be defined as an almost complex structure on the product  $B \times D$  ( $D$  is the unit disk) and this almost complex structure can be defined by forms (0.1), (0,2) of type (1,0). There  $c$  is a smooth vector-function,  $\mu$  is a smooth function and we have the estimates*

$$|\mu| \leq d < 1,$$

$$|\mu_{w^k, \bar{w}^l}(w, z)| \leq C(1 - |w|)^{-(k+l)}, \quad (1.1)$$

$$|\partial^{(k)}\mu(w, z)| \leq C(1 - |w|)^{-A|(k)|^4}, \quad |\partial^{(k)}c_i(w, z)| \leq C(1 - |w|)^{-A(|(k)|^4+1)} \quad (1.2)$$

for any pair  $k, l$  and multi-index  $(k)$ . The constant  $C$  in these estimates depends on  $k + l$  or on  $|(k)|$ , the constant  $A$  doesn't depend.

In fact, we don't need in the exact exponent  $-|(k)|^4$  in estimate (1.2). It is enough only to know that this exponent is negative and depends only on  $|(k)|$ . From the other hand, exact estimate (1.1) is essential.

Applying Theorem ACS, we can reformulate Theorem 1.

**Theorem 1** (second formulation). *Suppose  $X$  is a compact complex manifold of dimension  $n+1$  and  $\mathcal{F}$  is a holomorphic foliation of  $X$  with negative  $T_{\mathcal{F}}$ . Suppose that the singular set  $\Sigma$  is finite and in some neighborhood of each singular point the vector field locally defining the foliation is analytically linearizable and the linear part is diagonalizable. Let  $M$  be a manifold of universal coverings with a simply connected base  $B$ . Then for any  $p \geq 1$  the manifold  $M$  is diffeomorphic by a  $p$ -smooth fiberwise diffeomorphism to a domain  $\tilde{M} \subset B \times \mathbb{C}$  having continuous boundary and fibered by topological disks  $K_z$ . The domain  $\tilde{M}$  is an image of  $B \times D$  under the diffeomorphism  $f$  satisfying estimates (0.3) - (0.5). As a complex manifold  $M$  is biholomorphic to the manifold  $\tilde{M}$  supplied with an almost complex structure defined by the forms*

$$\begin{aligned} dz_i, \quad i = 1, \dots, n, \\ dw + \langle c, d\bar{z} \rangle, \end{aligned}$$

where  $z_i, w, c$  have the same sense as in (0.1), (0.2) and the vector-function  $c = \{c_1, \dots, c_n\}$  satisfies the estimates

$$|\partial^{(k)} c_i(w, z)| \leq C[1 - \text{dist}(w, \partial K_z)]^{-N}$$

for  $|(k)| \leq p$  with the constants  $C$  and  $N \geq 0$  depending only on  $p$ .

Theorem 1 in either formulation reduces to the next theorem about the Beltrami equation with parameters:

**Theorem 2** *Suppose  $\mu$  is a  $p$ -smooth function of a variable  $z \in D$  and a vector variable  $t = \{t_1, \dots, t_n\}$  belonging to some domain  $B \subset \mathbb{C}^n$ . Let  $\mu$  satisfies the estimates*

$$|\mu| \leq d < 1, \tag{1.3}$$

$$|\mu_{z^k, \bar{z}^l}(z, t)| \leq C(1 - |z|)^{-(k+l)}, \tag{1.4}$$

$$|\partial^{(k)} \mu(z, t)| \leq C(1 - |z|)^{-N}, \tag{1.5}$$

for  $k+l \leq p, |(k)| \leq p$  with constants  $C$  and  $N \geq 0$  depending only on  $p$ . Then there exists a solution  $f$  to the Beltrami equation

$$f_{\bar{z}} = \mu f_z$$

that is continuous in  $C_0(\mathbb{C})$  as a function of  $t$ , is  $p$ -smooth with respect to all variables, at every  $t$  maps  $D$  homeomorphically onto some bounded subdomain of  $\mathbb{C}$ , and satisfies the estimates

$$|f_z(z, t)| \leq C(1 - |z|)^{-\alpha}, |f_{\bar{z}}(z, t)| \leq C(1 - |z|)^{-\alpha}, 0 \leq \alpha < 1, \quad (1.6)$$

$$|f_{z^k \bar{z}^l}(z, t)| / |f_z(z, t)| \leq C(1 - |z|)^{-(k+l-1)}, k + l \leq p, \quad (1.7)$$

$$|\partial^{(k)} f(z, t)| \leq C(1 - |z|)^{-N}, k \leq p. \quad (1.8)$$

The constants  $C$  and  $N \geq 0$  depend only on  $p$ .

The proof of this theorem starts in Section 2. Now we present some motivations for the below considerations and outline main steps of the proof.

Remind at first the classical construction of homeomorphic solutions to the Beltrami equation  $f_{\bar{z}} = \mu f_z$  for a compactly supported  $\mu$  (see, for example [Al] or [As]). Recall the definition of the classical integral operators acting on functions  $f \in C_0^\infty(\mathbb{C})$ : the Cauchy transform

$$\mathcal{C}f(z) = \frac{1}{\pi} \int \frac{f(\tau)}{z - \tau} dS_\tau$$

and the Beorling transform

$$\mathcal{S}f(z) = -\frac{1}{\pi} \int \frac{f(\tau)}{(z - \tau)^2} dS_\tau.$$

Here  $dS_\tau$  is the usual measure on the  $\tau$ -plane and the second integral we understand in terms of its principal value. The Cauchy transform is right inverse to the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} \mathcal{C}f = f$$

and

$$\mathcal{S}f = \frac{\partial}{\partial z} \mathcal{C}f.$$

If  $1 < q < 2 < p < \infty$  is a Holder conjugate pair, then the Cauchy transform extends to a bounded linear mapping from  $L^p(\mathbb{C}) \cap L^q(\mathbb{C})$  into  $C_0(\mathbb{C})$ . The Beorling transform extends to a continuous operator from  $L^p(\mathbb{C})$  to  $L^p(\mathbb{C})$  for all  $1 < p < \infty$ . The norm of this operator tends to 1 as  $p \rightarrow 2$  (the Calderon-Zygmund inequality).

Suppose  $\mu$  has a compact support and  $|\mu(z)| \leq k < 1$ . For every  $\varphi \in L^p(\mathbb{C})$  with a compact support there exists a unique solution  $\sigma$  to the inhomogeneous Beltrami equation

$$\sigma_{\bar{z}} = \mu\sigma_z + \varphi \quad (1.9)$$

with derivatives in  $L^p(\mathbb{C})$  and decay  $f(z) = O(1/z)$  at infinity. We obtain this solution in the following way. The operator  $(\text{Id} - \mu\mathcal{S})^{-1}$  defined by the Neumann series

$$(\text{Id} - \mu\mathcal{S})^{-1} = \text{Id} + \mu\mathcal{S} + \mu\mathcal{S}\mu\mathcal{S} + \dots \quad (1.10)$$

is bounded in  $L^p(\mathbb{C})$  for  $p$  close enough to 2. It is easy to see that

$$\sigma = \mathcal{C}(\text{Id} - \mu\mathcal{S})^{-1}\varphi \quad (1.11)$$

is a solution to (1.9) and this solution has the required properties. We obtain a solution to the Beltrami equation if we put  $\varphi = \mu$  in (1.11) and set  $f(z) = z + \sigma(z)$ . It is an unique solution to the Beltrami equation with  $f_z$  belonging to  $L^p_{loc}(\mathbb{C})$  for  $2 \geq p$  close enough to 2 and normalized by the condition  $f(z) = z + O(1/z)$  as  $z \rightarrow \infty$ . Such solution is called *principal solution*. We have

$$f(z) = z + \mathcal{C}(\mu + \mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu + \dots)(z). \quad (1.12)$$

In fact, the principal solution is a homeomorphism of the complex plane. At first suppose that  $\mu \in C_0^\infty(\mathbb{C})$ . There is the solution  $\sigma \in L^p(\mathbb{C})$  to equation (1.9) with  $\varphi = \mu_z$ . We put

$$F(z) = z + \mathcal{C}(\mu e^\sigma)(z). \quad (1.13)$$

Since  $\sigma(z) = O(1/z)$  near  $\infty$ , it follows that  $e^\sigma - 1 \in L^p(\mathbb{C})$  and  $\mu e^\sigma \in L^p(\mathbb{C})$ . Hence,  $F_z = e^\sigma$  belongs to  $L^p_{loc}(\mathbb{C})$  and  $F(z) - z = O(1/z)$  near  $\infty$ .  $F$  satisfies the Beltrami equation  $F_{\bar{z}} = \mu F_z$  and, by uniqueness of the principal solution, we have  $F = f$ . Further,  $F_z = e^\sigma$  and  $F$  is a local homeomorphism. Since we can extend  $f$  to  $\hat{\mathbb{C}}$  by setting  $f(\infty) = \infty$ , we find that  $f$  is a local homeomorphism  $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  and, hence,  $f$  is a global homeomorphism by the monodromy theorem. For compactly supported measurable  $\mu$  we approximate by convolutions  $\mu_\varepsilon \rightarrow \mu$  in  $L^p(\mathbb{C})$  for proper  $p$  and obtain the principal solution as a limit of smooth conformal mappings.

We can remove the restriction that  $\mu$  is compactly supported and find a homeomorphism satisfying the Beltrami equation as a composition of solutions to the equations with the coefficients having supports in  $D$  and in the closer of  $\hat{\mathbb{C}} \setminus D$ .

Now suppose  $\text{supp} \mu \subset D$ . Applying the extension of  $\mu$  by symmetry we obtain a unique  $\mu$ -quasiconformal homeomorphism  $f : D \rightarrow D$  normalized by the conditions

$$f(0) = 0, \quad f(1) = 1.$$

We call this map a *normal solution* or a *normal mapping*.

If  $\mu$  smoothly depends on a parameter  $t$ , then the principal solution and the normal solution aren't necessarily  $t$ -differentiable. The normal solution has  $t$ -derivative only when  $\mu$  has uniformly bounded  $t$ -derivative ([AIB] or [Al]). Indeed, in general case we can't differentiate, for example, series (1.10). The integrals  $\mathcal{C}\mu_t$  and  $\mathcal{S}\mu_t$  aren't defined if  $\mu_t$  grows sufficiently rapidly near the boundary of  $D$ .

However, when derivatives of  $\mu$  satisfy estimates (1.4), (1.5), we can attempt to find  $t$ -differentiable solutions to the Beltrami equation if we replace the transforms  $\mathcal{C}$  and  $\mathcal{S}$  by integral operators with counter-items. Suppose  $\text{supp} f \subset D$ . We define

$$\begin{aligned} \mathcal{C}_m f(z) &= \frac{1}{\pi} \int_D f(\zeta) \left[ \frac{1}{z - \zeta} - \frac{1}{z - \bar{\zeta}^{-1}} - \dots - \frac{(\zeta - \bar{\zeta}^{-1})^{m-1}}{(z - \bar{\zeta}^{-1})^m} \right] dS_\zeta = \\ &= \frac{1}{\pi} \int_D \frac{f(\zeta)}{z - \zeta} \left( \frac{\zeta - \bar{\zeta}^{-1}}{z - \bar{\zeta}^{-1}} \right)^m dS_\zeta = \frac{1}{\pi} \int_D \frac{f(\zeta)}{z - \zeta} \left( \frac{1 - |\zeta|^2}{1 - \bar{\zeta}z} \right)^m dS_\zeta. \end{aligned}$$

Here we used the identity

$$\left( \frac{\zeta - \bar{\zeta}^{-1}}{z - \bar{\zeta}^{-1}} \right)^{k-1} \left( \frac{1}{z - \zeta} - \frac{1}{z - \bar{\zeta}^{-1}} \right) = \frac{1}{z - \zeta} \left( \frac{\zeta - \bar{\zeta}^{-1}}{z - \bar{\zeta}^{-1}} \right)^k.$$

Define also

$$\begin{aligned} \mathcal{S}_m f(z) &= -\frac{1}{\pi} \int_D f(\zeta) \left[ \frac{1}{(z - \zeta)^2} - \frac{1}{(z - \bar{\zeta}^{-1})^2} - \dots - \frac{(\zeta - \bar{\zeta}^{-1})^{m-1}}{(z - \bar{\zeta}^{-1})^{m+1}} \right] dS_\zeta = \\ &= -\frac{1}{\pi} \int_D \frac{f(\zeta)}{z - \zeta} \left( \frac{1 - |\zeta|^2}{1 - \bar{\zeta}z} \right)^m \left[ \frac{1}{z - \zeta} - \frac{m\bar{\zeta}}{1 - \bar{\zeta}z} \right] dS_\zeta. \end{aligned}$$

Again  $\mathcal{C}_m$  is right-inverse to the Cauchy-Riemann operator on  $D$  and  $\mathcal{S}_m f(z) = (\mathcal{C}_m f)_z(z)$ .



**Definition 1** We say that a function  $f$  on  $D$  belongs to  $L_s^p(D)$ ,  $0 \leq s < \infty$  if the function  $f(z)(1-|z|)^s$  belongs to  $L^p(D)$ . We denote by  $\|f\|_{p,s}$  the  $L^p$ -norm of the function  $f(z)(1-|z|)^s$ . A function  $f$  belongs to  $C_s^0$  if  $f(z)(1-|z|)^s$  is uniformly bounded. We denote by  $\|f\|_{0,s}$  the  $C^0$ -norm of the function  $f(z)(1-|z|)^s$ .

If  $2 < p < \infty$ ,  $m \geq s$ , then  $\mathcal{C}_m$  is a bounded mapping from  $L_s^p(D)$  into  $C_s^0(D)$ . The transform  $\mathcal{S}_m$  acts as continuous operator from  $L_s^p(D)$  to  $L_s^p(D)$  for all  $1 < p < \infty$ ,  $m \geq s$ . In what follows we shall prove these estimates in more general setting.

If  $\mu$  satisfies estimates (1.4), (1.5) we can seek solutions to the Beltrami equation analogous to (1.12) or (1.13) replacing the operators  $\mathcal{C}$  and  $\mathcal{S}$  by  $\mathcal{C}_m$  and  $\mathcal{S}_m$  correspondingly. That is, we can write

$$f(z) = z + \mathcal{C}_m(\mu + \mu\mathcal{S}_m\mu + \mu\mathcal{S}_m\mu\mathcal{S}_m\mu + \dots)(z). \quad (1.14)$$

For any  $l > 0$   $t$ -derivatives or mixed derivatives up to the order  $l$  of the items of series (1.14) will be defined if  $m$  is large enough. But there appear two difficulties.

First, though the operator  $\mathcal{S}_m$  is bounded in  $L_s^p(D)$ , its norm isn't close to 1 if  $m > 0$ . It implies that estimate (1.3) isn't enough for convergence of series (1.14). The constant  $d$  in (1.3) must be small enough.

Second, even if we shall find a locally homeomorphic solution analogously to (1.13), we can't extend this solution to  $\hat{\mathbb{C}}$  and apply the topological argument to prove its global univalence.

We apply some results of theory of univalent functions to overcome this obstacle. A function  $h$  holomorphic on  $D$  and mapping 0 to 0 is univalent if

$$|h''(z)/h'(z)| \leq (1-|z|)^{-1}. \quad (1.15)$$

(See [Pom]). If  $f$  is a solution to the equation  $f_{\bar{z}} = \mu f_z$ , then  $g = f_{z\bar{z}}/f_z$  satisfies the equation

$$g_{\bar{z}} = \mu g_z + \mu_z g_z + \mu_{z\bar{z}}. \quad (1.16)$$

Conversely, if we find a solution to this equation, then we can find a solution to the Beltrami equation by integration. It appears, we can find a solution to equation (1.16) with the estimate  $|g(z)| \leq b$  for any  $b > 0$  if the constant  $C$  in the right side of (1.4) is small enough for several first  $k+l$ . Further, for any  $\mu$ -quasiholomorphic function  $f$  on  $D$  we have the decomposition into the

product  $f = h \circ f_\mu$ , where  $f_\mu$  is the normal mapping and  $h$  is holomorphic. It implies that if we have sufficiently good estimates for the derivatives of  $f_\mu$  and for  $|g| = |f_{zz}/f_z|$ , then  $h$  satisfies estimate (1.15). Thus  $h$  will be univalent and  $f$  homeomorphic.

Now we outline the main steps of the proof of Theorem 2. First, in Section 2 we obtain estimates for derivatives of the normal solutions when  $\mu$  satisfies estimates (1.3), (1.4). In Section 3 we obtain estimates for the differences of these derivatives when we have the normal solutions with the complex dilatations  $\mu_1$  and  $\mu_2$ . For family  $\mu(z, t)$  satisfying estimates (1.5) we obtain Heolder estimates for differences of  $z$ -derivatives. In application to foliations we can consider this result as some generalization of the result of [DNS] on existence of Heolder estimates for the Poincare metrics on the leaves.

In Section 4, applying the obtained estimates, we approximate the family of normal mappings  $f_\mu$  with  $\mu(z, t)$  satisfying estimates (1.4), (1.5) by a finitely smooth family  $f_t$  of  $\mu_t$ -quasiconformal homeomorphisms with  $\mu_t$  approximating  $\mu(\cdot, t)$  in terms of  $\|\cdot\|_{0,k+l}$ -norms. The family  $f_t$  maps  $B \times D$  onto some domain  $\Omega \subset B \times \mathbb{C}$  fibered by topological disks  $\Omega_t$ . For functions on  $\Omega_t$  we define the spaces  $L_s^p$  and  $C_s^0$  as in Definition 1 replacing the difference  $1 - |z|$  by  $\text{dist}(z, \partial\Omega_t)$ . The mappings  $f_\mu(\cdot, t)$  decompose into the products  $g_t \circ f_t$ , where  $g_t$  are  $\tilde{\mu}_t$ -quasiconformal mappings with  $\tilde{\mu}_t$  small and having derivatives up to some finite order small in terms of  $\|\cdot\|_{0,k+l}$ -norms. As a result, we reduce Theorem 2 to the analogous theorem with  $\mu$  defined on  $\Omega_t$  and small with derivatives in terms of  $\|\cdot\|_{0,k+l}$ -norms.

In Section 5 we define integral operators analogous to  $\mathcal{C}_m$  and  $\mathcal{S}_m$  on the domains  $\Omega_t$ . In Sections 6 and 7 we obtain estimates for these operators in appropriate norms.

To obtain univalence we must find a solution  $f$  to equation (1.16) on  $\Omega_t$  satisfying the uniform estimate  $|f(z, t)| \leq c \text{dist}(z, \partial\Omega_t)^{-1}$  with sufficiently small  $c$ . It appears possible if  $\mu_t$  is small in terms of  $\|\cdot\|_{0,k+l}$ -norms. It is essential that appearing singular integral operators are bounded in appropriate Holder norms.

## 2 Estimates for derivatives of normal mappings

In the following estimates we shall often use the expression "uniform constant" in a sense that we shall specify in each case. In what follows  $c$  or  $C$  often means an indeterminate uniform constant. For example, in inequalities of the type  $|(\cdot)| \leq C(\cdot) \leq C(\dots)$  in the right side  $C$  in two cases isn't necessary the same.

In this section  $(k)$  is a multi-index of type  $(k) = (k_0, k_{\bar{0}})$  and  $f_{(k)}$  is the derivative  $f_{z^{k_0} \bar{z}^{k_{\bar{0}}}}$ .

**Lemma 1** *Suppose  $f$  is a  $\mu$ -quasiconformal normal mapping and for  $w \in D$  we have the estimates*

$$|\mu(w)| \leq d < 1,$$

$$|\mu_{(k)}(w)| \leq \frac{b_{|(k)|}}{(1 - |w|)^{|(k)|}}.$$

*Then we have the estimates*

$$a \leq |f_z(w)| \leq A \tag{2.1}$$

*with uniform  $a, A$ , and we can put these constants tending to 1 as  $d$  and  $b_1$  tend to 0.*

$$|f_{(k)}(w)| \leq \frac{B}{(1 - |w|)^{|(k)|-1}} \tag{2.2}$$

*with  $B$  depending only on  $d$  and  $b_{|(l)|}$ ,  $|l| \leq |(k)|$ . If  $d, b_1, \dots, b_k$  are small enough, then*

$$|f_{(k)}(w)| \leq C \frac{d + b_1 + \dots + b_k}{(1 - |w|)^{k-1}} \tag{2.3}$$

*for  $|k| \geq 2$  with some uniform  $C$  independent of  $d, b_1, \dots, b_k$ .*

In this section we say that an estimate or a constant is uniform if it depends only on the constants  $d, b_1, \dots, b_k$  of Lemma 1.

**Proof of Lemma 1.** 1) *Reduction to the case  $w = 0$ ,  $\mu(0) = 0$ .*

The map

$$\varphi_w(z) = \frac{w - z}{1 - z\bar{w}}$$

maps  $D \rightarrow D$  and the point  $w$  to zero. Note that

$$\varphi_w^{-1}(z) = \frac{w - z}{1 - z\bar{w}}$$

also.

Define  $w' = f(w)$ ,  $f_w = \varphi_{w'} \circ f \circ \varphi_w^{-1}$ . Then  $f_w$  maps zero to zero. There is the useful inequality

$$c_1(1 - |w|) \leq 1 - |w'| \leq c_2(1 - |w|) \quad (2.4)$$

for some uniform  $c_1, c_2$  depending only on  $d$ . Indeed, if  $(1 + |\mu|)/(1 - |\mu|) \leq K$ , then, by distortion theorems for quasiconformal mappings, (see, for example, [L])

$$|w|^K \leq |w'| \leq |w|^{1/K}$$

and we can put in (2.4)  $c_1 = 1/2K$ ,  $c_2 = 2K$ . Notice that we can put the bounds  $c_1, c_2$  independent of  $d$  for  $d \leq B$  for any  $B < 1$ .

We need in estimates for derivatives of  $\mu_{f_w}$ .

**Proposition 1** *We have*

$$|(\mu_{f_w})_{(k)}(z)| \leq \frac{C(d + \dots + b_{|k|})}{(1 - |z|)^{|k|}(1 - |\bar{w}z|)^{|k|}} \quad (2.5)$$

with  $C$  independent of  $d, b_1, \dots, b_{|k|}$ . In particular,

$$|(\mu_{f_w})_{|k|}| \leq C(b + \dots + b_{|k|}) \quad (2.6)$$

on every disk  $|z| \leq a < 1$ , where  $C$  depends on  $a$  but not depends on  $w$ .

**Proof.** We have

$$\mu_{f_w}(z) = \mu \circ \varphi_w^{-1}(z) \frac{(\varphi_w)_z}{(\varphi_w)_z} = \mu \circ \varphi_w^{-1}(z) \left( \frac{1 - \bar{z}w}{1 - z\bar{w}} \right)^2. \quad (2.7)$$

Denote  $\mu_0 = \mu_{f_w}(0) = \mu(w)$ . We have

$$(\mu_{f_w})_z = \mu_z \circ \varphi_w^{-1}(z) \left( \frac{1 - w\bar{z}}{1 - \bar{w}z} \right)^2 \frac{1 - |w|^2}{(1 - \bar{w}z)^2} + 2\mu \circ \varphi_w^{-1}(z) \frac{\bar{w}(1 - w\bar{z})^2}{(1 - \bar{w}z)^3}.$$

Since

$$|\mu_z \circ \varphi_w^{-1}(z)| \leq b_1(1 - |\varphi_w^{-1}(z)|)^{-1} = b_1 \frac{|1 - \bar{w}z|}{\left| |1 - \bar{w}z| - |w - z| \right|}, \quad (2.8)$$

we obtain

$$|(\mu_{f_w})_z| \leq (d + b_1) \left( \frac{1 - |w|^2}{\left| |1 - \bar{w}z| - |w - z| \right| |1 - \bar{w}z|} + \frac{|w|}{|1 - \bar{w}z|} \right)$$

We need in an estimate of the difference  $\left| |1 - \bar{w}z| - |w - z| \right|$  from above. Note that

$$\left| |1 - \bar{w}z| - |w - z| \right| \geq \left| |1 - \bar{w}z|^2 - |w - z|^2 \right| / 2.$$

Let  $\theta$  be the angle between  $z$  and  $w$ ,  $\rho = |w|$ ,  $r = |z|$ . We have

$$\begin{aligned} |1 - \bar{w}z|^2 &= (1 - \rho r \cos \theta)^2 + \rho^2 r^2 \sin^2 \theta = 1 - 2\rho r \cos \theta + \rho^2 r^2, \\ |w - z|^2 &= (\rho - r \cos \theta)^2 + r^2 \sin^2 \theta = \rho^2 - 2\rho r \cos \theta + r^2. \end{aligned}$$

Hence,

$$\left| |1 - \bar{w}z|^2 - |w - z|^2 \right| = (1 - \rho^2)(1 - r^2)$$

and

$$\left| |1 - \bar{w}z| - |w - z| \right| \geq (1 - \rho^2)(1 - r^2) / 2 \quad (2.9)$$

From this estimate and (2.8) we obtain

$$|(\mu_{f_w})_z| \leq (d + b_1) \left( \frac{1 - |w|^2}{(1 - |w|)(1 - |z|)|1 - \bar{w}z|} + \frac{|w|}{|1 - \bar{w}z|} \right) \leq \frac{2(d + b_1)}{(1 - |z|)(1 - |w|)}.$$

We get an analogous estimate for  $(\mu_{f_w})_{\bar{z}}$ . We obtained estimate (2.5) for the derivatives of first order.

Now we shall get estimates for derivatives of higher orders. We have

$$\begin{aligned} (\mu_{f_w})^{(k)}(z) &= \mu^{(k)} \circ \varphi_w^{-1}(z) \left( \frac{1 - w\bar{z}}{1 - \bar{w}z} \right)^2 \left( \frac{1 - |w|^2}{(1 - \bar{w}z)^2} \right)^{|(k)|} + \\ &+ \sum \mu^{(l)} \circ \varphi_w^{-1}(z) (\varphi_w^{-1})_{z^{l_1}}(z) \dots (\varphi_w^{-1})_{z^{l_p}}(z) \overline{(\varphi_w^{-1})_{z^{\bar{l}_1}}(z)} \dots \overline{(\varphi_w^{-1})_{z^{\bar{l}_q}}(z)} \left[ \left( \frac{1 - w\bar{z}}{1 - \bar{w}z} \right)^2 \right]_{(m)}. \end{aligned}$$

Here in the second line we have the sum of items with  $(|l| < |k|)$ ,  $p = l_0$ ,  $q = \bar{l}_0$ ,  $l_1 + \dots + l_p + \bar{l}_1 + \dots + \bar{l}_q + |m| = |k|$ . We estimate these items by induction. Denote by  $O(1)$  terms having uniform estimates.

The multiple  $\mu_{(l)} \circ \varphi_w^{-1}(z)$  has the estimate (we apply (2.8) and (2.9))

$$b_{|l|}(1 - |\varphi_w^{-1}(z)|)^{-|l|} \leq cb_{|l|} \frac{|1 - \bar{w}z|^{|l|}}{(1 - |z|^2)^{|l|}(1 - |w|^2)^{|l|}}.$$

with  $c$  depending only on  $|l|$ . After differentiation we obtain the multiple with the estimate

$$\begin{aligned} & cb_{|l|+1} \frac{|1 - \bar{w}z|^{|l|+1}}{(1 - |z|^2)^{|l|+1}(1 - |w|^2)^{|l|+1}} \frac{1 - |w|^2}{|1 - \bar{w}z|^2} = \\ & = cb_{|l|+1} \frac{|1 - \bar{w}z|^{|l|}}{(1 - |z|^2)^{|l|}(1 - |w|^2)^{|l|}} \frac{1}{(1 - |z|^2)|1 - \bar{w}z|} \end{aligned}$$

Any multiple of type  $(\varphi_w^{-1})_{z^l}(z)$  has the estimate

$$\frac{1 - |w|^2}{|1 - \bar{w}z|^{l+1}} O(1)$$

and the multiple  $[(1 - w\bar{z})^2/(1 - \bar{w}z)^2]_{(m)}$  is of the type

$$|1 - \bar{w}z|^{-|m|} O(1).$$

Thus at every differentiation there appears either the multiple  $|1 - \bar{w}z|^{-1}$  either the multiple  $|1 - \bar{w}z|^{-1}(1 - |z|^2)^{-1}$ , and in the estimates we must replace  $b_{|l|} \rightsquigarrow b_{|l|+1}$ . It finishes the proof of estimate (2.5).  $\square$

Let  $g_w$  be the map  $z \mapsto z + \mu_0 \bar{z}$ . Then

$$g_w^{-1}(z) = \frac{z - \mu_0 \bar{z}}{1 - |\mu_0|^2}$$

The map  $g_w$  maps  $D$  onto some ellipsis. Let  $H_w$  be the conformal map mapping this ellipsis onto  $D$  and having real derivative at zero. Let  $Z_w$  be the composition  $Z_w = H_w \circ g_w$ . We have

$$\mu_{Z_w}(z) = \mu_{g_w}(z) = \mu_0.$$

Define the map

$$h_w = f_w \circ Z_w^{-1}.$$

The complex dilatation  $\mu_{h_w}$  is

$$\mu_{h_w} = \frac{\mu_{f_w} - \mu_0}{1 - \bar{\mu}_0 \mu_{f_w}} \frac{Z_{w,z}}{Z_{w,z}} \circ Z_w^{-1}. \quad (2.10)$$

In particular,  $\mu_{h_w}(0) = 0$ . The map  $Z_w$  and its inverse  $Z_w^{-1}$  have all derivatives bounded uniformly with respect to  $w$ . It follows that for derivatives of  $\mu_{h_w}$  we have estimates analogous to the estimates of Proposition 1.

In what follows we adopt the notation  $Z = Z_w^{-1}(z)$ .

**Proposition 2** . *We have the estimates*

$$|(\mu_{h_w})^{(k)}(z)| \leq \frac{C(d + \dots + b_{|(k)|})}{(1 - |Z|)^{|(k)|}(1 - |\bar{w}Z|)^{|(k)|}}. \quad (2.11)$$

Here  $C$  doesn't depend on  $d, \dots, b_{|(k)|}$  if  $d \leq B$  for any  $B < 1$ , for example, if  $d \leq 1/2$ . In particular,

$$|(\mu_{h_w})^{(k)}| \leq C(d + \dots + b_{|(k)|}). \quad (2.12)$$

on any disk  $|z| \leq a < 1$ .

**Proof** Almost all assertions were already proved. The assertion about constant  $C$  in (2.11) holds because  $Z_w$  is quasiconformal with the complex dilatation  $\mu_0$ ,  $|(Z_w)_z|$  is bounded from below and from above by constants depending only on  $d$ , and  $(Z_w)_z$  tends to 1 uniformly as  $d$  tends to zero, all other derivatives of  $Z_w$  are also bounded by constants depending only on  $d$ , and we can put these constants arbitrary small as  $d$  tends to zero.  $\square$

Now we return to the original map  $f$ .

**Proposition 3** a) *We have the estimates*

$$a|(h_w)_z(0)| \leq |f_z(w)| \leq A|(h_w)_z(0)| \quad (2.13)$$

with some uniform  $a, A$ . These constants depend on  $d$  but we can put them independent of  $d$  for  $d \leq B$  for any  $B < 1$ .

$$|f^{(k)}(w)| \leq C \frac{1 - |w'|}{(1 - |w|)^{|(k)|}} \sum_{s_1|(l_1)| + \dots + s_j|(l_j)| \leq |(k)|} |((h_w)_{(l_1)}(0))^{s_1} \dots ((h_w)_{(l_j)}(0))^{s_j}| \quad (2.14)$$

with some uniform  $C$ .

**Proof.** We have

$$f = \varphi_w^{-1} \circ h_w \circ Z_w \circ \varphi_w. \quad (2.15)$$

We adopt the notations:  $\partial$  is the derivative of a function with respect to its analytic argument and  $\bar{\partial}$  is the derivative with respect to the conjugate variable. We have

$$\begin{aligned} f_z &= \partial(\varphi_{w'}^{-1}) \circ h_w \circ Z_w \circ \varphi_w [\partial(h_w) \circ Z_w \circ \varphi_w \cdot (\partial Z_w) \circ \varphi_w \cdot \partial\varphi_w + \\ &\quad + \bar{\partial}(h_w) \circ Z_w \circ \varphi_w \cdot \bar{\partial}Z_w \circ \varphi_w \cdot \partial\varphi_w]. \end{aligned} \quad (2.16)$$

In particular, recalling that  $\bar{\partial}h_w(0) = 0$ , we obtain

$$f_z(w) = \frac{1 - |w'|^2}{1 - |w|^2} \partial h_w(0) \partial Z_w(0).$$

The fraction  $\frac{1 - |w'|^2}{1 - |w|^2}$  is uniformly bounded from below and from above by (2.4) and we can set the bounds independent of  $d$  for  $d \leq B$  for any  $B < 1$ . The same holds for the value  $\partial Z_w(0)$ . We obtain (2.13).

Now differentiating (2.16) we see that the derivative  $\frac{\partial^2 f}{\partial z^2}(w)$  is the sum of terms of the types

$$\begin{aligned} \partial\partial\varphi_{w'}^{-1}(0)[\partial h_w(0)\partial Z_w(0)\partial\varphi_w(w)]^2 &= [\partial h_w(0)]^2 O((1 - |w|^2)^{-1}), \\ \partial\varphi_{w'}^{-1}(0)\partial\partial h_w(0)[\partial Z_w(0)\partial\varphi_w(w)]^2 &= \partial\partial h_w(0) O((1 - |w|^2)^{-1}), \\ \partial\varphi_{w'}^{-1}(0)\partial h_w(0)\partial\partial Z_w(0)[\partial\varphi_w(w)]^2 &= \partial h_w(0) O((1 - |w|^2)^{-1}) \\ \partial\varphi_{w'}^{-1}(0)\partial h_w(0)\partial Z_w(0)\partial\partial\varphi_w(w) &= \partial h_w(0) O((1 - |w|^2)^{-1}) \end{aligned}$$

and of analogous terms containing  $\bar{\partial}$  derivatives of  $h_w$  and  $Z_w^{-1}$ . All these terms have estimates  $O((1 - |w|^2)^{-1})$ .

At further differentiations we obtain each time either the multiple  $\partial\varphi_w(w) = 1/(1 - |w|^2)$  either the term, where the multiple  $\partial^k\varphi_w(w)$  is replaced by the multiple  $\partial^{k+1}\varphi_w(w)$ , which also results in multiplication by  $(1 - |w|^2)^{-1}$ . Also, if we had some item containing a product  $((h_w)_{(l_1)}(0))^{s_1} \dots ((h_w)_{(l_j)}(0))^{s_j}$ , then, after differentiation, we obtain the term of the same type with the sum  $s_1|l_1| + \dots + s_j|l_j|$  higher no more, than by 1. By induction we see that the derivative  $f_{(k)}(w)$  at  $|k| \geq 1$  can be represented as a sum of items of the types

$$\frac{1 - |w'|^2}{(1 - |w|^2)^m} ((h_w)_{(k_1)}(0))^{s_1} \dots ((h_w)_{(k_j)}(0))^{s_j} ((Z_w)_{(q_1)}(0))^{r_1} \dots ((Z_w)_{(q_i)}(0))^{r_i} O(1),$$



where  $m \leq |(k)|, s_1|(k_1)| + \dots s_j|(k_j)| + r_1|(q_1)| + \dots r_i|(q_i)| \leq |(k)|$  and  $O(1)$  is a multiple uniformly bounded and independent of  $t$  in conditions of point b). Estimate (2.14) follows immediately.  $\square$

We see that to prove Lemma 1 it is enough to show that  $(h_w)_z(0)$  is uniformly bounded from below and from above and that derivatives  $(h_w)_{(k)}(0)$ ,  $|k| \geq 2$  are uniformly bounded.

2) *Transition to the logarithmic chart*

Define the logarithmic coordinates

$$\zeta = \log z = \xi + i\varphi, \quad \omega = \log h_w(z) = \eta + i\theta.$$

We shall use the notation  $F_w$  for  $h_w$  represented in the logarithmic coordinates  $\omega = F_w(\zeta)$ . The complex dilatation  $\mu_{F_w}$  is

$$\mu_{F_w}(\zeta) = \frac{e^{\bar{\zeta}}}{e^{\zeta}} \mu_{h_w}(e^{\zeta}). \quad (2.17)$$

Consider the first terms of the power series decomposition for  $h_w$

$$h_w(z) = a_0 z + b_{20} z^2 + b_{1\bar{1}} z \bar{z} + b_{0\bar{2}} \bar{z}^2 + \dots \quad (2.18)$$

In the logarithmic coordinates we have the decomposition at  $\xi = -\infty$  (we assume  $z = r e^{i\varphi}$ ,  $\xi = \log r$ )

$$\begin{aligned} \omega = F_w(\zeta) &= \eta + i\theta = \log[a_0 r e^{i\varphi} + r^2(b_{20} e^{2i\varphi} + b_{1\bar{1}} + b_{0\bar{2}} e^{-2i\varphi}) + \dots] = \\ &= \xi + i\varphi + \log a_0 + \frac{e^{\xi}}{a_0} (b_{20} e^{i\varphi} + b_{1\bar{1}} e^{-i\varphi} + b_{0\bar{2}} e^{-3i\varphi}) + O(e^{2\xi}). \end{aligned} \quad (2.19)$$

Consider the higher terms. If the term of order  $k$  in the decomposition of  $h_w$  is

$$b_{k0} z^k + b_{(k-1)\bar{1}} z^{k-1} \bar{z} + \dots + b_{0\bar{k}} \bar{z}^k,$$

then the term of order  $e^{(k-1)\xi}$  in the decomposition of  $F_w$  is

$$a_0^{-1} e^{(k-1)\xi} (b_{k0} e^{i(k-1)\varphi} + b_{(k-1)\bar{1}} e^{i(k-3)\varphi} + b_{0\bar{k}} e^{-(k+1)\varphi}) + S, \quad (2.20)$$

where  $S$  is a sum of products of multiples containing only the coefficients of the decomposition  $h_w$  of degrees less than  $k$ . Thus uniform estimates for coefficients of decomposition (2.18) follow from uniform estimates for

coefficients of decomposition (2.20). Note also that we can write expression (2.20) as

$$a_0^{-1} e^{(k-1)\xi} (c_{k0} e^{i(k-1)\varphi} + c_{(k-1)\bar{1}} e^{i(k-3)\varphi} + c_{0\bar{k}} e^{-(k+1)i\varphi}). \quad (2.21)$$

Indeed, each term  $z^{k-1-p} \bar{z}^p$  in the decomposition of  $\log(h_w(z)/z)$  yields the term  $e^{(k-1)\xi} e^{i(k-1-2p)\varphi}$ . We see also that if we obtain uniform estimates for the coefficients of decomposition (2.21), then we also obtain uniform estimates for the coefficients  $b_{k0}, b_{(k-1)\bar{1}}, \dots$

3) *Integral operators in the logarithmic chart.*

Let  $H$  be the stripe  $-\pi \leq \varphi \leq \pi$  and  $L_H^p$  be a space of  $\varphi$ -periodic functions on  $H$  with usual  $L^p$ -norm. We adopt the notations  $L_{H_-}^p$  for the space of functions belonging to  $L_H^p$  with a support in the left half-stripe  $H \cap \{\xi \leq 0\}$  and  $L_{H_a}^p$  for the space of functions supported in the set  $H \cap \{\xi \leq a\}$ .

We define an integral transform  $P_H$  acting on  $\varphi$ -periodic functions on  $H$

$$P_H f(\zeta) = \frac{1}{2\pi i} \int_H f(\tau) \left( \frac{e^\tau}{e^\tau - e^\zeta} - 1 \right) d\tau d\bar{\tau} = \frac{e^\zeta}{2\pi i} \int_H \frac{f(\tau)}{e^\tau - e^\zeta} d\tau d\bar{\tau}.$$

**Proposition 4** *The transform  $P_H$  is right inverse to the operator  $\partial/\partial\bar{\zeta}$  for smooth functions belonging to  $L_{H_a}^p$  at  $a < \infty$ . Suppose  $e^{-\xi} f \in L_{H_a}^p$ ,  $2 < p \leq \infty$ . Then*

$$\|P_H f\|_C \leq C_{ap} \|e^{-\xi} f\|_p \quad (2.22)$$

with  $C_{ap}$  depending on  $a$  and  $p$ .

**Proof.** At first we prove that the operator  $P_H$  is right inverse to the Cauchy-Riemann operator. We have

$$\begin{aligned} P_H f(\zeta) &= \frac{1}{2\pi i} \int_H f(\tau) \left( \frac{e^\tau}{e^\tau - e^\zeta} - 1 \right) d\tau d\bar{\tau} = \\ &= \frac{1}{2\pi i} \int_H \frac{f(\tau) e^{-\bar{\tau}}}{e^\tau - e^\zeta} d(e^\tau) d(e^{\bar{\tau}}) + h(\zeta) = \frac{1}{2\pi i} \int_D \frac{f(\log t)}{t(t - e^\zeta)} dt d\bar{t} + h(\zeta). \end{aligned}$$

where  $h$  is a holomorphic function. Hence, we have

$$\frac{\partial P_H f}{\partial(e^\zeta)}(\zeta) = \frac{f(\zeta)}{e^\zeta},$$

and

$$\frac{\partial P_H f}{\partial\bar{\zeta}}(\zeta) = f(\zeta).$$

To proof estimate (2.22) it is enough to show that  $e^{\zeta \frac{\theta_a(\tau)e^\tau}{e^\tau - e^\zeta}}$  considered as a function of  $\tau$  has a uniform estimate with respect to  $\zeta$  in  $L_H^q$  for  $1 \leq q < 2$ . Here  $\theta_a(\tau)$  is the "step":  $\theta(\tau) = 1$  if  $\text{Re}\tau \leq a$  and  $\theta(\tau) = 0$  if  $\text{Re}\tau > a$ .

Suppose at first  $\xi = \text{Re}\zeta \leq 0$ . The integral

$$\int_{|\tau - \zeta| \leq 1} \frac{|e^{q\tau}|}{|e^\tau - e^\zeta|^q} d\tau d\bar{\tau} = \int_{|\tau - \zeta| \leq 1} \frac{d\tau d\bar{\tau}}{|1 - e^{\zeta - \tau}|^q}$$

is uniformly bounded and the integral

$$\int_{\{|\tau - \zeta| > 1, \text{Re}\tau \leq a\}} \frac{|e^{q\tau}|}{|e^\tau - e^\zeta|^q} d\tau d\bar{\tau}$$

has the estimate  $c_a(e^{-q\xi})$  with some  $c_a$  depending only on  $a$ . Hence we obtain an uniform estimate for the kernel.

If  $\xi > 0$ , then the integral over the domain  $|\tau - \zeta| \leq 1$  is obviously uniformly bounded (we should consider only the integral over the intersection of this disk with the semi-plane  $\{\xi \leq a\}$ ). From the other hand, the integral

$$\int_{H \cap \{|\tau - \zeta| > 1, \text{Re}\tau \leq a\}} \frac{|e^{q(\zeta + \tau)}|}{|e^\tau - e^\zeta|^q} d\tau d\bar{\tau}$$

at  $\xi \geq 1$  also has an uniform estimate because the function  $\frac{|e^\zeta|}{|e^\tau - e^\zeta|}$  is uniformly bounded.  $\square$

Now we define

$$T_H f = \frac{\partial P_H f}{\partial f} = \frac{1}{2\pi i} e^\zeta \int_H \frac{f(\tau) e^\tau}{(e^\tau - e^\zeta)^2} d\tau d\bar{\tau},$$

where the integral is defined in terms of its principal value.

**Proposition 5** *The operator  $T_H$  is bounded in  $L_H^p$  at  $1 < p < \infty$  and its  $L_p$ -norm tends to 1 when  $p$  tends to 2.*

**Proof.** If  $f$  is a  $\varphi$ -periodic function with support in some vertical stripe, then  $T_H f$  at large positive or negative  $\xi$  decreases as  $e^{-|\xi|}$ . In the following standard calculations all possible integrals over the boundary equal zero because  $P_H$  transforms periodic functions into periodic ones.

$$\int_H |T_H f|^2 dS_{zeta} = -\frac{1}{2i} \int_H (P_H f)_\zeta \overline{(P_H f)_{\bar{\zeta}}} dS_\zeta = \frac{1}{2i} \int_H P_H f \overline{(P_H f)_{\bar{\zeta}}} dS_\zeta =$$

$$= \frac{1}{2i} \int_H (P_H f) \bar{f}_\zeta dS_\zeta = -\frac{1}{2i} \int_H \bar{f} (P_H f)_\zeta dS_\zeta = \int_H |f|^2 dS_\zeta$$

Now we prove the  $L_H^p$ -boundedness. We have

$$\begin{aligned} T_H f(\zeta) &= \frac{1}{2\pi i} \int_H \frac{f(\tau) e^{\tau-\zeta}}{(e^{\tau-\zeta} - 1)^2} d\tau d\bar{\tau} = \frac{1}{2\pi i} \int_H \frac{f(\tau) d\tau d\bar{\tau}}{(\tau - \zeta)^2} + \\ &\quad + \frac{1}{2\pi i} \int_{|\tau-\zeta| \leq 1} f(\tau) \left[ \frac{e^{\tau-\zeta}}{(e^{\tau-\zeta} - 1)^2} - \frac{1}{(\tau - \zeta)^2} \right] d\tau d\bar{\tau} + \\ &\quad + \frac{1}{2\pi i} \int_{H \setminus \{|\tau-\zeta| \leq 1\}} f(\tau) \left[ \frac{e^{\tau-\zeta}}{(e^{\tau-\zeta} - 1)^2} - \frac{1}{(\tau - \zeta)^2} \right] d\tau d\bar{\tau} \end{aligned} \quad (2.23)$$

The first integral is the usual Beurling transform of a function  $f$  with support in  $H$  and has an estimate in  $L^p(\mathbb{C})$  by the Calderon-Zigmund theorem. The second integral is a convolution of the function  $f$  with the function

$$\left( \frac{e^t}{(e^t - 1)^2} - \frac{1}{t^2} \right) \chi_D(t),$$

where  $\chi_D$  is the characteristic function of the unit disk. This function belongs to  $L^1$  and the integral has an  $L^p(\mathbb{C})$ -estimate by the Yung inequality. The  $L_H^p$ -norms of these integrals are no greater than its  $L^p(\mathbb{C})$ -norms. The last integral can be considered as a bounded operator on  $L_H^1$  and on  $L_H^\infty$ . Indeed, if we denote by  $K(\tau, \zeta)$  the kernel (i.e., the function in the square brackets), then we can easily see that

$$\int_{H \setminus \{|\tau-\zeta| \leq 1\}} K(\tau, \zeta) dS_\tau$$

has a uniform estimate as a function of  $\zeta$  and, by symmetry, the integral

$$\int_{H \setminus \{|\tau-\zeta| \leq 1\}} K(\tau, \zeta) dS_\zeta$$

also has a uniform estimate as a function of  $\tau$ . We obtain an  $L^1$ -estimate of the last integral in (2.23)

$$\int_H \left| \int_{H \setminus \{|\tau-\zeta| \leq 1\}} f(\tau) K(\tau, \zeta) dS_\tau \right| dS_\zeta \leq$$

$$\leq \int_H |f(\tau)| \sup_{\tau} \int_{H \setminus \{|\tau - \zeta| \leq 1\}} |K(\tau, \zeta)| dS_{\zeta} dS_{\tau} \leq C \|f\|_{L_H^1}.$$

for some  $C$ . Analogously,

$$\left| \int_{H \setminus \{|\tau - \zeta| \leq 1\}} f(\tau) K(\tau, \zeta) dS_{\tau} \right| \leq \sup_{\tau} |f(\tau)| \sup_{\zeta} \int_{H \setminus \{|\tau - \zeta| \leq 1\}} |K(\tau, \zeta)| dS_{\tau} \leq C \|f\|_{L_H^{\infty}}.$$

We obtain  $L_H^p$ -boundedness of the last integral in (2.23) for  $1 \leq p < \infty$  by the Riesz-Thorin interpolation theorem.

The  $L_H^p$ -norm of the operator  $T_H$  tends to 1 as  $p$  tends to 2 also by the Riesz-Thorin theorem.  $\square$

4) *Solution to the Beltrami equation in the logarithmic chart.*

Suppose  $\nu(\zeta)$  is some  $\varphi$ -periodic function with the properties:

- a)  $|\nu(\zeta)| \leq d < 1$  for some  $d$ ,
- b)  $\nu$  has the support in some domain  $\xi \leq a$ ,
- c)  $|e^{-\xi} \nu(\xi)| \leq c$  for some  $c < \infty$ .

Since the function  $e^{\xi}$  is bounded in  $L_{H_a}^p$ , the series

$$h = (Id - T_H \nu)^{-1}(1) = 1 + T_H \nu + T_H \nu T_H \nu + \dots,$$

converge in  $L_H^p$  for  $p > 2$  sufficiently close to 2, as in the classical case, and we have for the  $L_H^{(p)}$ -norm of the function  $h - 1$  the estimate  $C_a c / (1 - d)$  with some  $C_a$  depending only on  $a$ . By Proposition 4,  $|P_H(\nu h)| \leq C_a c^2 / (1 - d)$  with, possibly, some new  $C_a$ . We see that

$$f_{\nu}(\zeta) = \zeta + P_H[\nu(Id - T_H \nu)^{-1}(1)](\zeta)$$

is a solution to the Beltrami equation with the Beltrami coefficient  $\nu$ . We call this solution *the principal logarithmic solution*.

**Proposition 6** *The principal logarithmic solution is a homeomorphism of the plane satisfying the estimate*

$$|f_{\nu}(\zeta) - \zeta| \leq \frac{C_a c^2}{1 - d}. \quad (2.24)$$

**Proof** Estimate (2.24) follows from Propositions 4 and 5. Prove that  $f_{\nu}$  is a homeomorphism.

The map  $\tilde{f}_\nu = \exp \circ f_\nu \circ \log$  is a quasiholomorphic function on the punctured plane with the compactly supported complex dilatation. From (2.24) follows that the estimate

$$c|z| \leq |\tilde{f}_\nu(z)| \leq C|z|, \quad 0 < c, C < \infty \quad (2.25)$$

holds at zero and at infinity. From the other hand,  $\tilde{f}_\nu$  extends as a quasiholomorphic function to zero. Indeed, after the change of the chart by a quasiconformal homeomorphism we obtain a holomorphic function with a removable singularity. Further, since  $\tilde{f}_\nu$  is holomorphic outside some disk, we obtain from (2.25) that at infinity this map has the asymptotics  $cz + b + O(1/z)$ ,  $c \neq 0$ . Subtracting  $b$  and dividing by  $c$  we obtain a quasiholomorphic map with a compactly supported complex dilatation and with the asymptotics  $z + O(1/z)$  at infinity. It implies that it is the principal solution to the corresponding Beltrami equation, which is unique and homeomorphic. It follows that  $f_\nu$  also is a homeomorphism.  $\square$

**Proposition 7** *The coefficient  $\mu_{F_w}$  satisfy conditions a) - c) for  $a = 0$  with estimates uniform with respect to  $w$ . The operator  $Id - T_H \mu_{F_w}$  is invertible in  $L_H^p$  if  $2 \leq p \leq P_{d+b_1}$  for some  $P_{d+b_1}$ , which can be made arbitrary large if  $d, b_1$  are small enough. Here  $d$  and  $b_1$  are the constants from the formulation of Lemma 1. Moreover, we have the estimate*

$$|f_{\mu_{F_w}}(\zeta) - \zeta| \leq C \frac{(d + b_1)^2}{1 - d}, \quad (2.26)$$

where  $C$  doesn't depend  $d, b_1$ .

**Proof.** If  $\xi \leq -\log 2$  we have, applying (2.17) and estimate (2.12)

$$\begin{aligned} |\mu_{F_w}(\zeta)| &= |\mu_{h_w}(e^\xi) \frac{e^{\bar{\zeta}}}{e^\zeta}| \leq \\ &\leq e^\xi \sup_{D_{1/2}} (|(\mu_{h_w})_z| + |(\mu_{h_w})_{\bar{z}}|) \leq C(d + b_1)e^\xi. \end{aligned}$$

Also,

$$|\mu_{F_w}(\zeta)| \leq Cd$$

with some uniform  $C$  if  $\xi > -\log 2$ . We see that  $\|\mu_{F_w}\|_p \leq C(d + b_1)$  and

$$\|(Id - T_H \mu_{F_w})^{-1}(1) - 1\|_p \leq C \frac{d + b_1}{1 - d}.$$

Analogously, we can see that the function  $e^{\zeta \frac{\theta(\tau)|\mu_{F_w}(\tau)|}{e^\tau - e^\zeta}}$  has  $L_H^q$ -norm no greater, than  $C(d + b_1)$ . By definition of  $f_{\mu_{F_w}}$  and applying estimate (2.24), we obtain (2.26).  $\square$

5) *The proof of estimate (2.1)* We shall show that the map  $F_w$  has a representation in terms of principal logarithmic solutions.

Define the map

$$\tilde{f}_{\mu_{F_w}}(\zeta) = -\overline{f_{\mu_{F_w}}(-\bar{\zeta})}.$$

This map is "symmetrical" to  $f_{\mu_{F_w}}$  with respect to the imaginary axis. Its complex dilatation is  $\tilde{\mu}_{F_w}(\zeta) = \overline{\mu_{F_w}(-\bar{\zeta})}$ . Now we define the function

$$\lambda_{F_w} = \mu_{F_w} \frac{\tilde{f}_{\mu_{F_w}, \zeta}}{f_{\mu_{F_w}, \zeta}} \circ \tilde{f}_{\mu_{F_w}}^{-1} = \mu_{F_w} \circ \tilde{f}_{\mu_{F_w}}^{-1} \frac{\overline{f_{\mu_{F_w}, \zeta}}}{f_{\mu_{F_w}, \zeta}} \circ (-\tilde{f}_{\mu_{F_w}}^{-1}). \quad (2.27)$$

**Proposition 8** *The coefficient  $\lambda_{F_w}$  satisfies conditions a) - c) with uniform constants. More, we have the estimate*

$$|f_{\lambda_{F_w}}(\zeta) - \zeta| \leq C \frac{(d + b_1)^2}{1 - d},$$

where  $d, b_1$  are from the formulation of Lemma 1 and  $C$  doesn't depend on  $d, b_1$  if  $d \leq 1/2$ .

**Proof.** Obviously  $|\lambda_{F_w}| = |\mu_{F_w} \circ \tilde{f}_{\mu_{F_w}}^{-1}|$ . We see that  $\lambda_{F_w}$  has a support in some half-plane  $\xi \leq a$  with an uniform  $a$  and, since for  $\text{id} - \tilde{f}_{\mu_{F_w}}^{-1}$  we have the estimate of Proposition 6, we have the estimate  $|\lambda_{F_w}(\zeta)| \leq C(b + b_1)e^\xi$  at  $\xi \leq -\log 2$  with some uniform  $C$ . We finish the proof analogously to the proof of Proposition 7.  $\square$

Consider the map

$$\tilde{F}_w = f_{\lambda_{F_w}} \circ \tilde{f}_{\mu_{F_w}}$$

In the chart  $z$  it is the map

$$\tilde{h}_w = \exp \circ \tilde{F}_w \circ \log.$$

**Proposition 9**

$$h_w(z) = \frac{\varphi_w'(1)}{\tilde{h}_w \circ Z_w^{-1} \circ \varphi_w(1)} \tilde{h}_w(z), \quad (2.28)$$

for  $z \in D$ .

**Proof.**  $\tilde{F}_w$  is a quasiconformal map with the complex dilatation  $\mu_{F_w} + \tilde{\mu}_{F_w}$ . The map  $\tilde{h}_w$ , obtained after transition to the chart  $z = e^\zeta$ , is a solution to the Beltrami equation on the punctured plane with the Beltrami coefficient symmetrical with respect to the unit circle and equal to  $\mu_{h_w}$  if  $|z| \leq 1$ . Also, this solution satisfies the uniform estimate  $b|z| \leq |\tilde{h}_w(z)| \leq B|z|$ , where for  $|\log |b||$  and  $|\log |B||$  we have the estimates

$$C \frac{(d + b_1)^2}{1 - d} \quad (2.29)$$

according to Propositions 6,7 and 8. As in the proof of Proposition 6 we can extend  $\tilde{h}_w$  to zero and obtain the quasiconformal map fixing zero and infinity. Suppose  $\tilde{h}_w(1) = a_w$ . Then  $|a_w|$  has uniform estimates from below and from above. Dividing by  $a_w$  we obtain a map fixing 0,  $\infty$  and 1, i.e., the normal map. This map is unique and symmetrical with respect to the unit circle. It means that on  $D$  it coincides with  $h_w$  up to, possibly, some rotation ( $h_w$  doesn't map necessarily 1 to 1).

We see that the map  $h_w$  differs from the map  $\tilde{h}_w$  at  $|z| \leq 1$  only by a constant multiple. We can find this multiple from the condition

$$1 = f(1) = \varphi_w^{-1} \circ h_w \circ Z_w^{-1} \circ \varphi_w(1),$$

i.e.,  $h_w \circ Z_w^{-1} \circ \varphi_w(1) = \varphi_w(1)$ . We obtain

$$h_w(z) = \frac{h_w \circ Z_w^{-1} \circ \varphi_w(1) \tilde{h}_w(z)}{\tilde{h}_w \circ Z_w^{-1} \circ \varphi_w(1)} = \frac{\varphi_w(1)}{\tilde{h}_w \circ Z_w^{-1} \circ \varphi_w(1)} \tilde{h}_w(z).$$

□

**Corollary.** *If  $d$  and  $b_1$  are small enough, then*

$$|h_w(z) - (h_w)_z(0)z| \leq C(d + b_1)^2$$

with some uniform  $C$ .

**Proof.** It follows from Propositions 7 and 8 and (2.28). □

**Proof of estimate (2.1).** Since  $\varphi_w(1)$  and  $\tilde{h}_w \circ Z_w^{-1} \circ \varphi_w(1)$  equal to 1 by modulus, and for  $\tilde{h}_w$  we have estimate  $b|z| \leq |\tilde{h}_w(z)| \leq B|z|$ , where  $|\log |b||$  and  $|\log |B||$  satisfy estimate (2.29), we obtain the estimate

$$\exp \frac{c(d + b_1)^2}{1 - d} \leq |(h_w)_z(0)| \leq \exp \frac{C(d + b_1)^2}{1 - d} \quad (2.30)$$



with uniform  $c, C$  don't depending on  $d, b_1$ . Now estimate (2.1) follows from estimate (2.15) of Proposition 3 because  $(1 - |w'|)/(1 - |w|)$  tends to 1 as  $d \rightarrow 0$ .  $\square$

6) *Estimates for derivatives of the mappings  $h_w, f_{\mu_{F_w}}$  and  $f_{\lambda_{F_w}}$ .* For convenience we place here these estimates, which we shall use in the next section.

**Proposition 10** a)

$$c \leq |(h_w)_z(z)| \leq \frac{C}{|1 - \bar{w}Z_w^{-1}(z)|^2}, \quad (2.31)$$

with some uniform  $c, C$ , and we have for  $|(h_w)_{\bar{z}}|$  an estimate analogous to the right inequality.

b)

$$c|\xi| \leq |f_{\mu_{F_w}, \zeta}(\zeta)| \leq C|\xi|^{-3}. \quad (2.32)$$

with some uniform  $c, C$ , and  $f_{\mu_{F_w}, \zeta}$  is uniformly bounded at  $\xi < -1$ . We have the same estimates for  $(\tilde{f}_{\mu_{F_w}})_\zeta$ .

c)

$$|(f_{\lambda_{F_w}})_\zeta(\zeta)| \leq C|\operatorname{Re} \tilde{f}_{\mu_{F_w}}^{-1}(\zeta)|^{-3} \quad (2.33)$$

with some uniform  $c, C$  at  $|\operatorname{Re} \tilde{f}_{\mu_{F_w}}^{-1}(\zeta)| \leq 1$  and  $(f_{\lambda_{F_w}})_\zeta(\zeta)$  is uniformly bounded at  $\operatorname{Re} \tilde{f}_{\mu_{F_w}}^{-1}(\zeta) < -1$ .

**Proof.** a) Remind that  $h_w = f_w \circ Z_w^{-1}$ , where  $f_w = \varphi_{w'} \circ f \circ \varphi_w^{-1}$ . The map  $Z_w$  is quasiconformal with the complex dilatation  $\mu_0$ ,  $|(Z_w)_z|$  is bounded from below and from above by constants depending only on  $d$ . Thus it is enough to prove estimate analogous to (2.31) for  $f_w$  if we replace  $Z_w^{-1}(z)$  by  $z$ . We have

$$\begin{aligned} (f_w)_z(z) &= (\varphi_{w'})_z \circ f \circ \varphi_w^{-1}(z) f_z \circ \varphi_w^{-1}(z) (\varphi_w^{-1})_z(z) = \\ &= \frac{1 - |w'|^2}{(1 - \bar{w}' f \circ \varphi_w^{-1}(z))^2} f_z \circ \varphi_w^{-1}(z) \frac{1 - |w|^2}{(1 - \bar{w}z)^2}. \end{aligned}$$

Applying (2.4) we see that

$$|1 - \bar{w}' f \circ \varphi_w^{-1}(z)| \geq c(1 - |w|).$$

Thus,

$$c|f_z \circ \varphi_w^{-1}(z)| \leq |(f_w)_z(z)| \leq \frac{C}{|1 - \bar{w}z|^2} |f_z \circ \varphi_w^{-1}(z)|.$$

Since  $|f_z|$  is uniformly bounded from below and from above, we obtain (2.31). The estimate for  $\bar{z}$ -derivative is analogous.

b) The map  $f_{\mu_{F_w}}$  written in the chart  $z$  is the map  $\hat{g}_w = \exp \circ f_{\mu_{F_w}} \circ \log$ . The function  $f_{\mu_{F_w}}(\zeta) - \zeta$  is uniformly bounded at  $\xi \rightarrow -\infty$ . It means that  $\hat{g}_w$  is a quasiconformal homeomorphism of  $\mathbb{C}$  with the complex dilatation  $\mu_{h_w}$  and with derivatives at zero uniformly bounded from below and from above. We can represent it on  $D$  as  $\hat{h}_w \circ h_w$ , where  $\hat{h}_w$  is a holomorphic univalent uniformly bounded function with the derivative at zero uniformly bounded from below and from above. For such functions we have estimates ([Pom])

$$c(1 - |z|) \leq |\hat{h}'_w(z)| \leq \frac{C}{1 - |z|}. \quad (2.34)$$

Also, for a variable  $w$ ,  $h_w$  is a family of quasiconformal mappings of  $D$  onto itself with uniformly bounded dilatations and, hence, there are the estimates

$$c(1 - |z|) \leq 1 - |h_w(z)| \leq C(1 - |z|) \quad (2.35)$$

. with uniform  $c, C$ .

Further, at  $\xi \leq 0$

$$(f_{\mu_{F_w}})_\zeta(\zeta) = (\hat{h}'_w \circ h_w \circ \exp \zeta)^{-1} \cdot (h_w)_z \circ \exp \zeta$$

By left inequality (2.31), estimate (2.35), and right inequality (2.34), we obtain

$$|f_{\mu_{F_w}, \zeta}(\zeta)| \geq c|\xi|$$

at  $|\xi| \leq 1$ . Analogously, right inequality (2.31) and left inequality (2.34) together with (2.35) yield right estimate (2.32).

On  $\mathbb{C} \setminus D$  the functions  $\hat{g}_w$  are univalent holomorphic, uniformly bounded away from zero and having uniformly bounded derivatives at infinity. Thus there is also the uniform estimates  $c(|z| - 1)|\hat{g}_w(z)_z| \leq C(|z| - 1)^{-1}$ . We obtain estimates (2.32) also at  $0 < |\xi| \leq 1$ .

The case of  $(f_{\mu_{F_w}})_\zeta$  is analogous.

c) From (2.28) follows that we have for the derivatives of  $\tilde{h}_w$  the estimates analogous to (2.31). Since  $\tilde{h}_w = \exp \circ f_{\lambda_{F_w}} \circ \tilde{f}_{\mu_{F_w}} \circ \log$  we have

$$|(f_{\lambda_{F_w}})_\zeta \circ \tilde{f}_{\mu_{F_w}}(\zeta)| |(\tilde{f}_{\mu_{F_w}})_\zeta(\zeta)| \leq C|\xi|^{-2},$$

and by left estimate (2.32) applied to  $\tilde{f}_{\mu_{F_w}}$ , we obtain

$$|(f_{\lambda_{F_w}})_\zeta(\zeta)| \leq C |\operatorname{Re} \tilde{f}_{\mu_{F_w}}^{-1}(\zeta)|^{-3}$$

at  $|\operatorname{Re} \tilde{f}_{\mu_{F_w}}^{-1}(\zeta)| \leq 1$ . The rest of the proposition is obvious.  $\square$

7) *Inductive change of variables.* Now we pass to estimates of derivatives of higher orders. Consider the decomposition of  $F_w$  at  $\xi \rightarrow -\infty$

$$F_w(\zeta) = \zeta + \log a_0 + a_0^{-1} e^\xi R_1(\varphi) + \dots + a_0^{-1} e^{k\xi} R_k(\varphi) + \dots (2.36)$$

Let  $\chi(\xi)$  be a function equal to 1 at  $\xi < 0$  and to 0 at  $\xi > 2$  with the derivative less than 1 by modulus. We define some successive change of variables. Put  $\chi_0(\xi) = \chi((4|\log a_0|)^{-1}(\xi - c_0))$  if  $|\log a_0| \geq 1/4$  and  $\chi_0(\xi) = \chi(\xi - c_0)$  if  $|\log a_0| < 1/4$  with some  $c \leq -2$  large enough by modulus in both cases and define the new variable  $\zeta_0$

$$\zeta = \sigma_0(\zeta_0) = s_0(\xi_0 + i\varphi_0) = \zeta_0 - \log a_0 \chi_0(\xi_0). \quad (2.37)$$

The derivative of the function  $|\log a_0| \chi_0$  is less than  $1/4$ ,  $\log a_0$  has uniform bounds independent of  $w$ , and we can set  $c_0$  also independently of  $w$ . We obtain the estimates  $|(\sigma_0)_{\zeta_0} - 1| < 1/4$ ,  $|(\sigma_0)_{\bar{\zeta}_0}| < 1/4$ . Hence,  $\sigma_0$  is a homeomorphism of the left half-plane onto itself. We get the new map  $F_{0w}$  with the asymptotics

$$\begin{aligned} F_{0w}(\zeta_0) &= F_w(\sigma_0(\zeta_0)) = \zeta_0 + a_0^{-1} e^{\xi_0} R_1(\varphi_0) + \dots + a_0^{-k} e^{k\xi_0} R_k(\varphi_0) + \dots = \\ &= \zeta_0 + e^{\xi_0} R_{01}(\varphi_0) + \dots + e^{k\xi_0} R_{0k}(\varphi_0) + \dots \end{aligned}$$

Suppose we have uniform estimates for the coefficients of the expansion of  $h_w$  at zero up to order  $k \geq 1$ . Define the variable  $\zeta_1$

$$\zeta_0 = \sigma_1(\zeta_1) = \sigma_1(\xi_1 + i\varphi_1) = \zeta_1 - e^{\xi_1} R_{01}(\varphi_1) \chi(\xi_1 - c_1). \quad (2.38)$$

We get the map

$$F_{1w}(\zeta_1) = F_{0w}(\sigma_1(\zeta_1)) = \zeta_1 + e^{2\xi_1} R_{12}(\varphi_1) + \dots$$

Analogously, we define the variable  $\zeta_l$ ,  $l \geq 2$

$$\zeta_{l-1} = \sigma_l(\zeta_l) = \sigma_l(\xi_l + i\varphi_l) = \zeta_l - e^{2\xi_l} R_{(l-1)l}(\varphi_l) \chi(\xi_l - c_l).$$

After  $k$  successive changes of variables we obtain the map  $F_{(k-1)w} = F_w \circ \sigma_0 \circ \sigma_1 \circ \dots \circ \sigma_{k-1}$

$$F_{(k-1)w}(\zeta) = \zeta + e^{k\xi} R_{(k-1)k}(\varphi) + \dots$$

Here we returned to the notation  $\zeta$  for the variable.

In the chart  $z$  we get the transformations

$$z = s_0(z_0) = \exp(\sigma_0(\log z_0)) = z_0 \exp[-\log a_0 \chi_0(\log |z_0|)], \quad (2.39)$$

...

$$z_{l-1} = \exp(\sigma_l(\log z_l)), \quad (2.40)$$

...

and the resulting mapping  $h_w^{(k)} = h_w \circ s_0 \dots \circ s_{k-1}$ ,

$$h_w^{(k)}(z) = z + c_{(k+1)\bar{0}} z^{k+1} + \dots + c_{0\overline{(k+1)}} \bar{z}^{k+1} + \dots = z + c_{(k+1)\bar{0}} z^{k+1} + P_k(z) + \dots$$

By the inductive assumption, we have uniform estimates for the coefficients of the expansion of  $h_w$  at zero up to order  $k$ . It is easy to see that to obtain estimates for the derivatives at zero of order  $k+1$  it is enough to estimate the coefficients  $c_{(k+1)\bar{0}}, \dots, c_{0\overline{(k+1)}}$ . Indeed, if we have decomposition (2.18) for  $h_w$ , then  $c_{(k+1)\bar{0}}$  (and analogously other coefficients at the terms of order  $k+1$ ) has the representation

$$c_{(k+1)\bar{0}} = b_{(k+1)\bar{0}}/a_0^{k+1} + p_{(k+1)\bar{0}},$$

where  $p_{(k+1)\bar{0}}$  is a polynomial from  $b_{i\bar{j}}/a_0^m$ ,  $i+j=m$  with  $m, k+1$ .

Let  $s$  be the composition  $s = s_0 \circ s_1 \dots \circ s_{k-1}$ . The complex dilatation of the map  $h_w^{(k)}$  is

$$\begin{aligned} \mu_{h_w^{(k)}}(z) &= \frac{\mu_{h_w} - \mu_s}{1 - \bar{\mu}_s \mu_{h_w}} \cdot \frac{s_z}{\bar{s}_z} \circ s^{-1}(z) = \\ &= H_{k+1}(z) + |z|^{k+2} h(z), \end{aligned}$$

where

$$H_{k+1}(z) = c_{k\bar{1}} z^k + \dots + (k+1) c_{0\overline{(k+1)}} \bar{z}^k.$$

The coefficients  $c_{k\bar{1}}/a_0, \dots, c_{0\overline{(k+1)}}/a_0$  can be represented as polynomials in the variables  $a_0^{-1}$ , the coefficients of the expansion of  $h_w$  at zero up to order  $k$ , and the coefficients of order  $k+1$  of  $\mu_{h_w}$ . All these coefficients are uniformly bounded either by the inductive assumption either by Proposition 2. It

follows that  $c_{k\bar{1}}/a_0, \dots, c_{0\overline{(k+1)}}/a_0$  are uniformly bounded. Thus it is enough only to estimate the coefficient  $c_{(k+1)\bar{0}}$ . Define the new transformation

$$z = \tilde{s}_k(z_1) = z_1[1 - z_1^{-1}P_k(z_1)\chi(\log|z_1| - c_k)]. \quad (2.41)$$

Again adopting the notation  $z$  for the chart we obtain the map

$$g_w^{(k)}(z) = h_w^{(k)} \circ \tilde{s}_k(z) = z + c_{(k+1)\bar{0}}z^{k+1} + O(|z|^{k+2})$$

with the complex dilatation  $\mu_{g_w^{(k)}}(z) = O(|z|^{k+1})$ .

Note that we obtained the relations

$$(g_w^{(k)})_{z^{k+1}}(0) = (h_w)_{z^{k+1}}(0) + p_k(\{(h_w)_{(l)}(0)\}), \quad (2.42)$$

where  $p_k$  is a polynomial in  $(h_w)_{z^l}(0)$  with  $l \leq k$  and in  $(\mu_{h_w})_{(r)}(0)$ ,  $|r| \leq k$ . From (2.42) follow the inverse relations

$$(h_w)_{(m)}(0) = q_{(m)}(\{(g_w^{(l)})_{z^{l+1}}(0), (\mu_{h_w})_{(r)}(0)\}). \quad (2.43)$$

Here  $|m| = k + 1$  and  $q_{(m)}$  is a polynomial in  $(g_w^{(l)})_{z^{l+1}}(0)$ ,  $(\mu_{h_w})_{(r)}(0)$  with  $l \leq k$ ,  $|r| \leq k$ .

**Proposition 11** *Suppose we have uniform estimates for the coefficients of the expansion of  $h_w$  at zero up to order  $k \geq 1$ . Then we can put  $c_0, c_1, \dots, c_k$  such that  $s_0, s_1, \dots, \tilde{s}_k$  will be homeomorphisms with uniformly bounded derivatives of order up to  $k + 1$ . The first derivatives  $(g_w^{(k)})_z, (g_w^{(k)})_{\bar{z}}$  and all derivatives  $(\mu_{g_w^{(k)}})_{(l)}$ ,  $2 \leq |(l)| \leq k + 1$  will be uniformly bounded on the disk  $D_{1/2}$ .*

*If the coefficients of the expansion of  $h_w$  at zero of order  $l$ ,  $l \leq k$  have the estimates  $C(d + b_1 + \dots + b_l)$  with some uniform  $C$  and  $d, b_1, \dots, b_k$  are small enough with some uniform estimates, we can put  $c_0, c_1, \dots, c_k$  such that*

$$|(g_w^{(k)})_z - 1| \leq C(d + b_1 + \dots + b_k), |(g_w^{(k)})_{\bar{z}}| \leq C(d + b_1 + \dots + b_k) \quad (2.44)$$

$$|(\mu_{g_w^{(k)}})_{(l)}| \leq C(d + b_1 + \dots + b_k), 2 \leq |(l)| \leq k + 1 \quad (2.45)$$

on  $D_{1/2}$  with uniform  $C$  independent of  $d, b_1, \dots, b_k$ .

**Proof.** We already proved that  $s_0$  is homeomorphic and here we give only a more explicit estimate. By (2.39), (2.38),

$$(s_0)_z(z_0) = \exp\{-\log a_0\chi((4|\log a_0|)^{-1}(\log|z_0| - c_0))\} \times$$

$$\left\{1 - \frac{z_0}{4} \chi'((4|\log a_0|)^{-1}(\log |z_0| - c_0)) \bar{z}_0^{1/2} / (2z_0^{3/2})\right\} \quad (2.46)$$

if  $|\log a_0| \geq 1/4$ , and

$$(s_0)_z(z_0) = \exp\{-\log a_0 \chi(\log |z_0| - c_0)\} \left\{1 - z_0 \log a_0 \chi'(\log |z_0| - c_0) \bar{z}_0^{1/2} / (2z_0^{3/2})\right\} \quad (2.47)$$

if  $|\log a_0| \leq 1/4$ . Since  $|\chi'| < 1$ , we can see that

$$\exp(-|\log a_0|)/2 \leq |(s_0)_z(z_0)| \leq 3/2 \max\{1, \exp(|\log a_0|)\}.$$

Also, applying estimate (2.30), we see that

$$|(s_0)_z(z_0) - 1| \leq C(d + b_1)^2 \quad (2.48)$$

with some uniform  $C$  for  $d + b_1$  small enough. Analogously,

$$(s_0)_{\bar{z}}(z_0) = \exp\{-\log a_0 \chi((4|\log a_0|)^{-1}(\log |z_0| - c_0))\} \times \frac{z_0}{4} \chi'((4|\log a_0|)^{-1}(\log |z_0| - c_0)) z_0^{1/2} / (2\bar{z}_0^{3/2}) \quad (2.46')$$

if  $|\log a_0| \geq 1/4$ , and

$$(s_0)_{\bar{z}}(z_0) = \exp\{-\log a_0 \chi(\log |z_0| - c_0)\} z_0 \log a_0 \chi'(\log |z_0| - c_0) z_0^{1/2} / (2\bar{z}_0^{3/2}) \quad (2.47')$$

if  $|\log a_0| \leq 1/4$ . We obtain the estimate

$$|(s_0)_{\bar{z}}(z_0)| \leq C[(d + b_1)^2] \exp[C(d + b_1)^2] \leq 2C(d + b_1)^2 \quad (2.48')$$

with some uniform  $C$  at  $d + b_1$  small enough.

Note that we can chose  $c_0$  independent of  $d$  and  $b_1$ , for example, we can put  $c_0 = -5$ .

Now, by (2.39) for  $l \leq k - 1$ ,

$$(s_l)_z(z_l) = 1 - (P_l)_{z_1}(z_1) \chi(\log |z_l| - c_l) - P_l(z_l) \chi'(\log |z_l| - c_l) \bar{z}_l^{1/2} / (2z_l^{3/2}). \quad (2.49)$$

Here  $P_l$  is a homogeneous polynomial in the variables  $z_1, \bar{z}_1$  of order  $l + 1$ . Its coefficients have uniform estimates by the inductive assumption. Moreover, these coefficients are polynomials in  $(h_w)_{(j)}(0)$  with  $2 \leq |j| \leq l$  without a term of zero order. By the second inductive assumption, we can estimate them as  $C(d + b_1 + \dots + b_l)$  if  $d, b_1, \dots, b_l$  are small enough. Remind that  $\chi(\log |z_l| - c_l) \neq 0$  only if  $\log |z_l| \leq c_l$ . Suppose the coefficients

of  $P_l$  are bounded by the constant  $M$ . Then  $|(P_l)_{z_1}(z_l)| \leq M(l+2)^2|z_l|^l$ ,  $|P_l(z_l)\bar{z}_l^{1/2}/(2\bar{z}_l^{3/2})| \leq M(l+2)|z_l|^l$ . If we put  $c_l < \log[-8M(l+2)^2]$ , then we obtain  $|(s_l)_z(z_l) - 1| \leq 1/4$ . If  $d, b_1, \dots, b_l$  are small enough, then we can put  $c_l = -5$  and obtain the estimate

$$|(s_l)_z(z_l) - 1| \leq C(d + b_1 + \dots + b_l). \quad (2.50)$$

Also,

$$(s_l)_{\bar{z}}(z_l) = (P_l)_{\bar{z}}(z_l)\chi(\log|z_l| - c_l) + |P_l(z_l)\chi'(\log|z_l| - c_l)z_l^{1/2}/(2\bar{z}_l^{3/2})| \quad (2.49')$$

and we obtain the estimate  $|(s_l)_{\bar{z}}(z_l)| \leq 1/4$  in the general case and the estimate

$$|(s_l)_{\bar{z}}(z_l)| \leq C(d + b_1 + \dots + b_l) \quad (2.50')$$

if  $d, b_1, \dots, b_l$  are small enough.

Consider at last the derivative

$$(\tilde{s}_k)_z(z_k) = 1 - (\tilde{P}_k)_{z_k}(z_k)\chi(\log|z_k| - c_k) - \tilde{P}_k(z_k)\chi'(\log|z_k| - c_k)\bar{z}_k^{1/2}/(2z_k^{3/2}). \quad (2.51)$$

The polynomial  $\tilde{P}_k$  is the sum  $\tilde{c}_{z^k\bar{1}}(0)z^k\bar{z} + \dots + \tilde{c}_{0\bar{k+1}}z^{k+1}$  and the coefficients have the representation  $c_{l\bar{k+1-l}} = \alpha_{l\bar{k+1-l}}(\mu_{h_w})_{z^l\bar{z}^{k-l}}(0) + p_{l\bar{k+1-l}}$ , where  $p_{l\bar{k+1-l}}$  is a polynomial in  $(h_w)_{(j)}(0)$  with  $2 \leq |(j)| \leq k$  without a term of zero order. Applying Proposition 2 and the inductive assumptions we obtain for these coefficients an uniform estimate in the general case and the estimate  $C(d + b_1 + \dots + b_k)$  if  $d, b_1, \dots, b_k$  are small enough. As above, we obtain the estimate  $|(\tilde{s}_k)_z(z_k) - 1| \leq 1/4$  at an appropriate  $c_k$ . If  $d, b_1, \dots, b_k$  are small enough, we can put  $c_k = -5$  and obtain

$$|(\tilde{s}_k)_z(z_k) - 1| \leq C(d + b_1 + \dots + b_l). \quad (2.52)$$

Analogously,

$$(\tilde{s}_k)_{\bar{z}}(z_k) = (\tilde{P}_k)_{\bar{z}}(z_k)\chi(\log|z_k| - c_k) + |\tilde{P}_k(z_k)\chi'(\log|z_k| - c_k)z_k^{1/2}/(2\bar{z}_k^{3/2})| \quad (2.51')$$

and we obtain an uniform estimate in the general case and the estimate

$$|(\tilde{s}_k)_{\bar{z}}(z_k)| \leq C(d + b_1 + \dots + b_k) \quad (2.52')$$

if  $d, b_1, \dots, b_k$  are small enough.

We proved that  $s_0, \dots, \tilde{s}_k$  are homeomorphic if we set  $c_0, \dots, c_k$  as above. Put  $S^{(k)} = s_0 \circ s_1 \circ \dots \circ \tilde{s}_k$ . By definition,  $g_w^{(k)} = h_w \circ S^{(k)}$ , and we obtain estimates (2.44) from (2.48), (2.48'), (2.50), (2.50'), (2.52), (2.52') and corollary of Proposition 9.

Now consider derivatives of the right parts of (2.46), (2.47), (2.46') and (2.47'). In the first derivatives there appear the terms of the types  $O(1) \log a_0 \chi'(\log |z_0| - c_0) |z_0|^{-1}$  and  $O(1) \log a_0 \chi''(\log |z_0| - c_0) |z_0|^{-1}$ . Since derivatives of the function  $\chi$  don't equal zero only at  $0 < \log(|z_0|) - c_0 < 2$ , we obtain the estimates

$$|(s_0)_{(l)}(z_0)| \leq C(d + b_1)^2 e^{|\log |z_0||}, \quad |l| = 2, \quad |z_0| \leq 1/2.$$

Derivatives of order  $l$  are sums of items containing multiples of the types  $O(1) \log a_0, \chi^{(j)}(\log[|z_0|] - c_0), 1 \leq |j| \leq l$  and  $|z_0|^{-s}$  with some integer  $s$ . At each differentiation there appears no more than one multiple  $|z_0|^{-1}$ , and we see that we have the estimate

$$|(s_0)_{(j)}(z_0)| \leq C(d + b_1)^2 e^{(|j|-1)|\log |z_0||}, \quad |z_0| \leq 1/2$$

with some uniform  $C$ . Remind that we can put  $c_0 = -5$ . At small  $d$  and  $b_1$  we obtain the estimate  $C(d + b_1)^2$  with  $C$  independent of  $d$  and  $b_1$ .

We estimate derivatives of  $s_l$  and  $(\tilde{s}_k)$  analogously. Differentiating the right parts of (2.49), (2.49'), (2.51) and (2.51') we obtain terms of order  $O(1)(d + b_1 + b_l) e^{(|j|-l)|\log |z_0||}$  for derivatives of order  $j$ . At our choice of  $c_l$  we have uniform estimates in the general case and the estimate  $C(d + b_1 + \dots + b_l)$  if  $d, b_1, \dots, b_l$  are small.

Now we have

$$\mu_{g_w^{(1)}} = \frac{\mu_{h_w} - \mu_{S^{(k)}}}{1 - \bar{\mu}_{S^{(k)}} \mu_{h_w}} \cdot \frac{S_z^{(k)}}{S_z^{(k)}} \circ (S^{(k)})^{-1}.$$

Now the proposition follows from the estimates for derivatives of  $S^{(k)}$  and Proposition 2.  $\square$

8) *Estimates for higher derivatives.* From the next proposition we obtain by induction estimate (2.2) and estimate (2.3).

**Proposition 12** *a) Suppose that we have uniform bounds for the coefficients of the expansion of  $h_w$  at zero up to order  $k \geq 1$ . Then the derivative  $(g_w^{(k)})_{z^{k+1}}$  is uniformly bounded.*

*b) If the coefficients of the expansion of  $h_w$  at zero of order  $l, l \leq k$  have the estimates  $C(d + b_1 + \dots + b_l)$  with some uniform  $C$  and  $d, b_1, \dots, b_k$  are small enough with some uniform estimate, then  $|(g_w^{(k)})_{z^{k+1}}| \leq C(d + b_1 + \dots + b_{k+1})$ .*



**Proof.** a) Denote by  $\mu_w^{(k)}$  the Beltrami coefficient of the map  $g_w^{(k)}$ . We have

$$g_w^{(k)}(z) = \frac{1}{2\pi i} \int_{\partial D_{1/2}} \frac{g_w^{(k)}(t)}{t-z} dt + \frac{1}{\pi i} \int_{D_{1/2}} \frac{\mu_w^{(k)}(t) g_{w,t}^{(k)}(t)}{t-z} dS_t. \quad (2.53)$$

at  $|z| \leq 1/2$ . All derivatives in zero of the first integral in (2.53) are uniformly bounded. Now we have

$$\frac{1}{t-z} = \frac{1}{t} + \frac{z}{t^2} + \dots + \frac{z^{k+1}}{t^{k+2}} + \frac{z^{k+2}}{t^{k+2}(t-z)}.$$

The first derivatives of the function  $g_w^{(k)}$  and derivatives of  $\mu_w^{(k)}$  of order  $k+1$  are uniformly bounded on  $D_{1/2}$  by Proposition 11.

Now  $\mu_w^{(k)}(z) = |z|^{k+1} \gamma(z)$ , where

$$|\gamma(z)| \leq \frac{1}{(k+1)!} \max_{|z| \leq 1/2} \|\partial^{k+1} \mu_w^{(k)}\|,$$

where  $\|\partial^{k+1} \mu_w^{(k)}\|$  is the sum of modulus of derivatives of order  $k+1$ . Hence,  $|\gamma|$  is uniformly bounded on  $D_{1/2}$ .

The integrals

$$\int_{D_{1/2}} \frac{|t|^{k+1} \gamma(t) g_{w,t}^{(k)}(t)}{t^l} dS_t, \quad l \leq k+2 \quad (2.54)$$

are all uniformly bounded. We get

$$\int_{D_{1/2}} \frac{\mu_w^{(k)}(t) g_{w,t}^{(k)}(t)}{t-z} dS_t = c_0 + z c_1 + \dots + z^{k+1} c_{k+1} + z^{k+2} \int_{D_{1/2}} \frac{|t|^{k+1} \gamma(t) g_{w,t}^{(k)}(t)}{t^{k+2}(t-z)} dS_t. \quad (2.55)$$

From the other hand, we have the estimate

$$\int_{D_{1/2}} \frac{dS_t}{|t||t-z|} \leq C \|\log |z|\|$$

with some uniform  $C$ . We see that integral (2.55) has  $k+1$  derivatives at zero and its  $z$ -derivative of order  $k+1$  is equal to  $(k+1)! c_{k+1}$ .

b) Remind that  $g_w^{(k)}(z) = h_w(z)$  at  $|z| = 1/2$ . Applying corollary of Proposition 9, we see that all derivatives at zero of order higher than 1 of

the first integral in (2.53) have the estimate  $C(d + b_1)^2$ . To estimate the derivatives of the second integral in (2.52) we need to estimate integrals (2.54). We obtain the estimates by Proposition 11.  $\square$

Now we finish the proof of Lemma 1.

**Proof of estimate (2.3).** Applying inductive relations (2.43) we obtain the estimates for the derivatives at zero of  $h_w$  from the estimates for  $g_w^{(k)}$ . The corollary follows now from Proposition 3.  $\square$

We shall use in the next section the following estimate:

**Proposition 13**

$$|(f_{\mu_{F_w}})_{(k)}(\zeta)| \leq C e^\xi (1 + |\xi|^{-6}), \quad |(k)| = 2$$

with some uniform  $C$  at  $\xi \leq 1$ .

**Proof.** Estimate at first the second derivatives of the map  $h_w$ . Analogously to the proof of Proposition 10 a) it is enough to obtain the estimates for  $f_w$ . We have

$$\begin{aligned} (f_w)_{zz}(z) &= \frac{\partial}{\partial z} \left[ \frac{1 - |w'|^2}{(1 - \bar{w}' f \circ \varphi_w^{-1}(z))^2} f_z \circ \varphi_w^{-1}(z) \frac{1 - |w|^2}{(1 - \bar{w}z)^2} \right] = \\ &= \frac{2w'(1 - |w'|^2)}{(1 - \bar{w}' f \circ \varphi_w^{-1}(z))^3} (f_z \circ \varphi_w^{-1}(z))^2 \left[ \frac{1 - |w|^2}{(1 - \bar{w}z)^2} \right]^2 + \\ &+ \frac{1 - |w'|^2}{(1 - \bar{w}' f \circ \varphi_w^{-1}(z))^2} \left[ f_{zz} \circ \varphi_w^{-1}(z) \left( \frac{1 - |w|^2}{(1 - \bar{w}z)^2} \right)^2 + f_z \circ \varphi_w^{-1}(z) \frac{2w(1 - |w|^2)}{(1 - \bar{w}z)^3} \right]. \end{aligned}$$

Applying (2.4) we see that

$$|(f_w)_{zz}(z)| \leq C \left[ \frac{1 - |w|}{(1 - \bar{w}z)^4} |f_{zz} \circ \varphi_w^{-1}(z)| + \frac{1}{(1 - \bar{w}z)^4} |f_z \circ \varphi_w^{-1}(z)|^2 \right]$$

Applying estimates (2.1), (2.8), and (2.9) we obtain

$$|f_{zz} \circ \varphi_w^{-1}(z)| \leq \frac{C}{1 - |\varphi_w^{-1}(z)|} \leq C \frac{|1 - \bar{w}z|}{(1 - |w|)(1 - |z|)},$$

and, hence,

$$|(f_{2w})_{zz}(z)| \leq \frac{C}{(1 - |z|)^4}, \quad |(h_{2w})_{zz}(z)| \leq \frac{C}{(1 - |z|)^4} \quad (2.56)$$

Obviously we have analogous estimates for other derivatives of second order.

Return now to the map  $f_{\mu_{F_w}}$ . As in the proof of Proposition 10 b) we can represent it in the chart  $z$  on  $D$  as  $\hat{g}_w = \hat{h}_w \circ h_w$ , where  $\hat{h}_w$  is a holomorphic univalent uniformly bounded function with derivative at zero uniformly bounded from below and from above. For such functions we have estimates (2.34) and ([Pom])

$$|\hat{h}_w''(z)| \leq \frac{C}{(1-|z|)^2}. \quad (2.57)$$

Applying this estimate and (2.31), (2.35), (2.56), we obtain at  $\xi \leq 0$

$$\begin{aligned} |(f_{\mu_{F_w}})_{\zeta\zeta}| &\leq C e^\xi [ |(\hat{g}_w)_z \circ \exp \zeta|^2 + |(\hat{g}_w)_{zz} \circ \exp \zeta| ] \leq \\ &\leq C e^\xi [ (|\hat{h}_w' \circ h_w \circ \exp \zeta|^2 |(h_w)_z \circ \exp \zeta|^2 + |\hat{h}_w'' \circ h_w \circ \exp \zeta| |(h_w)_z \circ \exp \zeta|^2 + \\ &\quad + |\hat{h}_w' \circ h_w \circ \exp \zeta| |(h_w)_{zz} \circ \exp \zeta| ) ] \leq C e^\xi (1 + |\xi|^{-6}) \end{aligned}$$

at  $|\xi| \leq 1$  and analogous estimates for other second derivatives of  $f_{\mu_{F_w}}$ . Obviously, for derivatives of  $\tilde{f}_{\mu_{F_w}}$  we have the same estimates.

On  $\mathbb{C} \setminus D$  the functions  $\hat{g}_w$  are univalent holomorphic with uniformly bounded derivatives at infinity and uniformly bounded away from zero. Hence, there is the uniform estimate  $|\hat{g}_w(z)|_{zz} \leq C(|z| - 1)^{-2}$ . Thus  $|(f_{\mu_{F_w}})_{\zeta\zeta}(\zeta)| \leq C|\zeta|^{-2}$  at  $0 < |\xi| \leq 1$ . We obtain the estimates for other derivatives of second order analogously.  $\square$

In conclusion of this section we obtain some estimates for derivatives of the principal solutions. Remind the construction of the normal solutions (see, for example, [Ah]).

Let  $f_\mu$  be the principal solutions with the Beltrami coefficient  $\mu$  and put  $f_\mu^0(z) = f_\mu(z) - f_\mu(0)$ . Put  $\tilde{f}(z) = \overline{f_\mu^0(\bar{z}^{-1})}^{-1}$  and

$$\lambda = \left( \mu \frac{\tilde{f}_z}{f_z} \right) \circ \tilde{f}^{-1},$$

Let  $f_\lambda$  be the corresponding principal solution and  $f_\lambda^0 = f_\lambda - f_\lambda(0)$ . Define

$$f_c = f_\lambda^0 \circ \tilde{f}.$$

That is

$$f_c(z) = f_\lambda^0 \circ \overline{(f_\mu^0(\bar{z}^{-1}))}^{-1}. \quad (2.58)$$

Then

$$f = f_c / f_c(1) \quad (2.59),$$

where  $|f_c(1)| = 1$ .

**Proposition 14** *At assumptions of Lemma 1 we have the estimates*

$$|f_\mu^0(z)| \geq c|z|, \quad (2.60)$$

$$c|1 - |z|| \leq |(f_\mu^0)_z(z)| \leq C|1 - |z||^{-1} \quad (2.61)$$

with  $c, C$  depending only on  $d$  and  $b_1$ .

$$|(f_\mu^0)_{(k)}(z)| \leq C|1 - |z||^{-2}, \quad |(k)| = 2. \quad (2.62)$$

with  $C$  depending only on  $d, b_1, b_2$ . Analogous estimates hold for  $\tilde{f}$ .

$$c|1 - |\tilde{f}^{-1}(z)|| \leq |(f_\lambda^0)_z(z)| \leq C|1 - |\tilde{f}^{-1}(z)||^{-1} \quad (2.63)$$

also with  $c, C$  depending only on  $d$  and  $b_1$ .

**Proof.** We have the representation on  $D$ :  $f_\mu^0 = H \circ f$ , where  $H$  is a holomorphic univalent function mapping zero to zero and  $f$  is the normal mapping. Since  $f_\mu^0$  is bounded on  $D$  with a bound depending only on  $d$  and we have estimate (2.1), it follows that the function  $H$  is bounded and has the derivatives at zero bounded from above by a constant depending only on  $d$ . Let show that its  $z$ -derivative at zero is bounded also from below. It is enough to prove that  $(f_\mu^0)_z(0)$  is bounded from below, that is, to prove estimate (2.60).

Since  $f_c$  differs from  $f$  by the multiple equal to 1 by modulus, we see that  $|f_c(z)| \leq C|z|$  with  $C$  depending only on  $d$  and  $b_1$ . Also,  $|f_\lambda^0(z) - z| \leq C$  with  $C$  also depending only on  $d$  and  $b_1$ . It means that

$$|f_c(z) - \overline{f_\mu^0(\bar{z}^{-1})}^{-1}| \leq C$$

and, hence,

$$|(f_\mu^0(\bar{z}^{-1}))^{-1}| \leq C|z|.$$

It means that we obtain estimate (2.60). Thus  $|H'(0)|$  is bounded also from below and we can apply estimates (2.34) and (2.57) to  $H$ . Applying also (2.4) we obtain estimates (2.61), (2.62) at  $|z| \leq 1$ .

On  $\mathbb{C} \setminus D$  the function  $f_c$  is univalent holomorphic with uniformly bounded derivatives at infinity. Also, on this domain  $f_c$  is uniformly bounded away from zero. We obtain estimates (2.61) and (2.62) as in Propositions 10 and 13. The estimates for  $\tilde{f}$  follow by symmetry.

Solving the system

$$\begin{aligned}(f_c)_z &= (f_\lambda^0)_z \circ \tilde{f} \cdot \tilde{f}_z + (f_\lambda^0)_{\bar{z}} \circ \tilde{f} \cdot \overline{\tilde{f}_z} \\ (f_c)_{\bar{z}} &= (f_\lambda^0)_{\bar{z}} \circ \tilde{f} \cdot \overline{\tilde{f}_z} + (f_\lambda^0)_z \circ \tilde{f} \cdot \tilde{f}_z\end{aligned}$$

we obtain

$$(f_\lambda^0)_z \circ \tilde{f} = \frac{(f_c)_z}{\tilde{f}_z} \cdot \frac{1 - \bar{\mu}_{\tilde{f}}}{1 - \mu_{\tilde{f}} \bar{\mu}_{\tilde{f}}}$$

Estimates (2.63) follow from (2.61) and from boundedness from below and from above of  $(f_c)_z$   $\square$

### 3 Estimates for differences of derivatives of the normal mappings with different complex dilatations.

The proof of the lemma below is long and tedious but essentially simple. We adopt the notation  $P(d)$  for the supremum of  $p$  such that the series  $1 + \mathcal{S}\mu + \mathcal{S}\mu\mathcal{S}\mu + \dots$  converge in  $L^p$  if  $\|\mu\|_C \leq d$ .

**Lemma 2** *Let  $\mu_1, \mu_2$  be functions on  $D$  satisfying assumptions of Lemma 1 with the same  $d, b_1, \dots, b_k$ . Let  $f_1, f_2$  be the corresponding normal mappings. Then we have the estimates:*

$$a) \quad \|f_1 - f_2\|_C \leq C \|\mu_1 - \mu_2\|_p^\alpha, \quad (3.1)$$

where  $2 < p < P(d)$  and  $\alpha$  depends only on  $d$  and  $b_1$ .

b) *Let  $(k)$  be a multi-index,  $|(k)| \geq 1$ . Fix some  $0 < R < 1$ . Then, for  $z \in D$ ,*

$$\begin{aligned}|(f_1)_{(k)}(z) - (f_2)_{(k)}(z)| &\leq \frac{C}{(1 - |z|)^{|(k)|-1}} \min \left\{ 1, \left[ \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |z|} + \right. \right. \\ &\left. \left. + \sup_{D_R} |\mu_1 - \mu_2|^\alpha + [(1 - R)/(1 - |z|)]^\alpha + \sup_{0 \leq |(q)| \leq |(k)|} \sup_{D_{R_z}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}|^\alpha \right] \right\},\end{aligned} \quad (3.2)$$

where  $R_z < 1$  is such that  $1 - R_z \geq a(1 - |z|)$  for some  $a$  depending only on  $d$ ,  $C$  is some uniform constant and  $0 < \alpha < 1$  depends only on  $d$ ,  $2 < p < P(d)$ .

In the proof we shall use the terminology and the notations of the previous section.

**Proof of estimate 3.1.** We put, as in Proposition 14,  $f_{\mu_i}^0(z) = f_{\mu_i}(z) - f_{\mu_i}(0)$ ,  $\tilde{f}_i(z) = \overline{f_{\mu_i}^0(\bar{z}^{-1})}^{-1}$ ,

$$\lambda_i = \left( \mu_i \frac{\tilde{f}_{i,z}}{\tilde{f}_{i,z}} \right) \circ \tilde{f}_i^{-1}. \quad (3.3)$$

By definition,  $f_{\lambda_i}$  is the corresponding principal solution,  $f_{\lambda_i}^0 = f_{\lambda_i} - f_{\lambda_i}(0)$ ,  $f_{c_i} = f_{\lambda_i}^0 \circ \tilde{f}_i$ ,  $f_i = f_{c_i}/f_{c_i}(1)$ .

By Proposition 14,

$$c|1 - |z|| \leq |(f_{\mu_i}^0)_z(z)| \leq C|1 - |z||^{-1}, \quad (3.4)$$

$$c|1 - |\tilde{f}_i^{-1}(z)|| \leq |f_{\lambda_i, (1)}^0(z)| \leq C|1 - |\tilde{f}_i^{-1}(z)||^{-1}, \quad (3.5)$$

$$|f_{\mu_i, (k)}^0(z)| \leq C|1 - |z||^{-2}, \quad |(k)| = 2. \quad (3.6)$$

We must estimate the value

$$|f_{c_1}(z) - f_{c_2}(z)| |f_{c_1}(1)|^{-1} + |f_{c_2}(z)| |f_{c_1}(1)|^{-1} - f_{c_2}(1)|^{-1}|.$$

Since  $|f_{c_1}(1)| = 1$ , it is enough to estimate the difference

$$|f_{c_1}(z) - f_{c_2}(z)| \leq |f_{\lambda_1}^0 \circ \tilde{f}_1(z) - f_{\lambda_2}^0 \circ \tilde{f}_1(z)| + |f_{\lambda_2}^0 \circ \tilde{f}_1(z) - f_{\lambda_2}^0 \circ \tilde{f}_2(z)|. \quad (3.7)$$

**Proposition 15** *Let  $f_{\nu_i}$ ,  $i = 1, 2$  be the principal solutions corresponding to compactly supported Beltrami coefficients  $\nu_i$ . Then*

$$|f_{\nu_1}(z) - f_{\nu_2}(z)| \leq \|(\nu_1 - \nu_2)(|(f_{\nu_1})_z - 1| + |(f_{\nu_2})_z - 1|)\|_p$$

$$\|f_{\nu_1, z} - f_{\nu_2, z}\|_p \leq \|(\nu_1 - \nu_2)(|(f_{\nu_1})_z - 1| + |(f_{\nu_2})_z - 1|)\|_p$$

$$|f_{\nu_1}^{-1}(\zeta) - f_{\nu_2}^{-1}(\zeta)| \leq C\|(\nu_1 - \nu_2)(|(f_{\nu_1})_\zeta - 1| + |(f_{\nu_2})_\zeta - 1|)\|_p$$

for  $2 < p < P(d)$ .

**Proof.** Let  $h_i$  be the solution to the equation  $h_i - \mathcal{S}\nu_i h_i = 1$ . (Remind that here and below  $\mathcal{S}$  is the Beorling transform and  $\mathcal{C}$  is the Cauchy transform). We have

$$|f_{\nu_1}(z) - f_{\nu_2}(z)| = |\mathcal{C}\nu_1 h_1(z) - \mathcal{C}\nu_2 h_2(z)| \leq C_p(\|(\nu_1 - \nu_2)h_1\|_p + \|\nu_2\|_C \|h_1 - h_2\|_p). \quad (3.8)$$

Further,

$$h_1 - h_2 = \mathcal{S}[\nu_1(h_1 - h_2)] + \mathcal{S}[(\nu_1 - \nu_2)h_2].$$

That is,

$$h_1 - h_2 = (id - \mathcal{S}\nu_1)^{-1}\mathcal{S}[(\nu_1 - \nu_2)h_2],$$

and

$$\|h_1 - h_2\|_p \leq C_p(\|(\nu_1 - \nu_2)h_2\|_p).$$

with some constant  $C_p$  depending only on  $p$ . Since  $h_i = (f_{\nu_i})_z - 1$ , we obtain the first two estimates of the proposition from (3.8).

Prove the third estimate. Suppose  $f_{\nu_1}(z_1) = f_{\nu_2}(z_2)$ . Then

$$|z_1 - z_2| = \mathcal{C}\nu_1 h_1(z_1) - \mathcal{C}\nu_2 h_2(z_2) \leq C_p(\|(\nu_1 - \nu_2)h_1\|_p + \|\nu_2\|_C \|h_1 - h_2\|_p).$$

□

We can apply this proposition to  $f_{\mu_i}, f_{\lambda_i}$  and, hence, to  $f_{\mu_i}^0, f_{\lambda_i}^0$ . Also we have

$$\begin{aligned} |\tilde{f}_1(z) - \tilde{f}_2(z)| &= |f_{\mu_1}^0(\bar{z}^{-1})f_{\mu_2}^0(\bar{z}^{-1})|^{-1} \|f_{\mu_1}^0(\bar{z}^{-1}) - f_{\mu_2}^0(\bar{z}^{-1})\| \leq \\ &\leq C|z|^2 \|(\mu_1 - \mu_2)(|(f_{\mu_1})_z - 1| + |(f_{\mu_2})_z - 1|)\|_p. \end{aligned} \quad (3.9)$$

Here we applied estimate (2.60).

We must estimate the right part of inequality (3.7). We proceed in several steps. In all inequalities below all constants such as  $c$  or  $\alpha$  depend only on  $d$  and  $b_1$ .

1)

$$|f_{\mu_1}^0 - f_{\mu_2}^0|_C \leq C\|(\mu_1 - \mu_2)\|_p^\alpha,$$

**Proof.** Fix some  $p < p' < P(d)$  and some  $r < 1$ . Applying Proposition 15 and estimate (3.4) we can write

$$\begin{aligned} \|(\mu_1 - \mu_2)((f_{\mu_i})_z - 1)\|_p &\leq C \left[ r^{-1} \left( \int_{\|z|-1 \geq r} |\mu_1 - \mu_2|^p dS_z \right)^{1/p} + \right. \\ &\quad \left. + \left( \int_{\|z|-1 \leq r} |(f_{\mu_i})_z - 1|^p dS_z \right)^{1/p} \right] \end{aligned}$$

The second integral we estimate by the Heolder inequality  $\|fg\|_p \leq \|f\|_{p'}\|g\|_{q'}$ ,  $q'^{-1} + p'^{-1} = p^{-1}$ . We obtain

$$\|(\mu_1 - \mu_2)((f_{\mu_i})_z - 1)\|_p \leq C[(1 - r)^{-1}\|(\mu_1 - \mu_2)\|_p +$$

$$+\|(f_{\mu_i})_z - 1\|_{p'}(\text{mes}\{r \leq |z| \leq 1\})^{\frac{1}{p}-\frac{1}{p'}}] \leq C[(1-r)^{-1}\|(\mu_1 - \mu_2)\|_p + r^{\frac{p'-p}{pp'}}].$$

If we define  $r$  from the equation  $(1-r)^{-1}\|(\mu_1 - \mu_2)\|_p = r^{\frac{p'-p}{pp'}}$ , i.e., if we put  $1-r = \|(\lambda_1 - \lambda_2)\|_p^{\frac{pp'}{pp'+p'-p}}$ , we get the estimate

$$|f_{\lambda_1} - f_{\lambda_2}| \leq C\|(\lambda_1 - \lambda_2)\|_p^{\frac{p'-p}{pp'+p'-p}}.$$

□

2) At  $|z| \leq 1$

$$|\tilde{f}_1(z) - \tilde{f}_2(z)| \leq C\|\mu_1 - \mu_2\|_p^\alpha,$$

**Proof.** It follows from step 1 and estimate (3.9). □

3)

$$\|(\tilde{f}_{1,z} - \tilde{f}_{2,z})\chi_D\|_p \leq C\|\mu_1 - \mu_2\|_p^\alpha,$$

Here  $\chi_D$  is the characteristic function of  $D$ .

**Proof.** The proof in step 1 depends only on the right side of the first inequality of Proposition 15. Since the second inequality has the same right part, we obtain our assertion from (3.9). □

4)

$$|f_{\lambda_1} - f_{\lambda_2}|_C \leq C\|(\lambda_1 - \lambda_2)\|_p^\alpha,$$

**Proof.** Again fix some  $p < p' < P(d)$  and some  $r < 1$ . Applying Proposition 15 and estimate (3.5) we can write

$$\begin{aligned} \|(\lambda_1 - \lambda_2)((f_{\lambda_i})_z - 1)\|_p &\leq C \left[ r^{-1} \left( \int_{\|\tilde{f}_i^{-1}(z)-1\| \geq r} |\lambda_1 - \lambda_2|^p dS_z \right)^{1/p} + \right. \\ &\quad \left. + \left( \int_{\|\tilde{f}_i^{-1}(z)-1\| \leq r} |(f_{\lambda_i})_z - 1|^p dS_z \right)^{1/p} \right] \end{aligned}$$

The second integral we again estimate by the Heolder inequality and obtain

$$\begin{aligned} \|(\lambda_1 - \lambda_2)((f_{\lambda_i})_z - 1)\|_p &\leq C[(1-r)^{-1}\|(\lambda_1 - \lambda_2)\|_p + \\ &\quad + \|(f_{\lambda_i})_z - 1\|_{p'}(\text{mes}\{|\tilde{f}_i^{-1}(z) - 1| \leq r\})^{\frac{1}{p}-\frac{1}{p'}}]. \end{aligned}$$

But

$$\text{mes}\{|\tilde{f}_i^{-1}(z) - 1| \leq r\} = \int_{\|z\|-1 \leq r} J(\tilde{f}_i)(z) dS_z,$$



where  $J(\tilde{f}_i)$  is the Jacobian of the transformation  $z \mapsto \tilde{f}_i(z)$ . Since  $(\tilde{f}_i)_z - 1$  belongs to  $L_p$ , we see that  $J(\tilde{f}_1)$  restricted on the ring  $\{|z| - 1| \leq r\}$  belongs to  $L^{p/2}$ . Hence, by the Geolder inequality

$$\text{mes}\{|\tilde{f}_i^{-1}(z) - 1| \leq r\} \leq Cr^{1-2/p}.$$

We obtain

$$\|(\lambda_1 - \lambda_2)((f_{\lambda_i})_z - 1)\|_p \leq C[(1-r)^{-1}\|(\lambda_1 - \lambda_2)\|_p + (1-r)^{\frac{(p'-p)(p-2)}{p^2 p'}}].$$

If we put  $1-r = \|(\lambda_1 - \lambda_2)\|_p^{\frac{p^2 p'}{(p'-p)(p-2)+p^2 p'}}$ , then we obtain the estimate

$$|f_{\lambda_1} - f_{\lambda_2}| \leq C\|(\lambda_1 - \lambda_2)\|_p^{\frac{(p'-p)(p-2)}{(p'-p)(p-2)+p^2 p'}}.$$

□

5)

$$\|\lambda_1 - \lambda_2\|_p \leq C\|\mu_1 - \mu_2\|_p^\alpha,$$

**Proof.** Since we have representation (3.3), we can see that

$$\begin{aligned} \|\lambda_1 - \lambda_2\|_p &\leq C_p \left[ \|\mu_1 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_1^{-1}\|_p + \|\mu_2 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_2^{-1}\|_p + \right. \\ &\quad \left. + \left\| \frac{\tilde{f}_{1,z}}{\tilde{f}_{1,z}} \circ \tilde{f}_1^{-1} - \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}} \circ \tilde{f}_1^{-1} \right\|_p + \left\| \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}} \circ \tilde{f}_1^{-1} - \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}} \circ \tilde{f}_2^{-1} \right\|_p \right]. \end{aligned} \quad (3.10)$$

We obtain estimates for all terms in the right side by the same method as in the steps above.

The first term we can write as

$$\|\mu_1 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_1^{-1}\|_p = \left( \int |\mu_1 - \mu_2|^p J(\tilde{f}_1)(z) dS_z \right)^{1/p},$$

Since  $\mu_1 - \mu_2 \neq 0$  only on  $D$ , we can apply the estimate of  $J(\tilde{f}_1)$  in  $L^{p/2}(D)$ . Also, from (3.4) follows the estimate  $|J(\tilde{f}_1)(\zeta)| \leq C(1-|\zeta|)^{-2}$  if  $\zeta \in D$ . Fix some  $r < 1$ . We obtain

$$\|\mu_1 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_2^{-1}\|_p \leq C(1-r)^{-2} \left( \int_{|z| \leq r} |\mu_1 - \mu_2|^p dS_z \right)^{1/p} + C \left( \int_{1-r \leq |z| \leq 1} J(\tilde{f}_1)(z) dS_z \right)^{1/p} \leq$$

$$\leq C[(1-r)^{-2}\|\mu_1 - \mu_2\|_p] + r^{1/p-2/p^2}.$$

We put  $1-r = \|\mu_1 - \mu_2\|_p^{\frac{p^2}{2p^2+p-2}}$  and obtain

$$\|\mu_1 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_1^{-1}\|_p \leq C\|\mu_1 - \mu_2\|_p^{\frac{p-2}{2p^2+p-2}}.$$

Consider the second term in (3.10). Remind that  $|(\mu_2)_{(k)}(z)| \leq b_1(1-|z|)^{-1}, |(k)| = 1$ . Again fix some  $r < 1$ . Applying step 2 and the third inequality of Proposition 15 we obtain

$$\begin{aligned} & \|\mu_2 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_2^{-1}\|_p \leq \\ & \leq C \left( (1-r)^{-p} \int_{r \leq |\tilde{f}_1^{-1}(z)| \leq 1} |\tilde{f}_1^{-1}(z) - \tilde{f}_2^{-1}(z)|^p dS_z \right)^{1/p} + C[\text{mes}\{|1-r \leq |\tilde{f}_1^{-1}(z)| \leq 1\}]^{1/p} \leq \\ & \leq C[(1-r)^{-1}\|\mu_1 - \mu_2\|_p^\alpha + \left( \int_{1-r \leq |z| \leq 1} J(\tilde{f}_1)(z) dS_z \right)^{1/p}] \leq . \\ & \leq C[(1-r)^{-1}\|\mu_1 - \mu_2\|_p^\alpha + (1-r)^{1/p-2/p^2}] \end{aligned}$$

Here  $\alpha$  is the same as in step 2) and we again recall that  $J(\tilde{f}_1)$  belongs to  $L^{p/2}$ .

We put  $1-r = \|\mu_1 - \mu_2\|_p^{\frac{\alpha p^2}{p^2+p-2}}$  and obtain

$$\|\mu_2 \circ \tilde{f}_1^{-1} - \mu_2 \circ \tilde{f}_2^{-1}\|_p \leq \|\mu_1 - \mu_2\|_p^{\frac{\alpha(p-2)}{p^2+p-2}}.$$

Consider the third term in (3.10). We have

$$\frac{\tilde{f}_{1,z}}{\tilde{f}_{1,z}} - \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}} = \frac{\tilde{f}_{1,z} - \tilde{f}_{2,z}}{\tilde{f}_{1,z}} - \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}} \frac{\tilde{f}_{1,z} - \tilde{f}_{2,z}}{\tilde{f}_{1,z}}.$$

We see that we must estimate  $|(\tilde{f}_{1,z} - \tilde{f}_{2,z}) \circ \tilde{f}_1^{-1}| |\tilde{f}_{1,z} \circ \tilde{f}_1^{-1}|^{-1}$ .

Applying left estimate (3.4) to  $\tilde{f}_{1,z}$  and acting as above we see that the third term in (3.10) is no greater than

$$C \left[ (1-r)^{-1} \left( \int_{r \leq |\tilde{f}_1^{-1}(z)| \leq 1} |(\tilde{f}_{1,z} - \tilde{f}_{2,z}) \circ \tilde{f}_1^{-1}(z)|^p dS_z \right)^{1/p} + (1-r)^{1/p-2/p^2} \right].$$

for any  $0 < r < 1$ . Here we use the uniform boundedness of the third term in (3.12) and again recall the estimate  $|J(\tilde{f}_1)(z)| \leq C(1 - |z|)^{-2}$ . We have

$$\begin{aligned} & \int_{r \leq |\tilde{f}_1^{-1}(z)| \leq 1} |(\tilde{f}_{1,z} - \tilde{f}_{2,z}) \circ \tilde{f}_1^{-1}(z)|^p dS_z = \\ & = \int_{r \leq |z| \leq 1} |\tilde{f}_{1,z} - \tilde{f}_{2,z}|^p |J(\tilde{f}_1)| dS_z \leq C(1-r)^{-2} \int_D |\tilde{f}_{1,z} - \tilde{f}_{2,z}|^p dS_\zeta. \end{aligned}$$

Applying step 3) we obtain for the third term the estimate

$$C(r^{-(1+2/p)} \|\mu_1 - \mu_2\|_p^\alpha + r^{1/p-2/p^2}),$$

where we set  $\alpha$  as in step 3). We put  $r = \|\mu_1 - \mu_2\|_p^{\frac{\alpha p^2}{p^2+3p-2}}$  and obtain the estimate

$$\left\| \frac{\tilde{f}_{1,z}}{\tilde{f}_{1,z}} \circ \tilde{f}_1^{-1} - \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}} \circ \tilde{f}_1^{-1} \right\|_p \leq C \|\mu_1 - \mu_2\|_p^{\frac{\alpha(p-2)}{p^2+3p-2}}.$$

Consider the last term in (3.10). Denote  $z_i = \tilde{f}_i^{-1}(z)$ . We must estimate  $p$ -norm of the sum

$$\frac{\tilde{f}_{2,z}(z_1) - \tilde{f}_{2,z}(z_2)}{\tilde{f}_{2,z}(z_1)} + \frac{\tilde{f}_{2,z}(z_2) - \tilde{f}_{2,z}(z_1)}{\tilde{f}_{2,z}(z_2)}.$$

Analogously to the previous case we get for this  $p$ -norm the estimate

$$C \left[ r^{-1} \left( \int_{r \leq |\tilde{f}_1^{-1}(z)| \leq 1} |\tilde{f}_{2,z}(z_1) - \tilde{f}_{2,z}(z_2)|^p dS_z \right)^{1/p} + r^{1/p-2/p^2} \right]$$

for any  $r > 0$ . Now at  $|z_1 - z_2| \leq (1-r)/2$ , applying estimate (3.6) and step 2), we have

$$\begin{aligned} & |\tilde{f}_{2,z}(z_1) - \tilde{f}_{2,z}(z_2)| \leq \\ & \leq \sup_{z \in [z_1, z_2], |(k)|=1} \|\tilde{f}_{2,z}^{(k)}(z)\| |z_1 - z_2| \leq Cr^{-2} \|\mu_1 - \mu_2\|_p^\alpha. \end{aligned} \quad (3.11)$$

Now we put  $1-r = \|\mu_1 - \mu_2\|_p^{\frac{\alpha p^2}{3p^2+p-2}}$ . In this case  $\|\mu_1 - \mu_2\|_p^\alpha$  and, hence,  $|\tilde{f}_1^{-1}(z) - \tilde{f}_2^{-1}(z)|$  is small by comparison with  $1-r$  and we, indeed, can apply estimate (3.11). We obtain the estimate

$$\left\| \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}}(z_1) - \frac{\tilde{f}_{2,z}}{\tilde{f}_{2,z}}(z_2) \right\|_p \leq C \|\mu_1 - \mu_2\|_p^{\frac{\alpha(p-2)}{3p^2+p-2}}.$$

□

We obtained the estimate of the first difference in the right side of inequality (3.7). We finish the proof of the proposition with estimation of the second difference.

6) For  $z \in D$

$$|f_{\lambda_2}^0 \circ \tilde{f}_1(z) - f_{\lambda_2}^0 \circ \tilde{f}_2(z)| \leq \|\mu_1 - \mu_2\|_p^\alpha$$

**Proof.** The estimate follows from step 2) and the well-known inequality  $|f_\nu(z_1) - f_\nu(z_2)| \leq C|z_1 - z_2|^{1-2/p}$ , which holds for any compactly supported  $\nu$ ,  $|\nu| \leq d$  and  $2 < p < P(d)$  (see, for example [Al]). □ □

Now we begin the proof of estimate (3.2). Analogously to the notations  $\mu_i, f_i$  we adopt the notations  $f_{wi}, Z_{wi}, h_{wi}$  and so on. At first we shall obtain estimates for the difference  $\mu_{h_{w1}} - \mu_{h_{w2}}$  and its derivatives.

**Proposition 16** a) Suppose  $|z| \leq R < 1$ . There is the estimate

$$\begin{aligned} & |(\mu_{h_{w1}} - \mu_{h_{w2}})_{(k)}(z)| \leq \\ & \leq \frac{C}{(1-R)^{2|k|}} \left[ \sup_{|l| \leq k} \sup_{D_{Rw}} |(\mu_1)_{(l)} - (\mu_2)_{(l)}| + \frac{1}{(1-R)^2} |\mu_1(w) - \mu_2(w)| \right], \end{aligned} \quad (3.12)$$

where

$$1 - R_w^2 = b(1 - |w|^2)(1 - R^2) \quad (3.13)$$

with some uniform  $b$  independent of  $R$ . In fact, it depends only on the maximal dilatation of  $f_i$  and we can put  $b = 1/3$  if  $\mu_1, \mu_2$  are small enough.

b) Let  $p \geq 1, R < 1$ . Then

$$\begin{aligned} \|\mu_{h_{w1}} - \mu_{h_{w2}}\|_p & \leq C \left[ \sup_{D_R} |\mu_1(z) - \mu_2(z)| + \left( \frac{1-R}{1-|w|} \right)^{1/p} + \right. \\ & \left. + |\mu_1(w) - \mu_2(w)|^{1/(2p+1)} \right]. \end{aligned} \quad (3.14)$$

In all these inequalities  $C$  is some uniform constant.

**Proof.** a) Recalling (2.10) we see that any derivative  $(\mu_{h_w})_{(k)}$  is a sum of items of the types

$$P_l(\mu_{f_w}, \bar{\mu}_0) \left( \frac{Z_{w,z}}{\bar{Z}_{w,z}} \right)_{(q)} (\mu_f)_{(l_1)}^{k_1} \cdots (\mu_f)_{(l_s)}^{k_s} \circ Z_w^{-1}(\cdot)_{(i_1)}^{j_1} \cdots (\cdot)_{(i_r)}^{j_r}, \quad (3.15)$$

where  $|(l_1)|k_1 + \dots + |(l_s)|k_s \leq |(k)|$ ,  $|(q)| \leq |(k)|$ ,  $|(i_1)|j_1 + \dots + |(i_r)|j_r \leq |(k)|$ ,  $P_l$  is the derivative of order  $l \leq |(k)|$  of the fraction

$$\frac{\mu_{f_w} - \mu_0}{1 - \bar{\mu}_0 \mu_{f_w}}$$

considered as a function of  $\mu_{f_w}$ . That is  $P_l$  is an uniformly bounded rational function of  $\mu_f, \bar{\mu}_0$ , and  $(\cdot)$  can be either  $Z_w^{-1}$  either  $\overline{Z_w^{-1}}$ . It follows that we can represent any difference  $(\mu_{h_{w1}})_{(k)} - (\mu_{h_{w2}})_{(k)}$  as a sum of terms of the types

$$(P_l(\mu_{f_{w1}}, \bar{\mu}_{1,0}) - P_l(\mu_{f_{w2}}, \bar{\mu}_{2,0}) \circ Z_{w1}^{-1}[\cdot]), \quad (3.16)$$

$$(P_l(\mu_{f_{w2}}, \bar{\mu}_{2,0}) \circ Z_{w1}^{-1} - P_l(\mu_{f_{w2}}, \bar{\mu}_{2,0}) \circ Z_{w2}^{-1})[\cdot], \quad (3.17)$$

$$((\mu_{f_{w1}})_{(l_i)}^{k_i} - (\mu_{f_{w2}})_{(l_i)}^{k_i}) \circ Z_{w1}^{-1}[\cdot], \quad (3.18)$$

$$((\mu_{f_{w2}})_{(l_i)}^{k_i} \circ Z_{w1}^{-1} - (\mu_{f_{w2}})_{(l_i)}^{k_i} \circ Z_{w2}^{-1}[\cdot]), \quad (3.19)$$

$$\left( \left( \frac{Z_{w1,z}}{Z_{w1,z}} \right)_{(l)} \circ Z_{w1}^{-1} - \left( \frac{Z_{w2,z}}{Z_{w2,z}} \right)_{(l)} \circ Z_{w2}^{-1} \right) [\cdot] \quad (3.20)$$

$$((Z_{w1}^{-1})_{(i_m)}^{j_m} - Z_{w2}^{-1})_{(i_m)}^{j_m}[\cdot], \quad (3.21)$$

where  $[\cdot]$  denotes each time a product of the multiples such as in (3.15) with omitted term corresponding to the written difference. All these multiples are either derivatives of  $Z_{wi}^{-1}$ ,  $i = 1, 2$ , either derivatives of  $\mu_{fi}$ ,  $i = 1, 2$  in the point  $Z_{wi}^{-1}(z)$ , either derivatives of  $P_l$  with respect to  $\mu_{fi}$ . The derivatives of  $Z_{wi}^{-1}$  and of  $P_l$  are uniformly bounded.

$Z_{wi}(z)$  is a real analytic function of  $\mu_{i,0}$  equal to  $z$  identically at  $\mu_{i,0} = 0$  and, hence, we can estimate the differences in (3.20) and (3.21) as  $C|\mu_{1,0} - \mu_{2,0}|$ . From the other hand, the products in the square brackets in these cases have the estimates

$$C(1 - |z|)^{-k_1|(l_1)| + \dots + k_s|(l_s)|} (|1 - wz|)^{-(k_1|(l_1)| + \dots + k_s|(l_s)|)} \leq C(1 - |z|)^{-2|(k)|}.$$

It follows from estimate (2.5). We obtain for terms (3.20), (3.21) the estimate

$$\frac{C}{(1 - R)^{2|(k)|}} |\mu_1(w) - \mu_2(w)| \quad (3.22)$$

The difference in (3.16) is no more than

$$C \sup_{D_R} |(\mu_{f_{w1}} - \mu_{f_{w2}}) \circ Z_{w1}^{-1}| = C \sup_{D_R} |(\mu_1 - \mu_2) \circ \varphi_w^{-1} \circ Z_{w1}^{-1}|.$$

Now we have

$$1 - |Z_{1,w}^{-1}(z)|^2 \geq C(1 - |z|^2) \quad (3.23)$$

with  $C$  depending only on  $d$ .

$$1 - |\varphi_w^{-1}(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2} \geq \frac{1}{2}(1 - |w|^2)(1 - |z|^2) \quad (3.24)$$

It implies

$$\sup_{D_R} |(\mu_{1,f_w} - \mu_{2,f_w}) \circ Z_{1,w}^{-1}| \leq C \sup_{D_{R_w}} |\mu_1 - \mu_2|,$$

where  $R_w$  is such that

$$1 - R_w^2 = b(1 - |w|^2)(1 - R^2)$$

with some  $b$  depending only  $C$  in the right side of (3.23), i.e. , on maximal dilatations of  $f_1, f_2$ . This constant  $C$  can be made close to 1 if  $\mu_1, \mu_2$  are small enough. In this case we can set  $b = 1/3$ .

The product in the square brackets in (3.16) has the same estimate as in (3.20), (3.21). We obtain for term (3.16) the estimate

$$\frac{C}{(1 - R)^{2|k|}} \sup_{D_{R_w}} |\mu_1 - \mu_2|. \quad (3.25)$$

Consider now term (3.17). The function  $P_l$  has bounded derivatives as a function of  $\mu_f$ , and  $Z_w^{-1}$  is an analytic function of  $\mu_0$  with a bounded derivative. Applying (2.5) and (3.23), we see that the difference in (3.17) is no more than

$$\begin{aligned} C \sup_{z \in D_R} \sup_{t \in [Z_{w_1}^{-1}(z), Z_{w_2}^{-1}(z)]} (|(\mu_{f_{w_2}})_t(t)| + |(\mu_{f_{w_2}})_{\bar{t}}(t)|) |\mu_{01} - \mu_{02}| &\leq \\ &\leq \frac{C}{(1 - R)^2} |\mu_1(w) - \mu_2(w)|. \end{aligned}$$

Since the product in the square brackets in (3.17) has the same estimate  $C(1 - |z|)^{-2k}$ , we obtain for term (3.17) the estimate

$$\frac{C}{(1 - R)^{2(k+1)}} |\mu_1(w) - \mu_2(w)|. \quad (3.26)$$

Considering term (3.18) we have

$$|((\mu_{f_{w1}})_{(l)}^k - (\mu_{f_{w2}})_{(l)}^k)| \leq C|p^{(k-1)}((\mu_{f_{w1}})_{(l)}, (\mu_{f_{w2}})_{(l)})| |(\mu_{f_{w1}})_{(l)} - (\mu_{f_{w2}})_{(l)}|,$$

where  $p^{(k-1)}$  is the homogeneous polynomial in the variables  $(\mu_{f_{w1}})_{(l)}, (\mu_{f_{w2}})_{(l)}$  of degree  $k-1$ . Thus for the difference in (3.18) we have the estimate

$$\frac{C}{(1-R)^{2|(l_i)|(k_i-1)}} \sup_{D_R} |(\mu_{f_{w1}})_{(l_i)} - (\mu_{f_{w2}})_{(l_i)} \circ Z_{w1}^{-1}|.$$

Now remind expression (2.7) for  $\mu_{f_w}$ . We have the obvious estimates

$$|(\varphi_w^{-1})_i(z)| \leq C|1 - z\bar{w}|^{-l}, |(r_w)_{(l)}(z)| \leq C|1 - z\bar{w}|^{-l}, \quad (3.27)$$

where we denote  $r_w = \left(\frac{1-\bar{z}w}{1-z\bar{w}}\right)^2$ .

We see that any derivative  $(\mu_{f_w})_{(l)}$  is a sum of items of the types

$$\cdot \mu_{(m)} \circ \varphi_w^{-1}(\cdot)_{(s_1)}^{r_1} \dots (\cdot)_{(s_q)}^{r_q} (r_w)_{(s)},$$

where  $|(m)| \leq |(l)| - |(s)|$ ,  $s_1 r_1 + \dots + s_q r_q = |(l)| - |(s)|$ , each  $(\cdot)$  can be either  $\varphi_w^{-1}$  either  $\varphi_w^{-1}$  and the corresponding derivative is either in  $z$  either in  $\bar{z}$ . Applying (3.27), we see that we can represent any difference  $(\mu_{f_{w1}})_{(l)}(z) - (\mu_{f_{w2}})_{(l)}(z)$  as a sum of terms with the estimates

$$C|1 - z\bar{w}|^{-|(l)|} \sup_{D_{Rw}} |(\mu_1)_{(m)} - (\mu_2)_{(m)}|, |(m)| \leq |(l)|.$$

Here we apply (3.24) and take into consideration that  $s_1 r_1 + \dots + s_q r_q + |(s)| = |(l)|$ . We obtain for the difference in (3.18) the estimate

$$C(1-R)^{-2|(l_i)|(k_i-1)-|(l_i)|} \sup_{m \leq l_i} \sup_{D_{Rw}} |(\mu_1)_{(m)} - (\mu_2)_{(m)}|$$

The product in the square brackets in (3.18) doesn't contain the term  $(\mu_{f_{wj}})_{(l_i)}^{k_i}$  and, hence, has the estimate:  $C(1-R)^{-2(|(k)|+2k_i|l_i|)}$ . We obtain for term (3.18) the estimate

$$\frac{C}{(1-R)^{(2|(k)|-|(l_i)|)}} \sup_{|(m)| \leq |(k)|} \sup_{D_{Rw}} |(\mu_1)_{(m)} - (\mu_2)_{(m)}| \quad (3.28)$$

At last, the difference in (3.19) is no greater, than

$$C|p^{(k_i-1)}((\mu_{f_{w2}})_{(l_i)} \circ Z_{w1}^{-1}, (\mu_{f_{w2}})_{(l_i)} \circ Z_{w2}^{-1})| |(\mu_{f_{w2}})_{(l_i)} \circ Z_{w1}^{-1} - (\mu_{f_{w2}})_{(l_i)} \circ Z_{w2}^{-1}|,$$

where  $p^{(k_i-1)}$  is the homogeneous polynomial in the variables  $(\mu_{f_{w_1}})_{(l_i)} \circ Z_{w_1}^{-1}, (\mu_{f_{w_2}})_{(l_i)} \circ Z_{w_2}^{-1}$  of degree  $k_i - 1$ . Thus we can estimate this difference as

$$\begin{aligned} & \frac{C}{(1-R)^{2|(l_i)|(k_i-1)}} \sup_{D_R} |(\mu_{f_{w_2}})_{(l_i)} \circ Z_{w_1}^{-1} - (\mu_{f_{w_2}})_{(l_i)} \circ Z_{w_2}^{-1}| \leq \\ & \leq \frac{C}{(1-R)^{2|(l_i)|(k_i-1)}} \sup_{z \in D_R} \sup_{[Z_{w_1}^{-1}(z), Z_{w_2}^{-1}(z)]} (|((\mu_{f_{w_2}})_{(l_i)})_t| + |((\mu_{f_{w_2}})_{(l_i)})_{\bar{t}}|) |\mu_1(w) - \mu_2(w)| \leq \\ & \leq \frac{C}{(1-R)^{2|(l_i)|(k_i-1)+2|(l_i)|+1}} |\mu_1(w) - \mu_2(w)| = \frac{C}{(1-R)^{2|(l_i)|k_i+1}} |\mu_1(w) - \mu_2(w)| \end{aligned}$$

Analogously to the previous case we obtain for term (3.19) the estimate

$$\frac{C}{(1-R)^{2(|(k)|+1)}} |\mu_1(w) - \mu_2(w)| \quad (3.29)$$

From (3.22), (3.25), (3.26), (3.28) and (3.29) follows (3.12).

b) Applying (2.10), we have

$$\begin{aligned} & \left( \int_D |\mu_{h_{w_1}}(z) - \mu_{h_{w_2}}(z)|^p dS_z \right)^{1/p} \leq C [|\mu_{01} - \mu_{02}| + \\ & + \left( \int_D |\mu_{f_{w_1}} \circ Z_{w_1}^{-1} - \mu_{f_{w_2}} \circ Z_{w_1}^{-1}|^p dS_z \right)^{1/p} + \left( \int_D |\mu_{f_{w_2}} \circ Z_{w_1}^{-1} - \mu_{f_{w_2}} \circ Z_{w_2}^{-1}|^p dS_z \right)^{1/p} + \\ & + \left( \int_D \left| \frac{Z_{w_1,z}}{Z_{w_1,z}} \circ Z_{w_1}^{-1} - \frac{Z_{w_2,z}}{Z_{w_2,z}} \circ Z_{w_2}^{-1} \right|^p dS_z \right)^{1/p} ]. \quad (3.30) \end{aligned}$$

Consider the first integral. We have

$$J_1 = \int_D |\mu_{f_{w_1}} \circ Z_{w_1}^{-1} - \mu_{f_{w_2}} \circ Z_{w_1}^{-1}|^p dS_z \leq C \int_D (|\mu_{f_{w_1}}(z) - \mu_{f_{w_2}}(z)|)^p dS_z,$$

since the Jacobian of  $Z_{w_1}$  is uniformly bounded from above and from below. Also, applying (2.7), we have

$$\int_D |\mu_{f_{w_1}}(z) - \mu_{f_{w_2}}(z)|^p dS_z = \int_D |\mu_1(z) - \mu_2(z)|^p J_{\varphi_w}(z) dS_z, \quad (3.31)$$

where  $J_{\varphi_w}$  is the Jacobian of the map  $\varphi_w$ . This Jacobian is equal to  $(1 - |w|^2)^2 / |1 - \bar{w}z|^4$  and there is the estimate

$$\int_{D \setminus D_R} |1 - \bar{w}z|^{-4} dS_z \leq 6\pi \frac{1 - R^2}{(1 - |w|^2)^3}.$$



Indeed,

$$\begin{aligned}
\int_{D \setminus D_R} |1 - \bar{w}z|^{-4} dS_z &= \int_R^1 \int_0^{2\pi} \frac{d\theta dr}{(1 + |w|^2 r^2 - 2|w|r \cos \theta)^2} = \\
&= 2\pi \int_R^1 \frac{r}{(1 - |w|^2 r^2)^2} \left(1 + \frac{2}{1 - |w|^2 r^2}\right) dr = \\
&= 2\pi(1 - R^2) \left( \frac{1}{(1 - |w|^2)(1 - R^2|w|^2)} + \frac{1}{(1 - |w|^2)(1 - R^2|w|^2)^2} + \frac{1}{(1 - |w|^2)^2(1 - R^2|w|^2)} \right) \leq \\
&\leq 6\pi \frac{1 - R^2}{(1 - |w|^2)^3}.
\end{aligned}$$

Further, the integral

$$\int_D J_{\varphi_w}(z) dS_z = (1 - |w|^2)^2 \int_D \frac{dS_z}{|1 - \bar{w}z|^4}$$

is uniformly bounded. Thus we obtain

$$\begin{aligned}
(J_1)^{1/p} &\leq C \sup_{D_R} |\mu_1(z) - \mu_2(z)| + C \left[ (1 - |w|^2)^2 \int_{D \setminus D_R} |1 - \bar{w}z|^{-4} dS_z \right]^{1/p} \leq \\
&\leq C \left[ \sup_{D_R} |\mu_1(z) - \mu_2(z)| + \left( \frac{1 - R}{1 - |w|} \right)^{1/p} \right] \quad (3.32)
\end{aligned}$$

for any  $R < 1$ . From the other hand, if  $|w| \leq 1/2$ , then from (3.31) follows the estimate

$$(J_1)^{1/p} \leq C \|\mu_1 - \mu_2\|_p. \quad (3.33)$$

Consider the second integral in (3.30). Applying estimate (2.5), we have for any  $r < 1$

$$\begin{aligned}
J_2 &= \int_D |\mu_{f_{w_2}} \circ Z_{w_1}^{-1} - \mu_{f_{w_1}} \circ Z_{w_2}^{-1}|^p dS_z \leq \\
&\leq C \int_{D_r} \sup_{[Z_{w_1}^{-1}(z), Z_{w_2}^{-1}(z)]} (|(\mu_{f_{w_2}})_t + (\mu_{f_{w_2}})_{\bar{t}}| |Z_{w_1}^{-1}(z) - Z_{w_2}^{-1}(z)|)^p dZ_z + C(1 - r) \leq \\
&\leq C((1 - r)^{-2p} |\mu_1(w) - \mu_2(w)|^p + (1 - r)).
\end{aligned}$$

That is,

$$(J_2)^{1/p} \leq C[(1 - 4)^{-2} |\mu_1(w) - \mu_2(w)| + (1 - R_1)^{1/p}].$$

If we put  $(1-r)^{-2}|\mu_1(w) - \mu_2(w)| = (1-r)^{1/p}$ , i.e.,  $1-r = |\mu_1(w) - \mu_2(w)|^{p/(2p+1)}$ , then we obtain

$$(J_2)^{1/p} \leq C|\mu_1(w) - \mu_2(w)|^{1/(2p+1)}. \quad (3.34)$$

At last, we estimate the third integral in (3.30) analogously to the difference in (3.20): it is no greater than

$$C|\mu_1(w) - \mu_2(w)|. \quad (3.35)$$

Collecting (3.30) and (3.33)-(3.35) we obtain (3.14).  $\square$

We also need an estimate for the norm of  $\mu_{F_{w_1}} - \mu_{F_{w_2}}$  in  $L_H^p$ . Remind that  $H$  is the stripe  $-\pi \leq \varphi \leq \pi$  and  $L_H^p$  is the space of  $\varphi$ -periodic functions on  $H$  with usual  $L^p$ -norm. In what follows we don't write the index  $H$  in our notations.

**Proposition 17** *Let  $p \geq 2$ ,  $R < 1$ . There is the estimate*

$$\begin{aligned} \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p \leq C & \left[ \sup_{D_{r_w}} (|\mu_{1,z} - \mu_{2,z}| + |\mu_{1,\bar{z}} - \mu_{2,\bar{z}}|) + \right. \\ & \left. + \sup_{D_R} |\mu_1(z) - \mu_2(z)| + \left( \frac{1-R}{1-|w|} \right)^{1/p} + |\mu_1(w) - \mu_2(w)|^{1/(2p+1)} \right], \quad (3.36) \end{aligned}$$

where  $1-r_w \geq b(1-|w|)$  with some  $b < 1$  depending only on  $d$ . If  $\mu_1, \mu_2$  are small enough, then we can take  $b = 1/4$ .

**Proof.** Remind equation (2.17). Since  $\mu_{h_{w_i}}(0) = 0$ ,  $i = 1, 2$ ,  $p > 2$  and the first derivatives of  $\mu_{h_{w_i}}$  are uniformly bounded on  $D_{1/2}$  (see (2.12)), we have

$$\begin{aligned} \int_H |\mu_{F_{w_1}}(\zeta) - \mu_{F_{w_2}}(\zeta)|^p dS_\zeta & \leq \int_D \frac{|\mu_{1,h_w}(z) - \mu_{2,h_w}(z)|^p}{|z|^2} dS_z \leq \\ & \leq \int_{|z| \leq 1/2} \sup_{|t| \leq |z|} (|\mu_{h_{w_1},t} - \mu_{h_{w_2},t}| + |\mu_{h_{w_1},\bar{t}} - \mu_{h_{w_2},\bar{t}}|)^p dS_z + \\ & \quad + \frac{1}{4} \int_{|z| \geq 1/2} |\mu_{h_{w_1}}(z) - \mu_{h_{w_2}}(z)|^p dS_z. \quad (3.37) \end{aligned}$$

The first integral in the right side has the obvious estimate

$$C \sup_{|z| \leq 1/2} (|\mu_{h_{w_1}, z} - \mu_{h_{w_2}, z}| + |\mu_{h_{w_1}, \bar{z}} - \mu_{h_{w_2}, \bar{z}}|).$$

Applying estimate (3.12) we see that the integral is no greater than

$$C[\sup_{D_{r_w}} (|\mu_{1,z} - \mu_{2,z}| + |\mu_{1,\bar{z}} - \mu_{2,\bar{z}}|) + |\mu_1(w) - \mu_2(w)|].$$

We apply estimates (3.14) to the second integral in (3.37). As a result we obtain estimate (3.36).  $\square$

Our next step will be the proof of estimate (3.2) for the derivatives of first order. It is a consequence of estimate (3.36) and the next proposition.

**Proposition 18** *Let  $f_i$ ,  $i = 1, 2$  be the normal solutions corresponding to the Beltrami coefficients  $\mu_1$  and  $\mu_2$  satisfying conditions of Theorem 1 with the same bounds  $b, b_1, b_2$ . There is the estimate*

$$|f_{1,z}(w) - f_{2,z}(w)| \leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{1w}} - \mu_{F_{2w}}\|_p^\alpha \right]$$

with some uniform  $C$  and some  $2 < p < P(d)$ ,  $\alpha < 1$  depending only on  $d, b_1, b_2$ . For the difference of  $\bar{z}$ -derivatives we have the analogous estimate.

**Proof.** By (2.15) and (2.28),

$$f_z(w) = \frac{1 - |w'|^2}{1 - |w|^2} (h_w)_z(0) (Z_w)_z(0) = \frac{\varphi_{w'}(1)}{\tilde{h}_w \circ Z_w^{-1} \circ \varphi_w(1)} \frac{1 - |w'|^2}{1 - |w|^2} (Z_w)_z(0) (\tilde{h}_w)_z(0). \quad (3.38)$$

Remind that  $w' = f(w)$  and  $\tilde{h}_w = \exp \circ \tilde{F}_w \circ \log$ , where  $\tilde{F}_w = f_{\lambda_{F_w}} \circ \tilde{f}_{\mu_{F_w}}$  is defined in subsection 5) of Section 2.

We can see that all multiples in (3.40) are uniformly bounded. For the multiple  $\frac{\varphi_{w'}(1)}{\tilde{h}_w \circ Z_w^{-1} \circ \varphi_w(1)}$  it follows from the notion that  $\varphi_w(1) = \frac{1-w}{1-\bar{w}}$  belongs to the unit circle and  $|\tilde{h}_w \circ Z_w^{-1}|$  is uniformly bounded from below on the unit circle. According to (3.38) we can represent the difference  $f_{1,z}(w) - f_{2,z}(w)$  as a sum of the terms

$$\frac{|w'_1|^2 - |w'_2|^2}{1 - |w|^2} O(1), \quad (3.39)$$

$$[(Z_{w_1}^{-1})_z(0) - (Z_{w_2})_z(0)] O(1), \quad (3.40)$$

$$(\varphi_{w'_1}(1) - \varphi_{w'_2}(1))O(1), \quad (3.41)$$

$$[\tilde{h}_{w1} \circ Z_{w1}^{-1} \circ \varphi_w(1) - \tilde{h}_{1w} \circ Z_{w2}^{-1} \circ \varphi_w(1)]O(1), \quad (3.42)$$

$$[\tilde{h}_{1w} \circ Z_{w2}^{-1} \circ \varphi_w(1)]^{-1} - \tilde{h}_{2w} \circ Z_{w2}^{-1} \circ \varphi_w(1)]O(1), \quad (3.43)$$

$$[(\tilde{h}_{w1})_z(0) - (\tilde{h}_{w2})_z(0)]O(1). \quad (3.44)$$

In fact, since the fractions  $(1 - |w'_i|)/(1 - |w|)$  are uniformly bounded, it is enough, instead of (3.41), to estimate

$$C \min \left[ 1, \frac{|w'_1 - w'_2|}{1 - |w|^2} \right].$$

The difference  $|w'_1 - w'_2| = f_1(w) - f_2(w)$  we estimate by inequality (3.1).

We can estimate term (3.40) as

$$C|\mu_1(w) - \mu_2(w)|.$$

For term (3.41) we have the estimate

$$C|w'_1 - w'_2|$$

To estimate term (3.42) we use the next proposition:

**Proposition 19** *Let  $F_\nu$  be the principal logarithmic solution corresponding to the coefficient  $\nu$ ,  $|\nu| \leq d < 1$ . Then*

$$|F_\nu(\zeta_1) - F_\nu(\zeta_2)| \leq C|\zeta_1 - \zeta_2|^{1-2/p}$$

for  $p \leq P(d)$ .

The proof is analogous to the proof for the principal solution in the classical case.  $\square$

Now, since  $\tilde{h}_w = \exp \circ f_{\lambda_{F_w}} \circ \tilde{f}_{\mu_{F_w}} \circ \log$  and  $|Z_{w1}^{-1}(z) - Z_{w2}^{-1}(z)| \leq C|\mu_1(w) - \mu_2(w)|$ , we obtain for term (3.42) the estimate

$$C|\mu_1(w) - \mu_2(w)|^{(1-2/p)^2}.$$

For term (3.43) we have the estimate

$$C \max_{\xi=0} |\tilde{F}_{w1}(\zeta) - \tilde{F}_{w2}(\zeta)|$$

and for term (3.44)

$$C \lim_{\xi \rightarrow -\infty} |\tilde{F}_{w1}(\zeta) - \tilde{F}_{w2}(\zeta)|.$$

We see that to prove Proposition 18 it remains to estimate the difference

$$\tilde{F}_{w1} - \tilde{F}_{w2} = (f_{\lambda_{F_{1w}}} \circ \tilde{f}_{\mu_{F_{1w}}} - f_{\lambda_{F_{2w}}} \circ \tilde{f}_{\mu_{F_{1w}}}) + (f_{\lambda_{F_{2w}}} \circ \tilde{f}_{\mu_{F_{1w}}} - f_{\lambda_{F_{2w}}} \circ \tilde{f}_{\mu_{F_{2w}}}). \quad (3.45)$$

We proceed analogously to the proof of estimate (3.1) but we use now the principal logarithmic solutions. Instead of Proposition 15 we have

**Proposition 20** *Let  $F_{\nu_1}$  and  $F_{\nu_2}$  be the principal logarithmic solutions corresponding to the coefficients  $\nu_1$  and  $\nu_2$ ,  $|\nu_i| \leq d < 1$ . Then, for  $\zeta \in H$ ,*

$$|F_{\nu_1}(\zeta) - F_{\nu_2}(\zeta)| \leq C \|(\nu_1 - \nu_2)(|(F_{\nu_1})_\zeta - 1| + |(F_{\nu_2})_\zeta - 1|\|_p,$$

$$\|(F_{\nu_1})_\zeta - (F_{\nu_2})_\zeta\|_p \leq C \|(\nu_1 - \nu_2)(|(F_{\nu_1})_\zeta - 1| + |(F_{\nu_2})_\zeta - 1|\|_p,$$

$$|F_{\nu_1}^{-1}(\zeta) - F_{\nu_2}^{-1}(\zeta)| \leq C \|(\nu_1 - \nu_2)(|(F_{\nu_1})_\zeta - 1| + |(F_{\nu_2})_\zeta - 1|\|_p$$

for  $2 \leq p \leq P(d)$ .

**Proof.** The proof of is completely analogous to the proof of Proposition 15, we only change  $\mathcal{C}$  and  $\mathcal{S}$  to  $P_h$  and  $T_H$  correspondingly.  $\square$

**Proposition 21** *Suppose  $|\mu_{F_{wi}}| \leq d$ ,  $|\lambda_{F_{wi}}| \leq d$ ,  $i = 1, 2$ . Then*

$$|f_{\lambda_{F_{w1}}} - f_{\lambda_{F_{w2}}}| \leq C \|\lambda_{F_{w1}} - \lambda_{F_{w2}}\|_p^\alpha,$$

$$\|(f_{\mu_{F_{w1}}})_\zeta - (f_{\mu_{F_{w2}}})_\zeta\|_p \leq C \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha,$$

$$|\tilde{f}_{\mu_{F_{w1}}} - \tilde{f}_{\mu_{F_{w2}}}| \leq C \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha,$$

$$|\tilde{f}_{\mu_{F_{w1}}}^{-1} - \tilde{f}_{\mu_{F_{w2}}}^{-1}| \leq C \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha,$$

where  $2 < p < P(d)$  and  $\alpha$  depend only on  $d$ , and  $C$  is uniform.

**Proof.** Consider the first estimate. By Proposition 20 we must estimate  $\|(\lambda_{F_{w1}} - \lambda_{F_{w2}})(f_{\lambda_{F_{wi}}})_\zeta - 1\|_p$ . But  $(f_{\lambda_{F_{wi}}})_\zeta - 1$  belongs to  $L^p$  with the norm bounded by a constant depending only on the maximal dilatation of  $\lambda_{F_{wi}}$  if  $p < P(d)$ . Also, we have estimate (2.33). Fix some  $r < 1$  and some  $p' > p, p' < P(d)$ . We have

$$\|(\lambda_{F_{w1}} - \lambda_{F_{w2}})(f_{\lambda_{F_{wi}}})_\zeta - 1\|_p \leq C \left[ r^{-3} \left( \int_{|\operatorname{Re}[\tilde{f}_{\mu_{F_{wi}}}^{-1}(\zeta)]| \geq r} |\lambda_{F_{w1}} - \lambda_{F_{w2}}|^p dS_\zeta \right)^{1/p} + \right.$$

$$\begin{aligned}
& + C(\|(f_{\lambda_{F_{w_i}}})_\zeta - 1\|_{p'}[\text{mes}\{H \cap \{|\text{Re}[\tilde{f}_{\mu_{F_{w_1}}^{-1}}(\zeta)]| \leq r\}\}]^{\frac{1}{p}(1-p/p')}) \leq \\
& \leq C \left[ r^{-3} \|\lambda_{F_{w_1}} - \lambda_{F_{w_2}}\|_p + \left( \int_{H \cap \{|\xi| \leq r\}} J(\tilde{f}_{\mu_{F_{w_1}}}, \zeta) dS_\zeta \right)^{\frac{p'-p}{pp'}} \right].
\end{aligned}$$

Since the restriction of the Jacobian  $J(\tilde{f}_{\mu_{F_{w_i}}})$  on the domain  $H \cap \{|\xi| \leq 1\}$  belongs to  $L^{p/2}$  with an uniform norm, we obtain for the last integral the estimate

$$\left( \int_H J^{p/2} \right)^{2/p} [\text{mes}\{H \cap \{-r \leq \xi \leq 0\}\}]^{1-2/p} \leq Cr^{1-2/p}.$$

Now we put  $r^{-3} \|\lambda_{F_{w_1}} - \lambda_{F_{w_2}}\|_p = r^{\frac{(p-2)(p'-p)}{p'p^2}}$ , i.e,  $r = (\|\lambda_{F_{w_1}} - \lambda_{F_{w_2}}\|_p)^{\frac{p'p^2}{3p'p^2 + (p-2)(p'-p)}}$ . We obtain the estimate

$$\|(\lambda_{F_{w_1}} - \lambda_{F_{w_2}})(f_{\lambda_{F_{w_1}}})_\zeta\|_p \leq C(\|\lambda_{F_{w_1}} - \lambda_{F_{w_2}}\|_p)^{\frac{(p-2)(p'-p)}{3p'p^2 + (p-2)(p'-p)}}.$$

Analogously, to obtain the second estimate of the proposition we proceed applying (2.34) and the second inequality of Proposition 20

$$\begin{aligned}
\|(f_{\mu_{F_{w_1}}})_\zeta - (f_{\mu_{F_{w_2}}})_\zeta\|_p & \leq C \left[ r^{-3} \left( \int_{|\xi| \leq r} |\mu_{F_{w_1}} - \mu_{F_{w_2}}|^p dS_\zeta \right)^{1/p} + \right. \\
& \left. + \|(f_{\mu_{F_{w_1}}})_\zeta - 1\|_{p'} r^{\frac{1}{p}(1-p/p')} \right] \leq C(r^{-3} \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p + r^{\frac{p'-p}{pp'}})
\end{aligned}$$

for some  $r < 1$  and some  $p' > p, p' < P(d)$ . Putting  $r = \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\frac{pp'}{3pp' + p' - p}}$  we obtain

$$\|(f_{\mu_{F_{w_1}}})_\zeta - (f_{\mu_{F_{w_2}}})_\zeta\|_p \leq C \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\frac{p'-p}{3pp' + p' - p}}.$$

The prove of the last two estimates of the proposition is analogous.  $\square$

Return now to the proof of Proposition 18. To estimate the first difference in (3.45) we apply Propositions 20 and 21. We must estimate the norm  $\|\lambda_{F_{w_1}} - \lambda_{F_{w_2}}\|_p$ . We have

$$\|\lambda_{F_{w_1}} - \lambda_{F_{w_2}}\|_p \leq C \left[ \|\mu_{F_{w_1}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1} - \mu_{F_{w_2}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1}\|_p + \|\mu_{F_{w_2}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1} - \mu_{F_{w_2}} \circ \tilde{f}_{\mu_{F_{w_2}}}^{-1}\|_p + \right.$$

$$+ \left\| \frac{\overline{f_{\mu_{F_{w_1}, \zeta}}}}{f_{\mu_{F_{w_1}, \zeta}}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}}) - \frac{\overline{f_{\mu_{F_{w_2}, \zeta}}}}{f_{\mu_{F_{w_2}, \zeta}}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}}) \right\|_p + \left\| \frac{\overline{f_{\mu_{F_{w_2}, \zeta}}}}{f_{\mu_{F_{w_2}, \zeta}}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}}) - \frac{\overline{f_{\mu_{F_{w_2}, \zeta}}}}{f_{\mu_{F_{w_2}, \zeta}}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_2}}^{-1}}}) \right\|_p \Bigg]. \quad (3.46)$$

Consider the first difference in the right side. Since  $\exp \circ \tilde{f}_{\mu_{F_{w_1}}} \circ \log$  is a homeomorphism of the plane, we can write

$$\begin{aligned} & \|\mu_{F_{w_1}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1} - \mu_{F_{w_2}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1}\|_p^p = \int_{\tilde{f}_{\mu_{F_{w_1}}}^{-1}(H)} |\mu_{F_{w_1}}(\zeta) - \mu_{F_{w_2}}(\zeta)|^p J(\tilde{f}_{\mu_{F_{w_1}}}, \zeta) dS_\zeta = \\ & = \int_{\mathbb{C}} |\mu_{F_{w_1}}(\log z) - \mu_{F_{w_2}}(\log z)|^p J(\tilde{f}_{\mu_{F_{w_1}}}, \log z) |z|^{-2} dS_z = \int_H |\mu_{F_{w_1}}(\zeta) - \mu_{F_{w_2}}(\zeta)|^p J(\tilde{f}_{\mu_{F_{w_1}}}, \zeta) dS_\zeta \end{aligned}$$

From (2.32) follows the estimate

$$|J(\tilde{f}_{\mu_{F_{w_1}}}, \zeta)| \leq C(|\xi|^{-6} + 1). \quad (3.47)$$

Also, restriction of  $J(\tilde{f}_{\mu_{F_{w_1}}})$  on the domain  $H \cap \{|\xi| \leq 1\}$  belongs to  $L^{p/2}$  with an uniform estimate. We apply our usual method. We have for any  $r < 1$

$$\begin{aligned} \|\mu_{F_{w_1}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1} - \mu_{F_{w_2}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1}\|_p^p & \leq C \left[ r^{-6} \int_{H \cap \{|\xi| \geq r\}} |\mu_{F_{w_1}}(\zeta) - \mu_{F_{w_2}}(\zeta)|^p dS_\zeta + \right. \\ & \left. + \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_C^p (\text{mes}\{H \cap \{|\xi| \leq r\}\})^{1-2/p} \right]. \end{aligned}$$

We determine  $r$  from the equation  $r^{-6} \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^p = r^{1-2/p}$ . I.e., we put  $r = \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\frac{p^2}{7p-2}}$  and obtain

$$\|\mu_{F_{w_1}} \circ \tilde{f}_{\mu_{F_{w_1}}}^{-1} - \mu_{F_{w_2}} \circ \tilde{f}_{\mu_{F_{w_2}}}^{-1}\|_p^p \leq C \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\frac{p(p-2)}{7p-2}}. \quad (3.48)$$

Consider the second term in the right side of (3.46). We can assume that  $\|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p$  is small with some uniform estimate that we shall specify below. Applying (2.11), (2.17) and the obvious estimate for  $Z = Z_w^{-1}(z)$ :  $c(1 - |z|) \leq 1 - |Z| \leq C(1 - |z|)$  with uniform  $c, C$ , we get the estimate  $|(\mu_{F_{w_i}})_{(l)}(\zeta)| \leq C e^\xi (1 + |\xi|^{-2})$  if  $|(l)| = 1$ ,  $i = 1, 2$ . By the last inequality of Proposition 21,

$$|\tilde{f}_{\mu_{F_{w_1}}}^{-1}(\zeta) - \tilde{f}_{\mu_{F_{w_2}}}^{-1}(\zeta)| \leq C \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^\alpha.$$

Suppose

$$2C\|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha \leq r < 1. \quad (3.49)$$

Applying estimate (3.47), we can write

$$\begin{aligned} & \int_{\operatorname{Re}[\tilde{f}_{\mu_{F_{w1}}}^{-1}(\zeta)] \leq -r} |\mu_{F_{w2}} \circ \tilde{f}_{\mu_{F_{w1}}}^{-1}(\zeta) - \mu_{F_{w2}} \circ \tilde{f}_{\mu_{F_{w2}}}^{-1}(\zeta)|^p dS_\zeta \leq \\ & \leq C \int_{H \cap \{\xi \leq -r\}} e^{p\xi} (1+|\xi|^{-2})^p (|\xi|^{-6}+1) |\tilde{f}_{\mu_{F_{w1}}}^{-1}(\zeta) - \tilde{f}_{\mu_{F_{w2}}}^{-1}(\zeta)|^p dS_\zeta \leq Cr^{-2p-5} \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^{p\alpha}, \end{aligned}$$

where  $\alpha$  is the same as in Proposition 21. If  $r$  satisfies conditions (3.49), we have

$$\begin{aligned} & \|\mu_{F_{w2}} \circ \tilde{f}_{\mu_{F_{w1}}}^{-1} - \mu_{F_{w2}} \circ \tilde{f}_{\mu_{F_{w2}}}^{-1}\|_p \leq \\ & \leq C[r^{-2-5/p} \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha + C[\operatorname{mes}\{H \cap \{|\operatorname{Re}[\tilde{f}_{\mu_{F_{w1}}}^{-1}(\zeta)]\} \leq 3r/2\}]]^{1/p}. \end{aligned}$$

Since the restriction of the Jacobian  $J(\tilde{f}_{\mu_{F_{w1}}}^{-1})$  on the domain  $H \cap \{|\xi| \leq 1\}$  belongs to  $L^{p/2}$ , we estimate the last integral as

$$\left( \int_H J^{p/2} \right)^{2/p} [\operatorname{mes}\{H \cap \{-r \leq \xi \leq 0\}\}]^{1-2/p} \leq Cr^{1-2/p}.$$

Now we put  $r^{-2-5/p} (\|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha = r^{1/p-2/p^2})$ , i.e.,  $r = \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^{\alpha \frac{p^2}{2p^2+6p-2}}$ . We see that conditions (3.49) are satisfied at  $\|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p$  small enough. We obtain

$$\|\mu_{F_{w2}} \circ \tilde{f}_{\mu_{F_{w1}}}^{-1} - \mu_{F_{w2}} \circ \tilde{f}_{\mu_{F_{w2}}}^{-1}\|_p \leq C \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^{\beta \frac{p-2}{2p^2+6p-2}}. \quad (3.50)$$

Consider the third term in (3.46). We have

$$\frac{\overline{f_{\mu_{F_{w1}, \zeta}}}}{f_{\mu_{F_{w1}, \zeta}}} - \frac{\overline{f_{\mu_{F_{w2}, \zeta}}}}{f_{\mu_{F_{w2}, \zeta}}} = \frac{\overline{f_{\mu_{F_{w1}, \zeta}} - f_{\mu_{F_{w2}, \zeta}}}}{f_{\mu_{F_{w1}, \zeta}}} - \frac{\overline{f_{\mu_{F_{w2}, \zeta}}}}{f_{\mu_{F_{w2}, \zeta}}} \frac{f_{\mu_{F_{w1}, \zeta}} - f_{\mu_{F_{w2}, \zeta}}}{f_{\mu_{F_{w1}, \zeta}}}.$$

For  $f_{\mu_{F_{w1}, \zeta}}$  we have estimate (2.34). Acting as above, we can see that the third term in (3.48) is no greater than

$$C \left[ r^{-1} \left( \int_{H \cap \{|\operatorname{Re}[\tilde{f}_{\mu_{F_{w1}}}^{-1}(\zeta)]\} \geq r\}} |(f_{\mu_{F_{w1}, \zeta}} - f_{\mu_{F_{w2}, \zeta}}) \circ (-\overline{\tilde{f}_{\mu_{F_{w1}}}^{-1}})|^p dS_\zeta \right)^{1/p} + r^{1/p-2/p^2} \right].$$



for any  $0 < r < 1$ . Applying estimate (3.47), we obtain

$$\begin{aligned} & \int_{H \cap \{\operatorname{Re}[\tilde{f}_{\mu_{F_{w_1}}^{-1}}(\zeta)] \geq r\}} |(f_{\mu_{F_{w_1}, \zeta}} - f_{\mu_{F_{w_2}, \zeta}}) \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}})|^p dS_\zeta = \\ &= \int_{H \cap \{|\xi| \geq r\}} |f_{\mu_{F_{w_1}, \zeta}} - f_{\mu_{F_{w_2}, \zeta}}|^p |J(\tilde{f}_{\mu_{F_{w_1}}, \zeta})| dS_\zeta \leq Cr^{-6} \int_H |f_{\mu_{F_{w_1}, \zeta}} - f_{\mu_{F_{w_2}, \zeta}}|^p dS_\zeta. \end{aligned}$$

Applying the second inequality of Proposition 21, we see that the third term in (3.46) satisfies the estimate

$$C(r^{-(1+6/p)} \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^\alpha + r^{1/p-2/p^2}),$$

where  $\alpha$  is the same as in Proposition 21. If we put  $r^{-(1+6/p)} \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^\alpha = r^{1/p-2/p^2}$ , i.e.,  $r = \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\frac{\alpha p^2}{p^2+7p-2}}$ , we obtain the estimate

$$\left\| \frac{\overline{f_{\mu_{F_{w_1}, \zeta}}}}{f_{\mu_{F_{w_1}, \zeta}}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}}) - \frac{\overline{f_{\mu_{F_{w_2}, \zeta}}}}{f_{\mu_{F_{w_2}, \zeta}}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}}) \right\|_p \leq C \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\beta \frac{p-2}{p^2+7p-2}} \quad (3.51).$$

Consider the last term in (3.46). Denote  $\zeta_i = -\overline{\tilde{f}_{\mu_{F_{w_i}}^{-1}}(\zeta)}$ ,  $i = 1, 2$ . We must estimate  $p$ -norm of the sum

$$\frac{\overline{f_{\mu_{F_{w_2}, \zeta}}(\zeta_1)} - \overline{f_{\mu_{F_{w_2}, \zeta}} \circ (\zeta_2)}}{f_{\mu_{F_{w_2}, \zeta}}(\zeta_1)} + \frac{\overline{f_{\mu_{F_{w_2}, \zeta}} \circ (\zeta_2)} f_{\mu_{F_{w_2}, \zeta}}(\zeta_2) - f_{\mu_{F_{w_2}, \zeta}}(\zeta_1)}{f_{\mu_{F_{w_2}, \zeta}}(\zeta_2) f_{\mu_{F_{w_2}, \zeta}}(\zeta_1)}.$$

Let  $r$  be such that

$$|\zeta_1 - \zeta_2| \leq r/2. \quad (3.52)$$

Analogously to the previous case we obtain for this  $p$ -norm the estimate

$$C \left[ r^{-1} \left( \int_{|\operatorname{Re}[\tilde{f}_{\mu_{F_{w_1}}^{-1}}(\zeta)]| \geq r} |f_{\mu_{F_{w_2}, \zeta}}(\zeta_1) - f_{\mu_{F_{w_2}, \zeta}}(\zeta_2)|^p dS_\zeta \right)^{1/p} + r^{1/p-2/p^2} \right]. \quad (3.53)$$

Now, by Proposition 21 and the estimate of Proposition 13,

$$\begin{aligned} & |f_{\mu_{F_{w_2}, \zeta}}(\zeta_1) - f_{\mu_{F_{w_2}, \zeta}}(\zeta_2)| \leq \\ & \leq \sup_{\zeta \in [\zeta_1, \zeta_2], |\ell|=2} |(f_{\mu_{F_{w_2}}(\zeta))(\ell)| |\zeta_1 - \zeta_2| \leq Ce^\xi (1 + r^{-6}) \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^\alpha. \end{aligned}$$

Applying (3.47), we obtain for the integral in (3.53) the estimate

$$Cr^{-7-6/p}\|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^\alpha.$$

Now we determine  $r$  from the equation  $r^{-7-6/p}\|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^\alpha = r^{1/p-2/p^2}$ , that is,  $r = \|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\alpha \frac{p^2}{7p^2+7p-2}}$ . In this case  $\|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p$  is small by comparison with  $r$  and condition (3.52) is satisfied. We obtain the estimate

$$\left\| \frac{\overline{f_{\mu_{F_{w_2}}, \zeta}}}{f_{\mu_{F_{w_2}}, \zeta}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_1}}^{-1}}}) - \frac{\overline{f_{\mu_{F_{w_2}}, \zeta}}}{f_{\mu_{F_{w_2}}, \zeta}} \circ (-\overline{\tilde{f}_{\mu_{F_{w_2}}^{-1}}}) \right\|_p \leq C\|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\alpha \frac{p-2}{7p^2+7p-2}}. \quad (3.54)$$

Collecting (3.48), (3.50), (3.51), and (3.54), we obtain the estimate for the first difference in the right side of (3.45).

It remains to estimate the second difference in (3.45). From Propositions 19 and 21 follows

$$\begin{aligned} & |f_{\lambda_{F_{w_2}}} \circ \tilde{f}_{\mu_{F_{w_1}}}(\zeta) - f_{\lambda_{F_{w_2}}} \circ \tilde{f}_{\mu_{F_{w_2}}}(\zeta)| \leq \\ & \leq C|\tilde{f}_{\mu_{F_{w_1}}}(\zeta) - \tilde{f}_{\mu_{F_{w_2}}}(\zeta)|^{1-2/p} \leq C(\|\mu_{F_{w_1}} - \mu_{F_{w_2}}\|_p^{\alpha(1-2/p)}). \end{aligned}$$

It finishes the proof of Proposition 18.  $\square$

Now we begin the prove of estimate (3.2) for derivatives of order higher than one. We use notations of step 7) of Section 2 with obvious modifications, in other words,  $g_{wi}^{(k)}$ ,  $i = 1, 2$  instead of  $g_w^{(k)}$ ,  $S_i^{(k)} = s_{0i} \circ s_{1i} \circ \dots \circ \tilde{s}_{ki}$  instead of  $S$  and so on. Remind that  $g_{wi}^{(k)}(z) = h_{wi}(z)$  at  $|z| \geq 1/2$ .

**Proposition 22** *There are the estimates*

a) *If  $|l| \leq k + 1$ ,  $|z| \leq 1/2$ , then*

$$\begin{aligned} & |(\mu_{g_{w_1}^{(k)}})_{(l)}(z) - (\mu_{g_{w_2}^{(k)}})_{(l)}(z)| \leq C(\sup_{D_{1/2}} \sup_{|(q)| \leq |l|} |(\mu_{h_{w_1}})_{(q)} - (\mu_{h_{w_2}})_{(q)}| + \\ & + \max_{|(m)| \leq k} |(h_{w_1})_{(m)}(0) - (h_{w_2})_{(m)}(0)| + \max_{|(m)|=k} |(\mu_{h_{w_1}})_{(m)}(0) - (\mu_{h_{w_2}})_{(m)}(0)|). \end{aligned}$$

b) *If  $|l| \leq k + 2$ , then*

$$|(S_1^{(k)})_{(l)} - (S_2^{(k)})_{(l)}| \leq C(\max_{|(m)| \leq k} |(h_{w_1})_{(m)}(0) - (h_{w_2})_{(m)}(0)| + \max_{|(m)|=k} |(\mu_{h_{w_1}})_{(m)}(0) - (\mu_{h_{w_2}})_{(m)}(0)|).$$

c) *For  $p \geq 1$*

$$\begin{aligned} & \|\mu_{g_{w_1}^{(k)}} - \mu_{g_{w_2}^{(k)}}\|_p \leq \|\mu_{h_{w_1}} - \mu_{h_{w_2}}\|_p + \\ & + C(\max_{|(l)| \leq k} |(h_{w_1})_{(l)}(0) - (h_{w_2})_{(l)}(0)| + \max_{|(l)|=k} |(\mu_{h_{w_1}})_{(l)}(0) - (\mu_{h_{w_2}})_{(l)}(0)|). \end{aligned}$$

**Proof.** a) and b) Remind that  $g_w^{(k)} = h_w \circ S^{(k)}$ . We have

$$\mu_{g_w^{(k)}} = \frac{\mu_{h_w} - \mu_{S^{(k)}}}{1 - \bar{\mu}_{S^{(k)}} \mu_{h_w}} \cdot \frac{(S^{(k)})_z}{(S^{(k)})_z} \circ (S^{(k)})^{-1}.$$

Any derivative  $(\mu_{g_w})_{(m)}$  is a sum of items of the types

$$P_a(\mu_{h_w}, \mu_{S^{(k)}}, \bar{\mu}_{S^{(k)}}) \left( \frac{(S^{(k)})_z}{(S^{(k)})_z} \right)_{(l)} (\mu_{[\cdot]}^{k_1})_{(l_1)} \dots (\mu_{[\cdot]}^{k_s})_{(l_s)} \circ (S^{(k)})^{-1} (\cdot)_{(i_1)}^{j_1} \dots (\cdot)_{(i_r)}^{j_r}, \quad (3.55)$$

where  $|l| + |l_1|k_1 + \dots + |l_s|k_s \leq |m|$ ,  $P_a$  is an uniformly bounded rational function of  $\mu_{h_w}, \mu_S, \bar{\mu}_S$ ,  $[\cdot]$  can be any function from the tuple  $h_w, S^{(k)}, \bar{S}^{(k)}$ , any  $(\cdot)$  can be either  $(S^{(k)})^{-1}$  either  $(\bar{S}^{(k)})^{-1}$ , and  $|(1_1)|j_1 + \dots + |(i_r)|j_r \leq |m|$ . It follows that we can represent any difference  $(\mu_{g_{w1}}^{(k)})_{(m)} - (\mu_{g_{w2}}^{(k)})_{(m)}$  as a sum of terms of the types

$$(P_a(\mu_{h_{w1}}, \mu_{S_1^{(k)}}, \bar{\mu}_{S_1^{(k)}}) - P_a(\mu_{h_{w2}}, \mu_{S_2^{(k)}}, \bar{\mu}_{S_2^{(k)}})) \circ (S_1^{(k)})^{-1}[\cdot], \quad (3.56)$$

$$(P_a(\mu_{h_{w2}}, \mu_{S_2^{(k)}}, \bar{\mu}_{S_2^{(k)}}) \circ (S_1^{(k)})^{-1} - P_a(\mu_{h_{w2}}, \mu_{S_2^{(k)}}, \bar{\mu}_{S_2^{(k)}}) \circ (S_2^{(k)})^{-1})[\cdot], \quad (3.57)$$

terms

$$((\mu_{h_{w1}})_{(l_i)}^{k_i} - (\mu_{h_{w2}})_{(l_i)}^{k_i}) \circ (S_1^{(k)})^{-1}[\cdot], \quad (3.58)$$

$$((\mu_{h_{w2}})_{(l_i)}^{k_i} \circ (S_1^{(k)})^{-1} - (\mu_{h_{w2}})_{(l_i)}^{k_i}) \circ (S_2^{(k)})^{-1}[\cdot], \quad (3.59)$$

$$((\mu_{S_1^{(k)}})_{(l_i)}^{k_i} - (\mu_{S_2^{(k)}})_{(l_i)}^{k_i}) \circ (S_1^{(k)})^{-1}[\cdot], \quad (3.60)$$

$$((\mu_{S_2^{(k)}})_{(l_i)}^{k_i} \circ (S_1^{(k)})^{-1} - (\mu_{S_2^{(k)}})_{(l_i)}^{k_i}) \circ (S_2^{(k)})^{-1}[\cdot], \quad (3.61)$$

the analogous terms with  $\bar{\mu}_{S_1^{(k)}}, \bar{\mu}_{S_2^{(k)}}$ , and

$$\left[ \left( \frac{(S_1^{(k)})_z}{(S_1^{(k)})_z} \right)_{(l)} \circ (S_1^{(k)})^{-1} - \left( \frac{(S_2^{(k)})_z}{(S_2^{(k)})_z} \right)_{(l)} \circ (S_2^{(k)})^{-1} \right] [\cdot] \quad (3.62)$$

$$(((S_1^{(k)})^{-1})_{(i_q)}^{j_q} - ((S_2^{(k)})^{-1})_{(i_q)}^{j_q})[\cdot], \quad (3.63)$$

where  $[\cdot]$  denotes each time the product of the multiples such as in (3.55), where we omit the term corresponding to the written difference. All these multiples are either derivatives of  $(S_j^{(k)})^{-1}, j = 1, 2$ , either derivatives of

$\mu_{h_{w_i}}, \mu_{S_i^{(k)}}, \bar{\mu}_{S_i^{(k)}}, i = 1, 2$  in the point  $(S_j^{(k)})^{-1}(z)$ , either derivatives of  $P_l$  with respect to  $\mu_{h_{w_j}}, \mu_{S_j^{(k)}}$  or  $\bar{\mu}_{S_j^{(k)}}$ . All these derivatives are uniformly bounded. For derivatives of  $\mu_{h_{w_j}}$  it holds because  $z$  belongs to the disk  $D_{1/2}$  and for derivatives of  $\mu_{S_i^{(k)}}, \bar{\mu}_{S_i^{(k)}}$  it follows from Proposition 11.

For terms (3.56) -(3.63) we have the estimates:

For term (3.56)

$$\leq C \sup_{D_{1/2}} (|\mu_{h_{w_1}} - \mu_{h_{w_2}}| + |\mu_{S_1^{(k)}} - \mu_{S_2^{(k)}}|). \quad (3.56')$$

For term (3.57)

$$\leq C \sup_{D_{1/2}} |(S_1^{(k)})^{-1}(z) - (S_2^{(k)})^{-1}(z)|. \quad (3.57')$$

For term (3.58)

$$\leq C \sup_{D_{1/2}} (|(\mu_{h_{w_1}})_{(l_i)} - (\mu_{h_{w_2}})_{(l_i)}|). \quad (3.58')$$

For term (3.59)

$$\leq C \sup_{D_{1/2}} |(S_1^{(k)})^{-1}(z) - (S_2^{(k)})^{-1}(z)|. \quad (3.59')$$

For term (3.60):

$$\leq C \sup_{D_{1/2}} |(\mu_{S_1^{(k)}})_{(l_i)} - (\mu_{S_2^{(k)}})_{(l_i)}|. \quad (3.60')$$

For term (3.61)

$$\leq C |(S_1^{(k)})^{-1}(z) - (S_2^{(k)})^{-1}(z)|. \quad (1.61')$$

For term (3.62)

$$\leq C \sup_{D_{1/2}} \left( \sup_{|(q)|=|l|+1} |(S_1^{(k)})_{(q)} - (S_2^{(k)})_{(q)}| + |(S_1^{(k)})^{-1}(z) - (S_2^{(k)})^{-1}(z)| \right). \quad (3.62')$$

For term (3.63)

$$\leq C \sup_{D_{1/2}} |((S_1^{(k)})^{-1})_{(i_q)} - ((S_2^{(k)})^{-1})_{(i_q)}|. \quad (3.63')$$

Here we take into consideration that all derivatives of  $S_i^{(k)}, (S_i^{(k)})^{-1}$  and  $\mu_{S_i^{(k)}}, \mu_{h_{w_i}}$  are uniformly bounded on  $D_{1/2}$ . Terms (3.56') and (3.58') yield the term  $\sup_{D_{1/2}} \sup_{|(q)| \leq |l|} |(\mu_{h_{w_1}})_{(q)} - (\mu_{h_{w_2}})_{(q)}|$  in the estimate of  $|(\mu_{g_{w_1}^{(k)}})_{(l)}(z) - (\mu_{g_{w_2}^{(k)}})_{(l)}(z)|$ .

We must estimate  $|((S_1^{(k)})^{-1})_{(l)} - ((S_2^{(k)})^{-1})_{(l)}|, |(l)| \leq k + 1$ ,  $|(S_1^{(k)})_{(l)} - (S_2^{(k)})_{(l)}|, |(l)| \leq k+2$ , and  $|(\mu_{S_1^{(k)}})_{(l)} - (\mu_{S_2^{(k)}})_{(l)}|, |(l)| \leq k+1$  on  $D_{1/2}$ . Consider at first the difference  $|(S_1^{(k)})_{(l)} - (S_2^{(k)})_{(l)}|$ .

The map  $S_j^{(k)}$  is the composition

$$S_j = s_{0j} \circ s_{1j} \dots \circ s_{(k-1)j} \circ \tilde{s}_{kj}.$$

We can write the difference  $(S_1^{(k)})_{(l)} - (S_2^{(k)})_{(l)}$  as a sum of terms of the types

$$((s_{i1})_{(l_i)} - (s_{i2})_{(l_i)}) \circ s_{(i+1)1} \circ \dots \circ \tilde{s}_{k1}[\cdot] \quad (3.64)$$

and

$$((s_{i2})_{(l_i)}) \circ s_{(i+1)1} \circ \dots \circ \tilde{s}_{k1} - ((s_{i2})_{(l_i)}) \circ s_{(i+1)2} \circ \dots \circ \tilde{s}_{k2}[\cdot], \quad (3.65)$$

where  $[\cdot]$  denotes products having uniform estimates in  $D_{1/2}$ . We don't write analogous terms containing differences of derivatives of  $\tilde{s}_{kj}$ ,  $j = 1, 2$ .

Consider at first difference (3.65). It has the estimate

$$C|s_{(i+1)1} \circ \dots \circ \tilde{s}_{k1}(z) - s_{(i+1)2} \circ \dots \circ \tilde{s}_{k2}(z)|.$$

We can represent the last difference as a sum of terms of the types

$$s_{(i+1)1} \circ \dots \circ \tilde{s}_{k1}(z) - s_{(i+1)2} \circ \dots \circ \tilde{s}_{k1}(z), \quad (3.66)$$

$$s_{(i+1)2} \circ \dots \circ s_{j1} \circ \dots \circ \tilde{s}_{k1}(z) - s_{(i+1)2} \circ \dots \circ s_{j2} \circ \dots \circ \tilde{s}_{k1}(z) \quad (3.66')$$

for  $j = i + 1, \dots, k$ .

The following considerations are analogous to the reasoning leading to relations (2.42), (2.43). In more details, let  $\{c_{q\bar{j}}\}, q + j = l + 1$  be the coefficients of the form  $R_{(l-1)l}$ . By (2.37) - (2.40), we can see that  $s_0$  is a function of  $a_0$ ;  $s_l, l < k$  are functions of  $c_{q\bar{j}}$ , and  $\tilde{s}_k$  is a function of  $c_{q\bar{j}}, q + j = k + 1, j \geq 1$ . All these functions have uniformly bounded derivatives of any order at  $|z| \leq 1/2$ . The same holds for  $s_l^{-1}$ . When we consider  $h_{wi}, i = 1, 2$  we denote by  $\{c_{q\bar{j}}^i\}, q + j = l + 1, i = 1, 2$  the coefficients of the corresponding forms  $R_{(l-1)l,i}$ . These coefficients are polynomial in  $a_0^{-1}$  and in derivatives in zero of function  $h_w$  and, hence, they are uniformly bounded. It follows that differences (3.66), (3.66') have estimates

$$\leq C \max\{|a_0^{-1} - a_0^2|, |c_{q\bar{j}}^1 - c_{q\bar{j}}^2|\}$$

for  $q + j = m, m \leq k$  or  $q + j = k + 1, j \geq 1$ .

Further, the coefficients  $c_{q\bar{j}}, q + j = l, l \leq k$  or  $q + j = k + 1, j \geq 1$  are functions of  $(h_w)_{z^p \bar{z}^q}(0)$  with  $p + q \leq k$  and  $(\mu_{h_w})_{(l)}(0), |l| = k$ . All these functions have bounded derivatives. We get for terms of type (3.65) the estimate

$$\leq C \left( \max_{|l| \leq k} |(h_{w1})_{(l)}(0) - h_{w2})_{(l)}(0)| + \max_{|l|=k} |(\mu_{h_{w1}})_{(l)}(0) - (\mu_{h_{w2}})_{(l)}(0)| \right). \quad (3.67)$$

From the other hand,  $z$ -derivatives of  $s_l$  are also functions of the same variables  $(h_w)_{z^p \bar{z}^q}(0), p + q \leq k$  and  $\mu_{w,z^q \bar{z}^{k-q}}^{(k-1)}(0)$ . We obtain for the terms of type (3.64) the same estimate (3.67). Thus we proved estimate b).

Obviously, we get analogous estimate for  $|((S_1^{(k)})^{-1})_{(l)} - ((S_2^{(k)})^{-1})_{(l)}|, |l| \leq k+1$ . Also,  $|(\mu_{S_1^{(k)}})_{(l)} - (\mu_{S_2^{(k)}})_{(l)}|, |l| \leq k+1$  is no greater, than  $\sup_{|l| \leq k+2} |((S_1^{(k)})_{(l)} - (S_2^{(k)})_{(l)})|$  and, hence, we also get for such term estimate (3.67). Since we have estimates (3.56') - (3.63'), we obtain a).

c) Since  $\mu_{g_{w_i}^{(k)}}(z) = \mu_{h_{w_i}}(z)$  at  $|z| \geq 1$ , the estimate follows from a).  $\square$

Our next step will be estimates for  $|h_{w1}(z) - h_{w2}(z)|$  and  $|(h_{w1})_z(z) - (h_{w2})_z(z)|$ . Also we obtain estimates for  $|g_{w1,z}^{(k)}(z) - g_{w2,z}^{(k)}(z)|$ .

**Proposition 23** *We have for some  $0 < \alpha < 1$  depending only on  $d$  and some uniform  $C$  a)*

$$|h_{w1}(z) - h_{w2}(z)| \leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + \frac{|\mu_1(w) - \mu_2(w)|}{(1 - |z|)^2} \right],$$

b)

$$\begin{aligned} |(h_{w1})_z(z) - (h_{w2})_z(z)| &\leq \frac{C}{(\min_{i=1,2} |1 - \bar{w} Z_{w_i}^{-1}(z)|)^2} \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + \right. \\ &\quad \left. + |\mu_1(w) - \mu_2(w)|^\alpha + \frac{|\mu_1(w) - \mu_2(w)|}{1 - |z|} + \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha \right], \end{aligned}$$

c) for  $|z| \leq 1/2$

$$\begin{aligned} |g_{w1,z}^{(k)}(z) - g_{w2,z}^{(k)}(z)| &\leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha + \right. \\ &\quad \left. + \max_{m \leq k-1} |(g_{w1}^{(m)})_{z^{m+1}}(0) - (g_{w2}^{(m)})_{z^{m+1}}(0)| + \max_{|m| \leq k} |(\mu_{h_{w1}})_{(m)}(0) - (\mu_{h_{w2}})_{(m)}(0)| \right]. \end{aligned}$$

**Proof.** a) Using the representation  $h_w = \varphi_{w'} \circ f \circ \varphi_w^{-1} \circ Z_w^{-1}$ , we see that we must estimate the sum

$$\begin{aligned} & |(\varphi_{w'_1} - \varphi_{w'_2}) \circ f_1 \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}| + |(\varphi_{w'_2} \circ f_1 - \varphi_{w'_2} \circ f_2) \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}| + \\ & + |\varphi_{w'_2} \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_1}^{-1} - \varphi_{2w'} \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}|. \end{aligned} \quad (3.68)$$

Considering  $\varphi'_w$  as a function of  $w'$  and applying inequality (2.4), we get that we can estimate the first term as

$$\sup_{[w'_1, w'_2]} (|(\varphi'_w)_{w'}| + |(\varphi'_w)_{\bar{w}'}|) |w'_1 - w'_2| \leq C \frac{|w'_1 - w'_2|}{1 - |w|}.$$

Applying estimate (3.1), we obtain

$$|(\varphi_{w'_1} - \varphi_{w'_2}) \circ f_1 \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z)| \leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} \right].$$

The second term in (3.68) we estimate as

$$\sup |(\varphi_{2w'})_z| |f_1 - f_2|_C \leq C(1 - |w|)^{-1} |f_1 - f_2|_C.$$

Again we obtain the estimate

$$|(\varphi_{w'_2} \circ f_1 - \varphi_{w'_2} \circ f_2) \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z)| \leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} \right].$$

Further, we have the estimate  $c \leq |(f_w)_z(z)| \leq C|1 - \bar{w}z|^{-2}$ , it is a particular case of estimate (2.31). We see that the third term in (3.68) we can estimate as

$$\begin{aligned} |f_{w_2} \circ Z_{w_1}^{-1}(z) - f_{w_2} \circ Z_{w_2}^{-1}(z)| & \leq C \min \left[ 1, \frac{|Z_{w_1}^{-1}(z) - Z_{w_2}^{-1}(z)|}{(1 - |z|)^2} \right] \leq \\ & \leq C_1 \min \left[ 1, \frac{|\mu_1(w) - \mu_2(w)|}{(1 - |z|)^2} \right] \end{aligned}$$

with some uniform  $C_1$ .

b) We must estimate the terms

$$|((\varphi_{w'_1})_z - (\varphi_{w'_2})_z) \circ f_1 \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z)| |(\varphi_{w'_1})_z \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z)| \cdot |(\varphi_{w'_2})_z \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z)| \cdot |(Z_{w_1}^{-1})_z(z)|, \quad (3.69)$$

$$|(\varphi_{w'_2})_z \circ f_1 \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z) - (\varphi_{w'_2})_z \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z)| | (f_1)_z \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z) \cdot (\varphi_w^{-1})_z \circ Z_{w_1}^{-1}(z) \cdot (Z_{w_1}^{-1})_z(z) |, \quad (3.70)$$

$$|(\varphi_{w'_2})_z \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z)| | ((f_1)_z - (f_2)_z) \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z) | | (\varphi_w^{-1})_z \circ Z_{w_1}^{-1}(z) \cdot (Z_{w_1}^{-1})_z(z) |, \quad (3.71)$$

$$|(\varphi_{w'_2})_z \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z)| | (f_2)_z \circ \varphi_w^{-1} \circ Z_{w_1}^{-1}(z) - (f_2)_z \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z) | | (\varphi_w^{-1})_z \circ Z_{w_1}^{-1}(z) \cdot (Z_{w_1}^{-1})_z(z) |, \quad (3.72)$$

$$|(\varphi_{w'_2})_z \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z) \cdot (f_2)_z \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z)| | (\varphi_w^{-1})_z \circ Z_{w_1}^{-1}(z) - (\varphi_w^{-1})_z \circ Z_{w_2}^{-1}(z) | | (Z_{w_1}^{-1})_z(z) |, \quad (3.73)$$

$$|(\varphi_{w'_2})_z \circ f_2 \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z) \cdot (f_2)_z \circ \varphi_w^{-1} \circ Z_{w_2}^{-1}(z)| | (\varphi_w^{-1})_z \circ Z_{w_2}^{-1}(z) | | (Z_{w_1}^{-1})_z(z) - (Z_{w_2}^{-1})_z(z) |. \quad (3.74)$$

We don't write the analogous terms containing  $\bar{z}$ -derivatives.

To estimate term (3.69) we proceed as in the case of the first difference in (3.68). The derivatives of  $(\varphi_{w'})_z$  with respect to  $w'$  have the estimate  $C(1 - |w|)^{-2}$ . The derivative  $(\varphi_w^{-1})_z \circ Z_{w_1}^{-1}(z)$  has the estimate  $C(1 - |w|^2)|1 - \bar{w}Z_{w_1}^{-1}(z)|^{-2}$ . Other multiples are uniformly bounded. Applying estimate (3.1), we obtain for term (3.69) the estimate

$$\frac{C}{|1 - \bar{w}Z_{w_1}^{-1}(z)|^2} \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} \right]. \quad (3.69')$$

For term (3.70) we obtain analogous estimate because the second derivative of  $\varphi_{2w'}$  has the estimate  $C(1 - |w|)^{-2}$ .

In all other terms there is the multiple  $(\varphi_{w'_2})_z = O(1 - |w|)^{-1}$ .

Thus for term (3.71) we have the estimate

$$\begin{aligned} & \frac{C}{1 - |w|} \| (f_1)_z - (f_2)_z \|_C \frac{1 - |w|^2}{|1 - \bar{w}Z_{w_1}^{-1}(z)|^2} \leq \\ & \leq \frac{C}{|1 - \bar{w}Z_{w_1}^{-1}(z)|^2} \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{1w}} - \mu_{F_{2w}}\|_p^\alpha \right] \end{aligned} \quad (3.71')$$

by Proposition 18.

Consider term (3.72). Denote  $z_i = \varphi_w^{-1} \circ Z_{w_i}^{-1}(z)$ ,  $i = 1, 2$ . We see that we have the estimate

$$\frac{C}{|1 - \bar{w}Z_{w_1}^{-1}(z)|^2} \sup_{t \in [Z_{w_1}^{-1}(z), Z_{w_2}^{-1}(z)], |(k)|=2} | (f_2)_{(k)} \circ \varphi_w^{-1}(z) \cdot (\varphi_w^{-1})_z(t) | | Z_{w_1}^{-1}(z) - Z_{w_2}^{-1}(z) |.$$



Now, if  $|k| = 2$ ,

$$\begin{aligned} |(f_2)_{(k)} \circ \varphi_w^{-1}(z)| &\leq C(1 - |\varphi_w^{-1}(z)|^2)^{-1} \leq C \frac{|1 - \bar{w}z|^2}{(1 - |w|^2)(1 - |z|^2)}, \\ &\leq |(\varphi_w^{-1})_z| = \frac{1 - |w|^2}{|1 - \bar{w}z|^2} \end{aligned}$$

Since  $c(1 - |z|) \leq 1 - |Z_{w_i}^{-1}(z)| \leq C(1 - |z|)$  for some uniform  $c$  and  $C$ , we obtain the estimate

$$\frac{C}{|1 - \bar{w}Z_{w_1}^{-1}(z)|^2} \min \left[ 1, \frac{|\mu_1(w) - \mu_2(w)|}{1 - |z|} \right]. \quad (3.72')$$

For term (3.73) we have the estimate

$$\frac{C}{1 - |w|} \sup_{[Z_{w_1}^{-1}(z), Z_{w_2}^{-1}(z)]} |(\varphi_w^{-1})_{zz}| |Z_{w_1}^{-1}(z) - Z_{w_2}^{-1}(z)| \leq C \frac{|Z_{w_1}^{-1}(z) - Z_{w_2}^{-1}(z)|}{(\min_{i=1,2} |1 - \bar{w}Z_{w_i}^{-1}(z)|)^3} \leq$$

$$\frac{C}{(\min_{i=1,2} |1 - \bar{w}Z_{w_i}^{-1}(z)|)^2} \frac{|\mu_1(w) - \mu_2(w)|}{\min_{i=1,2} |1 - \bar{w}Z_{w_i}^{-1}(z)|}$$

at  $|\mu_1(w) - \mu_2(w)| \leq 1 - |z|$ . We obtain the estimate

$$\frac{C}{(\min_{i=1,2} |1 - \bar{w}Z_{w_i}^{-1}(z)|)^2} \min \left[ 1, \frac{|\mu_1(w) - \mu_2(w)|}{1 - |z|} \right]. \quad (3.73')$$

At last, for term (3.74) we have the estimate

$$\frac{C}{|1 - \bar{w}Z_{w_2}^{-1}(z)|^2} |\mu_1(w) - \mu_2(w)|. \quad (3.74')$$

Collecting estimates (3.69') - (3.74') we obtain b).

c) From the representation:  $g_{w_i}^{(k)} = h_{w_i} \circ s_{0i} \circ \dots \circ \tilde{s}_{ki} = h_{w_i} \circ S_i^{(k)}$ ,  $i = 1, 2$ , follows that we must estimate the terms of the types

$$\begin{aligned} &(h_{w_1,z} - h_{w_2,z}) \circ S_1^{(k)}(z)[.], \\ &(h_{w_2,z} \circ S_1(z) - h_{w_2,z} \circ S_2^{(k)}(z))[.], \\ &((s_{j1})_z - (s_{j2})_z) \circ s_{(j+1)1} \circ \dots \circ \tilde{s}_{k1}(z)[.], \\ &((s_{j2})_z \circ s_{(j+1)1} \circ \dots \circ \tilde{s}_{k1}(z) - (s_{j2})_z \circ s_{(j+1)2} \circ \dots \circ \tilde{s}_{k2}(z))[.], \quad j \leq k - 1/ \end{aligned}$$

We don't write the analogous term with the difference  $\tilde{s}_{k1}(z) - \tilde{s}_{k2}(z)$ . Multiples in the square brackets are uniformly bounded because derivatives of  $s_{ij}$  are uniformly bounded and derivatives of  $h_{wi}$  are uniformly bounded on  $D_{1/2}$ .

For the first term we have estimate b)

$$C \max \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{1w}} - \mu_{F_{2w}}\|_p^\alpha \right].$$

The second term is no greater than

$$C|S_1(z) - S_2(z)|,$$

and, by point b) of Proposition 22, we get the estimate

$$C \left( \max_{|(l)| \leq k} |(h_{w1})_{(l)}(0) - h_{w2})_{(l)}(0)| + \max_{|(l)|=k} |(\mu_{h_{w1}})_{(l)}(0) - (\mu_{h_{w2}})_{(l)}(0)| \right).$$

The same estimate holds for the terms of third and fourth types. Indeed, these terms are the particular cases of terms (3.65) and (3.66). Applying (2.43) we obtain c).  $\square$

**Proof of estimate (3.2).** At first we estimate  $|g_{1w,z^{k+1}}^{(k)}(0) - g_{2w,z^{k+1}}^{(k)}(0)|$ . We have representations (2.53) and the decomposition analogous to (2.55) but cut on the term of order  $z^{k+1}$ . Thus,

$$|(g_{w1}^{(k)})_{z^{k+1}}(0) - (g_{w2}^{(k)})_{z^{k+1}}(0)| \leq C \left( \max_{\partial D_{1/2}} |h_{w1} - h_{w2}| + \int_{D_{1/2}} \frac{|(\gamma_1(g_{w1}^{(k)})_z - \gamma_2(g_{w2}^{(k)})_z)(z)|}{|z|} dS_z \right). \quad (3.75)$$

Here we take into consideration that  $g_{wi}^{(k)}(z) = h_{wi}(z)$  for  $|z| \geq 1/2$ .

For the first difference in the right side we have the first estimate of Proposition 23

$$\max_{\partial D_{1/2}} |h_{w1} - h_{w2}| \leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + |\mu_1(w) - \mu_2(w)| \right]. \quad (3.76)$$

We must estimate the integral in (3.75). We have

$$\int_{D_{1/2}} \frac{|(\gamma_1(g_{w1}^{(k)})_z - \gamma_2(g_{w1}^{(k)})_z)(z)|}{|z|} dS_z \leq$$

$$\leq \int_{D_{1/2}} \frac{|\gamma_1(z)| |((g_{w1}^{(k)})_z - (g_{w2}^{(k)})_z)(z)|}{|z|} dS_z + \int_{D_{1/2}} \frac{|(\gamma_1 - \gamma_2)(z)| |(g_{w2}^{(k)})_z(z)|}{|z|} dS_z \quad (3.77)$$

Consider the first integral in the right side. Applying third estimate of Proposition 23, we have

$$\begin{aligned} I_1 &= \int_{D_{1/2}} \frac{|\gamma_1(z)| |((g_{w1}^{(k)})_z - (g_{w2}^{(k)})_z)(z)|}{|z|} dS_z \leq \sup_{|z| \leq 1/2} |(g_{w1}^{(k)})_z - (g_{w2}^{(k)})_z| \int_{D_{1/2}} \frac{|\gamma_1(z)|}{|z|} dS_z \leq \\ &\leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{1w}} - \mu_{F_{2w}}\|_p^\alpha + \right. \\ &\left. + \max_{|(m)| \leq k} |(\mu_{h_{w1}})_{(m)}(0) - (\mu_{h_{w2}})_{(m)}(0)| + \max_{m \leq k-1} |(g_{w1}^{(m)})_{z^m}(0) - (g_{w2}^{(m)})_{z^m}(0)| \right]. \end{aligned} \quad (3.78)$$

Consider the second integral in the right side of (3.77). Remind that  $\gamma(z) = |z|^{-(k+1)} \mu_{g_w}^{(k)}(z)$ . It means that for any  $z = re^{i\varphi}$  we can write

$$|\gamma_1(z) - \gamma_2(z)| \leq \frac{1}{(k+1)!} \left| \frac{d^{k+1}}{dr^{k+1}} (\mu_{g_{w1}^{(k)}}(re^{i\varphi}) - \mu_{g_{w2}^{(k)}}(re^{i\varphi})) \right|.$$

By Proposition 22, we obtain

$$\begin{aligned} I_2 &= \int_{D_{1/2}} \frac{|\gamma_1 - \gamma_2|(z) |(g_{w2}^{(k)})_z(z)|}{|z|} dS_z \leq C \sup_{D_{1/2}} \sup_{|l|=k+1} |(\mu_{g_{w1}^{(k)}})_{(l)} - (\mu_{g_{w2}^{(k)}})_{(l)}| \int_{D_{1/2}} \frac{|(g_{w2}^{(k)})_z(z)|}{|z|} dS_z \leq \\ &\leq C (\sup_{D_{1/2}} \sup_{|(q)| \leq k+1} |(\mu_{h_{w1}})_{(q)} - (\mu_{h_{w2}})_{(q)}| + \max_{|(m)| \leq k} |(h_{w1})_{(m)}(0) - (h_{w2})_{(m)}(0)|). \end{aligned} \quad (3.79)$$

Gathering estimates (3.76), (3.78) and (3.79), we obtain by induction

$$\begin{aligned} |(g_{w1}^{(k)})_{z^{k+1}}(0) - (g_{w2}^{(k)})_{z^{k+1}}(0)| &\leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + \right. \\ &+ |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p^\alpha + \sup_{D_{1/2}} \sup_{|(q)| \leq k+1} |(\mu_{h_{w1}})_{(q)} - (\mu_{h_{w2}})_{(q)}| + \\ &\left. + \max_{|(m)| \leq k} |(h_{w1})_{(m)}(0) - (h_{w2})_{(m)}(0)| \right]. \end{aligned}$$

Now, if  $|(l)| = k + 1$ , we have once more by induction, applying relations (2.43),

$$\begin{aligned} |(h_{w1})_{(l)}(0) - (h_{w2})_{(l)}(0)| &\leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + \right. \\ &\left. + |\mu_1(w) - \mu_2(w)|^\alpha + \|\mu_{F_{1w}} - \mu_{F_{2w}}\|_p^\alpha + \sup_{D_{1/2}} \sup_{|(q)| \leq k+1} |(\mu_{h_{w1}})_{(q)} - (\mu_{h_{w2}})_{(q)}| \right] \end{aligned} \quad (3.80)$$

Further, by (3.12) we obtain

$$\begin{aligned} \sup_{D_{1/2}} \sup_{|q| \leq k+1} |(\mu_{h_{w1}})_{(q)} - (\mu_{h_{w2}})_{(q)}| &\leq C \min[1, \sup_{|q| \leq k+1} \sup_{D_{R_w}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}| + \\ &+ |\mu_1(w) - \mu_2(w)|], \end{aligned}$$

where  $1 - R_w^2 = b(1 - |w|^2)$  with some uniform  $b$ . Also, Proposition 17 yields

$$\begin{aligned} \|\mu_{F_{w1}} - \mu_{F_{w2}}\|_p &\leq C \min[1, \sup_{|q|=1} \sup_{D_{R_w}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}| + \\ &+ \sup_{D_R} |\mu_1(z) - \mu_2(z)| + [(1 - R)/(1 - |w|)]^{1/p} + |\mu_1(w) - \mu_2(w)|^{1/(2p+1)}], \end{aligned}$$

where we can take any  $R < 1$ . Thus we obtain

$$\begin{aligned} |(h_{1w})_{(k+1)}(0) - (h_{2w})_{(k+1)}(0)| &\leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} + \right. \\ &+ \sup_{|q| \leq k+1} \sup_{D_{R_w}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}| + |\mu_1(w) - \mu_2(w)|^\alpha + \sup_{D_R} |\mu_1 - \mu_2|^\alpha + \\ &\left. + [(1 - R)/(1 - |w|)]^\alpha + \sup_{|q|=1} \sup_{D_{R_w}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}|^\alpha \right]. \end{aligned} \quad (3.81)$$

with some new  $0 < \alpha < 1$ .

Now suppose that estimate (3.2) holds for multi-indexes  $(l)$  with  $|(l)| = k$ . Recalling representation (2.15) we see that we can represent the derivative  $f_{(l)}(w)$ ,  $|(l)| = k + 1$  as a sum of products, in which multiples have the types  $(\varphi_{w'}^{-1})_{l_1}(0) = c_{l_1}(1 - |w'|^2)\bar{w}'^{l_1-1}$ ,  $(h_w)_{(l_2)}(0)$ ,  $(Z_w)_{(l_3)}(0)$ ,  $(\varphi_w)_{(l_4)}(w) = c_{l_4}\bar{w}^{|(l_4)|-1}(1 - |w|^2)^{-|(l_4)|}$ , where  $l_1 \leq k + 1$ ,  $|(l_i)| \leq k + 1$ ,  $i \geq 2$ . It implies that we can represent the difference  $(f_1)_{(k+1)}(w) - (f_2)_{(k+1)}(w)$  as a sum of items of the types

$$\|w'_1\|^2 - |w'_2\|^2 |O((1 - |w|)^{-(k+1)})|, \quad (3.82)$$

$$|w'_1 - w'_2| O((1 - |w|)^{-k}), \quad (3.83)$$

$$|(Z_{w_1})_{(l)}(0) - Z_{w_2})_{(l)}(0)| O((1 - |w|)^{-k}), |l| \leq k + 1, \quad (3.84)$$

$$|(h_{w_1})_{(l)}(0) - (h_{w_2})_{(l)}(0)| O((1 - |w|)^{-k}), |l| \leq k + 1. \quad (3.85)$$

Items of type (3.82) have the estimate

$$C \min \left[ 1, \frac{|f_1(w) - f_2(w)|}{1 - |w|} (1 - |w|)^{-k} \right] \leq C \min \left[ 1, \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} \right] (1 - |w|)^{-k}.$$

Here we apply estimate (3.1) and recall that  $f_{(l)}(w) = O((1 - |w|)^{-k})$ . Also, term (3.83) has the estimate

$$C \|\mu_1 - \mu_2\|_p^\alpha (1 - |w|)^{-k}.$$

Term (3.84) we estimate as

$$C |\mu_1(w) - \mu_2(w)| (1 - |w|)^{-k}.$$

At last, for term (3.85) we have estimate (3.81). We obtain

$$\begin{aligned} |(f_1)_{(l)}(w) - (f_2)_{(l)}(w)| &\leq \frac{C}{(1 - |w|)^k} \inf \left\{ 1, \left[ \chi \left( \frac{\|\mu_1 - \mu_2\|_p^\alpha}{1 - |w|} \right) + \right. \right. \\ &+ \sup_{|q| \leq k+1} \sup_{D_{R_w}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}| + |\mu_1(w) - \mu_2(w)|^\alpha + \sup_{D_R} |\mu_1 - \mu_2|^\alpha + \\ &\left. \left. + [(1 - R)/(1 - |w|)]^\alpha + \sup_{|q|=1} \sup_{D_{R_w}} |(\mu_1)_{(q)} - (\mu_2)_{(q)}|^\alpha \right] \right\} \end{aligned}$$

if  $|l| = k + 1$ . Estimate (3.2) follows immediately.  $\square$

## 4 Approximation of a family of normal mappings by a smooth family

In this section we suppose that  $\mu$  depends on a vector parameter  $t \in \mathbb{C}^n$  and satisfies conditions of Theorem 2. In this section and in what follows the word "uniform" applied to an estimate or to a constant means, also, that it doesn't depend on  $t$ . We denote by  $z$  the chart on a fiber. We adopt the notations  $\partial^{(k,l)} f$  or  $f_{(k,l)}$  with double multi-indexes, as it was described in Section 1.

Also, we denote by  $f_{(k)}$  derivatives in  $z$ ,  $\bar{z}$  and by  $f_{(0,l)}$  derivatives in  $t$  with the multi-index  $(l)$ . We use the notations  $\mu_t(z) = \mu(z, t)$  or  $f_t(z) = f(z, t)$  when we aren't in danger to mix them with notations for derivatives.

The next proposition is a corollary of Lemma 2. It shows that for  $z$ -derivatives of the normal mappings we have a Holder continuity with respect to the parameters .

**Proposition 24** *In conditions of Theorem 2 let  $f_t$  be the  $\mu_t$ -quasiconformal normal mapping. Then*

$$|f_{(k)}(z, t) - f_{(k)}(z, t + \delta t)| \leq \frac{C}{(1 - |z|)^{|(k)|-1}} \min\{1, (\delta t)^\beta ((1 - |z|)^{-s})\}. \quad (4.1)$$

for uniform  $C$  and some  $0 < \beta < 1$  and  $s > 0$  depending only on  $d$  and  $|(k)|$ .

**Proof.** By estimate (3.2) of Lemma 2, we have

$$\begin{aligned} |f_{(k)}(z, t) - f_{(k)}(z, t + \delta t)| &\leq \frac{C}{(1 - |z|)^{|(k)|-1}} \min \left\{ 1, \frac{\|\mu_t - \mu_{t+\delta t}\|_p^\alpha}{1 - |z|} + \right. \\ &\left. + \sup_{|(q)| \leq |(k)|} \sup_{D_{R_z}} |(\mu_t)_{(q)} - (\mu_{t+\delta t})_{(q)}|^\alpha + \sup_{D_R} |\mu_t - \mu_{t+\delta t}|^\alpha + [(1 - R)/(1 - |z|)]^{\alpha/p} \right\}, \end{aligned} \quad (4.2)$$

where  $R$  can be arbitrary radius less than 1 and  $R_z$  is such that

$$1 - R_z \leq b(1 - |z|). \quad (4.3)$$

for some uniform  $b > 0$ . From the other hand, by inequality (1.5), we have

$$|(\mu_t)_{(q)}(z) - (\mu_{t+\delta t})_{(q)}(z)| \leq C \frac{\delta t}{(1 - |z|)^N}, \quad (4.4)$$

where  $N$  depends only on  $|(k)|$  if  $|(q)| \leq |(k)|$  and  $C$  is uniform.

Consider the right side of inequality (4.2). Set some  $r < 1$ . We have

$$\|\mu_t - \mu_{t+\delta t}\|_p \leq C \left[ \frac{\delta t}{(1 - r)^N} + (1 - r)^{1/p} \right].$$

Put  $r$  such that  $\delta t(1 - r)^{-N} = (1 - r)^{1/p}$ , i.e.,  $1 - r = (\delta t)^{p/(Np+1)}$ . We obtain

$$\|\mu_t - \mu_{t+\delta t}\|_p \leq C(\delta t)^{\frac{1}{Np+1}}.$$

Also,

$$\sup_{D_R} |\mu_t - \mu_{t+\delta t}|^\alpha \leq C \frac{\delta t^\alpha}{(1-R)^{N\alpha}}.$$

Put  $R$  such that  $[\delta t/(1-R)^N]^\alpha = [(1-R)/(1-|z|)]^{\alpha/p}$ , i.e,  $1-R = (\delta t)^{p/(Np+1)}(1-|z|)^{1/(Np+1)}$ . We obtain

$$\sup_{D_R} |\mu_t - \mu_{t+\delta t}|^\alpha + [(1-R)/(1-|z|)]^{\alpha/p} \leq C \delta t^{\frac{\alpha}{Np+1}} (1-|z|)^{-\frac{N\alpha}{Np+1}}.$$

At last, by (4.4) and (4.3),

$$\sup_{|q|\leq|k|} \sup_{D_{Rz}} |(\mu_t)_{(q)} - (\mu_{t+\delta t})_{(q)}|^\alpha \leq C \frac{\delta t^\alpha}{(1-|z|)^{N\alpha}}$$

We obtain for the sum in the right part of inequality (4.2) the estimate  $(\delta t)^\beta ((1-|z|)^{-s})$ , where we can put  $\beta = \alpha/(Np+1)$ ,  $s = N\alpha$ .  $\square$

The main result of this section is the next lemma about approximations:

**Lemma 3** *In the above assumptions for every  $\varepsilon > 0$  and natural  $m$  there exists a family of mappings  $f_{at}$  smoothly depending on  $z$  and approximating the family  $f_t$  up to derivatives of order  $m$*

$$|(f_{at})_{(l)} - (f_t)_{(l)}| \leq \varepsilon, |l| \leq m \quad (4.5)$$

on  $D$ . The maps  $f_{at}$  are quasiconformal with complex dilatations  $\mu_{at}$  and map  $D$  homeomorphically onto some domain  $\Omega_t$ . The mapping  $B \rightarrow C^0(D) : t \mapsto f_{at}$  is continuous.

There exists the decomposition

$$f_{at} = h_{at} \circ f^{\mu_{at}}, \quad (4.6)$$

where  $f^{\mu_{at}}$  is the normal map with with the complex dilatation  $\mu_{at}$  and  $h_{at}$  is a holomorphic univalent function on  $D$  satisfying the estimate

$$|h'_{at} - 1| \leq \varepsilon, |h''_{at}| \leq \varepsilon(1-|t|)^{-1} \quad (4.7)$$

at  $m \geq 2$ . Derivatives of  $f_{at}$  satisfy estimates analogous to (1.5)

$$|(f_{at})_{(k,l)}(z)| \leq C(1-|z|)^{N_{|l|+|k|}}, \quad (5.8)$$

where  $C$  is uniform and  $N_{|l|+|k|}$  doesn't depend on  $\varepsilon$  (though it can depend on  $m$ ).

**Proof.** Let  $h(z)$  be a "cap",  $\text{supp} h \subset \{|t| \leq 1\}$ ,  $\int h dV_t = 1$ . We shall consider the approximations

$$f_{(l),\delta}(z, t) = \delta^{-2n} \int f_{(l)}(z, \zeta) h\left(\frac{t - \zeta}{\delta}\right) dV_\zeta = \int f_{(l)}(z, t - \delta\zeta) h(\zeta) dV_\zeta. \quad (4.9)$$

Since  $h(\zeta) = 0$  at  $|\zeta| \geq 1$ , we obtain by Proposition 24

$$|f_{(l),\delta}(z) - f_{(l)}(z)| \leq C \frac{\delta^{\beta_k}}{(1 - |z|)^{s_k + |(l)| - 1}}, \quad (4.10)$$

for some  $\beta_k$  and  $s_k$  if  $|(l)| \leq k$  and  $\delta^{\beta_k}/(1 - |z|)^{s_k} \leq 1$ .

Now we make  $\delta$  depending on  $|z|$ . Namely we pick some  $0 < b < 1$  and define

$$\delta \leq [bC^{-1}(1 - |z|)^{s_{2m} + 2m - 1}]^{\frac{1}{\beta_{2m}}}.$$

Then estimate (4.10) yields

$$|f_{(l),\delta}(z) - f_{(l)}(z)| \leq b \quad (4.11)$$

if  $|(l)| \leq 2m$ .

Introduce also approximations of the functions  $f_{(l)}(0, t)$  for  $|(l)| \leq m$

$$G_{(l)0,\delta}(t) = \int f_{(l)}(0, t - \delta\zeta) h(\zeta) dV_\zeta.$$

As a particular case of (4.11) we have the estimates

$$|G_{(l)0,\delta}(t) - f_{(l)}(0, t)| \leq b. \quad (4.12)$$

at  $|(l)| \leq m$ .

Now we describe the construction of our approximation. In what follows  $z = re^{-\theta}$ . We adopt the notations  $(l) = (j\bar{k})$ ,  $|(l)| = j + k$ ,  $f_{j\bar{k}} = f_{z^j \bar{z}^k}$ .

We define the functions  $f_{(l),\delta}$  as the approximations of  $f_{(l)}$  for  $|(l)| = m$ . Now, if  $j + k = m - 1$ , we put

$$\begin{aligned} g_{(j\bar{k}),\delta}(z, t) &= G_{(j\bar{k})0,\delta}(t) + \int_{[0,|z|]} f_{(j+1,\bar{k}),\delta} dz + f_{(j,\bar{k}+1),\delta} d\bar{z} = \\ &= G_{(j\bar{k})0,\delta}(t) + \int_0^r [e^{i\theta} f_{(j+1,\bar{k}),\delta}(r, \theta, t) + e^{-i\theta} f_{(j,\bar{k}+1),\delta}(r, \theta, t)] dt. \end{aligned}$$



Analogously, define by induction at  $g + k = q$ ,  $0 \leq q \leq m - 2$ ,

$$g_{(j\bar{k}),\delta}(z, t) = G_{(j\bar{0}),\delta}(z) + \int_{[0,|z|]} g_{(j+1,\bar{k}),\delta} dz + g_{(j,\overline{k+1}),\delta} d\bar{z}.$$

In particular,

$$f_\delta(z, t) = g_{(0\bar{0}),\delta}(z, t) = \int_{[0,|z|]} g_{(1\bar{0}),\delta} dz + g_{(0\bar{1}),\delta} d\bar{z}.$$

We prove that  $f_\delta$  at small enough  $d$  satisfies all conditions of the lemma.

At first, applying (4.11) and (4.12), we see that, if at  $j + k = q$  there holds the inequality  $|g_{(j\bar{k}),\delta}(z, t) - f_{(j\bar{k})}(z, t)| \leq m_q b$ , then at  $j + k = q - 1$

$$|g_{(j\bar{k}),\delta}(z, t) - f_{(j\bar{k})}(z, t)| \leq b + \int_{[0,|z|]} [|g_{(j+1,\bar{k}),\delta} - f_{j+1,\bar{k}}| + |g_{(j,\overline{k+1}),\delta} - f_{j,\overline{k+1}}|] dr \leq (2m_q + 1)b.$$

It follows that  $m_{q-1} \leq 2m_q + 1$ , and by induction

$$|g_{(j\bar{k}),\delta}(z, t) - f_{(j\bar{k})}(z, t)| \leq (3^{m-j-k} + 1)b. \quad (4.13)$$

In particular,

$$|f_\delta - f| \leq (3^m + 1)b. \quad (4.14)$$

Show now that  $z$ -derivatives of  $f_\delta$  approximate the corresponding  $z$ -derivatives of  $f$  up to degree  $m$ . In the calculations below it is essential that  $\delta$  depends only on  $r$  and, hence, there don't appear "large" derivatives originating from  $\delta^{-2n} h\left(\frac{t-\zeta}{\delta}\right)$ . Consider at first  $(f_\delta)_z$ . We have

$$\begin{aligned} (f_\delta)_z(z, t) - f_z(z, t) &= \frac{e^{-i\theta}}{2} \left( \frac{\partial}{\partial r} + \frac{1}{ir} \frac{\partial}{\partial \theta} \right) \int_0^r [e^{i\theta}(g_{(1\bar{0}),\delta} - f_z) + e^{-i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})] dr = \\ &= \frac{1}{2} [g_{(1\bar{0}),\delta} - f_z + e^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})] + \frac{1}{2r} \int_0^r [g_{(1\bar{0}),\delta} - f_z - e^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})] dr + \\ &\quad + \frac{1}{2ir} \int_0^r \left[ \frac{\partial}{\partial \theta} (g_{(1\bar{0}),\delta} - f_z) + e^{-2i\theta} \frac{\partial}{\partial \theta} (g_{(0\bar{1}),\delta} - f_{\bar{z}}) \right] dr. \end{aligned} \quad (4.15)$$

Thus, applying (4.13) at  $j + k = 1$ , we obtain

$$|(f_\delta)_z - f_z| \leq 2(3^{m-1} + 1)b + \sup[|(g_{(1\bar{0}),\delta} - f_z)_\theta| + |g_{(0\bar{1}),\delta} - f_{\bar{z}}|]. \quad (4.16)$$

Now,

$$\begin{aligned} (g_{(1\bar{0}),\delta} - f_z)_\theta &= i \int_0^r [e^{i\theta}(g_{(2\bar{0}),\delta} - f_{2\bar{0}}) - e^{-i\theta}(g_{(1\bar{1}),\delta} - f_{1\bar{1}})] dr + \\ &+ \int_0^r [e^{i\theta}(g_{(2\bar{0}),\delta} - f_{2\bar{0}})_\theta + e^{-i\theta}(g_{(1\bar{1}),\delta} - f_{1\bar{1}}) - \theta] dr. \end{aligned} \quad (4.17)$$

Again applying (4.13), we see that

$$|(g_{(1\bar{0}),\delta} - f_z)_\theta| \leq 2(3^{m-2} + 1)b + \sup[|(g_{(2\bar{0}),\delta} - f_{2\bar{0}})_\theta| + |g_{(1\bar{1}),\delta} - f_{1\bar{1}})_\theta|]. \quad (4.18)$$

Proceeding in the same way we obtain at last

$$|(g_{(m-1,\bar{0}),\delta} - f_{m-1,\bar{0}})_\theta| \leq b + \sup[|(f_{(m\bar{0}),\delta} - f_{m\bar{0}})_\theta| + |f_{(m-1,\bar{1}),\delta} - f_{m-1,\bar{1}})_\theta|], \quad (4.19)$$

and we have analogous estimates for other differences  $(g_{(j\bar{k}),\delta} - f_{j\bar{k}})_\theta$ ,  $j + k = m - 1$ . But, by definition of  $f_{(m\bar{0}),\delta}$ ,

$$[(f_{(m\bar{0}),\delta} - f_{m\bar{0}})_\theta](z, t) = \int [(f_{m\bar{0}})_\theta(z, t - \delta\zeta) - (f_{m\bar{0}})_\theta(z, t)] h(\zeta) dV_\zeta. \quad (4.20)$$

and, by (4.13) with  $j + k = m + 1$ ,

$$|[(f_{(m\bar{0}),\delta} - f_{m\bar{0}})_\theta]| \leq b. \quad (4.21)$$

Evidently, we have analogous estimates for other derivatives, appearing in the process. Collecting (4.15) - (4.21), we obtain

$$|(f_\delta)_z - f_z| \leq K_m b,$$

where  $K_m$  is some integer-valued function of  $m$ , which we don't specify here because it is not essential for us. The analogous estimate we have for  $|(f_\delta)_z - f_{\bar{z}}|$ .

Now consider the  $z$ -derivative of difference (4.15). We apply to the right part the operator  $\frac{\partial}{\partial z} = \frac{e^{-i\theta}}{2} \left( \frac{\partial}{\partial r} + \frac{1}{ir} \frac{\partial}{\partial \theta} \right)$ . We have

$$\begin{aligned} g_{(1\bar{0}),\delta} - f_z &= \varphi(t) + \int_{[0,|z|]} (g_{(2\bar{0}),\delta} - f_{2\bar{0}}) dw + (g_{(1\bar{1}),\delta} - f_{1\bar{1}}) d\bar{z} = \\ &= \varphi(t) + \int_0^r [e^{i\theta}(g_{(2\bar{0}),\delta} - f_{2\bar{0}}) + e^{-i\theta}(g_{(1\bar{1}),\delta} - f_{1\bar{1}})] dr, \end{aligned}$$

where  $\varphi(t)$  is the value at zero. When we apply the operator  $\partial/\partial z$ , we obtain the terms

$$e^{i\theta}(g_{(2\bar{0}),\delta} - f_{2\bar{0}}) + e^{-i\theta}(g_{(1\bar{1}),\delta} - f_{1\bar{1}})], \quad (4.22)$$

$$\frac{1}{2r} \int_0^r [g_{(2\bar{0}),\delta} - f_{2\bar{0}} - e^{-2i\theta}(g_{(1\bar{1}),\delta} - f_{1\bar{1}})] dr \quad (4.23)$$

and

$$\frac{1}{2ir} \int_0^r [(g_{(2\bar{0}),\delta} - f_{2\bar{0}})_\theta + e^{-2i\theta}(g_{(1\bar{1}),\delta} - f_{1\bar{1}})_\theta] dr \quad (4.24)$$

We consider these terms exactly as we considered the right side of (4.15). We again reduce the problem to the estimate  $|(f_{(j\bar{k}),\delta} - f_{j\bar{k}})_\theta| \leq Cb$  at  $j+k = m+1$ . In the same way we proceed with derivatives of the term  $e^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})$ . The derivative of the multiple  $e^{-2i\theta}$  yields only the item  $-2ie^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})$ .

At differentiation of the second term in (4.15)) (the first integral) we obtain the terms

$$\frac{e^{-i\theta}}{4r} [g_{(1\bar{0}),\delta} - f_z - e^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})], \quad (4.25)$$

$$\frac{e^{-i\theta}}{4r^2} \int_0^r [g_{(1\bar{0}),\delta} - f_z \pm e^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})] dr \quad (4.26)$$

and

$$-\frac{e^{-i\theta}}{4ir^2} \int_0^r [(g_{(1\bar{0}),\delta} - f_z)_\theta - e^{-2i\theta}(g_{(0\bar{1}),\delta} - f_{\bar{z}})_\theta] dr \quad (4.27)$$

All our derivatives are regular at zero. It means that terms of the type  $r^{-1}\varphi(t)$ , originating from initial values at zero, must annihilate. Subtracting these initial items we obtain for terms (4.25), (4.26) the estimates

$$\sup \left[ \left| \frac{\partial}{\partial r}(g_{(1\bar{0}),\delta} - f_z) \right| + \left| \frac{\partial}{\partial r}(g_{(0\bar{1}),\delta} - f_{\bar{z}}) \right| \right] \leq$$

$$\leq C \sup(|g_{(2\bar{0}),\delta} - f_{2\bar{0}}| + |g_{(1\bar{1}),\delta} - f_{1\bar{1}}| + |g_{(0\bar{2}),\delta} - f_{0\bar{2}}|) \leq C'b$$

with some uniform  $C'$  by definition of  $g_{(j\bar{k}),\delta}$

Considering term (4.27) we must, analogously, estimate the mixed derivatives of  $(g_{(1\bar{0}),\delta} - f_z)_{r\theta}$  and  $(g_{(0\bar{1}),\delta} - f_{\bar{z}})_{r\theta}$ . Estimate, for example, the first difference. By (4.17), we see that we must estimate  $g_{(2\bar{0}),\delta} - f_{2\bar{0}}$ ,  $g_{(1\bar{1}),\delta} - f_{1\bar{1}}$ ,  $(g_{(2\bar{0}),\delta} - f_{2\bar{0}})_\theta$  and  $(g_{(1\bar{1}),\delta} - f_{1\bar{1}})_\theta$ . The first two differences are of order  $b$  by (4.13) and, proceeding as after (4.17), we can see that we can estimate these

terms through  $|f_{(j,k),\delta} - f_{(j,k)}|$  with  $j + k = m + 1$ , i.e., these terms also are of order  $b$ .

When we apply operator  $\partial/\partial w$  to the third term in (4.15) (the second integral) we obtain the terms

$$\frac{e^{-i\theta}}{4r} [(g_{(1\bar{0}),\delta} - f_z)_\theta - e^{-2i\theta} (g_{(0\bar{1}),\delta} - f_{\bar{z}})_\theta],$$

$$\frac{e^{-i\theta}}{4r^2} \int_0^r [(g_{(1\bar{0}),\delta} - f_z)_\theta \pm e^{-2i\theta} (g_{(0\bar{1}),\delta} - f_{\bar{z}})_\theta] dr,$$

and

$$-\frac{e^{-i\theta}}{4ir^2} \int_0^r [(g_{(1\bar{0}),\delta} - f_z)_{\theta^2} - e^{-2i\theta} (g_{(0\bar{1}),\delta} - f_{\bar{z}})_{\theta^2}] dr.$$

To estimate the first two terms we must estimate the  $r$ -derivatives of  $(g_{(1\bar{0}),\delta} - f_z)_\theta$  and  $(g_{(0\bar{1}),\delta} - f_{\bar{z}})_\theta$ . We already made it when we considered term (4.27). To estimate the last term we must estimate the the  $r$ -derivatives of  $(g_{(1\bar{0}),\delta} - f_z)_{\theta^2}$  and  $(g_{(0\bar{1}),\delta} - f_{\bar{z}})_{\theta^2}$ . Differentiating (4.17) with respect to  $r$  and  $\theta$  we obtain the terms analogous to already considered and the term

$$e^{i\theta} ((g_{(2\bar{0}),\delta} - f_{2\bar{0}})_{\theta^2} + e^{-i\theta} (g_{(1\bar{1}),\delta} - f_{1\bar{1}})_{\theta^2})$$

Again analogously to (4.19) - (4.21) we reduce estimation to the inequality  $|(f_{(j\bar{k}),\delta} - f_{j\bar{k}})_\theta| \leq Cb$  at  $j + k = m + 2$ .

The case of derivatives of higher degree is analogous. Applying, for example, operator  $(\partial/\partial z)^l$  to the right side of (4.15) we obtain the terms of the types

$$e^{is\theta} [g_{(j\bar{k}),\delta} - f_{(j\bar{k})}], \quad j + k = l + 1, \quad (4.28)$$

$$e^{is\theta} r^{-q} [g_{(j\bar{k}),\delta} - f_{(j\bar{k})}], \quad q \leq l, \quad j + k = l + 1 - q, \quad (4.29)$$

$$e^{is\theta} r^{-q} [g_{(j\bar{k}),\delta} - f_{(j\bar{k})}]_{\theta^p}, \quad q \leq l, \quad p \leq q, \quad j + k = l + 1 - q, \quad (4.30)$$

where  $s$  is some integer depending on the term. Also, we obtain integrals of the types

$$e^{is\theta} r^{-q} \int_0^r [g_{(j\bar{k}),\delta} - f_{(j\bar{k})}] dr, \quad q \leq l + 1, \quad j + k = l + 2 - q, \quad (4.31)$$

and

$$e^{is\theta} r^{-q} \int_0^r [g_{(j\bar{k}),\delta} - f_{(j\bar{k})}]_{\theta^p} dr, \quad q \leq l + 1, \quad p \leq q, \quad j + k = l + 2 - q. \quad (4.32)$$

Again the singular parts must annihilate and terms (4.28) - (4.30) have estimates

$$\sup \left| \frac{\partial^q}{\partial r^q} [g_{(j\bar{k}),\delta} - f_{j\bar{k}}] \right| \leq \sum_{j+k=l+1} \sup |(g_{(j\bar{k}),\delta} - f_{j\bar{k}})| \quad (4.33)$$

and

$$\sum_{j+k=l+1} \sup |(g_{(j\bar{k}),\delta} - f_{j\bar{k}})_{\theta^p}| \quad (4.34)$$

Integrals (4.31), (4.32) have the estimates analogous to (4.33), (4.34), only  $q$  and  $p$  could be equal to  $l+1$ . Analogously to the considerations after (4.17) (see (4.18) - (4.21)) we reduce estimates for right sides of (4.33), (4.34) to estimates (4.13) for  $|f_{(j\bar{k}),\delta} - f_{j\bar{k}}|$  at  $j+k \leq m+q$ ,  $q \leq m$ . The maximal degree  $j+k=2m$  occurs for the terms

$$r^{-m} \int_0^r [g_{(j\bar{k}),\delta} - f_{(j\bar{k})}]_{\theta^m} dr, \quad j+k=m.$$

We proved estimate (4.5).

To show that the mapping  $t \mapsto f_{at}$  is continuous it is enough to prove it for the mapping  $t \mapsto f_{(l),\delta}(\cdot, t)$ . By (4.9), this follows from continuity of the mapping  $t \mapsto f_{(l)}(\cdot, t)$ . But the last follows from estimate (4.1) of Proposition 24: for any  $z \in D$  we have  $f_{(l)}(z, t) \rightarrow f_{(l)}(z, t_0)$  as  $t \rightarrow t_0$ .

Suppose now  $m \geq 2$ . We have decomposition (4.6) with some holomorphic  $h_{at}$ . Consider the map  $f_{at} \circ (f^{\mu_t})^{-1} = h_{at} \circ f^{\mu_{at}} \circ (f^{\mu_t})^{-1}$ . Let  $\mu_{ct}$  be the Beltrami coefficient of the map  $f^{\mu_{at}} \circ (f^{\mu_t})^{-1}$ . By uniqueness, the normal map  $f^{\mu_{ct}}$  coincides with  $f^{\mu_{at}} \circ (f^{\mu_t})^{-1}$ . Denote  $\tilde{\mu}_{at} = \mu_{at} \circ (f^{\mu_t})^{-1} / ((f^{\mu_t})^{-1})_z$ . We have

$$\mu_{ct} = \frac{\tilde{\mu}_{at} + \mu_{(f^{\mu_t})^{-1}}}{1 - \overline{\mu_{(f^{\mu_t})^{-1}}} \tilde{\mu}_{at}} = \frac{\mu_{at} \circ (f^{\mu_t})^{-1} - \mu_t \circ (f^{\mu_t})^{-1} \overline{((f^{\mu_t})^{-1})_z}}{1 - \overline{\mu_{(f^{\mu_t})^{-1}}} \tilde{\mu}_{at}} \frac{\overline{((f^{\mu_t})^{-1})_z}}{((f^{\mu_t})^{-1})_z}$$

since

$$\mu_{(f^{\mu_t})^{-1}} = \mu \circ (f^{\mu_t})^{-1} \frac{f_z^{\mu_t}}{f_z^{\mu_t}} \circ (f^{\mu_t})^{-1} = -\mu_t \circ (f^{\mu_t})^{-1} \frac{\overline{((f^{\mu_t})^{-1})_z}}{((f^{\mu_t})^{-1})_z}$$

We see that, if we fix any  $\varepsilon$ , than at appropriate  $b$  (i.e., at appropriate approximation of  $f^{\mu_t}$ )

$$|\mu_{ct}| \leq \varepsilon, \quad |(\mu_{ct})_z| \leq \frac{\varepsilon}{1-|z|}, \quad |(\mu_{ct})_{\bar{z}}| \leq \frac{\varepsilon}{1-|z|},$$

$$|(\mu_{ct})_{(l)}| \leq \frac{\varepsilon}{(1-|z|)^2}, |l| = 2.$$

By Lemma 1 for any  $\varepsilon$  at appropriate  $b$

$$|(f^{\mu_{ct}})_z - 1| \leq \varepsilon, |(f^{\mu_{ct}})_{(l)}| \leq \frac{\varepsilon}{1-|z|}, |l| = 2.$$

From the other hand, since  $f_{at}$  approximate  $f^{\mu_t}$  up to second derivatives, we obtain analogous estimates for  $f_{at} \circ (f^{\mu_t})^{-1} = h_{at} \circ f^{\mu_{ct}}$ . Thus we obtain estimates (4.7). Hence,  $h_{at}$  is an univalent function ([Pom]) and  $f_{at}$  is a homeomorphism.

At last consider estimate (4.8). It is enough to prove it for the derivatives of approximation (4.9). We have, for example,

$$(f_{(l),\delta})(z, t)_t = \frac{1}{\delta^{2n+1}} \int f_{(l)}(z, \zeta) h_t \left( \frac{t-\zeta}{\delta} \right) dV_\zeta.$$

It follows that (see (4.10), (4.11))

$$|(f_{(l),\delta})(z, t)_t| \leq \frac{C}{(1-|t|)^{\frac{(s_{2m}+2m-1)(2n+1)}{\beta_{2m}}}}$$

Also,

$$\begin{aligned} (f_{(l),\delta})(z, t)_z &= -\frac{(2n+1)\delta_z}{\delta^{2n+2}} \int f_{(l)}(z, \zeta) h \left( \frac{t-\zeta}{\delta} \right) dV_\zeta + \\ &+ \frac{1}{\delta^{2m+1}} \int \left[ f_{(l)}(z, \zeta)_z h \left( \frac{t-\zeta}{\delta} \right) - h_z \left( \frac{z-\zeta}{\delta} \right) \frac{\delta_w}{\delta^2} \right] dV_\zeta. \end{aligned}$$

We again obtain an estimate of type (4.8). We obtain estimates for higher derivatives analogously.  $\square$

**Corollary 1** *The form  $dz + \mu d\bar{z}$  after the change of the variable  $z = f_{at}^{-1}(z_1)$  and division by  $(f_{at}^{-1})_{z_1}$  transforms into the form  $dz_1 + \tilde{\mu} d\bar{z}_1$  defined on  $\cup \Omega_t$ , where  $\tilde{\mu}$  satisfies the estimates*

$$|\tilde{\mu}| \leq d, |(\tilde{\mu})_{(k)}(z_1)| \leq \frac{b_{|k|}}{(\text{dist}(z_1, \partial\Omega_t))^{|k|}}, |k| \leq m,$$

where  $d$  and  $b_k$  can be made arbitrary small at appropriate approximation  $f_{at}$  and we have the estimate

$$|(\tilde{\mu})_{(k,l)}(z_1)| \leq \frac{C}{(\text{dist}(z_1, \partial\Omega_t))^{N_{|k|+|l|}}}$$

with some  $N_{|k|+|l|}$ ,  $|k| \leq m$ .

**Proof.** It is obvious that there are the estimates

$$|\tilde{\mu}| \leq d, |(\tilde{\mu})_{(k)}| \leq \frac{b_{|k|}}{(1 - |f_{at}^{-1}(z_1)|)^{|k|}}, |k| \leq m,$$

$$|(\tilde{\mu})_{(k,l)}(z_1)| \leq \frac{C}{(1 - |f_{at}^{-1}(z_1)|)^{N_{|k|+|l|}}}.$$

But

$$c_1 \text{dist}(z_1, \partial\Omega_t) \leq (1 - |f_{at}^{-1}(z_1)|) \leq c_2 \text{dist}(z_1, \partial\Omega_t)$$

for some uniform  $c_1, c_2$ . Indeed, there is decomposition  $f_{at} = h_{at} \circ f_{\mu_{at}}$ , where for  $f_{\mu_{at}}$  we have inequalities (2.4) and  $h_{at}$  has the derivative close to 1.  $\square$

As a result we obtained the important reduction. To prove Theorem 2 it is enough to prove the next theorem:

**Theorem 2'.** *Let  $\Omega \subset B \times \mathbb{C}$ ,  $B \subset \mathbb{C}^n$  be a domain fibered by topological disks  $\Omega_t, t \in B$  and suppose  $\Omega_t = g_t(D)$ , where  $g_t$  is a  $\nu_t$ -quasiconformal map with  $|\nu_t|$  uniformly bounded away from 1, and the mapping  $t \mapsto g_t$  is continuous as a mapping from  $B$  to  $C^0(D)$ . We suppose that  $g_t$  satisfies the estimates*

$$c \leq |(g_t)_z(z)| \leq C \quad (4.35)$$

for some uniform  $c, C$ ,

$$|(g_t)_{(k)}(z)| \leq \frac{B}{(1 - |z|)^{|k|-1}} \quad (4.36)$$

at  $|k| \leq P, P \geq 4$ ,

$$|(g_t)_{(k,l)}(z)| \leq C(1 - |z|)^{-N} \quad (4.37)$$

at  $|k| \leq P, |l| \leq L$  with some uniform  $B, C$ , and  $N$ . Also, we suppose that there is a decomposition  $g_t = h_t \circ f^{\nu_t}$ , where  $f^{\nu_t}$  is the  $\nu_t$ -quasiconformal normal map and  $h_t$  is a holomorphic univalent function satisfying the estimates

$$|h'(z) - 1| \leq \varepsilon, |h''(z)| \leq \varepsilon(1 - |z|)^{-1} \quad (4.38)$$

with some uniform  $\varepsilon$ .

Let  $\mu$  be a function on  $\Omega$  satisfying the estimates

$$|\mu| \leq b < 1, \quad (4.39)$$

$$|\mu_{(k)}(z, t)| \leq b(\text{dist}(z, \partial\Omega_t))^{-|k|}, \quad (4.40)$$

at  $|k| \leq K$

$$|\partial^{(k,l)}\mu(z,t)| \leq C(\text{dist}(z, \partial\Omega_t))^{-N}, \quad (4.41)$$

at  $|k| \leq K, K \geq 4, |l| \leq L$ . The constant  $C$  here and in (4.37) can depend on  $B$  but the exponent  $N$  doesn't depend.

Then, if  $K \leq P - L$  and the constant  $b$  in (4.39), (4.40) is small enough, there exists a solution  $f$  to the Beltramy equation

$$f_{\bar{z}} = \mu f_z,$$

which is continuous in  $C_0(\mathbb{C})$  as a function of  $t$ , is finitely smooth with respect to all variable up some be-degree  $\{p, q\}$ , where  $p$  and  $q$  can be arbitrary large if  $K$  and  $L$  are large enough, at every  $t$  maps  $\Omega_t$  homeomorphically onto some bounded subdomain of  $\mathbb{C}$ , and satisfies the estimates

$$c(\text{dist}(z, \partial\Omega_t))^\alpha \leq |f_z(z,t)| \leq C(\text{dist}(z, \partial\Omega_t))^{-\alpha}, \quad |f_{\bar{z}}(z)| \leq C(\text{dist}(z, \partial\Omega_t))^{-\alpha},$$

$$0 \geq \alpha < 1, \quad (4.42)$$

$$|f_{(k)}(z)|/|f_z(z,t)| \leq C(\text{dist}(z, \partial\Omega_t))^{1-|k|}, \quad (4.43)$$

at  $|k| \leq p$ ,

$$|\partial^{(k,l)}f(z,t)| \leq C(\text{dist}(z, \partial\Omega_t))^{-M} \quad (4.44)$$

for some uniform  $C$  and  $M$  at  $|k| \leq p, |l| \leq q$ .

The conditions  $P \geq 4, K \geq 4$  are of technical character, we shall use them in Section 7.

## 5 Extension of quasiconformal mappings

In this section we consider a family of quasiconformal mappings  $g_t$  satisfying estimate (4.35) -(4.38) of Theorem 2'. Mostly we have deal with an individual map  $g$  and we shall omit dependence on  $t$ . Thus  $g$  is a  $\nu$ -quasiconformal map, mapping  $D$  onto the domain  $\Omega$  and

$$|\nu(z)| \leq b < 1, \quad (5.1)$$

$$|\nu_{(k)}(z)| \leq \frac{B}{(1 - |z|)^{|k|}} \quad (5.2)$$

at  $|k| \leq P$ ,

$$|\nu_{(k,l)}(z,t)| \leq C(1 - |z|)^{-N} \quad (5.3)$$



at  $|k| \leq P, |l| \leq L$  with some uniform  $B$  and  $C$ . The last two inequalities follow from (4.35) -(4.37). There is the decomposition  $g = h \circ f^\nu$  and, in addition to estimates (4.38), from lemma 1 and (4.36) follows

$$\left| \frac{d^k}{dz^k} h(z) \right| \leq C(1 - |z|)^{k-1} \quad (5.4)$$

at  $k \leq P$  with some uniform  $C$ . Also, we have the obvious estimates

$$a(1 - |z|) \leq \text{dist}(g(z), \partial\Omega) \leq A(1 - |z|) \quad (5.5)$$

for some uniform  $a, A$ .

In what follows we shall need estimates for derivatives of the normal mappings in the particular case when all derivatives of  $\mu$  are uniformly bounded. ]

**Proposition 25** *Suppose  $\mu$  is smooth, has the support in  $D$ , and smoothly depends on a vector parameter  $t \in B \subset \mathbb{C}^n$ . Suppose the derivatives  $\mu_{(k,l)}$  satisfy the estimate*

$$|\mu_{(k,l)}| \leq M$$

at  $|k| \leq K + L, |l| \leq L$  with some constant  $M$  uniform with respect to the parameters. Then the derivatives of the normal mapping  $(f^\mu)_{(k,l)}$  satisfy the estimate

$$|(f^\mu)_{(k,l)}| \leq CM^{10(|k|+|l|)^2}$$

at  $|k| \leq K, |l| \leq L$  with some uniform  $C$ .

**Proof.** At first we estimate the derivatives of the principal solution  $f_\mu$ . Consider the derivative of first order in  $t$ . Namely we must estimate  $t$ -derivatives of the function  $\mathcal{C}h_\mu$ , where  $h_\mu$  is the solution to the equation

$$h - \mu \mathcal{S}h = \mu. \quad (5.6)$$

Differentiating by  $t$  we obtain

$$h_t - \mu \mathcal{S}h_t = \mu_t \mathcal{S}h + \mu_t.$$

The  $p$ -norm of the right side is no greater than  $CM$  and for  $h_t$  we obtain the estimate  $\|h_t\|_p \leq C_p CM / (1 - d)$ . It follows that for  $|f_\mu|$  we also have the estimate  $CM$  with some uniform  $C$ .

At further differentiation we obtain the equation

$$h_{0,(l)} - \mu \mathcal{S}h_{0,(l)} = F,$$

where for  $F$  we obtain by induction the estimate  $\|F\|_p \leq CM^{(l)}$ . Thus  $|(f_\mu)_{(0,l)}| \leq CM^{(l)}$ .

Now remind that for a smooth compactly supported  $\mu$  we can represent the function  $(f_\mu)_z$  as  $e^h$ , where  $h$  satisfies the equation

$$h_{\bar{z}} = \mu h_z + \mu_z \quad (5.7)$$

and tends to zero at infinity, i.e.,  $h = \mathcal{C}H$ , where  $H$  is the unique solution to the equation

$$H - \mu \mathcal{S}H = \mu_z.$$

For  $H_t$  we obtain the equation

$$H_t - \mu \mathcal{S}H_t = \mu_t \mathcal{S}H + \mu_{zt}. \quad (5.8)$$

Since  $\|\mu_t \mathcal{S}h\|_p \leq CM^2$ , we obtain the estimate  $\|H_t\|_p \leq CM^2$ . For  $H_{(0,l)}$  we obtain by induction the equation analogous to (5.8) with the right side  $F$  such that  $\|F\|_p \leq CM^{(l)+1}$  and, hence, the same estimate for  $|(f_\mu)_z)_{(0,l)}| = |(e^{\mathcal{C}H})_{(0,l)}|$ .

Now,  $(f_\mu)_{z^2} = e^h h_z = e^h g$ , where  $g$  satisfies the equation obtained by differentiation of equation (5.7)

$$g_{\bar{z}} - \mu g_z = \mu_z g + \mu_{z^2}.$$

But  $f$  is holomorphic outside of  $D$  and  $f(z) - z$  tends to zero when  $z \rightarrow \infty$ . Hence, all derivatives of order higher than one also tend to zero at infinity and the same holds for the function  $g$ . It implies that  $g = \mathcal{C}G$ , where  $G$  is the unique solution to the equation

$$G - \mu \mathcal{S}G = \mu_z g + \mu_{z^2}.$$

For  $G_t$  we obtain the equation

$$G_t - \mu \mathcal{S}G_t = \mu_t \mathcal{S}G + \mu_{zt}g + \mu_z g_t + \mu_{z^2}t \quad (5.9)$$

Since  $\|\mu_z g\|_p = \|\mu_z h_z\|_p \leq CM^2$ , we see that  $\|G\|_p \leq CM^2$ , and the right side of equation (5.9) has the  $L_p$ -estimate  $CM^3$ . At further differentiation

by parameters we obtain for  $G_{(0,l)}$  the equation with the right part estimated in  $L^p$  as  $CM^{l(l)+2}$ .

When we consider derivatives with respect to  $z$  of higher order we analogously can see that we obtain equations with right sides having the estimate  $CM^s$  with the exponent rising by 1 at each differentiation. As a result, we obtain the estimate

$$|(f_\mu)_{(k,l)}| \leq CM^{l(k)+l(l)}. \quad (5.10)$$

Consider now the normal solution  $f = f^\mu$ . By (2.58), (2.59), we can represent  $f_{(k,l)}$  as a sum of terms of the types

$$a(f_\lambda)_{(0,p)}(1) - f_{lambda}(0,p)(0) \tilde{f}_{(0,q)}(1) (f_\lambda)_{(r,s)} \circ \tilde{f} \tilde{f}_{(k_1,l_1)}^{r_1} \dots \tilde{f}_{(k_j,l_j)}^{r_j}, \quad (5.11)$$

where  $r \leq |(k)|+|(l)|$ ,  $p+q+s \leq |(l)|$ ,  $r_1(|(k_1)|+|(l_1)|)+\dots+r_j(|(k_j)|+|(l_j)|) \leq |(k)|+|(l)|$ . For the product  $\tilde{f}_{(k_1,l_1)}^{r_1} \dots \tilde{f}_{(k_j,l_j)}^{r_j}$  according (5.10) we have the estimate  $CM^{l(k)+l(l)}$ . From the other hand,  $\lambda_{(k,l)}$  can be represented as a sum of items of the types

$$\mu_{(p,q)} \circ \tilde{f}^{-1}(\tilde{f}_z)_{r,s} \circ \tilde{f}^{-1}(\tilde{f}_{(k_1,l_1)}^{-1})^{r_1} \dots (\tilde{f}_{(k_j,l_j)}^{-1})^{r_j}[\cdot],$$

where  $[\cdot]$  is a multiple bounded by a constant independent of  $(k,l)$ ,  $p+r \leq |(k)|+|(l)|$ ,  $q+s+r_1(|(k_1)|+|(l_1)|)+\dots+r_j(|(k_j)|+|(l_j)|) \leq |(k)|+|(l)|$ . But  $|(\tilde{f}_{(k_1,l_1)}^{-1})^{r_1} \dots (\tilde{f}_{(k_j,l_j)}^{-1})^{r_j}| \leq CM^{l(k)+l(l)}$ ,  $|\mu_{(p,q)}| \leq CM$ ,  $|(\tilde{f}_z)_{r,s}| \leq CM^{r+s+1}$  according to (5.10). We obtain  $|\lambda_{(k,l)}| \leq CM^{3(l(k)+l(l))+2}$ . We apply estimate (5.10) to  $f_\lambda$  and obtain the estimates  $|(f_\lambda)_{(r,s)}| \leq CM^{2(l(k)+l(l))[3(l(k)+l(l))+2]} \leq CM^{7(l(k)+l(l))^2}$ .  $|f_\lambda)_{(0,p)}| \leq CM^{l(l)[3(l(k)+l(l))+2]}$  for the terms in (5.11). As a result, we obtain for product (5.11) the estimate  $CM^{10(l(k)+l(l))^2}$ .  $\square$

Now we shall motivate the following constructions of this section. When we defined the transforms  $\mathcal{C}_m$  and  $\mathcal{S}_m$  in Section 1, we introduced into the kernels the counter-items of the types

$$\frac{(\bar{\zeta}^{-1} - \zeta)^{l-1}}{(\bar{\zeta}^{-1} - z)^l}. \quad (5.12)$$

We want to solve the Beltrami equation on the domain  $\Omega$ , and we need to define analogous counter-items to neutralize the growth of derivatives near the boundary. Namely we must find some map replacing the mapping  $z \mapsto \bar{z}^{-1}$  when we deal with the domain  $\Omega$  instead of the disk  $D$ . For that we define some extension of the map  $g$  on the domain  $\mathbb{C} \setminus \Omega$ . It appears, we can find a sufficiently good extension if  $\varepsilon$  in (4.38) is sufficiently small.

**Lemma 4** . Let the family of maps  $g_t$  satisfy conditions of Theorem 2'. If  $\varepsilon$  is small enough we can define for each  $t$  an extension of  $g_t$  to a quasiconformal homeomorphisms of the plane  $G_t$ ; we denote by  $\hat{g}_t$  its restriction on  $\mathbb{C} \setminus D$ . For the map  $\hat{g}_t$  we have the estimates

$$\hat{c}_1 \leq |\hat{g}_z(z, t)| \leq \hat{c}_2, \quad (5.13)$$

$$|\hat{g}_{(k)}(z, t)| \leq B(|z| - 1)^{1-|k|} \quad (5.14)$$

at  $|k| \leq P - L$ ,

$$|\hat{g}_{(k),(l)}(z, t)| \leq C(|z| - 1)^{-M} \quad (5.15)$$

with some  $M$  depending only on  $N$  in inequality (4.37) at  $|k| \leq P - L$ ,  $|l| \leq L$ ,

$$\hat{a}(|z|^2 - 1) \leq \text{dist}(\hat{g}(z, t), \partial\Omega_t) \leq \hat{A}(|z|^2 - 1), \quad (5.16)$$

$$c(1 - |z|^2) \leq |g(z, t) - \hat{g}(\bar{z}^{-1}, t)| \leq C(1 - |z|^2). \quad (5.17)$$

All constants in these inequalities are uniform.

**Proof.** In most part of the proof we fix some  $t$  and omit  $t$ -dependence.

Let  $r_n > 1/2$ ,  $n \geq 1$  be some sequence tending to 1 and consider the sequence  $D_{r_n}$  of disks of radii  $r_n$  centered at zero. Let  $\eta$  be a smooth function on the real axis  $\eta(x) = 1, x \leq 0, \eta(x) = 0, x \geq 1$ . Let  $\nu_n, h_n$  be the functions  $\nu_n(z) = \nu(z)\eta((|z| - r_n)/(1 - r_n))$ ,  $h_n(z) = h(r_n z)$ . Let  $f^{\nu_n}$  be the normal  $\nu_n$ -quasiconformal homeomorphism mapping  $D$  onto itself and  $g_n$  be the map  $g_n = h_n \circ f^{\nu_n}$ . The homeomorphism  $g_n$  maps  $D$  onto some domain  $\Omega_n$  with a smooth boundary and the sequence  $g_n$  converges to  $g$  uniformly on compact subsets of  $D$ .

On any disk  $D_{r_n}$  the map  $g|_{D_{r_n}}$  can be represented as the composition  $g = f_n \circ f^{\nu_n}$ , where  $f_n$  is some function holomorphic on  $f^{\nu_n}(D_{r_n})$ .

From (5.1) - (5.4) and (4.38) immediately follow the estimates

$$|\nu_n(z)| \leq b < 1, \quad (5.18)$$

$$|(\nu_n)_{(k)}(z)| \leq B(1 - r_n|z|)^{-|k|} \quad (5.19)$$

at  $|k| \leq m$ ,

$$|(\nu_n)_{(k),(l)}(z, t)| \leq C(1 - |r_n|)^{-N} \quad (5.20)$$

at  $|k| \leq m, |l| \leq L$ ,

$$|h'_n(z) - 1| \leq 1 - r_n + 2\varepsilon, |h''_n(z)| \leq \varepsilon(1 - r_n^2|z|^2)^{-1}, \quad (5.21)$$

$$\left| \frac{d^k}{dz^k} h_n(z) \right| \leq C(1 - r_n|z|)^{k-1} \quad (5.22)$$

at  $k \leq m$ . Also, we have the estimate analogous to (4.35), (4.36), and (5.5)

$$c_1 \leq |(g_n)_z| \leq c_2, \quad (5.23)$$

$$|(g_n)_{(k)}(z)| \leq B(1 - |z|)^{1-|k|} \quad (5.24)$$

at  $|k| \leq m$ ,

$$a(1 - |z|^2) \leq \text{dist}(g_n(z), \partial\Omega_n) \leq A(1 - |z|^2). \quad (5.25)$$

All constants in these estimates are uniform and independent of  $n$  and  $r_n$ . Below in this section "uniform" means, in particular, that estimate or constant is independent of  $n$  and  $r_n$ .

From Proposition 25 we obtain

$$|f_{(k,l)}^{\nu_n}| \leq C(1 - |r_n|)^{-M}, \quad (5.26)$$

$$|(f_n)_{(k,l)} \circ f^{\nu_n}| \leq C(1 - |r_n|)^{-M} \quad (5.27)$$

with some  $M$  independent of  $n$  at  $|k| \leq m - L, |l| \leq L$ .

**Proposition 26** *There exists some uniform  $B$  such that at  $1 - |z| \geq B(1 - r_m)$*

a)

$$|f_z^{\nu_n}(z) - f_z^{\nu_m}(z)| \leq C \frac{(1 - r_m)^\beta}{1 - |z|}. \quad (5.28)$$

$$|f_z^\nu(z) - f_z^{\nu_m}(z)| \leq C \frac{(1 - r_m)^\beta}{1 - |z|}. \quad (5.29)$$

b)

$$|h'_m \circ f^{\nu_m}(z) - f'_m \circ f^{\nu_m}(z)| \leq C \frac{(1 - r_m)^\beta}{1 - |z|}, \quad (5.30)$$

$$|h''_m \circ f^{\nu_m}(z) - f''_m \circ f^{\nu_m}(z)| \leq \frac{(1 - r_m)^\beta}{(1 - |z|)^2}. \quad (5.31)$$

c)

$$|f'_n \circ f^{\nu_n}(z) - f'_m \circ f^{\nu_m}(z)| \leq C \frac{(1-r_m)^\beta}{1-|z|}, \quad (5.32)$$

$$|f''_n \circ f^{\nu_n}(z) - f''_m \circ f^{\nu_m}(z)| \leq \frac{(1-r_m)^\beta}{(1-|z|)^2}. \quad (5.33)$$

Everywhere  $\beta > 0$  depends only on  $b$  in (5.1).

**Proof.** a) The second inequality is a limit case of the first one. We apply inequality (3.2) of Lemma 2 to estimate  $|f_z^{\nu_n} - f_z^{\nu_m}|$ . We estimate different terms in the right-side. At first consider  $\|\nu_n - \nu_m\|_p$ .

The functions  $\nu_n$  and  $\nu_m$  differ only in the ring  $R_{m,n} = \{r_m \leq |z| \leq r_n + (1-r_n)/2 = (1+r_n)/2\}$ . Hence,

$$\begin{aligned} \|\nu_n - \nu_m\|_p &= \left( \int_{R_{m,n}} dS_z \right)^{1/p} \leq C_p (1+r_n - 2r_m)^{1/p} \leq \\ &\leq C_p |1-r_m|^{1/p}, \quad n > m, \end{aligned} \quad (5.34)$$

with some  $C_p$  depending on  $p$ . Thus,

$$\|\nu_n - \nu_m\|_p^\alpha \leq C |1-r_m|^{\alpha/p}$$

Now, since  $\nu_n - \nu_m = 0$  on  $D_{r_m}$ , we obtain putting  $R = r_m$

$$\sup_{D_R} |\nu_n - \nu_m|^\alpha + \left( \frac{1-R}{1-|z|} \right)^{\alpha/p} = \left( \frac{1-r_m}{1-|z|} \right)^{\alpha/p}.$$

At last,  $\sup_{D_{R_z}} |(\nu_n)_{(l)} - (\nu_m)_{(l)}|^\alpha = 0$  for  $z$  such that  $R_z \leq r_m$ . Thus we obtain (5.28) with  $\beta = \alpha/p$ .

b) Let prove inequality (5.30).

$$|h'_m \circ f^{\nu_m} - f'_m \circ f^{\nu_m}| \leq |h'_m \circ f^{\nu_m}(z) - h' \circ f^{\nu_m}(z)| + |h' \circ f^{\nu_m} - h' \circ f^\nu| + |h' \circ f^\nu - f'_m \circ f^{\nu_m}|.$$

But

$$\begin{aligned} |h'_m(z) - h'(z)| &= |r_m h'(r_m z) - h'(z)| \leq (1-r_m) |h'(r_m z)| + |h'(r_m z) - h'(z)| \leq \\ &\leq c(1-r_m) + \varepsilon \frac{|r_m z - z|}{1-|z|} \leq C \frac{1-r_m}{1-|z|}. \end{aligned}$$

Here we applied estimates (4.38). Hence,

$$|h'_m \circ f^{\nu_m}(z) - h' \circ f^{\nu_m}(z)| \leq C \frac{1-r_m}{1-|f^{\nu_m}(z)|} \leq C \frac{1-r_m}{1-|z|} \quad (5.35)$$

according estimates (2.4).

Also, applying (4.38), (2.4), estimate (3.1) of Lemma 2, and (5.34), we have

$$|h' \circ f^{\nu_m}(z) - h' \circ f^{\nu}(z)| \leq \sup_{[f^{\nu_m}(z), f^{\nu}(z)]} |h''| |f^{\nu_m}(z) - f^{\nu}(z)| \leq \frac{C}{1-|z|} (1-r_m)^\beta \quad (5.36)$$

for some  $\beta$  depending only on  $b$  in (5.1).

At last,

$$h \circ f^{\nu} = f_m \circ f^{\nu_m} \quad (5.37)$$

on  $D_{r_m}$  and, hence,  $h' \circ f^{\nu} \cdot (f^{\nu})_z = f'_m \circ f^{\nu_m} \cdot (f^{\nu_m})_z$ . Thus,

$$|h' \circ f^{\nu} - f'_m \circ f^{\nu_m}| = |h' \circ f^{\nu}| |1 - (f^{\nu})_z / (f^{\nu_m})_z| \leq C(1-r_m)^\beta / (1-|z|) \quad (5.38)$$

by (5.29), if  $1-|z| \geq B(1-r_m)$ . Taking into consideration (5.35) and (5.36), we obtain (5.30).

The proof of inequality (5.31) is analogous. We have

$$|h''_m \circ f^{\nu_m} - f''_m \circ f^{\nu_m}| \leq |h''_m \circ f^{\nu_m}(z) - h'' \circ f^{\nu_m}(z)| + |h'' \circ f^{\nu_m} - h'' \circ f^{\nu}| + |h'' \circ f^{\nu} - f''_m \circ f^{\nu_m}|.$$

Applying (4.38) and (5.4), we can see that

$$\begin{aligned} |h''_m(z) - h''(z)| &= |r_m^2 h''(r_m z) - h''(z)| \leq (1-r_m^2) |h''(r_m z)| + |h''(r_m z) - h''(z)| \leq \\ &\leq c\varepsilon \frac{1-r_m}{1-r_m|z|} + C \frac{|r_m z - z|}{(1-|z|)^2} \leq C \frac{1-r_m}{(1-|z|)^2}. \end{aligned}$$

Hence,

$$|h''_m \circ f^{\nu_m}(z) - h'' \circ f^{\nu_m}(z)| \leq C \frac{1-r_m}{(1-|f^{\nu_m}(z)|)^2} \leq C \frac{1-r_m}{(1-|z|)^2} \quad (5.39)$$

Also, applying (5.4), (2.4), estimate (3.1) of Lemma 2, and (5.34), we obtain

$$|h'' \circ f^{\nu_m}(z) - h'' \circ f^{\nu}(z)| \leq \sup_{[f^{\nu_m}(z), f^{\nu}(z)]} |h'''| |f^{\nu_m}(z) - f^{\nu}(z)| \leq \frac{C}{(1-|z|)^2} (1-r_m)^\beta \quad (5.40)$$

for some  $\beta$  depending only on  $b$  in (5.1).

As above, from (5.37) we have  $h'' \circ f^\nu \cdot ((f^\nu)_z)^2 + h' \circ f^\nu \cdot (f^\nu)_{z^2} = f_m'' \circ f^{\nu_m} \cdot ((f^{\nu_m})_z)^2 + f_m' \circ f^{\nu_m} \cdot ((f^{\nu_m})_{z^2})$ . Thus,

$$|h'' \circ f^\nu - f_m'' \circ f^{\nu_m}| \leq |h'' \circ f^\nu| \left| 1 - \frac{((f^\nu)_z)^2}{((f^{\nu_m})_z)^2} \right| +$$

$$+ |(f^{\nu_m})_z|^{-2} (|h' \circ f^\nu - f_m' \circ f^{\nu_m}| |(f^\nu)_{z^2}| + |f_m' \circ f^{\nu_m}| |(f^\nu)_{z^2} - (f^{\nu_m})_{z^2}|).$$

Consider the right side.

$$|h'' \circ f^\nu(z)| \left| 1 - \frac{((f^\nu)_z)^2}{((f^{\nu_m})_z)^2} \right| \leq C_\varepsilon \frac{(1 - r_m)^\beta}{(1 - |z|)^2}$$

by (4.38), (2.4) and (5.29).

$$|h' \circ f^\nu(z) - f_m' \circ f^{\nu_m}(z)| |(f^\nu)_{z^2}| \leq C(1 - |z|)^{-2} (1 - r_m)^\beta$$

by Lemma 1 and (5.38). At last, the difference  $|(f^\nu)_{z^2} - (f^{\nu_m})_{z^2}|$  we estimate by inequality (3.2) of Lemma 2. From, in fact, the same considerations as in the proof of inequality (5.28), we can see that

$$|(f^\nu)_{z^2} - (f^{\nu_m})_{z^2}| \leq C(1 - r_m)^\beta (1 - |z|)^{-2}$$

Thus,

$$|h'' \circ f^\nu - f_m'' \circ f^{\nu_m}| \leq C(1 - r_m)^\beta (1 - |z|)^{-2}. \quad (5.41)$$

Collecting inequalities (5.39) - (5.41) we obtain (5.31).

c) We have

$$|f_n' \circ f^{\nu_n} - f_m' \circ f^{\nu_m}| \leq |f_n' \circ f^{\nu_n} - h' \circ f^\nu| + |h' \circ f^\nu - f_m' \circ f^{\nu_m}|,$$

$$|f_n'' \circ f^{\nu_n} - f_m'' \circ f^{\nu_m}| \leq |f_n'' \circ f^{\nu_n} - h'' \circ f^\nu| + |h'' \circ f^\nu - f_m'' \circ f^{\nu_m}|.$$

The required estimates follow from (5.38) and (5.41).  $\square$ .

Now we shall prove the lemma itself. We proceed in several steps.

1) *First extension of the map  $g_n$ .*

We use the modified construction of the Loewner chains from the theory of univalent functions. For a function  $f(z, t)$ , ( $z \in \mathbb{C} \setminus D, 0 \leq t < \infty$ ) we write  $\dot{f} = \partial f / \partial t$ .



We say that the family of functions  $f(z, t) = f_t(z)$  ( $z \in \mathbb{C} \setminus D, 0 \leq t < \infty$ ) is a Loewner chain if there exists a function  $p(z, t)$ , ( $z \in \mathbb{C} \setminus D, 0 \leq t < \infty$ ) such that  $\text{Re}p(z, t) > 0$  and

$$f_t(z, t) = [zf_z(z, t) - \bar{z}f_{\bar{z}}(z, t)]p(z, t). \quad (5.42)$$

For any  $z, |z| = r$  vector  $zf_z(z, t) - \bar{z}f_{\bar{z}}(z, t)$  is the vector of the outer normal at a boundary point of the set  $\{f_t(z) : |z| \leq r\}$ . Condition (5.42) means that

$$|\arg f_t(z, t) - \arg [zf_z(z, t) - \bar{z}f_{\bar{z}}(z, t)]| = |\arg p(z, t)| < \pi/2, \quad (5.43)$$

and this means that the velocity vector  $f_t$  on the boundary points out of this set.

Instead of conditions (5.42), (5.43) we shall use the equivalent condition

$$\left| \frac{p-1}{p+1} \right| = \left| \frac{f_t - zf_z + \bar{z}f_{\bar{z}}}{f_t + zf_z - \bar{z}f_{\bar{z}}} \right| < 1. \quad (5.44)$$

In what follows we denote by  $D_{\pm}$  the operators

$$D_{\pm} = \frac{\partial}{\partial t} \pm \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right).$$

For  $z : |z| \geq 1$  define the function  $\hat{f}^{\nu_n}(z) = 1/\overline{f^{\nu_n}(1/\bar{z})}$ . It is the symmetrical extension of  $f^{\nu_n}$ .

Now define for  $z : |z| \geq 1$

$$\hat{h}_n(z, t) = h_n \circ f^{\nu_n}(e^{-t}/\bar{z}) + [\hat{f}^{\nu_n}(e^t/\bar{z}) - f^{\nu_n}(e^{-t}/\bar{z})]h'_n \circ f^{\nu_n}(e^{-t}/\bar{z}).$$

Let check for this function condition (5.44) at  $|e^t/\bar{z}| \leq 2$ . We denote by  $w$  the chart in the image of maps  $f^{\nu_n}$  and  $\hat{f}^{\nu_n}$  and by  $\omega$  the chart in the preimage of this map. We have

$$\begin{aligned} & D_- \hat{h}_n(z, t) = \\ & = h_{n,w} \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \left[ -f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} - f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} + f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} - f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} \right] + \\ & \quad + \left[ \hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}} + \hat{f}_{\bar{\omega}}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{z} + \hat{f}_{\bar{\omega}}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{z} - \hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}} + \right. \\ & \quad \left. + f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} + f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} - f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} + f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} \right] h'_n \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) + \end{aligned}$$

$$\begin{aligned}
& + \left[ \hat{f}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) - f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \right] h_n'' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \times \\
& \times \left[ -f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} - f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} + f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} - f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} \right] = \\
& = 2h_n' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \hat{f}_{\bar{\omega}}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{z} - \\
& - 2 \left[ \hat{f}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) - f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \right] h_n'' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}} \\
& D_+ \hat{h}_n(z, t) = -2h_n' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} + \\
& + \left[ 2\hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}} + 2f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} \right] h_n' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) - \\
& - 2 \left[ \hat{f}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) - f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \right] h_n'' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z} = \\
& = 2\hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}} h_n' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) - \\
& - 2 \left[ \hat{f}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) - f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \right] h_n'' \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{z}
\end{aligned}$$

Applying inequalities (2.4) and estimate (2.1) of lemma 1 to  $f^{\nu_n}$ , we obtain

$$|f^{\nu_n}(e^{-t}z) - f^{\nu_n}(e^{-t}/\bar{z})| \leq Ce^{-t}(|z|^2 - 1)/|z|$$

for some  $C$  independent of  $n$  at  $|e^t/\bar{z}| \leq 2$ . Hence,

$$\begin{aligned}
& \left| \hat{f}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) - f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \right| \leq \\
& \leq \left| \frac{1}{f^{\nu_n}(e^{-t}z)} - f^{\nu_n}(e^{-t}z) \right| + |f^{\nu_n}(e^{-t}z) - f^{\nu_n}(e^{-t}/\bar{z})| \leq \\
& \leq C[(1 - |f^{\nu_n}(e^{-t}z)|^2) + e^{-t}(1 - |z|^2)/|z|] \leq C'(1 - |e^{-t}/\bar{z}|^2) \quad (5.45)
\end{aligned}$$

for some  $C'$  independent of  $n$ . The last inequality holds since  $1 - |e^{-t}z|^2 \leq 1 - |e^{-t}/\bar{z}|^2$  at  $|z| \geq 1$ ,  $|e^{-t}z| \leq 1$  and  $e^{-t}(|z|^2 - 1)/|z| \leq 1 - e^{-t}/|z| \leq 1 - |e^{-t}/\bar{z}|^2$ .

Also,

$$\hat{f}_z^{\nu_n}(z) = \frac{1}{f^{\nu_n}(1/\bar{z})^2} \overline{f_z^{\nu_n}(1/\bar{z})} \frac{1}{z^2}, \quad (5.46)$$

$$\hat{f}_{\bar{z}}^{\nu_n}(z) = \frac{1}{f^{\nu_n}(1/\bar{z})^2} \overline{f_{\bar{z}}^{\nu_n}(1/\bar{z})} \frac{1}{\bar{z}^2}. \quad (5.47)$$

Hence,

$$\hat{f}_{\bar{\omega}}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{z} = \bar{\nu}_n(e^{-t}z) \frac{z}{\bar{z}} \hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}}.$$

Also,

$$\left| \frac{f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right) \frac{e^{-t}}{\bar{z}}}{\hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}}} \right| = \left| \bar{z}^{-2} (f^{\nu_n}(e^{-t}z))^2 \frac{f_{\omega}^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right)}{f_{\omega}^{\nu_n}(e^{-t}z)} \right| \leq M$$

at  $|z| \geq 1$  for some  $M$  independent of  $n$ . Thus dividing  $D_{\pm} \hat{h}_n(z, t)$  by  $\hat{f}_{\omega}^{\nu_n} \left( \frac{e^t}{\bar{z}} \right) \frac{e^t}{\bar{z}} h_{n,w} \circ f^{\nu_n} \left( \frac{e^{-t}}{\bar{z}} \right)$  and applying estimates (5.18) and (5.21) we obtain

$$\left| \frac{D_- \hat{h}_n(z, t)}{D_+ \hat{h}_n(z, t)} \right| \leq \frac{b + CM\varepsilon}{1 - CMb\varepsilon} \leq b + B\varepsilon$$

for some uniform  $B$ . We see that  $\hat{h}_n$  is a Loewner chain at small enough  $\varepsilon$  if  $|e^t/\bar{z}| \leq 2$ .

Now we define

$$\hat{h}_n(z) = h_n \circ f^{\nu_n}(1/\bar{z}) + [\hat{f}^{\nu_n}(z) - f^{\nu_n}(1/\bar{z})] h'_n \circ f^{\nu_n}(1/\bar{z}). \quad (5.48)$$

at  $|z| \geq 1$ .

We prove that  $\hat{h}_n$  is a quasiconformal homeomorphism of  $\mathbb{C}$  extending  $g_n$ . At first we note that  $\hat{h}_n(z) = \hat{h}_n(z', t)$  for  $z$  represented in the form  $z = e^t z'$ ,  $t \geq 1, |z'| = 1$ . Thus in some neighborhood of the unite circle the function  $\hat{h}_n$  maps the point  $e^t z'$  into a point on the trajectory of the vector field  $P(e^t z') = \hat{h}_{n,t}(z', t)$  starting at  $z'$ . As it follows from (5.18), (5.21), the map  $g_n$  extends to a  $C^1$ -diffeomorphism of the unit circle onto the boundary  $\partial\Omega_n$ . Our vector field is transversal to this boundary and we obtain homeomorphism of some neighborhood of  $\Omega_n$  extending  $g$ .

Now we prove that  $\hat{h}_n$  is a local homeomorphism at any point  $|z| > 1$  with the complex dilatation bounded by some constant less than 1 depending only

on  $\nu$ . We denote  $\omega = 1/\bar{z}$ . We have

$$\begin{aligned}
\hat{h}_{n,z}(z) &= -h_{n,w} \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{z^2} + \\
&+ \left[ \hat{f}^{\nu_n}(z) + f_{\bar{\omega}}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{z^2} \right] h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) - \\
&- \left[ \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right] h''_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{z^2} = \\
&= \hat{f}_z^{\nu_n}(z) h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) - \left[ \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right] h''_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{z^2}, \tag{5.49}
\end{aligned}$$

$$\begin{aligned}
\hat{h}_{n,\bar{z}}(z) &= \hat{f}_{\bar{z}}^{\nu_n}(z) h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) - \\
&- \left[ \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right] h''_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) f_{\omega}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{z^2}. \tag{5.50}
\end{aligned}$$

We have analogously to (5.45)

$$\begin{aligned}
&\left| \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| = \\
&= \left| \frac{1}{f^{\nu_n}(1/\bar{z})} - f^{\nu_n}(1/\bar{z}) \right| \leq C[1 - |f^{\nu_n}(1/\bar{z})|^2]. \tag{5.51}
\end{aligned}$$

Thus,

$$\left| \hat{h}_{n,z}(z) - \hat{f}_z^{\nu_n}(z) h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| \leq C\varepsilon, \tag{5.52}$$

$$\left| \hat{h}_{n,\bar{z}}(z) - \hat{f}_{\bar{z}}^{\nu_n}(z) h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| \leq C\varepsilon, \tag{5.53}$$

for some  $C$  depending only on  $\nu$ . Taking into consideration (5.46), (5.47), we see that  $|\hat{h}_{n,z}|$  is bounded away from zero and  $|\hat{h}_{n,\bar{z}}/\hat{h}_{n,z}|$  is bounded by some uniform constant  $b' < 1$ .

We don't consider behavior of our mapping at infinity because on the next step we modify it outside some  $D_r, r > 1$ .

2) *Modification of  $\hat{h}_n$ .*

At  $|z| \geq 1/r_n$  we define

$$\hat{f}_n(z) = f_n \circ f^{\nu_n}(1/\bar{z}) + [\hat{f}^{\nu_n}(z) - f^{\nu_n}(1/\bar{z})]f'_n \circ f^{\nu_n}(1/\bar{z}). \quad (5.54)$$

Let  $\rho(s)$ ,  $s \geq 0$  be some monotonic smooth function,  $\rho(s) = 1$ ,  $s \leq 1$ ,  $\rho(s) = 0$ ,  $s \geq 2$ ,  $|\rho'(s)| \leq 2$ . Set some  $d_n \geq 1 - r_n$  and define at  $|z| \geq 1$

$$\tilde{g}_n(z) = \hat{h}_n(z)\rho((d_n)^{-1}(|z| - 1)) + \hat{f}_n(z)[1 - \rho((d_n)^{-1}(|z| - 1))]. \quad (5.55)$$

**Proposition 27** *There exists some uniform  $M$  such that at*

$$d_n \geq M(1 - r_n)^\beta, \quad (5.56)$$

where  $\beta$  is the exponent from Proposition 26, the map  $\tilde{G}_n$  defined on  $D$  as  $g_n$  and on  $\mathbb{C} \setminus D$  as  $\tilde{g}_n$  is a quasiconformal homeomorphism of the plane with uniformly bounded dilatation and with derivative  $(\tilde{g}_n)_z$  uniformly bounded from below and from above.

**Proof.** At first we prove that we can find  $M$  such that on the domain  $B_M = \{|z| \geq 1 + M(1 - r_n)^\beta\}$  the map  $\hat{f}_n$  is a local homeomorphism.

Analogously to (5.49), (5.50) and applying (5.51), we obtain

$$\begin{aligned} \hat{f}_{n,z} &= \hat{f}_z^{\nu_n}(z)f'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) - \\ &- \left[ \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right] f''_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) f_{\bar{\omega}}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{z^2}, \end{aligned} \quad (5.57)$$

$$\begin{aligned} \hat{f}_{n,\bar{z}}(z) &= \hat{f}_{\bar{z}}^{\nu_n}(z)f'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) - \\ &- \left[ \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right] f''_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) f_{\omega}^{\nu_n} \left( \frac{1}{\bar{z}} \right) \frac{1}{\bar{z}^2}. \end{aligned} \quad (5.58)$$

Applying estimates (5.30), (5.31) and (5.51), we see that

$$\left| \hat{f}_{n,z}(z) - \hat{f}_z^{\nu_n}(z)h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| \leq c \frac{(1 - r_n)^\beta}{1 - |f^{\nu_n}(1/\bar{z})|^2}, \quad (5.59)$$

$$\left| \hat{f}_{n,\bar{z}}(z) - \hat{f}_{\bar{z}}^{\nu_n}(z)h'_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| \leq c \frac{(1 - r_n)^\beta}{1 - |f^{\nu_n}(1/\bar{z})|^2} \quad (5.60)$$

with some uniform  $c$ . Taking into consideration inequalities (2.4), we see that at appropriate uniform  $M$  the map  $\hat{f}_n$  is a local homeomorphism on the domain  $B_M$ .

To consider behavior of our mapping at infinity we put  $\zeta = 1/z$  and consider the function  $\frac{\hat{f}_n^1(\zeta)}{\hat{f}_n^1(\zeta)} = 1/\hat{f}_n(1/\zeta)$ . It is easy to see that  $\hat{f}_n^1(\zeta) \rightarrow 0$  as  $\zeta \rightarrow 0$  and  $\hat{f}_{n,\zeta}^1 \rightarrow f_z^{\nu_n}(0)f_{n,z}(0)$ ,  $\hat{f}_{n,\bar{\zeta}}^1 \rightarrow \overline{f_{\bar{z}}^{\nu_n}(0)}f_{n,z}(0)$ . It implies that  $\hat{f}_n$  extends as a local homeomorphism on infinity.

It remains to prove that  $\tilde{g}_n$  is a local homeomorphism on the domain  $\{d_n \leq |z| - 1 \leq 2d_n\}$ . Indeed, then the map  $\tilde{G}_n$  extends to a local homeomorphism of the sphere and, hence, is a homeomorphism by the monodromy theorem.

On the domain  $\{d_n \leq |z| - 1 \leq 2d_n\}$

$$\tilde{g}_{n,z} = \hat{h}_{n,z}\rho_n + \hat{f}_{n,z}(1 - \rho) + (\rho_n)_z(\hat{h}_n - \hat{f}_n),$$

where  $\rho_n(z) = \rho(d_n^{-1}(|z| - 1))$ . We have

$$|\hat{h}_{n,z}\rho_n + \hat{f}_{n,z}(1 - \rho_n) - \hat{h}_{n,z}| = (1 - \rho_n)|\hat{f}_{n,z} - \hat{h}_{n,z}|.$$

By (5.52) and (5.59),

$$|\hat{h}_{n,z}\rho_n + \hat{f}_{n,z}(1 - \rho_n) - \hat{f}_z^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c \frac{(1 - r_n)^\beta}{1 - |f^{\nu_n}(1/\bar{z})|^2}$$

Also, by (5.53) and (5.60),

$$|\hat{h}_{n,\bar{z}}\rho_n + \hat{f}_{n,\bar{z}}(1 - \rho_n) - \hat{f}_{\bar{z}}^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c \frac{(1 - r_n)^\beta}{1 - |f^{\nu_n}(1/\bar{z})|^2}$$

Thus in the domain  $\{d_n \leq |z| - 1 \leq 2d_n\}$

$$|\hat{h}_{n,z}\rho_n + \hat{f}_{n,z}(1 - \rho_n) - \hat{f}_z^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c/M. \quad (5.61)$$

$$|\hat{h}_{n,\bar{z}}\rho_n + \hat{f}_{n,\bar{z}}(1 - \rho_n) - \hat{f}_{\bar{z}}^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c/M \quad (5.62)$$

with some uniform  $C$ , if we set  $d_n$  as in (5.56).

Consider now  $|\hat{h}_n - \hat{f}_n|$ . Since  $f_n \circ f^{\nu_n} = h \circ f^\nu$ , we have

$$|h_n \circ f^{\nu_n}(z) - f_n \circ f^{\nu_n}(z)| \leq |h_n \circ f^{\nu_n}(z) - h \circ f^{\nu_n}(z)| + |h \circ f^{\nu_n}(z) - h \circ f^\nu(z)|.$$

But

$$|h_n \circ f^{\nu_n}(z) - h \circ f^{\nu_n}(z)| = |h[r_n f^{\nu_n}(z)] - h[f^{\nu_n}(z)]| \leq c(1 - r_n)$$

since  $h'$  is uniformly bounded. Also, taking into consideration (5.34),

$$|h \circ f^{\nu_n}(z) - h \circ f^\nu(z)| \leq \sup_{[f^{\nu_n}(z), f^\nu(z)]} |h'| |f^{\nu_n}(z) - f^\nu(z)| \leq c(1 - r_m)^\beta.$$

Thus,

$$|h_n \circ f^{\nu_n}(z) - f_n \circ f^{\nu_n}(z)| \leq c(1 - r_m)^\beta.$$

According to (5.48) and (5.54) and applying also estimate (5.30), we obtain

$$|\hat{h}_n(z) - \hat{f}_n(z)| \leq c(1 - r_m)^\beta.$$

Since  $|\rho'(d_n^{-1}(|z| - 1))| \leq 2/d_n$ , we see that if we set  $d_n$  as in (5.56), then

$$|(\rho_n)_z(\hat{h}_n - \hat{f}_n)| \leq c/M$$

and analogous estimate holds for  $|(\rho_n)_{\bar{z}}(\hat{h}_n - \hat{f}_n)|$ .

Recalling also (5.61), (5.62) we see that in the domain  $\{d_n \leq |z| - 1 \leq 2d_n\}$  we have the estimates

$$|\tilde{g}_{n,z} - \hat{f}_z^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c/M,$$

$$|\tilde{g}_{n,\bar{z}} - \hat{f}_{\bar{z}}^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c/M.$$

We see that at appropriate uniform  $M$   $\tilde{g}_n$  is a local homeomorphism. The assertions about the dilatation and the derivative  $\tilde{g}_{n,z}$  follow from estimates (5.52), (5.53) for  $\hat{h}_n$ , (5.59), (5.60) and the last estimates.  $\square$

3) *The final extension of  $g_n$  and the extension of  $g$ .*

Now we define new extensions  $\hat{g}_n(z)$ . Fix  $1/2 < r_1 < 1$  and put  $\hat{g}_1 = \tilde{g}_1$ . As on the previous step,  $\rho(s)$ ,  $s \geq 0$  is some monotonic smooth function,  $\rho(s) = 1$ ,  $s \leq 1$ ,  $\rho(s) = 0$ ,  $s \geq 2$ ,  $|\rho'(s)| \leq 2$ . Let  $a_n > 0$ ,  $a_n < a_{n-1}$  be some sequence of constants, which we shall specify later. We define

$$\hat{g}_n(z) = \tilde{g}_n(z)\rho(a_n^{-1}(|z| - 1)) + \tilde{g}_{n-1}(z)[1 - \rho(a_n^{-1}(|z| - 1))]$$

at  $1 \leq |z| \leq 1 + a_{n-1}$ ,

$$\hat{g}_n(z) = \tilde{g}_{n-1}(z)\rho(a_{n-1}^{-1}(|z| - 1)) + \tilde{g}_{n-2}(z)[1 - \rho(a_{n-1}^{-1}(|z| - 1))]$$

at  $1 + a_{n-1} \leq |z| \leq 1 + a_{n-2}$ ,

.....

$$\hat{g}_n(z) = \tilde{g}_1$$

at  $|z| \geq 1 + a_1$ .

Now we specify the sequences  $r_n$ ,  $d_n$  and  $a_n$ . We put

$$1 - r_n \leq \frac{1}{5^{1/\beta}}(1 - r_{n-1}), \quad d_n = C(1 - r_n)^\beta, \quad a_n = 2d_{n-1}, \quad (5.63)$$

where  $\beta$  and  $C$  are the constants from Proposition 27.

**Proposition 28** *If we define the sequences  $r_n$ ,  $d_n$  and  $a_n$  according to (5.63) with some appropriate uniform  $C$ , then the map  $g_n$  extended on  $\mathbb{C} \setminus D$  as  $\hat{g}_n$  will be a quasiconformal homeomorphism of the plane, and the derivative  $(\hat{g}_n)_z$  will be uniformly bounded from below and from above. On each domain  $1 + a_k \leq |z| \leq 1 + a_{k-1}$ ,  $k \leq n$*

$$\hat{g}_n(z) = \hat{f}_k(z)\rho(a_k^{-1}(|z| - 1)) + \hat{f}_{k-1}(z)[1 - \rho(a_k^{-1}(|z| - 1))]. \quad (5.64)$$

**Proof.** It is easy to see that, if we set  $a_n$  according to (5.63), the map  $\hat{g}_n$  will be described by expression (5.64) on each ring  $1 + a_k \leq |z| \leq 1 + a_{k-1}$ ,  $k \leq n$ . We must check only that it is a local diffeomorphism on any such domain. By induction, it is enough to check it for  $k = n$ .

We have

$$\hat{g}_{n,z} = \hat{f}_{n,z}\rho_n + \hat{f}_{n-1,z}(1 - \rho_n) + (\rho_n)_z(\hat{f}_n - \hat{f}_{n-1}),$$

where  $\rho_n(z) = \rho(a_n^{-1}(1 - |z|))$ . The analogous expression we have for  $\hat{g}_{n,\bar{z}}$ . We have

$$|\hat{f}_{n,z}\rho_n + \hat{f}_{n-1,z}(1 - \rho_n) - \hat{f}_{n,z}| = (1 - \rho_n)|\hat{f}_{n,z} - \hat{f}_{n-1,z}|.$$

By (5.57), (5.59), (5.31), and taking into consideration (2.4),

$$\begin{aligned} |\hat{f}_{n,z}\rho_n + \hat{f}_{n-1,z}(1 - \rho_n) - \hat{f}_z^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| &\leq c \frac{(1 - r_{n-1})^\beta}{|z|^2 - 1} + \\ &+ |\hat{f}_z^{\nu_n}(z)f'_n \circ f^{\nu_n}(1/\bar{z}) - \hat{f}_z^{\nu_{n-1}}(z)f'_{n-1} \circ f^{\nu_{n-1}}(1/\bar{z})| + \\ &+ \left| \left[ \hat{f}_z^{\nu_n}(z) - f^{\nu_n}\left(\frac{1}{\bar{z}}\right) \right] f''_n \circ f^{\nu_n}\left(\frac{1}{\bar{z}}\right) f_{\bar{\omega}}^{\nu_n}\left(\frac{1}{\bar{z}}\right) \frac{1}{z^2} - \right. \\ &\left. - \left[ \hat{f}_z^{\nu_{n-1}}(z) - f^{\nu_{n-1}}\left(\frac{1}{\bar{z}}\right) \right] f''_{n-1} \circ f^{\nu_{n-1}}\left(\frac{1}{\bar{z}}\right) f_{\bar{\omega}}^{\nu_{n-1}}\left(\frac{1}{\bar{z}}\right) \frac{1}{z^2} \right| \end{aligned} \quad (5.65)$$



Now, applying (5.28) and (5.32), we have

$$\begin{aligned} & |\hat{f}_z^{\nu_n}(z) f'_n \circ f^{\nu_n}(1/\bar{z}) - \hat{f}_z^{\nu_{n-1}}(z) f'_{n-1} \circ f^{\nu_{n-1}}(1/\bar{z})| \leq |\hat{f}_z^{\nu_n}(z) - \hat{f}_z^{\nu_{n-1}}(z)| |f'_n \circ f^{\nu_n}(1/\bar{z})| + \\ & + |\hat{f}_z^{\nu_{n-1}}(z)| (|f'_n \circ f^{\nu_n}(1/\bar{z}) - f'_{n-1} \circ f^{\nu_{n-1}}(1/\bar{z})|) \leq c \frac{(1-r_{n-1})^\beta}{|z|-1}. \end{aligned}$$

with some uniform  $c$ .

Now remind that  $|f'_n(z)|$  is uniformly bounded from below and from above since the first derivatives of  $f^{\nu_n}$  and  $g|_{D_{r_n}} = f_n \circ f^{\nu_n}$  are uniformly bounded from below and from above, and  $|f''_n(z)| \leq c(1-|z|)^{-1}$  since  $|f''_n(z)/f'(z)| \leq c(1-|z|)^{-1}$  by properties of univalent functions (see [Pom]).

We see that the second difference in the right side of (5.65) is no greater than the sum of the terms: first,

$$\begin{aligned} & \left| \hat{f}^{\nu_n}(z) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) - \hat{f}^{\nu_{n-1}}(z) + f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) \right| \left| f''_{n-1} \circ f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) \right| O(1) \leq \\ & \leq c \frac{(1-r_{n-1})^\beta}{|z|-1} \end{aligned}$$

since  $|f^{\nu_n}(z) - f^{\nu_{n-1}}(z)| \leq c(1-r_{n-1})^\beta$  (see (3.1) and (5.34)); second,

$$\begin{aligned} & \left| \hat{f}^{\nu_{n-1}}(z) - f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) \right| \left| f''_{n-1} \circ f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) \right| \left| f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) - f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| \leq \\ & \leq c \frac{(1-r_{n-1})^\beta}{|z|-1} \end{aligned}$$

by (5.51) and (5.28); and third,

$$\begin{aligned} & \left| \hat{f}^{\nu_{n-1}}(z) - f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) \right| \left| f''_{n-1} \circ f^{\nu_{n-1}} \left( \frac{1}{\bar{z}} \right) - f''_n \circ f^{\nu_n} \left( \frac{1}{\bar{z}} \right) \right| O(1) \leq \\ & \leq c \frac{(1-r_{n-1})^\beta}{|z|-1} \end{aligned}$$

by (5.51) and (5.32).

Thus,

$$|\hat{f}_{n,z} \rho_n + \hat{f}_{n-1,z} (1-\rho_n) - \hat{f}_z^{\nu_n}(z) h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c \frac{(1-r_{n-1})^\beta}{|z|-1}. \quad (5.66)$$

Analogously we obtain

$$|\hat{f}_{n,\bar{z}}\rho_n + \hat{f}_{n-1,\bar{z}}(1 - \rho_n) - \hat{f}_{\bar{z}}^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c \frac{(1 - r_{n-1})^\beta}{|z| - 1} \quad (5.67)$$

with some uniform  $c$ .

Now we shall estimate  $|\hat{f}_n - \hat{f}_{n-1}|$ . Since  $f_n \circ f^{\nu_n} = g|_{D_{r_n}}$ , we see that  $f_n \circ f^{\nu_n}(1/\bar{z}) - f_{n-1} \circ f^{\nu_{n-1}}(1/\bar{z}) = 0$  if  $|1/z| \leq r_{n-1}$ . We must only estimate the difference of the terms containing  $f'_n$  and  $f'_{n-1}$ . We have

$$|[\hat{f}^{\nu_n}(z) - f^{\nu_n}(1/\bar{z}) - \hat{f}^{\nu_{n-1}}(z) + f^{\nu_{n-1}}(1/\bar{z})]f'_n \circ f^{\nu_n}(1/\bar{z})| \leq c(1 - r_{n-1})^\beta$$

by (3.1) and (5.34), and

$$|[\hat{f}^{\nu_{n-1}}(z) - f^{\nu_{n-1}}(1/\bar{z})][f'_n \circ f^{\nu_n}(1/\bar{z}) - f'_{n-1} \circ f^{\nu_{n-1}}(1/\bar{z})]| \leq c(1 - r_{n-1})^\beta.$$

by (5.51) and (5.31).

Since  $|\rho'_n| \leq 2/a_n$ , we see that

$$|(\rho_n)_z(\hat{f}_n - \hat{f}_{n-1})| \leq c/M$$

in the ring  $1 + a_n \leq |z| \leq 1 + a_{n-1}$ . For  $|(\rho_n)_{\bar{z}}(\hat{f}_n - \hat{f}_{n-1})|$  we have an analogous estimate.

As a result, collecting (5.66), (5.67), and the last estimates, we obtain

$$|(\hat{g}_n)_z - \hat{f}_z^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c/M,$$

$$|(\hat{g}_n)_{\bar{z}} - \hat{f}_{\bar{z}}^{\nu_n}(z)h'_n \circ f^{\nu_n}(1/\bar{z})| \leq c/M,$$

where  $M$  is the constant from inequality (5.56) and  $c$  is some uniform constant independent of  $M$ . At appropriate  $C$  we obtain that  $\hat{g}_n$  is a quasiconformal map with uniformly bounded dilatation and the derivative  $(\hat{g}_n)_z$  uniformly bounded from below and from above.  $\square$

Now we define the extension  $\hat{g}$  as the limit of  $\hat{g}_n$ . By definition,  $\hat{g}(z) = \hat{g}_n(z)$  if  $|z| \geq 1 + a_n$ . Thus  $\hat{g}_n$  converge to a quasiconformal map on  $\hat{\mathbb{C}} \setminus D$  with derivative uniformly bounded from below and from above. Also, the map defined as  $g$  on  $D$  and as  $\hat{g}$  on  $\mathbb{C} \setminus D$  is a one-to-one mapping. Indeed, if  $g(z_1) = \hat{g}(z_2)$ , then the domains  $g_n(D)$  and  $\hat{g}_n(\mathbb{C} \setminus D)$  must intersect at great enough  $n$ . Thus we obtained a homeomorphism of the plane and proved estimate (5.13).

4) **Proof of estimates (5.14) and (5.15).**

It is enough to prove that these estimates hold for  $\hat{g}_n$  at  $1 + a_n \leq |z| \leq 1 + a_{n-1}$ .

The maps  $g$  and  $f^{\nu_n}$  satisfy the estimates  $|g_{(k)}(z)| \leq C(1 - |z|)^{1-|k|}$ ,  $|f_{(k)}^{\nu_n}(z)| \leq C(1 - |z|)^{1-|k|}$ . From the equation  $g|_{D_{r_n}} = f_n \circ f^{\nu_n}$  by successive differentiation we obtain the estimate

$$|(f_n)_{z^k} \circ f^{\nu_n}(z)| \leq c(1 - |f^{\nu_n}(z)|)^{1-k} \leq C(1 - |z|)^{1-k}.$$

Also, we have estimates (5.26), (5.27) for  $|f_{(k,l)}^{\nu_n}|$  and  $|(f_n)_{(k,l)}|$ .

According to representations (5.64), (5.54) the derivative  $(\hat{g}_n)_{(k)}$  is a sum of items containing the multiples  $z^{-j}$ ,  $\bar{z}^{-j}$ , which don't influence an order in  $|z| - 1$  and multiples of the types

$$(f_n)z^j \circ f^{\nu_n}(1/\bar{z}), f_{(l)}^{\nu_n}(1/\bar{z}), \hat{f}_{(m)}^{\nu_n}(z), \quad (5.68)$$

and analogous terms containing  $f_{n-1}$ ,  $f^{\nu_{n-1}}$ . Also, there can be the multiples

$$(\rho_n)_{(s)}(z). \quad (5.69)$$

At differentiation of each multiple of type (5.68) we increase in order in  $(|z| - 1)^{-1}$  by one. From the other hand, differentiation of multiple (5.69) results in the additional multiple  $a_n^{-1}$ . On our ring it is a value of order  $(|z| - 1)^{-1}$ . By induction, we obtain estimate (5.14).

Now derivatives with respect to the parameter of terms of types (5.67) all have estimates  $C(1 - r_n)^{-M}$  for some  $M$  by (5.26) and (5.27). But, if we set  $r_n$ ,  $d_n$  and  $a_n$  according to (5.63), we have

$$1 - r_n = 5^{1/\beta}(1 - r_{n-1}) = 5^{1/\beta} \left( \frac{d_{n-1}}{C} \right)^{1/\beta} = 5^{1/\beta} \left( \frac{a_n}{2C} \right)^{1/\beta}.$$

Again differentiation of terms with  $\rho_n$  results in multiples  $a_n^{-j}$  with some  $j$ . Thus on the ring  $1 + a_n \leq |z| \leq 1 + a_{n-1}$  we obtain estimate (5.15) with some uniform  $C$  and  $M$ .  $\square$

### 5). Proof of estimates (5.16), (5.17).

We know that  $\hat{g}_n$  is a quasiconformal homeomorphism mapping  $\hat{\mathbb{C}} \setminus D$  onto  $\hat{\mathbb{C}} \setminus \Omega$  with complex dilatation  $\hat{\nu}$ . We can represent  $\hat{g}$  as a composition  $\tilde{h} \circ f^{\hat{\nu}}$ , where  $f^{\hat{\nu}}$  is a normal homeomorphism mapping  $\hat{\mathbb{C}} \setminus D$  onto itself and  $\tilde{h}$  is an univalent holomorphic function. From estimates ((5.13) and (5.14) at  $|k| \leq 2$  follows that  $|\hat{\nu}_{(k)}(z)| \leq c(|z| - 1)^{-1}$  at  $|k| = 1$ . By Lemma 1,

$c_1 \leq |f'_z| \leq c_2$  for some constants  $c_1, c_2$ . Applying (5.13), we see that for  $\tilde{h}'(z)$  we have analogous estimates. But for any univalent function  $\tilde{h}$  we have

$$a_1 |\tilde{h}_z(z)| (|z|^2 - 1) \leq \text{dist}(\tilde{h}(z), \partial\Omega) \leq a_2 |\tilde{h}_z(w)| (|z|^2 - 1)$$

with some uniform  $a_1, a_2$  (see [Pom]). Since for  $f^{\hat{\nu}}$  we have estimates (2.4), we obtain (5.16).

Let prove (5.17). The left inequality follows from (5.16) because  $|g(z) - \hat{g}(\bar{z}^{-1})| \geq \text{dist}(g(z), \partial\Omega) + \text{dist}(\hat{g}(\bar{z}^{-1}), \partial\Omega)$ .

Suppose now that  $1 + a_n \leq |1/\bar{z}| \leq 1 + a_{n-1}$ . Now, by (5.64), we must estimate  $g(z) - \hat{f}_n(1/\bar{z})$  and  $\hat{f}_n(1/\bar{z}) - \hat{f}_{n-1}(1/\bar{z})$ . Applying (5.54) and (5.51), we obtain

$$|\hat{f}_n(1/\bar{z}) - f_n \circ f^{\nu_n}(z)| \leq c(1 - |z|).$$

But  $f_n \circ f^{\nu_n}(z) = g(z)$  at  $|z| \leq r_n$  and, hence,  $|g(z) - \hat{f}_n(1/\bar{z})| \leq c(1 - |z|)$ . Analogously, we can see that  $|g(z) - \hat{f}_{n-1}(1/\bar{z})| \leq c(1 - |z|)$ . Also,

$$\begin{aligned} \hat{f}_n(1/\bar{z}) - \hat{f}_{n-1}(1/\bar{z}) &= [\hat{f}^{\nu_n}(z) - f^{\nu_n}(1/\bar{z})] f'_n \circ f^{\nu_n}(1/\bar{z}) - \\ &\quad - [\hat{f}^{\nu_{n-1}}(z) - f^{\nu_{n-1}}(1/\bar{z})] f'_{n-1} \circ f^{\nu_{n-1}}(1/\bar{z}). \end{aligned}$$

By (5.51), this difference has the estimate  $c(1 - |z|)$ . As a result, we obtain right inequality (5.17).  $\square$

## 6 Integral operators. $L^p$ -estimates.

We adopt the notations of the previous section. For  $z \in D$ ,  $w = \hat{g}(z) \in \Omega$  we define

$$\hat{w} = \hat{g}(1/\bar{z}). \quad (6.1)$$

**Proposition 29** *There are the estimates*

$$c(1 - |z|^2) \leq |w - \hat{w}| \leq C(1 - |z|^2), \quad (6.2)$$

$$\left| \frac{\omega - \hat{\omega}}{w - \hat{w}} \right| \leq C, \quad (6.3)$$

$$|w - \omega| \geq c|z - \zeta|, \quad (6.4)$$

$$c|z - \bar{\zeta}^{-1}| \leq |w - \hat{\omega}| \leq C|z - \bar{\zeta}^{-1}| \quad (6.5)$$

with uniform  $c, C$ .

**Proof.** The first estimate follows immediately from (5.17). The second one, also, follows from (5.16) and (5.17), taking into consideration that

$$|w - \hat{\omega}| \geq \text{dist}(\hat{\omega}, \partial\Omega).$$

Prove left estimate (6.4). It is enough to prove that the map  $g^{-1}$  has uniformly bounded derivatives. But  $g^{-1} = (f^\nu)^{-1} \circ h^{-1}$ , where  $h^{-1}$  has a bounded derivative and  $(f^\nu)^{-1}$  is the normal map with the Beltrami coefficient  $\nu_{-1} = -\nu \circ (f^\nu)^{-1}$ , and by Lemma 1 and estimate (2.4)  $|(\nu_{-1})_{(k)}(z)| \leq c(1 - |z|)^{-1}$  at  $|k| = 1$ . Again by Lemma 1 we obtain that  $(f^\nu)_z^{-1}, (f^\nu)_{\bar{z}}^{-1}$  are uniformly bounded. Analogously,

$$|\hat{g}(z_1) - \hat{g}(z_2)| \geq c|z_1 - z_2|. \quad (6.6)$$

Now let  $\omega'$  be a point on  $\partial\Omega$  closest to  $\hat{\omega}$  (there can be several such points but it isn't essential). Suppose at first that  $|w - \omega'| \leq |\omega' - \hat{\omega}|$ . Then

$$|w - \hat{\omega}| \geq |\omega' - \hat{\omega}| \geq c(|\bar{\zeta}|^{-1} - 1) \geq c'|z - \bar{\zeta}^{-1}|$$

for some uniform  $c, c'$ . From the other hand, if  $|w - \omega'| \leq |\omega' - \hat{\omega}|$ , then, applying (6.5) and (6.6), we obtain

$$|w - \hat{\omega}| \geq |\omega' - w| \geq c|z - g^{-1}(\omega')| \geq c'|z - \bar{\zeta}^{-1}|$$

for some uniform  $c, c'$ .

From the other hand, according to (4.35) and (5.13), the map  $g$  extended on  $\mathbb{C} \setminus D$  as  $\hat{g}$  is a Lipschitz homeomorphism of the plane with an uniformly bounded Lipschitz constant. Thus we obtain right estimate (6.4).  $\square$

Now we define integral transforms, which allow us to find solutions to the Beltrami equation on  $\Omega$  with required estimates on the boundary. We define

$$\begin{aligned} P_m f(w) &= \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^m dS_\omega = \\ &= \frac{1}{\pi} \int_{\Omega_n} f(\omega) \left[ \frac{1}{w - \omega} - \frac{1}{w - \hat{\omega}} - \dots - \frac{(\omega - \hat{\omega})^{m-1}}{(w - \hat{\omega})^m} \right] dS_\omega. \end{aligned} \quad (6.7)$$

The last representation follows from the identity

$$\frac{1}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^m = \left( \frac{1}{w - \omega} - \frac{1}{w - \hat{\omega}} \right) \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^{m-1}.$$

Differentiating  $P_m f(w)$  in  $w$  we obtain the transform

$$\begin{aligned} T_m f(w) &= -\frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^m \cdot \left( \frac{1}{w - \omega} + \frac{m}{w - \hat{\omega}} \right) dS_{\omega} = \\ &= -\frac{1}{\pi} \int_{\Omega} f(\omega) \left[ \frac{1}{(w - \omega)^2} - \frac{1}{(w - \hat{\omega})^2} - \dots - \frac{(m-1)(\omega - \hat{\omega})^{m-1}}{(w - \hat{\omega})^{m+1}} \right] dS_{\omega}. \end{aligned} \quad (6.8)$$

Here we understand the integral in terms of its principal value. In the chart  $z$  we have

$$P_m f(z) = \frac{1}{\pi} \int_D \frac{f(g_n(\zeta))}{g_n(z) - g_n(\zeta)} \left( \frac{g_n(\zeta) - \hat{g}_n(\zeta)}{g_n(z) - \hat{g}_n(\zeta)} \right)^m |(g_n)_{\zeta}(\zeta)|^2 (1 - |\nu_n(\zeta)|^2) dS_{\zeta}, \quad (6.9)$$

$$\begin{aligned} T_m f(z) &= -\frac{1}{\pi} \int_D \frac{f(g_n(\zeta))}{g_n(z) - g_n(\zeta)} \left( \frac{g_n(\zeta) - \hat{g}_n(\zeta)}{g_n(z) - \hat{g}_n(\zeta)} \right)^m \left( \frac{1}{g_n(z) - g_n(\zeta)} + \right. \\ &\quad \left. + \frac{1}{g_n(z) - \hat{g}_n(\zeta)} \right) |(g_n)_{\zeta}(\zeta)|^2 (1 - |\nu_n(\zeta)|^2) dS_{\zeta} = \\ &= -\frac{1}{\pi} \int_D f(g_n(\zeta)) \left[ \frac{1}{(g_n(z) - g_n(\zeta))^2} - \frac{1}{(g_n(z) - \hat{g}_n(\zeta))^2} - \dots \right. \\ &\quad \left. - \frac{(m-1)(g_n(\zeta) - \hat{g}_n(\zeta))^{m-1}}{(g_n(z) - \hat{g}_n(\zeta))^{m+1}} \right] |(g_n)_{\zeta}(\zeta)|^2 (1 - |\nu_n(\zeta)|^2) dS_{\zeta}. \end{aligned} \quad (6.10)$$

**Definition 2** We say that a function  $f$  on  $\Omega$  belongs to  $L^p_s(\Omega)$  if the function  $f(w)(\text{dist}(w, \partial\Omega))^s$  belongs to  $L^p$ . We denote by  $\|f\|_{p,s}$  the  $L^p$ -norm of the function  $f(w)(\text{dist}(w, \partial\Omega))^s$ . A function  $f$  belongs to  $C^0_s$  if  $f(w)(\text{dist}(w, \partial\Omega))^s$  is uniformly bounded. We denote by  $\|f\|_{0,s}$  the  $C^0$ -norm of the function  $f(w)(\text{dist}(w, \partial\Omega))^s$ .

The equivalent conditions are:  $f(g(z))(1 - |z|^2)^s$  belongs to  $L^p(D)$  and  $f(g(z))(1 - |z|^2)^s$  is uniformly bounded on  $D$ . Corresponding norms are equivalent to the norms of the Definition.

We will need in the following estimates:

**Proposition 30** Define the integrals

$$J(w) = \frac{1}{\pi} \int_{\Omega} \frac{dS_{\omega}}{|w - \omega| |w - \hat{\omega}|^k},$$

$$\tilde{J}(\omega) = \frac{1}{\pi} \int_{\Omega} \frac{dS_w}{|w - \omega||w - \hat{\omega}|^k}.$$

Suppose  $k \geq 2$ . Then

$$J(w) \leq C(1 - |z|^2)^{k-1}, \quad \tilde{J}(\omega) \leq C(1 - |\zeta|^2)^{k-1}, \quad (6.11)$$

where  $w = g(z)$ ,  $\omega = g(\zeta)$ .

**Proof.** a) Applying estimates (6.4) and (6.5) we can see that we must show that the integral

$$I(z) = \int_D \frac{dS_{\zeta}}{|z - \zeta||z - \bar{\zeta}^{-1}|^2} = \int_D \frac{|\bar{\zeta}| dS_{\zeta}}{|z - \zeta||1 - z\bar{\zeta}|^k}.$$

has the estimate

$$I(z) \leq C(1 - |z|^2)^{k-1} \quad (6.12)$$

and that the analogous integral

$$\tilde{I}(\zeta) = \int_D \frac{dS_z}{|z - \zeta||z - \bar{\zeta}^{-1}|^k} = \int_D \frac{|\bar{\zeta}| dS_z}{|z - \zeta||1 - z\bar{\zeta}|^2}$$

has the estimate

$$\tilde{I}(\zeta) \leq C(1 - |\zeta|^2)^{k-1}. \quad (6.13)$$

Let  $\varphi_z$  be the map

$$\varphi_z(\tau) = \frac{z - \tau}{1 - \bar{z}\tau}.$$

We have

$$|z - \zeta| = \frac{|\tau|(1 - |z|^2)}{|1 - \bar{z}\tau|},$$

$$|z - \bar{\zeta}^{-1}| = \frac{1 - |z|^2}{|\bar{z} - \bar{\tau}|}.$$

The Jacobian of the change of the variable  $\zeta = \varphi_z(\tau)$  is

$$\frac{(1 - |z|^2)^2}{|1 - \bar{z}\tau|^4}.$$

After this change of the variable we can write integral (6.12) as

$$(1 - |z|^2)^{1-k} \int_D \frac{|z - \tau|^k}{|\tau||1 - \bar{z}\tau|^3} dS_{\tau}. \quad (6.14)$$

But  $|z - \tau| \leq c|1 - \bar{z}\tau|$  for some uniform  $c$ , and we see that it is enough to show that the integral

$$\int_D \frac{dS_\tau}{|\tau||1 - \bar{z}\tau|}$$

is uniformly bounded. But we can write this integral as a sum of the integral over the disk  $D_{1/2}$  and of the integral over the ring  $D \setminus D_{1/2}$ . The first integral is no greater, than

$$C \int_D \frac{dS_\tau}{|\tau|}$$

And the second one is no greater, than

$$C \int_D \frac{dS_\tau}{|1 - \bar{z}\tau|}.$$

Both these integral are uniformly bounded and, hence, integral (6.14) has the estimate  $C(1 - |z|^2)^{-1}$ .

Now notice that  $\tilde{I}$  also reduces to integral (6.14), and we obtain estimate (6.13) exactly as (6.12).  $\square$

The following estimate is a corollary of this proposition.

**Proposition 31** *Let  $f$  belongs to  $C_2^0$ . Then*

$$\|P_m f\|_{0,1} \leq C \|f\|_{0,2}$$

*with some uniform  $C$ .*

**Proof.** By (6.3), we have

$$|P_m f(w)| \leq \frac{1}{\pi} \int_\Omega \left| \frac{f(\omega)}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^m \right| dS_\omega \leq \frac{1}{\pi} \sup_{\omega \in \Omega} |f(\omega)(\omega - \hat{\omega})^2| \int_\Omega \frac{dS_\omega}{|w - \omega||w - \hat{\omega}|^2}.$$

We apply first estimate (6.11).  $\square$

**Proposition 32** *Define the operators:*

$$L_\beta f(\omega) = \frac{1}{\pi} \int_\Omega f(\omega) \frac{\beta(w, \omega)}{(w - \hat{\omega})^2} dS_\omega,$$

*and*

$$\tilde{L}_\beta f(\omega) = \frac{1}{\pi} \int_\Omega f(\omega) \frac{\beta(w, \omega)}{(\hat{w} - \omega)^2} dS_\omega,$$

*where  $\beta$  is uniformly bounded function. These operators are bounded in  $L^p(\Omega)$ ,  $1 < p < \infty$ . As a consequence, the operators  $T_m$  and  $\tilde{T}_m$  are bounded in  $L^p(\Omega)$ ,  $1 < p < \infty$ .*



**Proof.** Consider the operator  $L_\beta$ . From boundedness of the derivatives of  $g$  and estimates (6.4), (6.5) follows that we can write

$$w - \omega = g(z) - g(\tau) = h_1(z, \tau)(z - \tau), \quad w - \hat{\omega} = h_2(z, \tau)(z - \hat{\tau}^{-1}),$$

where  $h_1$  and  $h_2$  are uniformly bounded from below and from above.

The operator  $L_\beta$  obviously is bounded in any  $L^p$  for functions supported in any domain  $|g^{-1}(\omega)| \leq c < 1$ . Suppose that the support of  $f$  is contained in the domain  $1/2 \leq |g^{-1}(\omega)| \leq 1$ . Go to the chart  $t$  on the disk  $D$ . Recalling that the Jacobian of the transformation  $\tau \mapsto \omega$  is uniformly bounded, we see that we must estimate the norm of the operator

$$L'_\beta f(z) = \frac{1}{\pi} \int_D f(\tau) \frac{\beta(z, \tau)}{(z - \bar{\tau}^{-1})^2} dS_\tau. \quad (6.15)$$

Introduce a new variable  $t = \bar{\tau}^{-1}$ . Then  $\tau = \bar{t}^{-1}$ . Integral (6.15) transforms to

$$\frac{1}{\pi} \int \tilde{f}(t) \frac{\tilde{\beta}(z, t)}{(z - t)^2} dS_t,$$

where  $\tilde{f}(t) = f(\bar{t}^{-1})$ ,  $\tilde{\beta} = \beta(z, \bar{t}^{-1})J(t)$  and  $J$  is the Jacobian of the transformation  $t \mapsto \tau$ . Here we take the integral over  $\mathbb{C}$ . Remind that support  $\tilde{f}$  is contained in the domain  $\mathbb{C} \setminus D$ , the Jacobian  $J$  is uniformly bounded on this domain, and  $z \in D \setminus \partial D$ . The below considerations follow the method described in [Ah, Ch. 5, D].

Let be  $t - z = \rho e^{i\theta}$ . Our integral is of the type

$$\int_0^{2\pi} \int_0^\infty \tilde{f}(z + \rho e^{i\theta}) \frac{h(z, z + \rho e^{i\theta})}{\rho} e^{-2i\theta} d\rho d\theta,$$

where  $|h(z, z + \rho e^{i\theta})|/\rho \leq c$  We have

$$\|L'_\beta f(z)\|_p \leq 2\pi \max_\theta \left\| \int_0^\infty \tilde{f}(z + \rho e^{i\theta}) \frac{h(z, z + \rho e^{i\theta})}{\rho} d\rho \right\|_p$$

We can suppose that the functions  $f$  and  $h$  are real. We have

$$\left| \int_0^\infty \tilde{f}(z + \rho e^{i\theta}) \frac{h(z, z + \rho e^{i\theta})}{\rho} d\rho \right| \leq c \int_0^\infty \frac{|\tilde{f}(z + \rho e^{i\theta})|}{\rho} d\rho$$

for some  $c$  and, hence,

$$\|L'_\beta f(z)\|_p \leq 2\pi c \max_\theta \left\| \int_0^\infty \frac{|\tilde{f}(z + \rho e^{i\theta})|}{\rho} d\rho \right\|_p$$

Let this maximum corresponds to  $\theta = \Theta$ . The norm in the right side doesn't change if we replace  $z$  by  $ze^{i\Theta}$ . If we denote  $\tilde{f}_\Theta(t) = \tilde{f}(te^{i\Theta})$ , the integral in the right side becomes

$$H\tilde{f}_\Theta(w) = \int_0^\infty \frac{|\tilde{f}_\Theta(z + \rho)|}{\rho} d\rho.$$

The point  $z = x + iy$  doesn't belong to the support of  $\tilde{f}_\Theta$  and the value of the integral doesn't change if we extend the function of  $\rho$ :  $\tilde{f}_\Theta(z + \cdot)$  as zero on the domain  $\rho < 0$ , extend the integral over this domain, and take the principal value. Hence,

$$\int |H\tilde{f}_\Theta(x + iy)|^p dx \leq A_p \int |\tilde{f}_\Theta(x + iy)|^p dx$$

for some  $A_p$  by one-dimensional Calderon-Zigmund inequality. Now we get for the two-dimensional norm

$$\begin{aligned} \|H\tilde{f}_\Theta\|_p^p &= \int \int |H\tilde{f}_\Theta(x + iy)|^p dx dy \leq \\ &\leq A_p^p \int \int |\tilde{f}_\Theta(x + iy)|^p dx dy = A_p^p \|\tilde{f}_\Theta\|_p^p. \end{aligned}$$

Since the functions  $\tilde{f}$  and  $\tilde{f}_\Theta$  have equal  $L^p$ -norms and the  $L^p$ -norm of  $\tilde{f}$  can be estimated through  $L^p$ -norm of  $f$ , we obtain the required estimate.

Consider now the operator  $T_m$ . The first item under the integral in the right side of (6.8) is the Beurling transform bounded in  $L^p$ . The other items are the integrals

$$\frac{1}{\pi} \int_\Omega f(\omega) \frac{(\omega - \hat{\omega})^k}{(w - \hat{\omega})^{k+2}}.$$

These integrals are of the type considered above for  $\beta(w, \omega) = (\omega - \hat{\omega})^k / (w - \hat{\omega})^k$ .

The case of the operators  $\tilde{L}_\beta$  and  $\tilde{T}_m$  is analogous. It is even simpler because we don't need now in the change of the variables  $\tau \mapsto t$ . The proof with this exception repeats the proof for  $L_\beta$ .  $\square$

**Proposition 33** *Suppose an operator  $L$  satisfies the estimate*

$$Lf(w) \leq C|w - \hat{w}| \int_{\Omega} |f(\omega)| \frac{|\omega - \hat{\omega}|}{|w - \omega||w - \hat{\omega}|^3} dS_{\omega}.$$

*Then this operator is bounded in  $L^p(\Omega)$  for  $1 \leq p \leq \infty$ .*

**Proof** We shall prove the boundedness in  $L^1$  and  $L^{\infty}$ . Then the general case will follow from the Riesz-Thorin interpolation theorem (see, for example, [RS]). We have

$$\begin{aligned} \|Lf\|_1 &\leq C \int_{\Omega} |w - \hat{w}| \int_{\Omega} |f(\omega)| \frac{|\omega - \hat{\omega}|}{|w - \omega||w - \hat{\omega}|^3} dS_{\omega} dS_w \leq \\ &\leq C \int_{\Omega} |f(\omega)| |\omega - \hat{\omega}| \int_{\Omega} \frac{dS_w}{|w - \omega||w - \hat{\omega}|^2} dS_{\omega} \leq C \|f\|_1 \end{aligned}$$

by the second estimate of Proposition 30.

From the other hand,

$$\begin{aligned} |Lf(w)| &\leq C|w - \hat{w}| \|f\|_{\infty} \int_{\Omega} \frac{|\omega - \hat{\omega}|}{|w - \omega||w - \hat{\omega}|^3} dS_{\omega} \leq \\ &\leq C \|f\|_{\infty} |w - \hat{w}| \int_{\Omega} \frac{dS_{\omega}}{|w - \omega||w - \hat{\omega}|^2} \leq C \|f\|_{\infty} \end{aligned}$$

by the first estimate of Proposition 30.  $\square$

**Proposition 34** *Let  $I_{m,s}$ ,  $\tilde{I}_{m,s}$ ,  $J_{m,s}$  be the operators*

$$\begin{aligned} I_{m,s}f(w) &= (w - \hat{w})^s \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{w - \omega} \frac{(\omega - \hat{\omega})^{m-s}}{(w - \hat{\omega})^{m+1}} dS_{\omega}, \\ \tilde{I}_{m,s}f(w) &= (w - \hat{w})^s \frac{1}{\pi} \int_{\Omega} f(\omega) \frac{\hat{w} - \hat{\omega}}{(w - \omega)^2} \frac{(\omega - \hat{\omega})^{m-s}}{(\hat{w} - \omega)^{m+1}} dS_{\omega}, \\ J_{m,s}f(w) &= (w - \hat{w})^s \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w - \omega)^2} \frac{(\omega - \hat{\omega})^{m-s}}{(w - \hat{\omega})^m} dS_{\omega}, \end{aligned}$$

*. These operators are bounded in  $L^p(\Omega)$  for  $1 < p < \infty$  at  $m \geq 0, s \leq m$ .*

**Proof.** Suppose at first that  $m \geq 2$ ,  $1 \leq s \leq m - 1$ . Then the operators satisfy the estimate of the previous proposition.

Suppose that  $m \geq 1$ ,  $s = m$ . We have

$$\begin{aligned} I_{m,m}f(w) &= (w - \hat{w})^m \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{w - \omega} \frac{dS_{\omega}}{(w - \hat{\omega})^{m+1}} = \\ &= (w - \hat{w})^m \frac{1}{\pi} \int_{\Omega} f(\omega) \left( \frac{\omega - \hat{\omega}}{(w - \omega)(w - \hat{\omega})^{m+2}} + \frac{1}{(w - \hat{\omega})(w - \hat{\omega})^{m+2}} \right) dS_{\omega} = \\ &= I_{m+1,m}f(w) + L_m f(w), \end{aligned}$$

where  $L_m$  is an operator of the type considered in Proposition 22, and we obtain the estimate in  $L^p$ .

Now suppose  $m \geq 1$ ,  $s = 0$ . We have

$$\begin{aligned} I_{m,0}f(w) &= \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{w - \omega} \frac{(\omega - \hat{\omega})^m}{(w - \hat{\omega})^{m+1}} dS_{\omega} = \\ &= \frac{1}{\pi} \int_{\Omega} f(\omega) \frac{(\omega - \hat{\omega})^m}{(w - \hat{\omega})^{m+1}(\hat{w} - \omega)} \left( 1 - \frac{w - \hat{w}}{w - \omega} \right) dS_{\omega}. \end{aligned}$$

In the right side we have the sum of two integrals, where the first one is of the type considered in Proposition 22 and the second one satisfies estimate of Proposition 23.

Suppose now  $m = 0$ . We have

$$I_{0,0}f(w) = \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w - \omega)(w - \hat{\omega})} dS_{\omega} = \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w - \hat{\omega})^2} \left( 1 + \frac{\omega - \hat{\omega}}{w - \omega} \right) dS_{\omega}.$$

Again we obtain the sum of two integrals, where the first one is of the type of Proposition 22 and the second one is  $I_{1,0}$ .

The case of  $\tilde{I}_{m,s}$  is analogous. We prove estimate only for  $\tilde{I}_{m,0}$ , which we shall apply below.

$$\begin{aligned} \tilde{I}_{m,0}f(w) &= \frac{1}{\pi} \int_{\Omega} f(\omega) \frac{\hat{w} - \hat{\omega}}{(w - \omega)^2} \frac{(\omega - \hat{\omega})^m}{(\hat{w} - \omega)^{m+1}} dS_{\omega} = \\ &= \frac{1}{\pi} \int_{\Omega} f(\omega) \frac{\hat{w} - \hat{\omega}}{w - \omega} \frac{(\omega - \hat{\omega})^m}{(w - \hat{\omega})^{m+1}(\hat{w} - \omega)} \left( 1 - \frac{w - \hat{w}}{w - \omega} \right) dS_{\omega}. \end{aligned}$$

Again we have the term satisfying conditions of Proposition 22 and the term satisfying the estimate of Proposition 23.

Consider now the operator  $J_{m,s}$ . We can write  $J_{m,s}$  in the form

$$J_{m,s}f(w) = \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w-\omega)^2} dS_{\omega} + (w-\hat{w})^s \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w-\omega)^2} \left[ \frac{(\omega-\hat{w})^{m-s}}{(w-\hat{w})^m} - \frac{1}{(w-\hat{w})^s} \right] dS_{\omega}$$

The first integral is bounded in  $L^p$  by the Calderon-Zigmund inequality. The second integral after multiplying by  $(w-\hat{w})^s$  is

$$\begin{aligned} & \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w-\omega)^2} \frac{(\omega-\hat{w})^{m-s}[(w-\hat{w})^s - (w-\hat{w})^s] + (w-\hat{w})^s[(\omega-\hat{w})^{m-s} - (w-\hat{w})^{m-s}]}{(w-\hat{w})^m} dS_{\omega} = \\ & = \sum_{k=0}^{s-1} \frac{1}{\pi} \int_{\Omega} f(\omega) \frac{\hat{w}-\hat{\omega}}{(w-\omega)^2} (\omega-\hat{\omega})^{m-s} \frac{(w-\hat{w})^{s-1-k}}{(w-\hat{\omega})^{m-k}} dS_{\omega} + \\ & \quad + \sum_{k=0}^{m-s-1} \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{w-\omega} \frac{(\omega-\hat{\omega})^{m-s-1-k}}{(w-\hat{\omega})^{m-s-k}} dS_{\omega}. \end{aligned}$$

All items of the both sums are of the types considered in Proposition 34. For example, the last item of the first sum is  $\tilde{I}_{m-s+1,0}$  and the last item of the second sum is  $I_{0,0}$ .

For the completeness we consider also the case of non-integer  $s$ , though it isn't very essential. It is easy to see that boundedness in  $L^p$  of the operator  $J_{m,s}$  is equivalent to boundedness in  $L^p$  of the operator  $J_{m,s}^+$ , where

$$J_{m,s}^+f(w) = |w-\hat{w}|^s \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w-\omega)^2} \left( \frac{\omega-\hat{\omega}}{w-\hat{\omega}} \right)^m \frac{dS_{\omega}}{|\omega-\hat{\omega}|^s}.$$

. Indeed,  $\|J_{m,s}^+f\|_p = \|J_{m,s}f^+\|_p$ , where  $f^+(w) = f(w)(w-\hat{w})^s|w-\hat{w}|^{-s}$ .

Define the family of operators  $J_{m,s,z}^+$ ,  $z = t + iy$ ,  $0 \leq t \leq 1$

$$J_{m,s,z}^+f(w) = |w-\hat{w}|^{sz} \frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{(w-\omega)^2} \left( \frac{\omega-\hat{\omega}}{w-\hat{\omega}} \right)^m \frac{dS_{\omega}}{|\omega-\hat{\omega}|^{sz}}.$$

It is an analytic family of operators. On the left and right boundaries of the strip  $0 \leq t \leq 1$  the  $L^p$ -norms of  $J_{m,s,z}^+$  are uniformly bounded. Indeed, these operators differ from the cases already considered only by the multiple  $|w-\hat{\omega}|^{-isy}$  under the integral, which we can include in the function  $f$ , and

by the multiple  $|w - \hat{w}|^{isy}$ , which doesn't change the  $L^p$ -norm. Thus the conditions of the Stein interpolation theorem (see, for example, [RS]) for this family of operators are satisfied, and we obtain  $L^p$ -estimates for all  $2 \leq s \leq m - 1$ .  $\square$

**Proposition 35** *The operators  $T_m$  are bounded in  $L_s^p(\Omega)$  for  $2 \leq p < \infty$ ,  $m \geq 3$ ,  $2 \leq s \leq m - 1$ .*

**Proof.** Denote  $f_s(w) = f(w)(w - \hat{w})^s$ . It is enough to prove the estimate

$$\frac{1}{\pi} \int_{\Omega} |w - \hat{w}|^{sp} |T_m f(w)|^p dS_w \leq \|f_s\|_p^p.$$

Recalling (6.8), we see that the estimates for  $T_m f$  in  $L_s^p$  follow from the estimates for the integrals  $I_{m,s} f_s$  and  $J_{m,s} f_s$  obtained in the previous proposition.  $\square$

## 7 The operator $\tilde{T}_m$ and uniform estimates.

Now we return to the program described in Section 1. Our purpose is to prove Theorem 2'. We can find a solution to the Beltrami equation with  $\mu$  satisfying conditions of Theorem 2' analogously to the classical method replacing the transforms  $\mathcal{C}$  and  $\mathcal{S}$  by the transforms  $P_m$  and  $T_m$ . In such a way we obtain the solution with estimates of its derivatives with respect to the parameters, but this solution isn't necessary a homeomorphism mapping of  $\Omega$  onto its image.

But we can obtain a  $\mu$ -quasiconformal homeomorphism if we find a solution to the Beltrami equation  $F$ , which is  $\mu$ -quasiholomorphic and satisfies the estimate

$$|F_{ww}(w)/F_w(w)| \leq \delta / \text{dist}(w, \partial\Omega) \quad (7.1)$$

with sufficiently small  $\delta$ . Indeed,  $\Omega = h(D)$ , where  $h$  satisfies estimates (4.38), and  $F \circ h$  is quasiholomorphic on  $D$  with the complex dilatation  $\tilde{\mu}(z) = \mu \circ h(z) h_z(z) / \overline{h_z(z)}$ . It isn't difficult to check that we can write  $F \circ h$  as  $\tilde{h} \circ f^{\tilde{\mu}}$ , where  $f^{\tilde{\mu}}$  is the  $\tilde{\mu}$ -normal map and  $h$  is holomorphic, and  $|\tilde{h}''(z)/\tilde{h}'(z)| \leq 1/(1 - |z|)$  if the constants  $\epsilon$  in (4.38) and  $\delta$  in (7.1) are small enough. Here we don't give details because we shall return to these matters in the next section. It follows that  $\tilde{h}$  is univalent (see [Pom]) and, hence,  $F$  is a homeomorphism.

Now, if  $F$  is a  $\mu$ -quasiholomorphic map, then  $f = (\log F_w)_w = F_{ww}/F_w$  satisfies the equation

$$f_{\bar{w}} = \mu f_w + \mu_w f + \mu_{ww}. \quad (7.2)$$

We can solve this equation by iteration method. On the first step we find a function  $f_1$  satisfying the equation  $(f_1)_{\bar{w}} = \mu(f_1)_w + \mu_{ww}$ . If  $f_1$  is such that  $|(f_1)_w(w)| \leq b/\text{dist}(w, \partial\Omega)$  with sufficiently small  $b$ , then we shall solve the equation  $(f_2)_{\bar{w}} = \mu(f_2)_w + \mu_w f_1 + \mu_{ww}$  and so on. On each step we must solve the equation

$$f_{\bar{w}} - \mu f_w = G,$$

where  $|G(w)| \leq C/(\text{dist}(w, \partial\Omega))^2$  for some  $C$ . We can hope that there exist solutions represented as

$$P_m(\text{Id} - \mu T_m)^{-1}G = P_m G + P_m \mu T_m G + P_m \mu T_m \mu T_m G \dots \quad (7.3)$$

with appropriate  $m$  and that these solutions have the estimate  $|f(w)| \leq c/\text{dist}(w, \partial\Omega)$ . There is a difficulty, the transform  $T_m$  hasn't good uniform estimates. The key observation is that we can change the order of integration in each term of series (7.3) and write these series as

$$\begin{aligned} & \frac{1}{\pi} \int_{\Omega} \mathcal{P}_w(\omega) G(\omega) dS_{\omega} + \frac{1}{\pi} \int_{\Omega} \tilde{T}_m \mu \mathcal{P}_w(\omega) G(\omega) dS_{\omega} + \\ & + \frac{1}{\pi} \int_{\Omega} \tilde{T}_m \mu \tilde{T}_m \mu \mathcal{P}_w(\omega) G(\omega) dS_{\omega} + \dots, \end{aligned}$$

where  $\mathcal{P}_w(\omega)$  is the kernel of the operator  $P_m$

$$\mathcal{P}_w(\omega) = \frac{1}{w - \omega} \left( \frac{\omega - \hat{w}}{w - \hat{w}} \right)^m \quad (7.4)$$

and  $\tilde{T}_m$  is the transform

$$\tilde{T}_m f(w) = -\frac{1}{\pi} \int_{\Omega} \frac{f(\omega)}{\omega - w} \left( \frac{w - \hat{w}}{\omega - \hat{w}} \right)^m \left( \frac{1}{\omega - w} + \frac{m}{\omega - \hat{w}} \right) dS_{\omega}. \quad (7.5)$$

This operator has the same kernel as  $T_m$  but with transposed  $w$  and  $\omega$ .

The main reason, why it is useful to change the order of integration is that  $\tilde{T}_m$  contains the multiple  $(w - \hat{w})^m$ , and  $w$  isn't a variable of integration. As a result,  $\tilde{T}_m$  has better uniform estimates than  $T_m$ . For example, the integral

$$|w - \hat{w}| \int_{\Omega} \frac{dS_{\omega}}{|\omega - \hat{w}|^2 |\omega - w|}$$

is bounded and the integral

$$\int_{\Omega} \frac{|w - \hat{w}|}{|\omega - \hat{w}|^2 |\omega - w|} dS_w$$

isn't.

Thus we can write the sum of series (7.3) (defined at this moment only formally) as

$$\frac{1}{\pi} \int_{\Omega} G(\omega) f_w(\omega) dS_w, \quad (7.6)$$

where  $f_w$  satisfy the equation

$$f_w - \tilde{T}_m \mu f_w = P_w. \quad (7.7)$$

Suppose this equation has a solution that we can represent as

$$\frac{1}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^2 g_w(\omega), \quad (7.8)$$

where  $g_w$  is an uniformly bounded function. Then, applying the estimates of Proposition 29 and the first estimate of Proposition 30, we obtain for integral (7.6) the estimate  $C/\text{dist}(w, \partial\Omega)$ . Thus we must prove that for the solution to equation (7.7) there exists representation (7.8).

In fact, we shall prove a more general assertion.

**Lemma 5** *Suppose  $\Omega$  and  $\mu$  satisfy the conditions of Theorem 2'. Suppose a function  $P_w$  can be written in the form*

$$P_w(\omega) = \frac{1}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^l L_w(\omega), \quad l \leq m - 6, \quad (7.9)$$

where  $L_w(\omega)$  is uniformly bounded and

$$|L_w(\omega) - L_w(\omega_0)| \leq c \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}_0|} \quad (7.10)$$

with some uniform  $c$ . Then at  $m \geq l + 6$  the equation

$$f_w - \tilde{T}_m \mu f_w = P_w \quad (7.11)$$

has a unique solution representable in the form

$$f_w(\omega) = \frac{1}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^l g_w(\omega), \quad (7.12)$$

where  $g_w(\omega)$  is uniformly bounded.



**Proposition 36** *The function  $\mathcal{P}_w$  satisfies conditions (7.9), (7.10).*

**Proof.** The only nontrivial part is to prove estimate (7.10) for

$$L_w(\omega) = \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^{m-l}.$$

Note at first that  $|(L_w)_\omega(\omega)| \leq c|w - \hat{\omega}|^{-1} \leq c'|\omega - \hat{\omega}|^{-1}$  for some  $c'$  and we have an analogous estimate for  $|(L_w)_{\bar{\omega}}|$ . From estimate (6.2) follows that  $|\omega - \hat{\omega}| \geq \delta \text{dist}(\omega, \partial\Omega)$  for some  $\delta$ . Hence, if  $|\omega - \omega_0| \leq \delta|\omega_0 - \hat{\omega}_0|/2$ , then for some uniform  $C$  the derivatives  $|(L_w)_\omega|$  and  $|(L_w)_{\bar{\omega}}|$  are no greater than  $C|\omega - \hat{\omega}_0|^{-1}$  on the segment  $[\omega, \omega_0]$ , and we obtain estimate (7.10).

Suppose now that  $|\omega - \omega_0| > \delta|\omega_0 - \hat{\omega}_0|/2$ . Then  $|\omega - \hat{\omega}_0| \leq |\omega_0 - \hat{\omega}_0| + |\omega - \omega_0| \leq C|\omega - \omega_0|$  for some  $C$  and inequality (7.10) is trivial because  $L_w(\omega)$  is uniformly bounded.  $\square$ .

We shall prove Lemma 5 in the next section. Now we shall obtain some estimates necessary to the proof.

Apply the transform  $\tilde{T}_m$  to a function  $f_w$  of type (7.12). We have

$$\tilde{T}_m f_w(\omega) = \frac{1}{\pi} \int \frac{g_w(t)}{(w-t)(t-\omega)} \left( \frac{t-\hat{t}}{w-\hat{t}} \right)^l \left( \frac{1}{t-\omega} + \frac{m}{t-\hat{\omega}} \right) \left( \frac{\omega-\hat{\omega}}{t-\hat{\omega}} \right)^{m-l} dS_t.$$

Since

$$\frac{1}{(w-t)(t-\omega)} = \frac{1}{w-\omega} \left( \frac{1}{w-t} + \frac{1}{t-\omega} \right),$$

we see that, if  $f_w$  satisfies equation (7.11), then  $g_w$  must satisfy the equation

$$g_w - \mathcal{T}_w \mu g_w = L_w, \quad (7.13)$$

where  $\mathcal{T}_w$  is the transform

$$\begin{aligned} \mathcal{T}_w f(\omega) = & \frac{1}{\pi} \int f(t) \left( \frac{\omega-\hat{\omega}}{t-\hat{\omega}} \right)^{m-l} \left( \frac{t-\hat{t}}{t-\hat{\omega}} \right)^l \left( \frac{w-\hat{\omega}}{w-\hat{t}} \right)^l \\ & \left( \frac{1}{(t-\omega)^2} - \frac{1}{(t-\omega)(t-w)} + \frac{m}{(t-\omega)(t-\hat{\omega})} - \frac{m}{(t-w)(t-\hat{\omega})} \right) dS_t. \end{aligned} \quad (7.14)$$

We denote by  $\mathcal{K}_w$  the kernel of this operator. In the rest of this section we study the operator  $\mathcal{T}_w$ .

In what follows we denote by  $\chi_\Omega$  the characteristic function of the domain  $\Omega$ . As we shall show below, even at the action of  $\mathcal{K}_w$  on  $\chi_\Omega$  there appears the term  $(\bar{\omega} - \bar{w})/(\omega - w)$  with singular derivatives. Therefore, when we consider the action of the transform  $\mathcal{K}_w$ , we must distinguish the "bad" part and study the action of  $\mathcal{K}_w$  on this part. The next proposition describes the situation.

**Proposition 37** *Suppose  $\Omega$  and  $\mu$  satisfy the conditions of Theorem 2'. Let  $r_{wk}$ ,  $k \geq 0$  be the functions*

$$r_{wk}(\omega) = \left( \frac{\bar{\omega} - \bar{w}}{\omega - w} \right)^k \chi_\Omega(\omega). \quad (7.15)$$

Suppose  $m \geq l + 6$ . Then

$$\mathcal{T}_w \mu r_{wk}(\omega) = \mu(w) r_{w,k+1}(\omega) + F_{wk}(\omega), \quad |F_{wk}(\omega)| \leq ckb, \quad (7.16)$$

with some uniform  $c$  independent of  $k$  ( $b$  is the constant from estimate (4.40)).

Fix a point  $\omega_0 \in \Omega$ . Then

$$F_{wk}(\omega) - F_{wk}(\omega_0) = (\omega - \omega_0) F_{w\omega_0}(\omega),$$

where  $F_{w\omega_0}$  satisfies the estimate

$$|F_{w\omega_0}(\omega)| \leq k^2 \frac{cb}{|\omega - \hat{\omega}_0|}, \quad (7.17)$$

with some uniform  $c$  independent of  $k$ .

**Proof.** In what follows we denote

$$K_w(t, \omega) = \left( \frac{\omega - \hat{\omega}}{t - \hat{\omega}} \right)^{m-l} \left( \frac{t - \hat{t}}{t - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \hat{t}} \right)^l$$

We have the representations

$$K_w(t, \omega) = 1 + (t - \omega) R_w(t, \omega), \quad K_w(t, \omega) = a_w(\omega) + (t - w) R_w(t, \omega), \quad (7.18)$$

where

$$a_w(\omega) = K_w(w, \omega) = \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^{m-l}, \quad a_w(w) = 1.$$

There exists some uniform  $0 < d_1 \leq d_2$  such that the disks  $D_\omega$  and  $D_w$  centered at  $\omega$  and  $w$  of radii  $d_1|\omega - \hat{\omega}| \leq r_\omega \leq d_2|\omega - \hat{\omega}|$  and  $d_1 \leq |w - \hat{w}|r_w \leq d_2|w - \hat{w}|$  correspondingly are contained in  $\Omega$ . In what follows we shall specify  $r_\omega$  and  $r_w$ .

We need in some estimates, which we collect in the next proposition.

**Proposition 38** *For  $t \in D_\omega$  there are the estimates*

$$|R_\omega(t, \omega)| \leq \frac{C}{|\omega - \hat{\omega}|}, \left| \frac{\partial}{\partial \omega} R_\omega(t, \omega) \right| \leq \frac{C}{|\omega - \hat{\omega}|^2}. \quad (7.19)$$

$$|(a_w)_{(j)}(\omega)| \leq C|\omega - \hat{\omega}|^{-|j|}, \quad |j| \leq 3. \quad (7.20)$$

For  $t \in D_w, \omega \in D_w$  we have the estimates

$$|R_w(t, \omega)| \leq \frac{C}{|w - \hat{w}|}, \left| \frac{\partial}{\partial \omega} R_w(t, \omega) \right| \leq \frac{C}{|w - \hat{w}|^2}, \left| \frac{\partial}{\partial t} R_w(t, \omega) \right| \leq \frac{C}{|w - \hat{w}|^2}, \quad (7.21)$$

$$\left| \frac{\partial^2}{\partial t \partial \omega} R_w(t, \omega) \right| \leq \frac{C}{|w - \hat{w}|^3}, \left| \frac{\partial^2}{\partial t^2} R_w(t, \omega) \right| \leq \frac{C}{|w - \hat{w}|^3}, \quad (7.22)$$

$$\left| \frac{\partial^3}{\partial t^3} R_w(t, \omega) \right| \leq \frac{C}{|w - \hat{w}|^4}, \quad (7.23)$$

and analogous estimates we have for multi-indices containing  $\bar{\omega}$  and  $\bar{t}$ -derivatives. All constants are uniform.

**Proof.** Denote by  $\hat{h}(\omega)$  the function  $\omega \mapsto \hat{\omega}$ . By (5.13), (5.14), we have the estimates

$$c_1 \leq |\hat{h}_{(k)}(\omega)| \leq c_2, \quad |k| = 1, \quad |\hat{h}_{(k)}(\omega)| \leq \frac{C}{|\omega - \hat{\omega}|^{|k|-1}}, \quad |k| \geq 2. \quad (7.24)$$

We prove only estimates (7.19) - (7.23). The cases of  $\bar{\omega}$  and  $\bar{t}$ -derivatives are analogous.

To prove inequalities (7.19) we must estimate  $(K_w)_t$  and  $(K_w)_{t\omega}$  at  $t \in D_\omega$ . In what follows we denote by  $O$  fractions of the type

$$\left( \frac{\omega - \hat{\omega}}{t - \hat{\omega}} \right)^{j_1} \left( \frac{t - \hat{t}}{t - \hat{\omega}} \right)^{j_2} \left( \frac{w - \hat{w}}{w - \hat{t}} \right)^{j_3},$$

where  $j_1 \leq m - l, j_2 \leq l, j_3 \leq l$ .

The derivative  $(K_w)_t$  contains the terms of the types

$$(t - \hat{\omega})^{-1}O, \frac{1 - \hat{h}_t(t)}{t - \hat{\omega}}O, \frac{\hat{h}_t(t)}{w - \hat{t}}O.$$

The first two terms obviously have estimates  $C|\omega - \hat{\omega}|^{-1}$  (we apply first estimate (7.24)). The last term has estimate  $C|t - \hat{t}|^{-1}$ . At  $t \in D_\omega$  it is equivalent to the estimate  $C|\omega - \hat{\omega}|^{-1}$ .

Consider now  $(K_w)_{t\omega}$ . There appear the terms

$$\frac{\hat{h}_\omega(\omega)}{(t - \hat{\omega})^2}O, \frac{1 - \hat{h}_\omega(\omega)}{(t - \hat{\omega})^2}O, \frac{\hat{h}_\omega(\omega)}{(t - \hat{\omega})(w - \hat{t})}O, \quad (7.25)$$

$$\frac{(1 - \hat{h}_t(t))\hat{h}_\omega(\omega)}{(t - \hat{\omega})^2}O, \frac{(1 - \hat{h}_t(t))(1 - \hat{h}_\omega(\omega))}{(t - \hat{\omega})^2}O, \frac{(1 - \hat{h}_t(t))\hat{h}_\omega(\omega)}{(t - \hat{\omega})(w - \hat{t})}O, \quad (7.26)$$

$$\frac{1 - \hat{h}_\omega(\omega)}{(t - \hat{\omega})(w - \hat{t})}O. \quad (7.27)$$

Again applying first estimate (7.24) and taking into consideration that  $t \in D_\omega$ , we obtain for all these terms the estimate  $C|\omega - \hat{\omega}|^{-2}$ .

Estimate (7.20) easily follows from (7.24).

First two estimates (7.21) follow from (7.19) if we put  $\omega = w$ . To prove the third estimate we consider the derivative  $(t_w)_{t^2}$  at  $t \in D_\omega$ . Analogously to (7.25) - (7.27), there appear the terms

$$\frac{1}{(t - \hat{\omega})^2}O, \frac{1 - \hat{h}_t(t)}{(t - \hat{\omega})^2}O, \frac{\hat{h}_t(t)}{(t - \hat{\omega})(w - \hat{t})}O, \quad (7.28)$$

$$\frac{\hat{h}_{t^2}(t)}{t - \hat{\omega}}O, \frac{(1 - \hat{h}_t(t))\hat{h}_t(t)}{(t - \hat{\omega})(w - \hat{t})}O, \frac{(\hat{h}_t(t))^2}{(w - \hat{t})^2}O, \frac{\hat{h}_{t^2}(t)}{w - \hat{t}}O. \quad (7.29)$$

As above, we see that all these terms have the estimate  $C|\omega - \hat{\omega}|^{-2}$  (here we must apply also second estimate (7.24)).

To prove inequalities (7.22) we estimate  $\omega$  and  $t$ -derivatives of terms (7.28), (7.29). Again, when we differentiate  $O$ , there appears the multiples

$$\frac{1}{t - \hat{\omega}}, \frac{\hat{h}_\omega(\omega)}{t - \hat{\omega}}, \frac{1 - \hat{h}_\omega(\omega)}{t - \hat{\omega}}, \frac{\hat{h}_t(t)}{t - \hat{\omega}}, \frac{\hat{h}_t(t)}{w - \hat{t}}, \frac{\hat{h}_\omega(\omega)}{w - \hat{t}}.$$

All these multiples are of order  $|\omega - \hat{\omega}|^{-1}$ . Other multiples appearing at differentiation are

$$\begin{aligned}
& \frac{1}{(t - \hat{\omega})^3}, \frac{\hat{h}_\omega(\omega)}{(t - \hat{\omega})^3}, \frac{(1 - \hat{h}_t(t))\hat{h}_\omega(\omega)}{(t - \hat{\omega})^3}, \\
& \frac{1 - \hat{h}_t(t)}{(t - \hat{\omega})^3}, \frac{\hat{h}_{t^2}(t)}{(t - \hat{\omega})^2}, \frac{\hat{h}_t(t)}{(t - \hat{\omega})^2(w - \hat{t})}, \\
& \frac{(\hat{h}_t(t))^2}{(t - \hat{\omega})(w - \hat{t})^2}, \frac{\hat{h}_{t^2}(t)}{(t - \hat{\omega})(w - \hat{t})}, \frac{\hat{h}_{t^3}(t)}{t - \hat{\omega}}, \\
& \frac{\hat{h}_{t^2}(t)\hat{h}_\omega(\omega)}{(t - \hat{\omega})^2}, \frac{(1 - \hat{h}_t(t))\hat{h}_t(t)}{(t - \hat{\omega})^2(w - \hat{t})}, \frac{(1 - \hat{h}_t(t))\hat{h}_t(t)\hat{h}_\omega(\omega)}{(t - \hat{\omega})^2(w - \hat{t})}, \\
& \frac{(1 - \hat{h}_t(t))(\hat{h}_t(t))^2}{(t - \hat{\omega})(w - \hat{t})^2}, \frac{(1 - \hat{h}_t(t))\hat{h}_{t^2}(t)}{(t - \hat{\omega})(w - \hat{t})}, \frac{h_t(t)\hat{h}_{t^2}(t)}{(t - \hat{\omega})(w - \hat{t})}, \\
& \frac{(\hat{h}_t(t))^3}{(w - \hat{t})^3}, \frac{2\hat{h}_t(t)\hat{h}_{t^2}(t)}{(w - \hat{t})^2}, \frac{\hat{h}_{t^3}(t)}{w - \hat{t}}.
\end{aligned}$$

As above, we obtain the estimate  $C|\omega - \hat{\omega}|^{-3}$  at  $t \in D_\omega$ .

To prove inequality (7.23) we must estimate  $(K_w)_{t^4}$ . As above, at each differentiation we either obtain the additional multiples  $(t - \hat{\omega})^{-1}O(1)$  or  $(w - \hat{t})^{-1}O(1)$  either replace a derivative of the function  $\hat{h}$  by a derivative of order higher by 1. In either case we obtain an expression of order  $|\omega - \hat{\omega}|^{-4}$ .  $\square$

Remark. The fact that we must differentiate up to the fourth order and use estimate (7.23) explains the conditions  $P \geq 4$ ,  $L \geq 4$  of Theorem 2'.

Return now to Proposition 37. We shall consider the cases corresponding to various items in the kernel of integral (7.14).

1) Consider at first the integral

$$J_{k1}(\omega) = \frac{1}{\pi} \int_{\Omega} \mu(t)r_{wk}(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)^2}.$$

**Proof of representation (7.16) for  $J_{k1}$ .** Suppose at first that  $\omega \in D_w$  and consider the integral over the domain  $D_w \cup D_\omega$ . We have representation (7.18) and the representation

$$\mu(t) = \mu(w) + (t - w)\mu_w(t),$$

where  $\mu_w$  satisfies the estimates

$$|\mu_w(t)| \leq cb|w - \hat{w}|^{-1}, |(\mu_w)_{(j)}(t)| \leq cb|w - \hat{w}|^{-2}, |(j)| = 1 \quad (7.30)$$

for some uniform  $c$ .

Thus we consider the integral

$$\frac{1}{\pi} \int_{D_\omega \cup D_w} (\mu(w) + (t - w)\mu_w(t))(a_w(\omega) + (t - w)R_w(t, \omega))r_{wk}(t) \frac{dS_t}{(t - \omega)^2}. \quad (7.31)$$

At first we estimate the integral

$$I_{kw}(\omega) = \frac{1}{\pi} \mu(w)a_w(\omega) \int_{D_w} r_{wk}(t) \frac{dS_t}{(t - \omega)^2} + \frac{1}{\pi} \mu(w)a_w(\omega) \int_{D_\omega \setminus D_w} r_{wk}(t) \frac{dS_t}{(t - \omega)^2}. \quad (7.32)$$

The first integral in the right side up to the multiple  $\mu(w)a_w(\omega)$  is the value of the Beurling transform of the function  $r_{wk}\chi_{D_w}$  in the point  $\omega$ . But it is easy to obtain this transform in the explicit form. Indeed, consider the function equals to

$$\frac{(\bar{t} - \bar{w})^{k+1}}{(k+1)(t - w)^k}$$

in  $D_w$  and to

$$\frac{r_w^{2k+2}}{(k+1)(t - w)^{2k+1}}$$

in  $\mathbb{C} \setminus D_w$ . This function is continuous, tends to zero at infinity, and its  $\bar{t}$ -derivative (in the sense of distributions) equals to  $r_{wk}\chi_{D_w}$ . Hence,  $t$ -derivative of this function is the Beurling transform of  $r_{wk}\chi_{D_w}$ . Thus,

$$\mathcal{S}r_{wk}\chi_{D_w}(\omega) = -\frac{k}{k+1} \left( \frac{\bar{\omega} - \bar{w}}{\omega - w} \right)^{k+1} = -\frac{k}{k+1} r_{w,k+1}(\omega)$$

at  $\omega \in D_w$ . We see that we can write the first integral in the right side of (7.32) as

$$-\frac{k}{k+1} \mu(w)a_w(\omega)r_{w,k+1}(\omega) = -\frac{k}{k+1} \mu(w)r_{w,k+1}(\omega) + \mu(w)A_w(\omega),$$

where  $A_w$  is uniformly bounded. Below we shall prove that  $A_w$  satisfies estimate (7.17).

Now we can estimate the second integral in the right side of (7.32) as

$$\left| \int_{D_\omega} (r_{wk} \chi_{D_\omega \setminus D_w} - r_{wk}(\omega)) \frac{dS_t}{(t-\omega)^2} \right| \leq \max_{D_\omega} (|(r_{wk})_t| + |(r_{wk})_{\bar{t}}|) \int_{D_\omega} \frac{dS_t}{|t-\omega|} \leq C_1 k$$

for some uniform  $C_1$ . We obtain for  $I_{kw}$  the representation

$$I_{kw}(\omega) = -\pi \frac{k}{k+1} r_{w,k+1}(\omega) + k f_w(\omega)$$

with uniformly bounded  $f_w$ .

Now to finish with integral (7.31) we must estimate the integral

$$\int_{D_\omega \cup D_w} g_w(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1}} \frac{dS_t}{(t-\omega)^2}, \quad (7.33)$$

where  $|g_w(t, \omega)| \leq cb|w - \hat{w}|^{-1}$ ,  $|(g_w)_t(t, \omega)| \leq cb|w - \hat{w}|^{-2}$ , and we have analogous estimate for  $|(g_w)_{\bar{t}}$  according to (7.30), (7.20). Consider at first the integral over  $D_\omega$ . Under the integral we have the function

$$\frac{\tilde{g}_w(t, \omega)}{(t-\omega)^2},$$

where  $t$ -derivatives of the function  $\tilde{g}_w$  we can estimate as

$$cb|w - \hat{w}|^{-2} r_w + kcb|w - \hat{w}|^{-1} \leq Ckb|w - \hat{w}|^{-1} \leq Ckb|\omega - \hat{\omega}|^{-1}. \quad (7.34)$$

Thus we can estimate integral (7.33) over  $D_\omega$  as

$$\left| \int_{D_\omega} \frac{\tilde{g}_w(t, \omega)}{(t-\omega)^2} dS_t \right| \leq \frac{Ckb}{|\omega - \hat{\omega}|} \int_{D_\omega} \frac{dS_t}{|t-\omega|} \leq C'kb$$

for some  $C'$  since  $D_\omega$  is a disk of radii  $d|\omega - \hat{\omega}|$ . We estimate integral (7.33) over  $D_w \setminus D_\omega$  analogously to the second integral in (7.32) applying estimate (7.34).

Now, let  $\omega \notin D_w$ , and consider the integral over  $D_\omega$ . Instead of (7.31) we use the representation

$$\int_{D_\omega} (\mu(\omega) + (t-\omega)\mu_\omega(t))(1 + (t-\omega)R_\omega(t, \omega))r_{wk}(t) \frac{dS_t}{(t-\omega)^2},$$

Here for  $\mu_\omega(t)$  and  $R_\omega(t, \omega)$  we have estimates (7.30) and (7.18). We act as above but this case is more simple, we don't need in estimates (7.20). We estimate the integral

$$\int_{D_\omega} r_{wk}(t) \frac{dS_t}{(t - \omega)^2}$$

as

$$\max_{D_\omega} (|(r_{wk})_t| + |(r_{wk})_{\bar{t}}|) \int_{D_\omega} \frac{dS_t}{|t - \omega|} \leq kc|\omega - w|^{-1}r_\omega \leq kC|\omega - \hat{\omega}|^{-1}r_\omega \leq kC$$

for some uniform  $C$ .

Instead of (7.33) we have the integral

$$\int_{D_\omega} g'_w(t, \omega) r_{wk}(t) \frac{dS_t}{(t - \omega)^2},$$

where  $|g'_w(t, \omega)| \leq cb|\omega - \hat{\omega}|^{-1}$ . Acting as above we obtain the estimate  $Ckb$ .

Now we estimate the integral over  $\Omega \setminus (D_w \cup D_\omega)$ .

We set  $\omega = g(\zeta)$ ,  $w = g(z)$ . Let  $\varphi_\zeta$  be the function

$$\varphi_\zeta(\tau) = \frac{\zeta - \tau}{1 - \bar{\zeta}\tau},$$

and define  $g_\zeta = g \circ \varphi_\zeta$ . Note that  $\zeta = \varphi_\zeta(0)$ .

Since  $|r_{wk}| = 1$  and  $|\mu| \leq \delta$ , it is enough to estimate the integral

$$\begin{aligned} \mathcal{I}_{k1}(\zeta) &= (1 - |\zeta|^2)^2 \int_{D \setminus g_\zeta^{-1}D_\omega} \left| \frac{g_\zeta(0) - \widehat{g_\zeta(0)}}{g_\zeta(\tau) - \widehat{g_\zeta(0)}} \right|^{m-l} \left| \frac{g_\zeta(\tau) - \widehat{g_\zeta(\tau)}}{g_\zeta(\tau) - \widehat{g_\zeta(0)}} \right|^l \left| \frac{g(z) - \widehat{g_\zeta(0)}}{g(z) - \widehat{g_\zeta(\tau)}} \right|^l \\ &\quad \frac{J_g(\varphi_\zeta(\tau))}{|g_\zeta(\tau) - g_\zeta(0)|^2} \frac{dS_\tau}{|1 - \bar{\zeta}\tau|^4}, \end{aligned}$$

where  $J_g(t)$  is the Jacobian of the change of the variable  $\omega \mapsto t = g^{-1}(\omega)$ .

Now, by inequality (6.4), we have

$$|g_\zeta(\tau) - g_\zeta(0)| \geq c|\varphi_\zeta(\tau) - \varphi_\zeta(0)|.$$

for some uniform  $c$ . But

$$\varphi_\zeta(\tau) - \varphi_\zeta(0) = \frac{|\tau|(1 - |\zeta|^2)}{|1 - \bar{\zeta}\tau|}. \quad (7.35)$$



From the other hand, the domain  $g_\zeta^{-1}D_\omega$  contains some domain

$$\{|\varphi_\zeta(\tau) - \zeta| \leq d'(1 - |\zeta|^2)\} = \left\{ \frac{|\tau|}{|1 - \bar{\zeta}\tau|} \leq d' \right\},$$

where  $d'$  is some uniform constant. Thus  $|\tau| \geq d'|1 - \bar{\zeta}\tau|$  on  $D \setminus g_\zeta^{-1}D_\omega$ . In particular, the domain  $g_\zeta^{-1}D_\omega$  contains some disk of radius uniformly bounded from below. Applying (7.35), we see that on  $D \setminus g_\zeta^{-1}D_\omega$

$$|g_\zeta(\tau) - g_\zeta(0)| \geq c(1 - |\zeta|^2) \quad (7.36)$$

for some uniform  $c$ . We see that it is enough to estimate the integral

$$|g(\zeta)(0) - \widehat{g(\zeta)(0)}|^{m-l} |g(z) - \widehat{g_\zeta(0)}|^l \int_D \frac{|g_\zeta(\tau) - \widehat{g_\zeta(\tau)}|^l dS_\tau}{|g_\zeta(\tau) - \widehat{g_\zeta(0)}|^m |g(z) - \widehat{g_\zeta(\tau)}|^l |1 - \bar{\zeta}\tau|^4}.$$

Applying (6.2) and (6.5), we see that we must estimate the integral

$$(1 - |\zeta|^2)^{m-l} |1 - z\bar{\zeta}|^l |\zeta|^{-m} \int_D \frac{|\varphi_\zeta(\tau) - \overline{\varphi_\zeta(\tau)}^{-1}|^l dS_\tau}{|\varphi_\zeta(\tau) - \bar{\zeta}^{-1}|^m |z - \overline{\varphi_\zeta(\tau)}^{-1}|^l |1 - \bar{\zeta}\tau|^4}. \quad (7.37)$$

We have

$$\varphi_\zeta(\tau) - \overline{\varphi_\zeta(\tau)}^{-1} = \frac{(1 - |\zeta|^2)(1 - |\tau|^2)}{(\bar{\zeta} - \bar{\tau})(1 - \tau\bar{\zeta})}, \quad (7.38)$$

$$\varphi_\zeta(\tau) - \bar{\zeta}^{-1} = \frac{|\zeta|^2 - 1}{\bar{\zeta}(1 - \tau\bar{\zeta})}, \quad (7.39)$$

$$(z - \overline{\varphi_\zeta(\tau)}^{-1})^{-1} = -\frac{\bar{\zeta} - \bar{\tau}}{1 - z\bar{\zeta} - \bar{\tau}(\zeta - z)} = \frac{\bar{\zeta} - \bar{\tau}}{1 - z\bar{\zeta}} \frac{1}{1 - \bar{\tau}z\zeta}, \quad (7.40)$$

where we denote

$$z_\zeta = (\varphi_\zeta)^{-1}(z) = \frac{\zeta - z}{1 - z\bar{\zeta}}. \quad (7.41)$$

We see that integral (7.35) equals to

$$\begin{aligned} (1 - |\zeta|^2)^m |1 - z\bar{\zeta}|^2 |\zeta|^{-m} \int_D \frac{(1 - |\tau|^2)^l}{|\zeta - \tau|^l |1 - \tau\bar{\zeta}|^l} \frac{|\zeta|^m |1 - \tau\bar{\zeta}|^m}{(1 - |\zeta|^2)^m} \frac{|\zeta - \tau|^l}{|1 - z\bar{\zeta}|^l (|1 - \bar{\tau}z\zeta|^l |1 - \tau\bar{\zeta}|^4)} dS_\tau &= \\ = \int_D \frac{(1 - |\tau|^2)^l |1 - \tau\bar{\zeta}|^{m-l-4}}{|1 - \bar{\tau}z\zeta|^l} dS_\tau. \end{aligned}$$

This integral is uniformly bounded at  $m \geq l + 4$ .  $\square$

For convenience we write here the "general term" that appears in various integrals when we make the change of the variable:  $t = g_\zeta(\tau)$  and neglect uniformly bounded multiples. We denote

$$G_z(\tau, \zeta) = (1 - |\zeta|^2)^2 \left[ \frac{g_\zeta(0) - \widehat{g_\zeta(0)}}{g_\zeta(\tau) - \widehat{g_\zeta(0)}} \right]^{m-l} \left[ \frac{g_\zeta(\tau) - \widehat{g_\zeta(\tau)}}{g_\zeta(\tau) - \widehat{g_\zeta(0)}} \right]^l \left[ \frac{g(z) - \widehat{g_\zeta(0)}}{g(z) - \widehat{g_\zeta(\tau)}} \right]^l \frac{J_g(\varphi_\zeta(\tau))}{|1 - \bar{\zeta}\tau|^4}.$$

We write our correspondence in the form

$$G_z(\tau, \zeta) \sim (1 - |\zeta|^2)^2 \frac{(1 - |\tau|^2)^l (|1 - \tau\bar{\zeta}|^{m-l-4})}{|1 - \bar{\tau}\zeta|^l}. \quad (7.42)$$

**Proof of estimate (7.17) for  $J_{k_1}$ .**

Fix some  $\omega_0 \in \Omega$ . Note that it is enough to prove estimate (7.17) when  $\omega \in D_{\omega_0}$ . Indeed,  $|\omega - \omega_0| > c|\omega - \hat{\omega}_0|$  with some uniform  $c$  if  $|\omega - \omega_0| > d|\omega_0 - \hat{\omega}_0|$  and we obviously have estimate (7.17) since  $|F_{wk}| \leq ckb$ .

Let  $D_{w_1}, D_{w_2}, D_{w_3}$  be three disks centered at  $w$  of radii  $r_{1w}, r_{2w} = 2r_{1w}, r_{3w} = 3r_{1w}$ . There can be the case when at least one of the points  $\omega$  or  $\omega_0$  belongs to  $D_{w_1}$ . We set  $r_\omega = r_{\omega_0} = r_{1w}$  in this case. The other point then belongs to  $D_{w_2}$  and both disks  $D_\omega$  and  $D_{\omega_0}$  are contained in  $D_w$ . We put  $D_w = D_{w_3}$  in this case and  $D_w = D_{w_1}$  if both  $\omega$  or  $\omega_0$  don't belong to  $D_{w_1}$ . Thus we can consider only the cases:

- a)  $D_w$  contains  $D_\omega$  and  $D_{\omega_0}$ ,  $r_\omega = r_{\omega_0} = r_w/3$ ,
- b) both  $\omega$  and  $\omega_0$  don't belong to  $D_w$ ,  $|w - \omega| > r_\omega = r_{\omega_0} < |w - \omega_0|$ .

In the following proof we shall consider cases a) and b) separately.

*Case a).* The difference of the integrals over  $D_w$ .

We estimate the difference of the integrals

$$\frac{1}{\pi} \int_{D_w} \mu(t) K_w(t, \omega) r_{wk}(t) \frac{dS_t}{(t - \omega)^2} - \frac{1}{\pi} \int_{D_w} \mu(t) K_w(t, \omega_0) r_{wk}(t) \frac{dS_t}{(t - \omega_0)^2}.$$

We represent this difference as

$$\begin{aligned} & \frac{1}{\pi} \int_{D_w} (\mu(w) + (t - w)\mu_w(t))(a_w(\omega) + (t - w)R_w(t, \omega)) r_{wk}(t) \frac{dS_t}{(t - \omega)^2} - \\ & - \frac{1}{\pi} \int_{D_w} (\mu(w) + (t - w)\mu_w(t))(a_w(\omega_0) + (t - w)R_w(t, \omega_0)) r_{wk}(t) \frac{dS_t}{(t - \omega_0)^2}. \end{aligned} \quad (7.43)$$

We shall estimate the difference of these integrals in several steps. The integral

$$\frac{1}{\pi} \int_{D_w} r_{wk}(t) \left( \frac{a_w(\omega)}{(t-\omega)^2} - \frac{a_w(\omega_0)}{(t-\omega_0)^2} \right) dS_t$$

equals to  $\pi k/(k+1)(a_w(\omega)r_{w,k+1}(\omega) - a_w(\omega_0)r_{w,k+1}(\omega_0))$ , as we already saw. Now  $a_w(\omega) = 1 + (\omega - w)b_w(\omega)$ , where  $|b_w(\omega)| \leq c|\omega - \hat{\omega}|^{-1}$  and  $|(b_w)_{(j)}(\omega)| \leq c|\omega - \hat{\omega}|^{-2}$  at  $\omega \in D_w$  and  $|(j)| = 1$ . It follows

$$\begin{aligned} & |(a_w(\omega)r_{w,k+1}(\omega) - a_w(\omega_0)r_{w,k+1}(\omega_0)) - (r_{w,k+1}(\omega) - r_{w,k+1}(\omega_0))| \leq \\ & \leq |b_w(\omega) - b_w(\omega_0)| \frac{|\bar{\omega} - \bar{w}|^{k+1}}{|\omega - w|^k} + |b_w(\omega_0)| \left| \frac{(\bar{\omega} - \bar{w})^{k+1}}{(\omega - w)^k} - \frac{(\bar{\omega}_0 - \bar{w})^k}{(\omega_0 - w)^{k-1}} \right| \leq \\ & \leq C|\omega - \omega_0| \left( \frac{|\bar{\omega} - \bar{w}|}{|\omega - \hat{\omega}|^2} + \frac{k}{|\omega - \hat{\omega}|} \right) \leq \frac{Ck}{|\omega - \hat{\omega}|}. \end{aligned}$$

Thus we obtained the representation

$$\mu(w)a_w(\omega) \int_{D_w} r_{wk}(t) \frac{dS_t}{(t-\omega)^2} = \pi \frac{k}{k+1} \mu(w)r_{w,k+1}(\omega) + A_w(\omega),$$

where

$$|A_w(\omega) - A_w(\omega_0)| \leq ck \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|}.$$

Now consider the integral

$$\begin{aligned} & \frac{1}{\pi} \int_{D_w} \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1}} [(\mu_w(t)a_w(\omega) + \mu(w)R_w(t, \omega) + (t-w)\mu_w(t)a_w(\omega)R_w(t, \omega)) \frac{1}{(t-\omega)^2} - \\ & - (\mu_w(t)a_w(\omega_0) + \mu(w)R_w(t, \omega_0) + (t-w)\mu_w(t)a_w(\omega_0)R_w(t, \omega_0)) \frac{1}{(t-\omega_0)^2}] dS_t. \end{aligned} \tag{7.44}$$

We represent the expression in the square brackets under the integral as

$$\begin{aligned} & [r_1(w, \omega) + (t-w)r_1'(t, \omega)] \left[ \frac{1}{(t-\omega)^2} - \frac{1}{(t-\omega_0)^2} \right] + \\ & (\omega - \omega_0)r_2(t, \omega, \omega_0) \frac{1}{(t-\omega_0)^2}, \end{aligned}$$

where, by (7.20), we have the estimates

$$|r_1(w, \omega)| \leq \frac{Cb}{|\omega - \hat{\omega}|}, |r_1'(t, \omega)| \leq \frac{Cb}{|\omega - \hat{\omega}|^2}, |r_2(t, \omega, \omega_0)| \leq \frac{Cb}{|\omega - \hat{\omega}|^2} \quad (7.45),$$

and, by (7.21), (7.22),

$$|\partial_{(j),t} r_1'(t, \omega)| \leq \frac{Cb}{|\omega - \hat{\omega}|^3}, |\partial_{(j),t} r_2(t, \omega, \omega_0)| \leq \frac{Cb}{|\omega - \hat{\omega}|^3}, |(j)| = 1, \quad (7.46)$$

$$|\partial_{(j),t} r_1'(t, \omega)| \leq \frac{Cb}{|\omega - \hat{\omega}|^4}, |(j)| = 1, \quad (7.47)$$

where  $\partial_{(j),t}$  means  $t$ -derivative with multi-index  $(j)$ .

Return now to integral (7.44). At first we consider the integral

$$r_1(w, \omega) \int_{D_w} \frac{1}{\pi} \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-1}} \left[ \frac{1}{(t - \omega)^2} - \frac{1}{(t - \omega_0)^2} \right] dS_t. \quad (7.48)$$

We must calculate the difference in the points  $\omega$  and  $\omega_0$  of the Beurling transform of the function  $(\bar{t} - \bar{w})^k / (t - w)^{k-1} \chi_{D_w}(t)$ . But this transform equals to

$$\frac{k-1}{k+1} \frac{(\bar{t} - \bar{w})^{k+1}}{(t - w)^k}.$$

if  $t \in D_w$ . Indeed, the Cauchy transform of the last function equals to

$$\frac{(\bar{t} - \bar{w})^{k+1}}{(k+1)(t - w)^{k-1}}$$

in  $D_w$  and

$$\frac{r_w^{2k+2}}{(k+1)(t - w)^{2k}}$$

in  $\mathbb{C} \setminus D_w$ . Indeed, this function is continuous, tends to zero at infinity, and its  $\bar{t}$ -derivative (in the sense of distributions) equals to  $(\bar{t} - \bar{w})^k / (t - w)^{k-1} \chi_{D_w}(t)$ . Hence,  $t$ -derivative of this function is the Beurling transform of  $(\bar{t} - \bar{w})^k / (t - w)^{k-1} \chi_{D_w}(t)$ . We see that integral (7.48) equals to

$$r_1(w, \omega) \frac{k-1}{k+1} \left[ \frac{(\bar{\omega} - \bar{w})^{k+1}}{(\omega - w)^k} - \frac{(\bar{\omega}_0 - \bar{w})^{k+1}}{(\omega_0 - w)^k} \right].$$

We obtain the estimate

$$Ck|r_1(w, \omega)||\omega - \omega_0| \leq \frac{Ckb}{|\omega - \hat{\omega}|}|\omega - \omega_0|,$$

since the first order derivatives of the function  $(\bar{\omega} - \bar{w})^{k+1}/(\omega - w)^k$  have the uniform estimate  $\leq k + 1$ , and for  $r_1$  we have estimate (7.45).

Now consider the difference

$$\int_{D_w} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}} \frac{1}{(t - \omega)^2} dS_t - \int_{D_w} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}} \frac{1}{(t - \omega_0)^2} dS_t.$$

We make the change of the variable  $t \mapsto t + \omega - \omega_0$  in the first integral and denote by  $D_{w_0}$  the domain  $\omega - \omega_0 + D_w$ . We get the sum

$$\begin{aligned} & \int_{D_w \cap D_{w_0}} \left[ r'_1(t + \omega - \omega_0, \omega) \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-2}} - r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}} \right] \frac{dS_t}{(t - \omega_0)^2} + \\ & + \int_{D_w \setminus D_{w_0}} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}} \frac{1}{(t - \omega)^2} dS_t + \int_{D_{w_0} \setminus D_w} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}} \frac{1}{(t - \omega_0)^2} dS_t. \end{aligned} \quad (7.49)$$

We represent the first integral as

$$\begin{aligned} & \int_{D_w \cap D_{w_0}} [r'_1(t + \omega - \omega_0, \omega) - r'_1(t, \omega)] \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-2}} \frac{dS_t}{(t - \omega_0)^2} + \\ & + \int_{D_w \cap D_{w_0}} r'_1(t, \omega) \left[ \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-2}} - \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}} \right] \frac{dS_t}{(t - \omega_0)^2} \end{aligned} \quad (7.50)$$

We consider the first integral in right side analogously to integral (7.33). We see that we must estimate

$$\begin{aligned} & |\omega - \omega_0| \left[ \sup_{t \in D_w \cap D_{w_0}, |(j)|=2} (|\partial_{(j),t} r'_1(t, \omega)| |t + \omega - \omega_0 - w|^2) + \right. \\ & \left. + \sup_{t \in D_w \cap D_{w_0}, |(j)|=1} (|\partial_{(1),t} r'_1(t, \omega)| k |t + \omega - \omega_0 - w|) \right] \int_{D_w \cap D_{w_0}} \frac{dS_t}{|t - \omega_0|} \end{aligned}$$

But  $|t + \omega - \omega_0 - w| \leq c|\omega - \hat{\omega}|$  for some  $c$ . We apply inequalities (7.46) and (7.47) and obtain the estimate  $Ckb|\omega - \hat{\omega}|^{-1}|\omega - \omega_0|$ . From the other hand, we can represent the difference

$$\frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-2}} - \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-2}}$$

as  $(\omega - \omega_0)\tilde{R}_w(t, \omega, \omega_0)$ , where for  $\tilde{R}_w$  we have the estimates  $|\tilde{R}_w(t, \omega, \omega_0)| \leq Ck|\omega - \hat{\omega}|$ ,  $|\partial_{(j),t}\tilde{R}_w(t, \omega, \omega_0)| \leq Ck^2$ ,  $|(j)| = 1$ . We see that we can estimate the second integral in the right side of (7.49) as

$$|\omega - \omega_0|C \left[ \sup_{t \in D_w \cap D_{w_0}, |(j)|=1} (|\partial_{(j),t}r'_1(t, \omega)|k|\omega - \hat{\omega}| + \sup_{t \in D_w \cap D_{w_0}} |r'_1(t, \omega)|k^2) \int_{D_w \cap D_{w_0}} \frac{dS_t}{|t - \omega_0|} \leq \frac{Ck^2b}{|\omega - \hat{\omega}|} |\omega - \omega_0|. \right.$$

Here we again apply estimates (7.45), (7.46).

The second and third integrals in (7.49) are over the lunules  $D_w \setminus D_{w_0}$  and  $D_{w_0} \setminus D_w$  of width  $\sim |\omega - \omega_0|$ . We estimate, for example, the second integral as

$$\left[ \sup_{t \in D_w \setminus D_{w_0}, |(j)|=1} (|\partial_{(j),t}r'_1(t, \omega)||t-w|^2) + k \sup_{t \in D_w \setminus D_{w_0}} (|r'_1(t, \omega)||t-w|) \right] \int_0^\pi \int_{r_{1\theta}}^{r_{2\theta}} d\rho d\theta \leq \leq \frac{Ckb}{|\omega - \hat{\omega}|} |\omega - \omega_0|$$

since  $|t - w| \leq c|\omega - \hat{\omega}|$  for some  $c$  and we have estimates (7.45), (7.46).

Now to finish with integral (7.44) we must estimate the integral

$$\int_{D_w} r_2(t, \omega, \omega_0) \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-1} (t - \omega_0)^2} dS_t.$$

Applying (7.45), (7.46), we obtain the estimate

$$\left[ \sup_{t \in D_w, |(j)|=1} (|\partial_{(j),t}r_2(t, \omega, \omega_0)||t-w|) + k \sup_{t \in D_w} (|r_2(t, \omega, \omega_0)|) \right] \int_{D_w} \frac{dS_t}{|t - \omega_0|} \leq \frac{Ckb}{|\omega - \hat{\omega}|}.$$

*Case a). Difference of the integrals over  $\Omega \setminus D_w$ .*

We put  $\tau = g_w^{-1}(t) = \varphi_z^{-1} \circ g^{-1}(t)$ . We must estimate the difference of the integrals

$$\begin{aligned} & \int_{D \setminus g_w^{-1}D_w} G_w(\tau, \omega) \mu \circ g_w(\tau) r_{wk} \circ g_w(\tau) \frac{dS_\tau}{(g_w(\tau) - \omega)^2} - \\ & - \int_{D \setminus g_w^{-1}D_w} G_w(\tau, \omega_0) \mu \circ g_w(\tau) r_{wk} \circ g_w(\tau) \frac{dS_\tau}{(g_w(\tau) - \omega_0)^2}, \end{aligned}$$

where

$$G_w(\tau, \omega) = K_w(g_w(\tau), \omega) J_{g_w}(\tau) = \left( \frac{\omega - \hat{\omega}}{g_w(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_w(\tau) - \widehat{g_w(\tau)}}{g_w(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \widehat{g_w(\tau)}} \right)^l J_{g_w}(\tau).$$

Here  $J_{g_w}$  is the Jacobian of the transformation  $t \mapsto \tau = \varphi_z^{-1} \circ g^{-1}(t)$ .

It is enough to estimate the  $\omega$ -derivative of the integral over  $\Omega \setminus D_w$ . Now we can differentiate under the integral, i.e., we must estimate the expression

$$\int \frac{((G_w)_\omega \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)^2} dS_\tau - 2 \int \frac{(G_w \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)^3} dS_\tau \quad (7.51)$$

We have

$$\begin{aligned} (G_w)_\omega(\tau, \omega) &= (m-l) \left( \frac{1 - \hat{h}_\omega(\omega)}{g_w(\tau) - \hat{\omega}} + \frac{\omega - \hat{\omega}}{(g_w(\tau) - \hat{\omega})^2} \hat{h}_\omega(\omega) \right) \\ &\quad \left( \frac{\omega - \hat{\omega}}{g_w(\tau) - \hat{\omega}} \right)^{m-l-1} \left( \frac{g_w(\tau) - \widehat{g_w(\tau)}}{g_w(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \widehat{g_w(\tau)}} \right)^l J_{g_w}(\tau) + \\ &\quad + l \frac{g_w(\tau) - \widehat{g_w(\tau)}}{(g_w(\tau) - \hat{\omega})^2} \hat{h}_\omega(\omega) \left( \frac{\omega - \hat{\omega}}{g_w(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_w(\tau) - \widehat{g_w(\tau)}}{g_w(\tau) - \hat{\omega}} \right)^{l-1} \left( \frac{w - \hat{\omega}}{w - \widehat{g_w(\tau)}} \right)^l J_{g_w}(\tau) - \\ &\quad - l \frac{\hat{h}_\omega(\omega)}{w - \widehat{g_w(\tau)}} \left( \frac{\omega - \hat{\omega}}{g_w(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_w(\tau) - \widehat{g_w(\tau)}}{g_w(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \widehat{g_w(\tau)}} \right)^{l-1} J_{g_w}(\tau) + \\ &= G_w(\tau, \omega) \left( (m-l) \frac{1 - \hat{h}_\omega(\omega)}{\omega - \hat{\omega}} + m \frac{\hat{h}_\omega(\omega)}{g_w(\tau) - \hat{\omega}} + l \frac{\hat{h}_\omega(\omega)}{w - \hat{\omega}} \right). \end{aligned} \quad (7.52)$$

Now it is easy to see that for  $G_w$  we have the same estimate (7.42) as for  $G_z$

$$G_w(\tau, \omega) \sim (1 - |\zeta|^2)^2 \frac{(1 - |\tau|^2)^l (|1 - \tau \bar{\zeta}|^{m-l-4})}{|1 - \bar{\tau} z_\zeta|^l}. \quad (7.53)$$

Applying (7.52) and (7.36), we see that difference (7.51) has the estimate  $C|1 - |\zeta|^2|^{-1}$  at  $m \geq l + 4$ .

*Case b).* The difference of the integrals over  $D_w$  and  $D_{w_0}$ .

This case is more simple. We put  $\tilde{K}_w(t, \omega) = K_w(t + \omega, \omega)$ . In the integrals over  $D_\omega$  and  $D_{\omega_0}$  we change the variable  $t \mapsto t + \omega$  and  $t \mapsto t + \omega_0$  correspondingly. Thus we must estimate the integral

$$\int_{|t| \leq r_\omega} [\tilde{K}_w(t, \omega) r_{wk}(t + \omega) \mu(t + \omega) - \tilde{K}_w(t, \omega_0) r_{wk}(t + \omega_0) \mu(t + \omega_0)] \frac{dS_t}{t^2}.$$

Applying estimate (7.19) and estimates for derivatives of  $r_{wk}$  we represent the difference in the square brackets as

$$(r_{wk}(\omega) \mu(\omega) - r_{wk}(\omega_0) \mu(\omega_0)) + t R_1(t, \omega, \omega_0),$$

where  $|R_1| \leq cbk^2 |\omega - \omega_0| |\omega_0 - \hat{\omega}_0|^{-2}$  with some uniform  $c$ . Further,

$$(r_{wk}(\omega) \mu(\omega) - r_{wk}(\omega_0) \mu(\omega_0)) \int_{|t| \leq r_\omega} \frac{dS_t}{t^2} = 0$$

and

$$\left| \int_{|t| \leq r_\omega} R_1(t, \omega, \omega_0) \frac{dS_t}{t} \right| \leq \frac{ck^2 b |\omega - \omega_0|}{|\omega_0 - \hat{\omega}_0|^2} \int_{D_\omega} \frac{dS_t}{|t|} \leq \frac{Cbk^2 |\omega - \omega_0|}{|\omega_0 - \hat{\omega}_0|}.$$

It proves the estimate in our case.

*Case b).* The difference of the integrals over  $\Omega \setminus D_{\omega_0}$  and  $\Omega \setminus D_\omega$ . At first we unify the domains of integration.

**Proposition 39** *Suppose, as above,  $\omega = g(\zeta)$ ,  $r_\omega = d|\omega - \hat{\omega}|^{-1}$ . If  $d$  is small enough with some uniform estimate there exists a homeomorphism  $g_\omega$  mapping the ring  $1/2 \leq |\tau| \leq 1$  onto the domain  $D \setminus D_\omega$ . We can represent the map  $g_\omega$  as a composition*

$$g_\omega = g \circ \varphi_\zeta \circ \tilde{g}_\omega,$$

where  $\tilde{g}_\omega$  is a homeomorphism mapping the ring  $1/2 \leq |\tau| \leq 1$  onto the domain  $D \setminus \varphi_\zeta^{-1} \circ g^{-1}(D_\omega)$ . Denote  $\tau'_\omega = \tilde{g}_\omega(\tau)$ . For the map  $g_\omega$  we get the estimates

$$|(g_\omega)_\tau(\tau)| \leq C \frac{1 - |\zeta|^2}{|1 - \bar{\zeta} \tau'_\omega|^2}, \quad |(g_\omega)_\omega(\tau)| \leq \frac{C}{|1 - \bar{\zeta} \tau'_\omega|^2}, \quad (7.54)$$

$$|(g_\omega)_{\tau\omega}(\tau)| \leq C \left( \frac{1}{|1 - \bar{\zeta} \tau'_\omega|^4} + \frac{1}{|1 - \bar{\zeta} \tau'_\omega|^2 (1 - |\tau'_\omega|^2)} \right), \quad (7.55)$$



**Proof.** The domain  $D_\omega$  with the boundary  $\{\omega + r_\omega e^{i\varphi}\}, 0 \leq \varphi \leq 2\pi$  transforms under the mapping  $g^{-1}$  onto some domain with the boundary, described by the equation

$$|g(\zeta + \rho e^{i\varphi}) - g(\zeta)|^2 = d^2|\omega - \hat{\omega}|^2. \quad (7.56)$$

If  $d$  is small enough with some uniform estimate, then this domain is star-like with respect to  $\zeta$ . Indeed, we have the estimates

$$|g_z| \leq b, |g_{\bar{z}}/g_z| \leq \delta < 1, |g_{(j)}(z)| \leq B|g(z) - \hat{g}(z)|^{-1}, |(j)| = 2.$$

Hence, we can write

$$g(\zeta + \rho e^{i\varphi}) - g(\zeta) = (g_z(\zeta)e^{i\varphi} + g_{\bar{z}}(\zeta)e^{-i\varphi})\rho + \Delta_1,$$

where  $|\Delta_1| \leq 4Bd\rho$  if  $\zeta + \rho e^{i\varphi}$  belongs to the curve described by equation (7.56). Also,

$$\frac{\partial}{\partial \rho} g(\zeta + \rho e^{i\varphi}) = (g_z(\zeta) + \Delta_2)e^{i\varphi} + (g_{\bar{z}}(\zeta) + \Delta_3)e^{-i\varphi},$$

where  $|\Delta_2| \leq 4Bd, |\Delta_3| \leq 4Bd$ . Thus we can write  $\rho$ -derivative of the left side of equation (7.56) in the form

$$\begin{aligned} 2\operatorname{Re}[(\overline{(g_z(\zeta)e^{i\varphi} + g_{\bar{z}}(\zeta)e^{-i\varphi})\rho + \Delta_1})((g_z(\zeta) + \Delta_2)e^{i\varphi} + (g_{\bar{z}}(\zeta) + \Delta_3)e^{-i\varphi})] = \\ = 2\rho(|g_z(\zeta)|^2 + |g_{\bar{z}}(\zeta)|^2) + 4\rho\operatorname{Re}[g_z(\zeta)\overline{g_{\bar{z}}(\zeta)}e^{i\varphi}] + g_z(\zeta)\Delta, \end{aligned} \quad (7.57)$$

where  $|\Delta| \leq cBd\rho$  with some uniform  $c$ . The right side of (7.57) by modulus is no less, than  $2\rho|g_z(\zeta)|^2(1 - \delta)^2 - b\Delta$  and non-equal to zero for sufficiently small  $d$ . It follows that the domain with boundary (7.56) is star-like.

Applying the change of the variable  $\tau = \varphi_\zeta^{-1} \circ g^{-1}$  we obtain the domain with the boundary described by the equation:

$$|g \circ \varphi_{g^{-1}(\omega)}(\rho e^{i\varphi}) - \omega|^2 - d^2|\omega - \hat{\omega}|^2 = 0. \quad (7.58)$$

Here we use the same notations in the chart  $\tau$ :  $\tau = \rho e^{i\varphi}$ . This domain is also star-like with respect to zero and we see that its boundary can be described also by the equation  $\rho = \tilde{h}(\varphi, \omega)$ . By differentiation of equation (7.58) we get the estimates for the derivatives

$$|\tilde{h}_\varphi| \leq C, |\tilde{h}_\omega| \leq C|\omega - \hat{\omega}|^{-1}, |\tilde{h}_{\varphi\omega}| \leq C|\omega - \hat{\omega}|^{-1}, |\tilde{h}_{\omega^2}| \leq C|\omega - \hat{\omega}|^{-2},$$

$$|\tilde{h}_{\varphi\omega^2}| \leq C|\omega - \hat{\omega}|^{-2}.$$

By homothety along radii we define a diffeomorphism  $\tilde{g}_\omega$  mapping the ring  $1/2 \leq |\tau| \leq 1$  onto the domain  $D \setminus \varphi_\zeta^{-1} \circ g^{-1}(D_\omega)$ . We use the same notation  $\tau$  for the chart in the preimage of  $\tilde{g}_\omega$ . The estimates for  $\tilde{h}$  yield following estimates for  $\tilde{g}_\omega$

$$|(\tilde{g}_\omega)_\tau| \leq C, |(\tilde{g}_\omega)_\omega| \leq C|\omega - \hat{\omega}|^{-1}, |(\tilde{g}_\omega)_{\tau\omega}| \leq C|\omega - \hat{\omega}|^{-1}, \quad (7.59)$$

and we have analogous estimates for derivatives containing  $\partial\bar{\tau}$  or  $\partial\bar{\omega}$ .

The map  $g_\omega = g \circ \varphi_\zeta \circ \tilde{g}_\omega$  maps the chart  $\tau$  onto the original chart  $t$ . We obtain estimates (7.53), (7.54) applying (7.59), estimates for derivatives of  $\varphi_\zeta$ , and the estimate

$$|g^{(k)} \circ \varphi_\zeta(\tau)| \leq C(1 - |\varphi_\zeta(\tau)|^2)^{1-|k|} = C \left( \frac{|1 - \bar{\zeta}\tau|^2}{(1 - |\zeta|^2)(1 - |\tau|^2)} \right)^{|k|-1}$$

at  $|k| \geq 1$ .  $\square$

Return now to the integral  $J_{k1}$  over the domain  $\Omega \setminus D_\omega$ . We can write it as

$$\int_{1/2 \leq |\tau| \leq 1} k_w(\tau, \omega) \mu \circ g_\omega(\tau) r_{wk} \circ g_\omega(\tau) \frac{dS_\tau}{(g_\omega(\tau) - \omega)^2},$$

where

$$k_w(\tau, \omega) = K_w(g_\omega(\tau), \omega) J_{g_\omega}(\tau) = \left( \frac{\omega - \hat{\omega}}{g_\omega(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_\omega(\tau) - \widehat{g_\omega(\tau)}}{g_\omega(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{w}}{w - \widehat{g_\omega(\tau)}} \right)^l J_{g_\omega}(\tau).$$

Here  $J_{g_\omega}$  is the Jacobian of the transformation  $t \mapsto \tau = \tilde{g}_\omega^{-1} \circ \varphi_\zeta^{-1} \circ g^{-1}(t)$ .

As above, to estimate the difference of the integrals over  $\Omega \setminus D_{\omega_0}$  and  $\Omega \setminus D_\omega$  it is enough to estimate  $\omega$ -derivative of this integral.

Again we can differentiate under the integral, i.e., we must estimate the integral

$$\begin{aligned} & \int \frac{((k_w)_\omega \mu \circ g_\omega r_{wk} \circ g_\omega)(\tau, \omega)}{(g_\omega(\tau) - \omega)^2} dS_\tau - \\ & - 2 \int \frac{(k_w \mu \circ g_\omega r_{wk} \circ g_\omega)(\tau, \omega)}{(g_\omega(\tau) - \omega)^3} ((g_\omega)_\omega(\tau) - 1) dS_\tau + \\ & + \int \frac{[k_w(\mu \circ g_\omega)_\omega r_{wk} \circ g_\omega + k_w \mu \circ g_\omega (r_{wk} \circ g_\omega)_\omega](\tau, \omega)}{(g_\omega(\tau) - \omega)^2} dS_\tau. \end{aligned} \quad (7.60)$$

We have

$$\begin{aligned}
(k_w)_\omega(t, \omega) &= (m-l) \left( \frac{1 - \hat{h}_\omega(\omega)}{g_\omega(\tau) - \hat{\omega}} - \frac{\omega - \hat{\omega}}{(g_\omega(\tau) - \hat{\omega})^2} ((g_\omega)_\omega(\tau) - \hat{h}_\omega(\omega)) \right) \\
&\quad \left( \frac{\omega - \hat{\omega}}{g_\omega(\tau) - \hat{\omega}} \right)^{m-l-1} \left( \frac{g_\omega(\tau) - \widehat{g_\omega(\tau)}}{g_\omega(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \widehat{g_\omega(\tau)}} \right)^l J_{g_\omega}(\tau) + \\
&\quad + l \left( \frac{(g_\omega)_\omega(\tau) - (\hat{h} \circ g_\omega)_\omega(\tau)}{g_\omega(\tau) - \hat{\omega}} - \frac{g_\omega(\tau) - \widehat{g_\omega(\tau)}}{(g_\omega(\tau) - \hat{\omega})^2} ((g_\omega)_\omega(\tau) - \hat{h}_\omega(\omega)) \right) \\
&\quad \left( \frac{\omega - \hat{\omega}}{g_\omega(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_\omega(\tau) - \widehat{g_\omega(\tau)}}{g_\omega(\tau) - \hat{\omega}} \right)^{l-1} \left( \frac{w - \hat{\omega}}{w - \widehat{g_\omega(\tau)}} \right)^l J_{g_\omega}(\tau) - \\
&\quad - l \left( \frac{\hat{h}_\omega(\omega)}{w - \widehat{g_\omega(\tau)}} - \frac{w - \hat{\omega}}{(w - \widehat{g_\omega(\tau)})^2} (\hat{h} \circ g_\omega)_\omega(\tau) \right) \\
&\quad \left( \frac{\omega - \hat{\omega}}{g_\omega(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_\omega(\tau) - \widehat{g_\omega(\tau)}}{g_\omega(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \widehat{g_\omega(\tau)}} \right)^{l-1} J_{g_\omega}(\tau) + \\
&\quad + \left( \frac{\omega - \hat{\omega}}{g_\omega(\tau) - \hat{\omega}} \right)^{m-l} \left( \frac{g_\omega(\tau) - \widehat{g_\omega(\tau)}}{g_\omega(\tau) - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \widehat{g_\omega(\tau)}} \right)^l (J_{g_\omega})_\omega(\tau) = \\
&= k_w(t, \omega) \left( (m-l) \frac{1 - \hat{h}_\omega(\omega)}{\omega - \hat{\omega}} - (m-2l) \frac{(g_\omega)_\omega(\tau) - \hat{h}_\omega(\omega)}{g_\omega(\tau) - \hat{\omega}} + l \frac{(g_\omega)_\omega(\tau) - (\hat{h} \circ g_\omega)_\omega(\tau)}{g_\omega(\tau) - \widehat{g_\omega(\tau)}} - \right. \\
&\quad \left. - l \frac{\hat{h}_\omega(\omega)}{w - \hat{\omega}} + l \frac{(\hat{h} \circ g_\omega)_\omega(\tau)}{w - \widehat{g_\omega(\tau)}} + \frac{(J_{g_\omega})_\omega(\tau)}{J_{g_\omega}(\tau)} \right). \tag{7.61}
\end{aligned}$$

Now we note that  $(k_w)(\tau, \omega) = G_z(\tau'_\omega, \zeta)$  and apply estimate (7.42). Also, we can apply formulas (7.35) and (7.38) - (7.40) with  $\tau = \tau'_\omega$ . For derivatives of  $g_\omega$  we have estimates (7.53), (7.54). We obtain

$$\begin{aligned}
(k_w)_\omega(t, \omega) &\sim (1 - |\zeta|^2)^2 \frac{|1 - |\tau'_\omega|^2|^l (|1 - \tau'_\omega \bar{\zeta}|^{m-l-4})}{|1 - \bar{\tau}'_\omega z_\zeta|^l} \\
&\quad \left[ \frac{1}{1 - |\zeta|^2} + \frac{|\bar{\zeta} - \bar{z}| |1 - \tau'_\omega \bar{\zeta}|}{(1 - |\zeta|^2)(1 - |\tau'_\omega|^2)} + \frac{|\bar{\zeta} - \bar{z}|}{|1 - z \bar{\zeta}| |1 - \bar{\tau}'_\omega z_\zeta|} + \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(1 - |\zeta|^2)} \left( \frac{1}{|1 - \tau'_\omega \bar{\zeta}|^2} + \frac{1}{1 - |\tau'_\omega|^2} \right) \Big] \leq \\
& \leq C(1 - |\zeta|^2) |1 - \tau'_\omega \bar{\zeta}|^{m-l-4} \left( \frac{1}{|1 - \bar{\tau}'_\omega z_\zeta|} + \frac{1}{|1 - \tau'_\omega \bar{\zeta}|^2} \right) \quad (7.62)
\end{aligned}$$

Again

$$|g_\omega(\tau) - \omega| \geq c(1 - |\zeta|^2)$$

in our domain analogously to estimate (7.36). We see that the first integral in (7.60) has the estimate

$$\frac{C}{1 - |\zeta|^2} \int_D \left( \frac{|1 - \tau'_\omega \bar{\zeta}|^{m-l-4}}{|1 - \bar{\tau}'_\omega z_\zeta|} + |1 - \tau'_\omega \bar{\zeta}|^{m-l-6} \right) dS_\tau.$$

We obtain estimate  $O(1 - |\zeta|^2)^{-1}$  at  $m \geq l + 6$ . Analogously, the second integral in (7.60) has the estimate (here we apply estimate (7.53))

$$\frac{C}{1 - |\zeta|^2} \int_D |1 - \tau'_\omega \bar{\zeta}|^{m-l-6} dS_\tau.$$

We also obtain estimate  $O(1 - |\zeta|^2)^{-1}$  at  $m \geq l + 6$ .

Consider the last integral in (7.60). Applying (7.53), we have

$$|(\mu \circ g_\omega)_\omega(\tau)| \leq \frac{cb}{1 - |\varphi_\zeta(\tau'_\omega)|^2} \frac{1}{|1 - \zeta \tau'_\omega|^2} \leq \frac{Cb}{(1 - |\zeta|^2)(1 - |\tau'_\omega|^2)}.$$

Again applying estimate (7.42) to  $k_w(\tau, \omega) = G_z(\tau'_\omega, \omega)$ , we obtain

$$\left| \int \frac{(k_w(\mu \circ g_\omega)_\omega r_{wk} \circ g_\omega)(\tau, \omega)}{(g_\omega(\tau) - \omega)^2} dS_\tau \right| \leq \frac{Cb}{1 - |\zeta|^2} \int_D \frac{|1 - \tau'_\omega \bar{\zeta}|^{m-l-4}}{|1 - \bar{\tau}'_\omega z_\zeta|} dS_\tau \leq \frac{Cb}{1 - |\zeta|^2}$$

at  $m \geq l + 4$ . Also,

$$(r_{wk} \circ g_\omega)_\omega(\tau, \omega) = k \left[ \frac{(\overline{g_\omega(\tau)} - \bar{w})^{k-1}}{(g_\omega(\tau) - w)^k} (\overline{g_\omega(\tau)})_\omega - \frac{(\overline{g_\omega(\tau)} - \bar{w})^k}{(g_\omega(\tau) - w)^{k+1}} (g_\omega(\tau))_\omega \right].$$

Applying (7.53), we get for this function the estimate

$$\frac{Ck}{|1 - \tau'_\omega \bar{\zeta}| |z - \varphi_\zeta(\tau'_\omega)|} = \frac{Ck}{|1 - \bar{\zeta} z|} \frac{1}{|\tau'_\omega - z_\zeta|}.$$

Again applying (7.42), we obtain

$$\left| \int \frac{(k_w \mu \circ g_\omega(r_{wk} \circ g_\omega)_\omega)(\tau, \omega)}{(g_\omega(\tau) - \omega)^2} dS_\tau \right| \leq \frac{Ckb}{|1 - \bar{\zeta}z|} \int_D \frac{|1 - \tau'_\omega \bar{\zeta}|^{m-l-4}}{|\tau'_\omega - z_\zeta|} dS_\tau \leq \frac{Ckb}{1 - |\zeta|^2}.$$

at  $m \geq l + 4$ . It finishes the proof of estimate (7.17) for  $J_{kw}$ .  $\square$

2)

$$J_{k2}(\omega) = \frac{1}{\pi} \int_\Omega \mu(t) r_{wk}(t) K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - w)}.$$

We follow the same steps as in case 1).

**Proof of representation (7.16) for  $J_{k2}$ .** At first we consider the case  $\omega \in D_w$  and the integral over  $D_w \cup D_\omega$ , i.e., we consider the integral

$$\frac{1}{\pi} \int_{D_\omega \cup D_w} (\mu(w) + (t - w)\mu_w(t))(a_w(\omega) + (t - w)R_w(t, \omega)) r_{wk}(t) \frac{dS_t}{(t - \omega)(t - w)}.$$

We set  $r_\omega = r_w/2$  and consider at first the integral over  $D_w$ .

The integral

$$\frac{1}{\pi} \mu(w) a_w(\omega) \int_{D_w} r_{wk}(t) \frac{dS_t}{(t - \omega)(t - w)}$$

up to the multiple  $\mu(w) a_w(\omega)$  is the Cauchy transform of the function  $r_{wk}(t)(t - w)^{-1} \chi_{D_w}(t)$ . This Cauchy transform equals to

$$\frac{1}{k+1} \frac{(\bar{\omega} - \bar{w})^{k+1}}{(\omega - w)^{k+1}} = \frac{1}{k+1} r_{w,k+1}(\omega)$$

at  $\omega \in D_w$  and

$$\frac{1}{k+1} \frac{r_w^{2k+2}}{(\omega - w)^{2k+2}}$$

at  $\omega \notin D_w$ . Indeed, this function is continuous, tends to zero at infinity, and its  $\omega$ -derivative (in the sense of distributions) equals to  $r_{wk}(\omega)(\omega - w)^{-1} \chi_{D_w}(\omega)$ .

Analogously to case 1) we obtain the representation

$$\mu(w) a_w(\omega) \frac{1}{\pi} \int_{D_w} r_{wk}(t) \frac{dS_t}{(t - \omega)(t - w)} = \frac{\pi}{k+1} \mu(w) r_{w,k+1}(\omega) + \mu(w) A'_w(\omega),$$

where  $A'_w$  is uniformly bounded and, as in case 1), we shall show that it satisfies estimate (7.17).

The integral

$$\mu(w)a_w(\omega)\frac{1}{\pi}\int_{D_\omega\setminus D_w}r_{wk}(t)\frac{dS_t}{(t-\omega)(t-w)}$$

we estimate as

$$\frac{\mu(w)}{r_w}\frac{1}{\pi}\int_{D_\omega}\frac{dS_t}{|t-\omega|}=2\mu(w)\frac{r_\omega}{r_w}=\pi\mu(w)$$

since  $r_w = 2r_\omega$ .

To estimate the part of the integral  $J_{k2}(\omega)$  over the domain  $D_w \cup D_\omega$  it is enough now to estimate the integral

$$\int_{D_w\cup D_\omega}g_w(t,\omega)r_{wk}(t)\frac{dS_t}{t-\omega},$$

where  $|g_w(t,\omega)| \leq cb|\omega - \hat{\omega}|^{-1}$ . This integral obviously has the estimate  $Cb$ .

If  $\omega \notin D_w$  we estimate the integral

$$\int_{D_\omega}\mu(t)r_{wk}(t)K_w(t,\omega)\frac{dS_t}{(t-\omega)(t-w)}$$

essentially as in case 1). We obtain the estimate

$$\frac{cb}{r_w}\int_{D_\omega}\frac{dS_t}{|t-\omega|}\leq Cb.$$

Consider now the integral over  $\Omega \setminus D_\omega$ . Analogously to case 1) we make the change of the variable  $t = g_\zeta(\tau) = g \circ \varphi_\zeta(\tau)$ . We have

$$\begin{aligned} |g_\zeta(\tau) - g(z)| &\geq \inf_{[z,\varphi_\zeta], |(j)|=1} |g_{(j)}| |\varphi_\zeta(\tau) - z| \geq C \left| \frac{\zeta - z - \tau(1 - z\bar{\zeta})}{1 - \bar{\zeta}\tau} \right| = \\ &= C \left| \frac{(1 - z\bar{\zeta})(\tau - z_\zeta)}{1 - \bar{\zeta}\tau} \right|, \end{aligned} \quad (7.63)$$

where  $z_\zeta$  is defined by (7.41). Applying (7.36), (7.42) and (7.63), we see that it is enough to estimate the integral

$$(1 - |\zeta|^2) \int_D \frac{(1 - |\tau|^2)^l |1 - \tau\bar{\zeta}|^{m-l-3}}{|1 - \tau z_\zeta|^l |1 - z\bar{\zeta}| |\tau - z_\zeta|} dS_\tau \leq C \int_D \frac{|1 - \tau\bar{\zeta}|^{m-l-3}}{|\tau - z_\zeta|} dS_\tau.$$

We obtain an uniform boundedness at  $m \geq l + 3$ .

We obtained the representation

$$J_{k2}(\omega) = \frac{\mu(w)}{k+1} r_{w,k+1}(\omega) + \mu(w) A'_w(\omega) + F'_{w2}(\omega),$$

where  $|A'_w(\omega)| \leq C$ ,  $|F'_{w2}(\omega)| \leq Cb$ . Note that the sum of the coefficients  $k/(k+1)$  and  $1/(k+1)$  of cases 1) and 2) equals 1.

**Proof of estimate (7.17) for  $J_{k2}$ .** We can assume, as above,  $\omega \in D_{\omega_0}$ .

Let  $D_{w1}, D_{w2}, D_{w3}, D_{w4}$  be four disks centered at  $w$  of radii  $r_{1w}, r_{2w} = 2r_{1w}, r_{3w} = 3r_{1w}, r_{4w} = 4r_{1w}$ . There can be the case when at least one of the points  $\omega$  or  $\omega_0$  belongs to  $D_{w2}$ . We put  $r_\omega = r_{\omega_0} = r_{1w}$  in this case. The other point then belongs to  $D_{w3}$  and both disks  $D_\omega$  and  $D_{\omega_0}$  are contained in  $D_w$ . We put  $D_w = D_{w4}$  in this case. If both  $\omega$  or  $\omega_0$  don't belong to  $D_{w2}$  we put  $D_w = D_{w1}$ . In the last case we can set  $r_\omega = r_{\omega_0}$  such that  $|\omega_0 - w| \geq r_{\omega_0}$  and  $|\omega - w| \geq r_{\omega_0}$ . Thus we can consider two different cases: .

a)  $\omega$  and  $\omega_0$  belong to  $D_w$ ,  $r_\omega = r_{\omega_0} = r_w/4$ ,

b) both  $\omega$  and  $\omega_0$  don't belong to  $D_w$ ,  $|w - \omega_0| > r_{\omega_0}$ ,  $|\omega - w| \geq r_{\omega_0}$ .

Again in the proof we consider cases a) and b) separately.

*Case a).* The difference of the integrals over  $D_w$ .

We consider the difference

$$\frac{1}{\pi} \int_{D_w} \mu(t) K_w(t, \omega) r_{wk}(t) \frac{dS_t}{(t-\omega)(t-w)} - \frac{1}{\pi} \int_{D_w} \mu(t) K_w(t, \omega_0) r_{wk}(t) \frac{dS_t}{(t-\omega_0)(t-w)}.$$

We represent this difference as

$$\begin{aligned} & \frac{1}{\pi} \int_{D_w} (\mu(w) + (t-w)\mu_w(t))(a_w(\omega) + (t-w)R_w(t, \omega)) r_{wk}(t) \frac{dS_t}{(t-\omega)(t-w)} - \\ & - \frac{1}{\pi} \int_{D_w} (\mu(w) + (t-w)\mu_w(t))(a_w(\omega_0) + (t-w)R_w(t, \omega_0)) r_{wk}(t) \frac{dS_t}{(t-\omega_0)(t-w)}. \end{aligned} \tag{7.64}$$

Consider at first the difference of the integrals over  $D_w$ . As in case 1), there is the term

$$\mu(w)(k+1)^{-1} [a_w(\omega) r_{w,k+1}(\omega) - a_w(\omega_0) r_{w,k+1}(\omega_0)] =$$

$$= \mu(w)(k+1)^{-1} (r_{w,k+1}(\omega) - r_{w,k+1}(\omega_0)) + \mu(w)(A'_w(\omega) - A'_w(\omega_0)),$$

and  $|A'_w(\omega) - A'_w(\omega_0)| \leq C|\omega - \omega_0|\omega - \hat{\omega}|^{-1}$ .

It remains to estimate the integral (see (7.44))

$$\begin{aligned} \int_{D_w} r_{wk}(t) \left[ (r_1(w, \omega) + (t-w)r'_1(t, \omega)) \left( \frac{1}{t-\omega} - \frac{1}{t-\omega_0} \right) + \right. \\ \left. + (\omega - \omega_0)r_2(t, \omega, \omega_0) \frac{1}{t-\omega_0} \right] dS_t, \end{aligned} \quad (7.65)$$

where  $r_1, r'_1, r_2$  satisfy estimates (7.45) - (7.47).

The integral

$$r_1(w, \omega) \int_{D_w} r_{wk}(t) \left( \frac{1}{t-\omega} - \frac{1}{t-\omega_0} \right) dS_t$$

up to the constant  $\pi r_1(w, \omega)$  is the difference in the points  $\omega$  and  $\omega_0$  of the Cauchy transform of the function  $r_{wk}\chi_{D_w}$ . As above, this difference equals to

$$\frac{1}{k+1} \left[ \frac{(\bar{\omega} - \bar{w})^{k+1}}{(\omega - w)^k} - \frac{(\bar{\omega}_0 - \bar{w})^{k+1}}{(\omega_0 - w)^k} \right].$$

and, as in case 1), we obtain the estimate  $Cb|\omega - \hat{\omega}|^{-1}|\omega - \omega_0|$ .

Now consider the difference

$$\int_{D_w} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1} t - \omega} dS_t - \int_{D_w} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1} t - \omega_0} dS_t.$$

As in case 1), we make the change of the variable:  $t \mapsto t + \omega - \omega_0$  in the first integral and denote by  $D_{w_0}$  the domain  $\omega - \omega_0 + D_w$ . We obtain the sum

$$\begin{aligned} \int_{D_w \cap D_{w_0}} \left[ r'_1(t + \omega - \omega_0, \omega) \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-1}} - r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1}} \right] \frac{dS_t}{t - \omega_0} + \\ + \int_{D_w \setminus D_{w_0}} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1} t - \omega} dS_t + \int_{D_{w_0} \setminus D_w} r'_1(t, \omega) \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1} t - \omega_0} dS_t. \end{aligned} \quad (7.66)$$

We again represent the first integral as

$$\begin{aligned} \int_{D_w \cap D_{w_0}} [r'_1(t + \omega - \omega_0, \omega) - r'_1(t, \omega)] \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-1} t - \omega_0} dS_t + \\ + \int_{D_w \cap D_{w_0}} r'_1(t, \omega) \left[ \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-1}} - \frac{(\bar{t} - \bar{w})^k}{(t-w)^{k-1}} \right] \frac{dS_t}{t - \omega_0} \end{aligned} \quad (7.67)$$



Applying the estimate

$$\left| \frac{(\overline{t + \omega - \omega_0} - \bar{w})^k}{(t + \omega - \omega_0 - w)^{k-1}} - \frac{(\bar{t} - \bar{w})^k}{(t - w)^{k-1}} \right| \leq Ck|\omega - \omega_0|$$

and acting as in case 1), we see that integral (7.67) is no greater, than

$$C|\omega - \omega_0| \left[ \sup_{t \in D_w \cup D_{w_0}, |(j)|=1} (|\partial_{(j),t} r'_1(t, \omega)| |t + \omega - \omega_0 - w|) + \sup_{t \in D_w \cup D_{w_0}} (|r'_1(t, \omega)| k) \right] \int_{D_w \cap D_{w_0}} \frac{dS_t}{|t - \omega_0|}$$

Applying (7.45), (7.46), we obtain for integral (7.67) the estimate

$$\frac{Ckb}{|\omega - \hat{\omega}|} |\omega - \omega_0|.$$

The second and third integrals in (7.66) we estimate analogously to the corresponding terms in (7.49). We estimate, for example, the second integral as

$$C \sup_{t \in D_w \setminus D_{w_0}} (|r'_1(t, \omega)| |t - w|) |\omega - \omega_0| \leq \frac{Ckb}{|\omega - \hat{\omega}|} |\omega - \omega_0|.$$

Now to finish with integral (7.65) we must estimate the integral

$$\int_{D_w} r_2(t, \omega, \omega_0) r_{wk}(t) \frac{dS_t}{t - \omega_0}.$$

Applying (7.45), we obtain the estimate

$$\sup_{t \in D_w} (|r_2(t, \omega, \omega_0)|) \int_{D_w} \frac{dS_t}{|t - \omega_0|} \leq \frac{Ckb}{|\omega - \hat{\omega}|}.$$

*Case a).* The difference of the integrals over  $\Omega \setminus D_w$ .

We act as in case 1). After the change of the variable  $t = g_w(\tau)$  we differentiate under the integral and analogously to (7.51) we get the integral

$$\int \frac{((G_w)_\omega \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)(g_w(\tau) - w)} dS_\tau - 2 \int \frac{(G_w \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)^2 (g_w(\tau) - w)} dS_\tau \quad (7.68)$$

We apply (7.52), (7.53), (7.36) and take into consideration that  $|g_w(\tau) - w| \geq r_\omega \geq d(1 - |\zeta|^2)$  for some uniform  $d$  if  $\omega \in \Omega \setminus D_w$ . We obtain the estimate  $Cb|\omega - \hat{\omega}|^{-1}$  for integral (7.68) at  $m \geq l + 4$ .

*Case b).*

There is a difference by comparison with case 1). We have an additional pole in  $w$  and, after transition to the chart  $\tau$  on  $D$ , we can't differentiate under the integral.

We use the decomposition

$$\frac{1}{(t - \omega)(t - w)} = \frac{1}{\omega - w} \left( \frac{1}{t - \omega} - \frac{1}{t - w} \right).$$

We define

$$\mathcal{J}_{kw}^1(\omega) = \int_{\Omega} K_w(t, \omega) \mu(t) r_{kw}(t) \frac{dS_t}{t - \omega}$$

and

$$\mathcal{J}_{kw}^2(\omega) = \int_{\Omega} K_w(t, \omega) \mu(t) r_{kw}(t) \frac{dS_t}{t - w}.$$

We write

$$\begin{aligned} J_{k2}(\omega) - J_{k2}(\omega_0) &= (\omega - w)^{-1} (\mathcal{J}_{kw}^1(\omega) - \mathcal{J}_{kw}^1(\omega_0)) + \\ &\quad + (\omega - w)^{-1} (\mathcal{J}_{kw}^2(\omega) - \mathcal{J}_{kw}^2(\omega_0)) + \\ &\quad + [(\omega - w)^{-1} - (\omega_0 - w)^{-1}] (\mathcal{J}_{kw}^1(\omega_0) - \mathcal{J}_{kw}^2(\omega_0)). \end{aligned} \quad (7.69)$$

In what follows we consider several cases corresponding to different terms in the right side.

*The difference of the integrals  $\mathcal{J}_{kw}^1(\omega)$  and  $\mathcal{J}_{kw}^1(\omega_0)$  over  $D_\omega$  and  $D_{\omega_0}$ .*

We must estimate the difference

$$\int_{D_\omega} K_w(t, \omega) \mu(t) r_{kw}(t) \frac{dS_t}{t - \omega} - \int_{D_{\omega_0}} K_w(t, \omega_0) \mu(t) r_{kw}(t) \frac{dS_t}{t - \omega_0}.$$

As in case 1), we change the variable  $t \mapsto t + \omega$  and  $t \mapsto t + \omega_0$  in the integrals over  $D_\omega$  and  $D_{\omega_0}$  correspondingly and denote  $\tilde{K}_w(t, \omega) = K_w(t + \omega, \omega)$ . We get the integral

$$\int_{|t| \leq r_\omega} [\tilde{K}_w(t, \omega) r_{wk}(t + \omega) \mu(t + \omega) - \tilde{K}_w(t, \omega_0) r_{wk}(t + \omega_0) \mu(t + \omega_0)] \frac{dS_t}{|t|}. \quad (7.70)$$

If  $|t| \leq r_\omega$  and  $s \in [\omega, \omega_0]$ , then at  $|(j)| = 1$  we use the estimates

$$|(\tilde{K}_w)_{(j),s}(t, s)| \leq c|\omega - \hat{\omega}|^{-1}, |(\mu_{(j)}(t + s))| \leq cb|\omega - \hat{\omega}|^{-1}, \quad (7.71)$$

$$|(r_{wk})_{(j)}(t + s)| \leq ck|t + s - w|^{-1} \leq Ck|\omega - \hat{\omega}|^{-1} \quad (7.72)$$

Thus for integral (7.70) we obtain the estimate

$$Ckb \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|} \int_{|t| \leq r_\omega} \frac{dS_t}{|t|} \leq Ckb. \quad (7.73)$$

The difference of the integrals  $\mathcal{J}_{kw}^1(\omega)$  and  $\mathcal{J}_{kw}^1(\omega_0)$  over  $\Omega \setminus D_\omega$  and  $\Omega \setminus D_{\omega_0}$ .

We proceed analogously to case 1) and apply Proposition 39. After change of the variable  $t = g_\omega(\tau)$  and differentiation we obtain the expression

$$\begin{aligned} & \int \frac{((k_w)_\omega \mu \circ g_\omega r_{wk} \circ g_\omega)(\tau, \omega)}{g_\omega(\tau) - \omega} dS_\tau - \\ & - \int \frac{(k_w \mu \circ g_\omega r_{wk} \circ g_\omega)(\tau, \omega)}{(g_\omega(\tau) - \omega)^2} ((g_\omega)_\omega(\tau) - 1) dS_\tau + \\ & + \int \frac{[k_w(\mu \circ g_\omega)_\omega r_{wk} \circ g_\omega + k_w \mu \circ g_\omega(r_{wk} \circ g_\omega)_\omega](\tau, \omega)}{g_\omega(\tau) - \omega} dS_\tau. \end{aligned}$$

The distinction from integrals (7.60) is that we replace the multiple  $(g_\omega(\tau) - \omega)^{-2}$  with the multiple  $(g_\omega(\tau) - w)^{-1}$  in the first and third integrals and replace the multiple  $(g_\omega(\tau) - \omega)^{-3}$  with the multiple  $(g_\omega(\tau) - \omega)^{-2}$  in the second integral. As a result, these integrals have the uniform estimate  $Ckb$  instead of  $Ckb(1 - |\zeta|^2)^{-1}$ . Together with estimate (7.73) it yields the estimate

$$|\mathcal{J}_{kw}^1(\omega) - \mathcal{J}_{kw}^1(\omega_0)| \leq Ckb|\omega - \omega_0|.$$

Since  $|\omega - w| \geq c|\omega - \hat{\omega}|$  for some  $c$  in our case, we obtain the estimate  $Ckb|\omega - \omega_0||\omega - \hat{\omega}|^{-1}$  for the first term in the right side of identity (7.69).

The difference  $\mathcal{J}_{2w}^2(\omega) - \mathcal{J}_{2w}^2(\omega_0)$ .

The singular term  $(t - w)^{-1}$  doesn't depend on  $\omega$  and we can differentiate under the integral. Namely we must show that the integral

$$\int_{\Omega} (K_w)_\omega(t, \omega) \mu(t) r_{kw}(t) \frac{dS_t}{t - w} \quad (7.74)$$

is uniformly bounded. After obvious calculations we can write

$$(K_w)_\omega(t, \omega) = m \frac{\hat{h}_\omega(\omega)}{t - \hat{\omega}} K_w(t, \omega) + \\ + (m - l) \frac{1 - \hat{h}_\omega(\omega)}{t - \hat{\omega}} K'_w(t, \omega) - l \frac{\hat{h}_\omega(\omega)}{w - \hat{\omega}} K_w(t, \omega), \quad (7.75)$$

where

$$K'_w(t, \omega) = \left( \frac{\omega - \hat{\omega}}{t - \hat{\omega}} \right)^{m-l-1} \left( \frac{t - \hat{t}}{t - \hat{\omega}} \right)^l \left( \frac{w - \hat{\omega}}{w - \hat{t}} \right)^l.$$

The last term in (7.75) by modulus is no greater than  $C|t - \hat{\omega}|^{-1}|K'_w(t, \omega)|$ . After the change of the variable  $t = g \circ \varphi_\zeta(\tau)$  applying (7.63), (7.42) and (7.39) we obtain the estimate for integral (7.74)

$$\sim \int_D \frac{|1 - \tau \bar{\zeta}|^{m-l-2}}{|\tau - z_\zeta|} dS_\tau + \int_D \frac{|1 - \tau \bar{\zeta}|^{m-l-3}}{|\tau - z_\zeta|} dS_\tau.$$

This integral is uniformly bounded at  $m \geq l + 3$ .

*The last term in identity (7.69).*

We have

$$[(\omega - w)^{-1} - (\omega_0 - w)^{-1}](\mathcal{J}_{kw}^1(\omega_0) - \mathcal{J}_{kw}^2(\omega_0)) = \\ = \frac{\omega - \omega_0}{((\omega - w)(\omega_0 - w))} \int_\Omega (K_w)_\omega(t, \omega_0) \mu(t) r_{kw}(t) \frac{\omega_0 - w}{(t - \omega_0)(t - w)} dS_t = \frac{\omega - \omega_0}{\omega - w} J_{k2}(\omega_0).$$

We already proved the uniform estimate for  $J_{k2}$ . Thus we estimate (7.17) for all terms in (7.69).  $\square$

3)

$$J_{k3}(\omega) = \int_\Omega K_w(t, \omega) \mu(t) r_{wk}(t) \frac{dS_t}{(t - \omega)(t - \hat{\omega})}.$$

After two "difficult" cases this case is "simple". We estimate the integral introducing our usual chart  $\tau = \varphi_\zeta^{-1} \circ g^{-1}(t)$ . Applying (7.35), (7.39), and (7.42), we see that we can estimate our integral as

$$Cb \int_D \frac{|1 - \tau \bar{\zeta}|^{m-l-2}}{|\tau|} dS_\tau.$$

We obtain the estimate  $Cb$  at  $m \geq l + 2$ .

**Proof of estimate (7.17) for  $J_{k3}(\omega) - J_{k3}(\omega_0)$ .**

If we set  $r_w$  and  $r_\omega = r_{\omega_0}$  as in case 1), we again can consider two cases:

a)  $D_w$  contains  $D_\omega$  and  $D_{\omega_0}$ ,  $r_\omega = r_{\omega_0} = r_w/3$ ,

b) both  $\omega$  and  $\omega_0$  don't belong to  $D_w$ ,  $|w - \omega| > r_\omega = r_{\omega_0} < |w - \omega_0|$ .

*Case a) The difference of the integrals over  $D_w$ .*

We represent the integral as

$$\begin{aligned} & \mu(w) \left[ \frac{1}{w - \hat{\omega}} \int_{D_w} \frac{r_{wk}(t)}{t - \omega} dS_t - \frac{1}{w - \hat{\omega}_0} \int_{D_w} \frac{r_{wk}(t)}{t - \omega_0} dS_t \right] + \\ & + \int_{D_w} r_{wk}(t)(t - w) \frac{r_1(w, \omega) + (t - w)r_1'(t, \omega)}{t - \hat{\omega}} \left( \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right) dS_t + \\ & + (\omega - \omega_0) \int_{D_w} r_{wk}(t)(t - w) \frac{r_2(t, \omega, \omega_0)}{t - \hat{\omega}_0} \frac{dS_t}{t - \omega_0} dS_t, \end{aligned} \quad (7.76)$$

where for  $r$ ,  $r_1'$  and  $r_2$  we have estimates (7.45), (7.46). As above, the first difference equals to

$$\frac{\mu(w)}{k} \left[ \frac{1}{w - \hat{\omega}} \frac{(\bar{\omega} - \bar{w})^{k+1}}{(\omega - w)^k} - \frac{1}{w - \hat{\omega}_0} \frac{(\bar{\omega}_0 - \bar{w})^{k+1}}{(\omega_0 - w)^k} \right]$$

and has the estimate  $C|\omega - \hat{\omega}|^{-1}|\omega - \omega_0|$ . In the second integral the expression before the brackets has the estimate  $cb|\omega - \hat{\omega}|^{-1}$  and its  $t$ -derivatives have the estimate  $ckb|\omega - \hat{\omega}|^{-2}$ . Following essentially the same considerations as for integral (7.66) we obtain the estimate  $Ckb|\omega - \hat{\omega}|^{-1}|\omega - \omega_0|$ . The estimate for the last integral in (7.76) follows from the estimate for  $|r_2(t, \omega, \omega_0)|$ .

*Case a). The difference of the integrals over  $\Omega \setminus D_w$ .*

Analogously to case 1) (integral (7.51)) we must estimate the expression

$$\begin{aligned} & \int \frac{((G_w)_\omega \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)(g_w(\tau) - \hat{\omega})} dS_\tau - \int \frac{(G_w \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)^2 (g_w(\tau) - \hat{\omega})} dS_\tau - \\ & - (\hat{h})_\omega(\omega) \int \frac{(G_w \mu \circ g_w r_{wk} \circ g_w)(\tau, \omega)}{(g_w(\tau) - \omega)(g_w(\tau) - \hat{\omega})^2} dS_\tau, \end{aligned}$$

where the integrals are over  $D \setminus g_w^{-1}(D_w)$ . The difference with integral (7.51) is that in the two first integrals we replace one multiple  $g_w(\tau) - \omega$  in the denominator by the multiple  $g_w(\tau) - \hat{\omega}$  and in the third integral the denominator is  $(g_w(\tau) - \omega)(g_w(\tau) - \hat{\omega})^2$  instead of  $(g_w(\tau) - \omega)^3$ . Such replacing

doesn't change the order in  $1 - |\zeta|^2$  and in  $|1 - \bar{\zeta}\tau|$  and we have the same estimate as in case  $C(1 - |\zeta|^2)^{-1}$  at  $m \geq l + 4$  as in case 1.

*Case b).*

We consider the difference of the integrals over  $D_\omega$  and  $D_{\omega_0}$  analogously to cases 1) and 2). We must estimate the integral

$$\int_{|t| \leq r_\omega} \left[ \frac{\tilde{K}_w(t, \omega) r_{wk}(t + \omega) \mu(t + \omega)}{t + \omega - \hat{\omega}} - \frac{\tilde{K}_w(t, \omega_0) r_{wk}(t + \omega_0) \mu(t + \omega_0)}{t + \omega_0 - \hat{\omega}_0} \right] \frac{dS_t}{|t|}. \quad (7.77)$$

Applying estimates (7.71), (7.72) and the estimate

$$\left| \frac{1}{t + \omega - \hat{\omega}} - \frac{1}{t + \omega_0 - \hat{\omega}_0} \right| \leq C \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|^2},$$

we obtain for integral (7.77) the estimate  $Ckb|\omega - \omega_0||\omega - \hat{\omega}|^{-1}$ .

We consider, as in case 1), the difference of the integrals over  $\Omega \setminus D_\omega$  and  $\Omega \setminus D_{\omega_0}$ . We apply Proposition 39 and again see that the difference with case 1) is that we replace the multiples  $g_\omega(\tau) - \omega$  or  $(g_\omega(\tau) - \omega)^2$  by the multiple  $g_\omega(\tau) - \hat{\omega}$  or  $(g_\omega(\tau) - \hat{\omega})^2$  in the denominators of the expressions under the integrals. As a result, we obtain the same estimate as in case 1):  $Ckb(1 - |\zeta|^2)^{-1}$  at  $m \geq l + 6$ .  $\square$ .

4)

$$J_{k4}(\omega) = \int_{\Omega} \mu(t) r_{wk}(t) K_w(t, \omega) \frac{dS_t}{(t - w)(t - \hat{\omega})}.$$

This case is most simple. We write the integral in the chart  $\tau = \varphi_\zeta^{-1} \circ g^{-1}(t)$ , Applying (7.39), (7.42), and (7.63), we obtain the estimate

$$Cb \int_D \frac{|1 - \tau \bar{\zeta}|^{m-l-2}}{|\tau - z_\zeta|} dS_\tau \leq C'b$$

at  $m \geq l + 2$ .

Now in the our case the differentiation with respect to  $\omega$  doesn't lead to new singularities and for estimation of the difference  $J_{k4}(\omega) - J_{k4}(\omega_0)$  it is enough to estimate the integral from the derivative of the expression under the integral. It follows that we must estimate the integral

$$\int_{\Omega} \frac{(K_w)_\omega(t, \omega) \mu(t) r_{kw}(t)}{(t - \hat{\omega})(t - w)} dS_t - \hat{h}_\omega(\omega) \int_{\Omega} \frac{(K_w)_\omega(t, \omega) \mu(t) r_{kw}(t)}{(t - \hat{\omega})^2(t - w)} dS_t.$$

Again introducing the chart  $\tau$  and applying (7.39), (7.42), (7.63), and (7.75), we get the estimate

$$\sim b(1 - |\zeta|^2)^{-1} \int_D (|1 - \tau\bar{\zeta}|^{m-l-1} + |1 - \tau\zeta|^{m-l-2}) \frac{dS_\tau}{|\tau - z_\zeta|}.$$

We obtain the estimate  $Cb(1 - |\zeta|^2)^{-1}$  at  $m \geq 2$ .  $\square$

**Proposition 40** *Let  $f$  be a function defined on  $\Omega$  and satisfying the estimates*

$$|f(\omega)| \leq a, \quad |f(\omega) - f(\omega_0)| \leq \frac{a_1}{|\omega_0 - \hat{\omega}_0|} |\omega - \omega_0|.$$

*Then for every  $0 < \alpha < 1$   $f$  satisfies the Holder estimate*

$$|f(\omega) - f(\omega_0)| \leq \frac{a_1^\alpha (2a)^{1-\alpha}}{|\omega_0 - \hat{\omega}_0|^\alpha} |\omega - \omega_0|^\alpha.$$

**Proof.** Define  $b = 2a(a_1/2a)^\alpha = a_1^\alpha (2a)^{1-\alpha} r = (2a/a_1)|\omega_0 - \hat{\omega}_0|$ . From the identity  $a_1|\omega_0 - \hat{\omega}_0|^{-1}r = b|\omega_0 - \hat{\omega}_0|^{-\alpha}r^\alpha = 2a$  it follows that

$$|f(\omega) - f(\omega_0)| \leq \frac{a_1}{|\omega_0 - \hat{\omega}_0|} |\omega - \omega_0| \leq \frac{b}{|\omega_0 - \hat{\omega}_0|^\alpha} |\omega - \omega_0|^\alpha$$

if  $|\omega - \omega_0| \leq r$ . From the other hand, if  $|\omega - \omega_0| \geq r$ , then  $|f(\omega) - f(\omega_0)| \leq 2a \leq b|\omega_0 - \hat{\omega}_0|^{-\alpha}|\omega - \omega_0|^\alpha$ .  $\square$

**Proposition 41** *Let  $f$  be a function defined on  $\Omega$  equal to zero in  $w$  and satisfying the estimates*

$$\begin{aligned} |f(\omega)| &\leq a, \\ |f(\omega) - f(\omega_0)| &\leq \frac{a_1}{|\omega_0 - \hat{\omega}_0|^\alpha} |\omega - \omega_0|^\alpha \end{aligned} \quad (7.78)$$

*for some  $\alpha > 0$ . Then*

$$\begin{aligned} |\mathcal{T}_w f(\omega)| &\leq C(a + a_1), \\ |\mathcal{T}_w f(\omega) - \mathcal{T}_w f(\omega_0)| &\leq \frac{C(a + a_1)}{|\omega_0 - \hat{\omega}_0|^\alpha} |\omega - \omega_0|^\alpha \end{aligned} \quad (7.79)$$

*for some uniform  $C$ .*

**Proof.** We apply notations of Proposition 37, in particular we use the notations  $D_w, D_\omega, r_w, r_\omega$ . Again we consider four cases.

1)

$$J_{1f}(\omega) = \int_{\Omega} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)^2}.$$

**Proof of uniform boundedness.** Consider at first the integral over  $D_\omega$ .  $K_w$  satisfies estimates of Proposition 40 with some uniform constants. We have

$$\begin{aligned} \left| \int_{D_\omega} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)^2} \right| &\leq \frac{C(a + a_1)}{|\omega - \hat{\omega}|^\alpha} \int_{D_\omega} \frac{dS_t}{|t - \omega|^{2-\alpha}} = \\ &= \frac{2\pi C(a + a_1)}{|\omega - \hat{\omega}|^\alpha} \int_0^{r_\omega} \frac{d\rho}{\rho^{1-\alpha}} \leq C'(a + a_1) \end{aligned}$$

with some uniform  $C'$  since  $r_\omega \leq d|\omega - \hat{\omega}|$  for some uniform  $d$ .

For the integral over  $\Omega \setminus D_\omega$  we can apply the estimate of Proposition 37, case 1) since  $f(t)$  is uniformly bounded. Thus we obtain  $|J_{1f}(\omega)| \leq C(a + a_1)$ .  $\square$

**Proof of estimate (7.79).** It is enough to check inequality (7.79) if  $|\omega - \omega_0| \leq r_\omega/2$ . Indeed, in the opposite case  $|\mathcal{T}_w f(\omega) - \mathcal{T}_w f(\omega_0)| \leq C(a + a_1) = C(a + a_1)|\omega_0 - \hat{\omega}_0|^{-\alpha}|\omega_0 - \hat{\omega}_0|^\alpha \leq C'|\omega_0 - \hat{\omega}_0|^{-\alpha}|\omega - \omega_0|^\alpha$  for some uniform  $C'$ .

Let  $D'_{\omega_0}$  be the disk of radius  $2r_{\omega_0}$  centered at  $\omega_0$ . Applying breaking of the identity we reduce the problem to estimations of integrals of the functions  $f_1$  and  $f_2$  with the supports in  $D'_{\omega_0}$  and  $\Omega \setminus D'_{\omega_0}$  and satisfying Holder estimates of type (7.79) with constants  $c(a + a_1)$ .

We have the representation

$$\int_{D'_{\omega_0}} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)^2} = \int_{D'_{\omega_0}} f(t) \frac{dS_t}{(t - \omega)^2} + \int_{D'_{\omega_0}} f(t)R_w(t, \omega) \frac{dS_t}{t - \omega}, \quad (7.80)$$

where  $R_w$  satisfies estimates (7.19), (7.20),

$$|R_w(t, \omega)| \leq \frac{c}{|\omega_0 - \hat{\omega}_0|}, \quad |(R_w)_\omega(t, \omega)| \leq \frac{c}{|\omega_0 - \hat{\omega}_0|^2}.$$

The first integral in right side of (7.80) is the Beurling transform of the function  $f_1$ , which is bounded with the norm  $c_\alpha$  in the Holder space  $C^\alpha$  [As].



Consider the difference

$$\int_{D'_{\omega_0}} f(t)R_w(t, \omega) \frac{dS_t}{t - \omega} - \int_{D'_{\omega_0}} f(t)R_w(t, \omega_0) \frac{dS_t}{t - \omega_0}.$$

Following the same lines as in the proof of Proposition 38 we change the variable in the first integral  $t \mapsto t + \omega - \omega_0$ . The disk  $D'_{\omega_0}$  transforms into the disk  $D'_{1\omega_0}$  and we get the sum of the integrals

$$\begin{aligned} & \int_{D'_{\omega_0} \cap D'_{1\omega_0}} [f(t + \omega - \omega_0)R_w(t + \omega - \omega_0, \omega) - f(t)R_w(t, \omega_0)] \frac{dS_t}{t - \omega_0} + \\ & + \int_{D'_{1\omega_0} \setminus D'_{\omega_0}} f(t)R_w(t, \omega) \frac{dS_t}{t - \omega_0} + \int_{D'_{\omega_0} \setminus D'_{1\omega_0}} [f(t\omega - \omega_0)R_w(t + \omega - \omega_0, \omega) \frac{dS_t}{t - \omega_0}]. \end{aligned} \quad (7.81)$$

The first integral is no greater by modulus than

$$\begin{aligned} & \int_{D'_{\omega_0} \cap D'_{1\omega_0}} |f(t + \omega - \omega_0) - f(t)|R_w(t + \omega - \omega_0, \omega) \left| \frac{dS_t}{|t - \omega_0|} \right| + \\ & + \int_{D'_{\omega_0} \cap D'_{1\omega_0}} |f(t)| |R_w(t + \omega - \omega_0, \omega) - R_w(t, \omega)| \left| \frac{dS_t}{|t - \omega_0|} \right| \leq \\ & \leq \frac{C(a + a_1)|\omega - \omega_0|^\alpha}{|\omega_0 - \hat{\omega}_0|^{1+\alpha}} \int_{D'_{\omega_0} \cap D'_{1\omega_0}} \frac{dS_t}{|t - \omega_0|} + \frac{Ca|\omega - \omega_0|}{|\omega_0 - \hat{\omega}_0|^2} \int_{D'_{\omega_0} \cap D'_{1\omega_0}} \frac{dS_t}{|t - \omega_0|} \leq \\ & \leq \frac{C(a + a_1)|\omega - \omega_0|^\alpha}{|\omega_0 - \hat{\omega}_0|^\alpha} + \frac{Ca|\omega - \omega_0|}{|\omega_0 - \hat{\omega}_0|}. \end{aligned}$$

But

$$\frac{|\omega - \omega_0|}{|\omega_0 - \hat{\omega}_0|} \leq \frac{|\omega - \omega_0|^\alpha}{|\omega_0 - \hat{\omega}_0|^\alpha}$$

at  $|\omega - \omega_0| \leq |\omega_0 - \hat{\omega}_0|$  and we obtain the estimate for the first integral in (7.81)  $C(a + a_1)|\omega_0 - \hat{\omega}_0|^{-\alpha}|\omega - \omega_0|^\alpha$ .

We estimate the second and third integrals in (7.81) as in the proof of Proposition 37 taking into consideration that the "width" of the lunules  $D'_{1\omega_0} \setminus D'_{\omega_0}$  and  $D'_{\omega_0} \setminus D'_{1\omega_0}$  is of order  $|\omega - \omega_0|$  and for  $|R_w|$  we have estimate (7.19). We obtain the estimate  $Ca|\omega_0 - \hat{\omega}_0|^{-1}|\omega - \omega_0|$  and, hence, as above,  $Ca|\omega_0 - \hat{\omega}_0|^{-\alpha}|\omega - \omega_0|^\alpha$ .

Now consider the integral

$$\int_{\Omega \setminus D_{\omega_0}} f(t) \left[ \frac{K_w(t, \omega)}{(t - \omega)^2} - \frac{K_w(t, \omega_0)}{(t - \omega_0)^2} \right] dS_t.$$

Applying Proposition 39 we see that we must estimate the difference

$$\begin{aligned} & \int_{1/2 \leq |\tau| \leq 1} k_w(\tau, \omega) f \circ g_w(\tau) \frac{dS_\tau}{(g_w(\tau) - \omega)^2} - \int_{1/2 \leq |\tau| \leq 1} k_w(\tau, \omega_0) f \circ g_{\omega_0}(\tau) \frac{dS_\tau}{(g_{\omega_0}(\tau) - \omega_0)^2} = \\ & , \\ & = \int_{1/2 \leq |\tau| \leq 1} (k_w(\tau, \omega) - k_w(\tau, \omega_0)) (f \circ g_w(\tau)) \frac{dS_\tau}{(g_w(\tau) - \omega)^2} + \\ & + \int_{1/2 \leq |\tau| \leq 1} k_w(\tau, \omega_0) (f \circ g_w(\tau) - f \circ g_{\omega_0}(\tau)) \frac{dS_\tau}{(g_{\omega_0}(\tau) - \omega_0)^2}. \end{aligned}$$

, The first integral in the right side up to the bounded multiples is the integral considered in case 1) of Proposition 37. We have for this term the estimate  $Ca|\omega_0 - \hat{\omega}_0|^{-1}|\omega - \omega_0| \leq Ca|\omega_0 - \hat{\omega}_0|^{-\alpha}|\omega - \omega_0|^\alpha$ . For the second integral we have the estimate

$$\begin{aligned} & a_1 \int_{1/2 \leq |\tau| \leq 1} |k_w(\tau, \omega_0)| \frac{|g_w(\tau) - g_{\omega_0}(\tau)|^\alpha}{|g_{\omega_0}(\tau) - \widehat{g_{\omega_0}(\tau)}|^\alpha} \frac{dS_\tau}{|g_{\omega_0}(\tau) - \omega_0|^2} \leq \\ & \leq Ca_1 \int_{1/2 \leq |\tau| \leq 1} \frac{|1 - \tau'_{\omega_0} \bar{\zeta}|^\alpha}{(1 - |\zeta|^2)^\alpha (1 - |\tau'_{\omega_0}|^2)^\alpha} \frac{|\omega - \omega_0|^\alpha}{|1 - \tau'_{\omega_0} \bar{\zeta}|^{2\alpha}} \frac{(1 - |\tau'_{\omega_0}|^2)^l (|1 - \tau'_{\omega_0} \bar{\zeta}|^{m-l-4})}{|1 - \bar{\tau}'_{\omega_0} z_\zeta|^l} \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega_0 - \hat{\omega}_0|^\alpha} \int_{1/2 \leq |\tau| \leq 1} \frac{|1 - \tau'_{\omega_0} \bar{\zeta}|^{m-l-4-\alpha}}{|1 - \bar{\tau}'_{\omega_0} z_\zeta|} \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega_0 - \hat{\omega}_0|^\alpha} \end{aligned}$$

at  $m \geq l + 5$ . Here we applied estimates (7.54) and (7.39), (7.42) with  $\tau = \tau'_{\omega_0}$ .  $\square$

2)

$$J_{2f}(\omega) = \int_{\Omega} f(t) K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - w)}.$$

**Proof of uniform boundedness.** As in the proof of Proposition 37 we can consider two cases:

- a)  $D_\omega \subset D_w$ ,  $r_\omega = r_w/2$ ,
- b)  $|\omega - w| \geq r_w$ .

Consider case a). We use the estimate

$$\int_{D_\rho} \frac{dS_t}{|t|^{1-\alpha}|t-\omega|} \leq 10\pi \frac{\rho^\alpha}{\alpha} \quad (7.82)$$

for some  $C$ . Here  $D_\rho$  is a disk of radius  $\rho$  centered at zero,  $\omega \in D_\rho$ . Indeed, if  $t$  lies outside of the union of the disks  $|t| \leq |\omega|/2$  and  $|t-\omega| \leq |\omega|/2$ , then  $|t-\omega| \geq |t|/3$  and we see that integral (7.82) is no greater than

$$\begin{aligned} & \frac{4\pi}{|\omega|} \int_0^{|\omega|/2} r^\alpha dr + \frac{4\pi}{|\omega|^{1-\alpha}} \int_0^{|\omega|/2} dr + 6\pi \int_{|\omega|/2}^\rho r^{\alpha-1} dr \leq \\ & \leq 4\pi|\omega|^\alpha + 6\pi\rho^\alpha/\alpha \leq 10\pi\rho^\alpha/\alpha. \end{aligned}$$

Now we consider integral  $J_{2f}$  over  $D_w$ . Since  $f(w) = 0$ , we get applying (7.82)

$$\begin{aligned} \left| \int_{D_w} f(t) K_w(t, \omega) \frac{dS_t}{(t-\omega)(t-w)} \right| & \leq \frac{c(a+a_1)}{|w-\hat{w}|^\alpha} \int_{D_w} \frac{dS_t}{|t-\omega||t-w|^{1-\alpha}} \leq \\ & \leq \frac{C(a+a_1)r_w^\alpha}{|w-\hat{w}|^\alpha} \leq C(a+a_1). \end{aligned}$$

Analogously, in case b)

$$\left| \int_{D_w} f(t) K_w(t, \omega) \frac{dS_t}{(t-\omega)(t-w)} \right| \leq \frac{ca}{|\omega-\hat{\omega}|} \int_{D_w} \frac{dS_t}{|t-\omega|} \leq Ca.$$

From the other hand, the integral over  $\Omega \setminus (D_\omega \cup D_w)$  has the same estimate as the integral over  $\Omega \setminus D_\omega$  in case 2 of Proposition 37, t.e., it is uniformly bounded.  $\square$

**Proof of estimate (7.79).** As in the proof of Proposition 37 we consider two cases:

- a)  $D_w$  contains  $D_\omega$  and  $D_{\omega_0}$ ,
- b)  $|\omega-w| \geq r_\omega$  and  $|\omega_w-w| \geq r_{\omega_0} = r_\omega$ .

In case a) we must consider the difference of the integrals over  $D_w$ . Remind the representation

$$K_w(t, \omega) = a_w(\omega) + (t-w)R_w(t, \omega),$$

where  $a_w(\omega) = ((\omega - \hat{\omega})/(w - \hat{\omega}))^{m-2}$ . We represent our difference as

$$\begin{aligned} (a_w(\omega) - a_w(\omega_0)) \int_{D_w} f(t) \frac{dS_t}{(t - \omega)(t - w)} + a_w(\omega_0) \int_{D_w} \frac{f(t)}{t - w} \left( \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right) dS_t + \\ + \int_{D_w} f(t) \left( \frac{R_w(t, \omega)}{t - \omega} - \frac{R_w(t, \omega_0)}{t - \omega_0} \right) dS_t. \end{aligned} \quad (7.83)$$

Applying (7.82) we estimate the first integral as

$$Ca_1 \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|^{1+\alpha}} \int_{D_w} \frac{dS_t}{|t - w|^{1-\alpha} |t - \omega|} \leq Ca_1 \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|} \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}.$$

Passing to the second integral in (7.83) we suppose at first that  $|\omega - \omega_0| \geq \max\{|\omega - w|, |\omega_0 - w|\}/2$ . Suppose, for certainty, that this maximum equals to  $|\omega - w|/2$ . Applying (7.82) and taking into consideration that  $|t - \omega|$  and  $|t - \omega_0|$  are no less than  $|t - w|/2$  at  $|t - w| \geq 2|\omega - w|$ , we get

$$\begin{aligned} \left| \int_{D_w} \frac{f(t)}{t - w} \left( \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right) dS_t \right| &\leq \frac{Ca_1}{|w - \hat{w}|^\alpha} \int_{|t - w| \leq 2|\omega - w|} \left( \frac{1}{|t - \omega|} + \frac{1}{|t - \omega_0|} \right) \frac{dS_t}{|t - w|^{1-\alpha}} + \\ &+ \frac{Ca_1}{|w - \hat{w}|^\alpha} |\omega - \omega_0| \int_{|t - w| \geq 2|\omega - w|} \frac{dS_t}{|t - w|^{1-\alpha} |t - \omega| |t - \omega_0|} \leq \\ &\leq Ca_1 \frac{|\omega - w|^\alpha}{|w - \hat{w}|^\alpha} + Ca_1 \frac{|\omega - w|}{|w - \hat{w}|^\alpha} \int_{|\omega - w|}^{r_w} \rho^{\alpha-2} d\rho \leq ca_1 \frac{|\omega - w|^\alpha}{|w - \hat{w}|^\alpha} \leq \\ &\leq C'a_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}. \end{aligned}$$

Suppose now that  $|\omega - \omega_0| \leq \max\{|\omega - w|, |\omega_0 - w|\}/2$ . Suppose again that this maximum equals to  $|\omega - w|$ . It also means that  $|\omega - \omega_0| \leq \min\{|\omega - w|, |\omega_0 - w|\} = |\omega_0 - w|$ . Denote by  $D_{0w}$  the disk  $\{|t - w| \leq |\omega_0 - w|/2\}$ , by  $D_{0\omega}$ , and by  $D_{0\omega_0}$  the disks  $\{|t - \omega| \leq |\omega_0 - w|/2\}$  and  $\{|t - \omega_0| \leq |\omega_0 - w|/2\}$  correspondingly. We estimate separately the integrals over  $D_{0w}$ ,  $D_{0\omega} \cup D_{0\omega_0}$  and  $D_w \setminus (D_{0w} \cup D_{0\omega} \cup D_{0\omega_0})$ . We have

$$\begin{aligned} \left| \int_{D_{0w}} \frac{f(t)}{t - w} \left( \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right) dS_t \right| &\leq \frac{Ca_1}{|w - \hat{w}|^\alpha} \int_{|t - w| \leq |\omega_0 - w|/2} \left| \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right| \frac{dS_t}{|t - w|^{1-\alpha}} \leq \\ &\leq Ca_1 \frac{|\omega - \omega_0|}{|w - \hat{w}|^\alpha} \frac{1}{|\omega_0 - w|^2} \int_0^{|\omega_0 - w|/2} \rho^\alpha d\rho \leq Ca_1 |\omega - \omega_0|^\alpha \frac{|\omega - \omega_0|^{1-\alpha}}{|\omega_0 - w|} \frac{|\omega_0 - w|^\alpha}{|w - \hat{w}|^\alpha} \leq \end{aligned}$$

$$\leq Ca_1 |\omega - \omega_0|^\alpha |\omega - \hat{\omega}|^{-\alpha}.$$

Now consider the difference

$$\int_{D_{0\omega} \cup D_{0\omega_0}} \frac{f(t)}{t-w} \frac{dS_t}{t-\omega} - \int_{D_{0\omega} \cup D_{0\omega_0}} \frac{f(t)}{t-w} \frac{dS_t}{t-\omega_0}.$$

We change the variable in the second integral  $t \mapsto t + \omega_0 - \omega$  and represent this difference as the sum of the integrals

$$\begin{aligned} & \int_{D_{0\omega}} \left[ \frac{f(t)}{t-w} - \frac{f(t+\omega_0-\omega)}{t+\omega_0-\omega-w} \right] \frac{dS_t}{t-\omega} + \\ & + \int_{D_{0\omega_0} \setminus D_{0\omega}} \frac{f(t)}{(t-w)(t-\omega)} dS_t - \int_{D_{0\omega} \setminus D_{0\omega_0}} \frac{f(t+\omega_0-\omega)}{(t+\omega_0-\omega-w)(t-\omega)} dS_t. \end{aligned} \quad (7.84)$$

The first integral we estimate as

$$\begin{aligned} & \int_{D_{0\omega}} |f(t) - f(t+\omega_0-\omega)| \frac{dS_t}{|t-w||t-\omega|} + \int_{D_{0\omega}} |f(t+\omega_0-\omega)| \left| \frac{1}{t-w} - \frac{1}{t+\omega_0-\omega-w} \right| \frac{dS_t}{|t-\omega|} \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha} \int_{D_{0\omega}} \frac{dS_t}{|t-w||t-\omega|} + C |\omega - \omega_0| \int_{D_{0\omega}} \frac{|f(t+\omega_0-\omega)| dS_t}{|t+\omega_0-\omega-w||t-w||t-\omega|} \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha} \frac{1}{|\omega_0 - w|} \int_0^{|\omega_0 - w|} d\rho + Ca_1 \frac{|\omega - \omega_0|}{|\omega_0 - w|} \frac{1}{|w - \hat{w}|^\alpha} \int_{D_{0\omega}} \frac{|t+\omega_0-\omega-w|^\alpha dS_t}{|t+\omega_0-\omega-w||t-\omega|} \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha} + Ca_1 \frac{|\omega - \omega_0|}{|\omega_0 - w|} \frac{1}{|w - \hat{w}|^\alpha} \int_{D_{0\omega}} \frac{dS_t}{|t+\omega_0-\omega-w|^{1-\alpha} |t-\omega|} \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha} + Ca_1 \frac{|\omega - \omega_0|}{|w - \hat{w}|^\alpha |\omega_0 - w|^{2-\alpha}} \int_0^{|\omega_0 - w|/2} d\rho \leq \end{aligned}$$

(since  $|t + \omega_0 - \omega - w| \geq |\omega_0 - w|/2$  if  $t \in D_{0\omega}$ )

$$\leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha} + Ca_1 \frac{|\omega - \omega_0|^\alpha}{|w - \hat{w}|^\alpha} \frac{|\omega - \omega_0|^{1-\alpha}}{|\omega_0 - w|^{1-\alpha}} \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}.$$

The second and third items in (7.84) are integrals over the lunules  $D_{0\omega_0} \setminus D_{0\omega}$  and  $D_{0\omega} \setminus D_{0\omega_0}$  of width  $\leq c|\omega - \omega_0|$ . They have the estimates

$$Ca_1 \frac{|\omega - \omega_0|}{|\omega_0 - w|^{1-\alpha} |w - \hat{w}|^\alpha} \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|w - \hat{w}|^\alpha} \frac{|\omega - \omega_0|^{1-\alpha}}{|\omega_0 - w|^{1-\alpha}} \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}.$$

Now let estimate the integral over  $D_w \setminus (D_{0w} \cup D_{0\omega} \cup D_{0\omega_0})$ . Note that if  $t$  belongs to this domain, then  $|t - \omega| \geq |t - w|/4$  and  $|t - \omega_0| \geq |t - w|/4$ . We have

$$\begin{aligned} & \left| \int_{D_w \setminus (D_{0w} \cup D_{0\omega} \cup D_{0\omega_0})} \frac{f(t)}{t - w} \left( \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right) dS_t \right| \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|}{|w - \hat{w}|^\alpha} \int_{D_w \setminus (D_{0w} \cup D_{0\omega} \cup D_{0\omega_0})} \frac{dS_t}{|t - w|^{1-\alpha} |t - \omega| |t - \omega_0|} \leq Ca_1 \frac{|\omega - \omega_0|}{|w - \hat{w}|^\alpha} \int_{|\omega_0 - w|/2}^{r_w} \rho^{\alpha-2} d\rho \leq \\ & \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|w - \hat{w}|^\alpha} \frac{|\omega - \omega_0|^{1-\alpha}}{|\omega_0 - w|^{1-\alpha}} \leq Ca_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{w}|^\alpha}. \end{aligned}$$

Consider now the third integral in (7.83). We represent it as the sum

$$\int_{D_w} (R_w(t, \omega) - R_w(t, \omega_0)) f(t) \frac{dS_t}{t - \omega} + \int_{D_w} R_w(t, \omega_0) f(t) \left( \frac{1}{t - \omega} - \frac{1}{t - \omega_0} \right) dS_t. \quad (7.85)$$

The first integral has the estimate

$$Ca \frac{|\omega - \omega_0|}{|w - \hat{w}|^2} \int_{D_w} \frac{dS_t}{|t - \omega|} \leq Ca \frac{|\omega - \omega_0|}{|w - \hat{w}|} \leq Ca \frac{|\omega - \omega_0|^\alpha}{|w - \hat{w}|^\alpha}.$$

The second integral in (7.85) we estimate by our usual method. After the change of the variable  $t \mapsto t + \omega_0 - \omega$  the domain  $D_w$  transforms into the domain  $D'_w$ . We must estimate the sum

$$\begin{aligned} & \int_{D_w \cap D'_w} (R_w(t, \omega_0) f(t) - R_w(t + \omega_0 - \omega, \omega_0) f(t + \omega_0 - \omega)) \frac{dS_t}{t - \omega} + \\ & + \int_{D_w \setminus D'_w} R_w(t, \omega_0) f(t) \frac{dS_t}{t - \omega} - \int_{D'_w \setminus D_w} R_w(t + \omega_0 - \omega, \omega_0) f(t + \omega_0 - \omega) \frac{dS_t}{t - \omega}. \end{aligned}$$

The first integral has the estimate

$$\begin{aligned} & C \left( a \frac{|\omega - \omega_0|}{|w - \hat{w}|^2} + a_1 \frac{|\omega - \omega_0|^\alpha}{|w - \hat{w}|^{1+\alpha}} \right) \int_{D_w \cap D'_w} \frac{dS_t}{|t - \omega|} \leq \\ & \leq C(a + a_1) \frac{|\omega - \omega_0|^\alpha}{|w - \hat{w}|^\alpha}. \end{aligned}$$

The second and third integrals are over the lunules of width of order  $|\omega - \omega_0|$  and have the estimates

$$Ca \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|} \leq Ca \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}.$$

Consider now case b) and estimate the difference of the integrals

$$\int_{D_\omega} f(t) K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - w)} - \int_{D_{\omega_0}} f(t) K_w(t, \omega_0) \frac{dS_t}{(t - \omega_0)(t - w)}.$$

We again apply the change of the variable  $t \mapsto t + \omega_0 - \omega$  in the second integral. Denote  $D'_{\omega_0} = D_{\omega_0} + \omega_0 - \omega$ . Applying the estimates for derivatives of  $K_w$  and recalling that now  $|t - w| \geq d|\omega - \hat{\omega}|$  for some uniform  $d$ , we obtain

$$\begin{aligned} & \left| \int_{D_{\omega_0} \cap D'_{\omega_0}} \left( \frac{f(t) K_w(t, \omega)}{t - w} - \frac{f(t + \omega_0 - \omega) K_w(t + \omega_0 - \omega, \omega_0)}{t + \omega_0 - \omega - w} \right) \frac{dS_t}{t - \omega} \right| \leq \\ & \leq C \left( a \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|^2} + a_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^{1+\alpha}} \right) \int_{D_{\omega_0} \cap D'_{\omega_0}} \frac{dS_t}{|t - \omega|} \leq \\ & \leq C(a + a_1) \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}, \\ & \left| \int_{D_{\omega_0} \setminus D'_{\omega_0}} \frac{f(t) K_w(t, \omega)}{t - w} \frac{dS_t}{t - \omega} \right| \leq Ca \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|} \leq Ca \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}. \end{aligned}$$

We have an analogous estimate for the integral over  $D'_{\omega_0} \setminus D_{\omega_0}$ .

At last consider the integrals

$$\int_{\Omega \setminus D_\omega} \frac{f(t)}{t - w} \left[ \frac{K_w(t, \omega)}{t - \omega} - \frac{K_w(t, \omega_0)}{t - \omega_0} \right] dS_t$$

in case a) and

$$\int_{\Omega \setminus (D_\omega \cup D_{\omega_0})} \frac{f(t)}{t - w} \left[ \frac{K_w(t, \omega)}{t - \omega} - \frac{K_w(t, \omega_0)}{t - \omega_0} \right] dS_t.$$

in case b). As in case 1) we can write these integral as sums of the terms that up to the bounded multiples are the integrals considered in case 2) of Proposition 37 and the terms containing differences of values of the function

$f$ . We estimate these terms as in case 1) and obtain the required estimate.

□

3)

$$J_{3f}(\omega) = \int_{\Omega} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - \hat{\omega})}.$$

Let prove the uniform boundedness. The integral over  $D_\omega$  has the estimate

$$\begin{aligned} \left| \int_{D_\omega} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - \hat{\omega})} \right| &\leq c(a + a_1) \int_{D_\omega} \frac{dS_t}{|t - \omega|^{1-\alpha}|t - \hat{\omega}|} \leq \\ &\leq c(a + a_1) \int_{D_\omega} \frac{dS_t}{|t - \omega|^{2-\alpha}} \end{aligned}$$

since  $|t - \hat{\omega}| \geq c|t - \omega|$  for some uniform  $c$ . Hence, the integral is bounded. The integral over  $\Omega \setminus D_\omega$  reduces to the integral of case 3) of Proposition 37.

Let prove estimate (7.79) in our case. The difference

$$\int_{D_{\omega_0}} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - \hat{\omega})} - \int_{D_{\omega_0}} f(t)K_w(t, \omega_0) \frac{dS_t}{(t - \omega_0)(t - \hat{\omega}_0)}$$

we estimate by the usual method, applying the change of the variable in the second integral. Denote by  $D'_{\omega_0}$  the disk  $D_{\omega_0} + \omega_0 - \omega$ . In the usual way, applying estimates for  $|f(t) - f(t + \omega_0 - \omega)|$ ,  $|K_w(t, \omega) - K_w(t + \omega_0 - \omega, \omega_0)|$  and  $|(t - \hat{\omega})^{-1} - (t + \omega_0 - \omega - \hat{\omega})^{-1}|$  we obtain

$$\begin{aligned} \left| \int_{D_{\omega_0} \cap D'_{\omega_0}} \left[ \frac{f(t)K_w(t, \omega)}{(t - \omega)(t - \hat{\omega})} - \frac{f(t + \omega_0 - \omega)K_w(t + \omega_0 - \omega, \omega_0)}{(t - \omega)(t + \omega_0 - \omega - \hat{\omega})} \right] dS_t \right| &\leq \\ &\leq C \left( a_1 \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^{1+\alpha}} + a \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|^2} \right) \int_{D_{\omega_0} \cap D'_{\omega_0}} \frac{dS_t}{|t - \omega|} \leq \\ &\leq C(a + a_1) \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha}, \end{aligned}$$

$$\left| \int_{D_{\omega_0} \setminus D'_{\omega_0}} f(t)K_w(t, \omega) \frac{dS_t}{(t - \omega)(t - \hat{\omega})} \right| \leq Ca \frac{|\omega - \omega_0|}{|\omega - \hat{\omega}|} \leq Ca \frac{|\omega - \omega_0|^\alpha}{|\omega - \hat{\omega}|^\alpha},$$

and for the integral over  $D'_{\omega_0} \setminus D_{\omega_0}$  we obtain an analogous estimate.

We obtain an estimate for the difference of the integrals over  $\Omega \setminus D_{\omega_0}$  and  $\Omega \setminus D_{\omega_0}$  analogously to the previous cases.



4)

$$J_{4f}(\omega) = \int_{\Omega} f(t)K_w(t, \omega) \frac{dS_t}{(t-w)(t-\hat{\omega})}.$$

Again applying the inequality  $|t - \hat{\omega}| \geq c|t - \omega|$  with some uniform  $c$ , we can see that this integral is no greater by modulus than

$$C(a + a_1) \int_{\Omega} \frac{dS_t}{|t-w|^{1-\alpha}|t-\hat{\omega}|} \leq C(a + a_1).$$

Estimation of the difference  $J_{4f}(\omega) - J_{4f}(\omega_0)$  again presents no difficulties since we can differentiate under the integral and we obtain the estimate analogously to case 4) of Proposition 37.  $\square$ .

## 8 Solutions to the Beltrami equation with estimates of derivatives

**Proof of Lemma 5.** Remind that to prove Lemma 5 means to show that equation (7.13) has an unique bounded solution. We represent the right side of this equation as

$$L_w(\omega) = A + R_w(\omega),$$

where  $R_w(w) = 0$

Remind the notation  $r_{wk}$ ,  $k \geq 1$  and put  $r_{w0} = 1$  identically. Introduce a linear space  $X$  such that elements of  $X$  are functions on  $\Omega$  of the type

$$f(\omega) = \sum c_k r_{wk}(\omega) + f_w(\omega),$$

where the sum  $\sum c_k r_{wk}(\omega)$  converges uniformly and the function  $f_w(\omega)$  equals to zero at  $\omega = w$ , has a finite  $C^0$ -norm, and satisfies the Holder condition  $|f_w(\omega) - f_w(\omega_0)| \leq c|\omega - \omega_0|^\alpha / |\omega_0 - \hat{\omega}_0|^\alpha$  with some  $\alpha < 1$ . The norm on this space is defined as

$$\| \sum c_k (r_{wk} \|_C + \|f_w\|_{C^0} + \|f_w\|_\alpha,$$

where  $\|f_w\|_\alpha$  is defined as

$$\sup_{\omega \in \Omega, \omega_0 \in \Omega} \frac{|f_w(\omega) - f_w(\omega_0)|}{|\omega - \omega_0|^\alpha} |\omega_0 - \hat{\omega}_0|^\alpha.$$

By Proposition 40 the function  $L_w = A + R_w$  belongs to  $X$ .

The operator  $\mathcal{T}_w\mu$  isn't, in general, contracting on  $X$  but, however, we can obtain the solution to equation (7.13) by an iterations process if we put

$$f_0 = L_w = A + R_w,$$

$$f_{k+1} = f_k + \mathcal{T}_w\mu f_k$$

at  $k \geq 0$  and prove that  $\|(\mathcal{T}_w\mu)^k f_0\|_X \leq a^k$  for some  $a < 1$ .

We shall prove by induction

$$(\mathcal{T}_w\mu)^k f_0 = \sum_{i=0}^k c_{ik} r_{wi} + f_{kw},$$

where  $f_{kw}(w) = 0$  and

$$\sum_{i=0}^k |c_{ik}| \leq (Cb)^k, \|f_{kw}\|_{C^0} \leq (Cb)^k, \|f_{kw}\|_{\alpha} \leq (Cb)^k \quad (8.1)$$

for some uniform  $C$ .

Note at first that, by Proposition 37 at  $i \geq 1$ ,

$$c_{i,k+1} = \mu(w)c_{i-1,k} \quad (8.2)$$

and

$$c_{0,k+1} = \mathcal{T}_w\mu f_{kw}(w) + \sum_{i=0}^k c_{ik} F_{wi}(w), \quad (8.3)$$

where, by Propositions 37 and 40,

$$\|F_{wi}\|_{C^0} \leq c(i+1)b, \|F_{wi}\|_{\alpha} \leq c(i+1)^2b \quad (8.4)$$

for some uniform  $c$ . Also, by Proposition 37,

$$f_{k+1,w} = \mathcal{T}_w\mu f_{kw} - \mathcal{T}_w\mu f_{kw}(w) + \sum_{i=0}^k c_{ik}(F_{wi} - F_{wi}(w)).$$

Applying Propositions 37, 40, and 41, we get

$$|c_{0,k+1}| \leq cb(\|f_{kw}\|_{C^0} + \|f_{kw}\|_{\alpha}) + cb \sum_{i=0}^k (i+1)|c_{ik}|, \quad (8.5)$$

$$\|f_{k+1,w}\|_{C^0} \leq cb(\|f_{kw}\|_{C^0} + \|f_{kw}\|_{\alpha}) + cb \sum_{i=0}^k (i+1)|c_{ik}|, \quad (8.6)$$

$$\|f_{k+1,w}\|_{\alpha} \leq cb(\|f_{kw}\|_{C^0} + \|f_{kw}\|_{\alpha}) + cb \sum_{i=0}^k (i+1)^2|c_{ik}| \quad (8.7)$$

with some uniform  $c$ . In what follows we denote by  $\beta$  the value  $cb$ .

From inductive relations (8.2) and (8.4) - (8.7) follows that the tuple  $(\{|c_{ik}|\}, \|f_{kw}\|_{C^0}, \|f_{kw}\|_{\alpha})$  is majorated by the tuple  $(\{d_{ik}\}, A_k, B_k)$ , for which we have the inductive relations

$$d_{i,k+1} = \beta d_{i-1,k}, \quad i \geq 1,$$

$$d_{0,k+1} = \beta \sum_{i=0}^k (i+1)d_{ik} + \beta(A_k + B_k),$$

$$A_{k+1} = \beta \sum_{i=0}^k (i+1)d_{ik} + \beta(A_k + B_k),$$

$$B_{k+1} = \beta \sum_{i=0}^k (i+1)^2 d_{ik} + \beta(A_k + B_k),$$

and we can put, replacing, if it is necessary, the constant  $c$ :  $d_{00} = A_0 = B_0 = 1$ . By induction we easy get

$$d_{ik} = \beta^i d_{0,k-i} \quad (8.8)$$

and, hence, we can rewrite our inductive relations as

$$d_{0k} = \beta \sum_{i=0}^{k-1} (i+1)\beta^i d_{0,k-1-i} + \beta(A_{k-1} + B_{k-1}),$$

$$A_k = \beta \sum_{i=0}^{k-1} (i+1)\beta^i d_{0,k-1-i} + \beta(A_{k-1} + B_{k-1}),$$

$$B_k = \beta \sum_{i=0}^{k-1} (i+1)^2 \beta^i d_{0,k-1-i} + \beta(A_{k-1} + B_{k-1}).$$

Now, at  $\beta$  small enough,  $(i+1)\beta^i \leq (2\beta)^i$  and  $(i+1)^2\beta^i \leq (2\beta)^i$ . We shall prove by induction that  $d_{0k} \leq (4\beta)^k$ ,  $A_k \leq (4\beta)^k$ ,  $B_k \leq (4\beta)^k$ . Indeed, we have

$$d_{0k} \leq \beta \sum_{i=0}^{k-1} 2^i \beta^i 4^{k-1-i} \beta^{k-1-i} + 2\beta 4^{k-1} \beta^{k-1} = 4^{k-1} \beta^k \left( 2 + \sum_{i=0}^{k-1} 1/2^i \right) \leq 4^k \beta^k.$$

We obtain the estimates for  $A_k$  and  $B_k$  analogously.

We get from (8.8)  $d_{ik} \leq 4^{k-i} \beta^k$  and, hence,

$$\sum_{i=0}^k c_{ik} \leq \sum_{i=0}^k d_{ik} \leq \frac{4^{k+1}}{3} \beta^k.$$

We obtain estimates (8.1) with  $C = 5c$ , where  $c$  is the constant from (8.4) - (8.6).

We obtained the bounded solution to equation (7.13) and, hence, a solution to equation (7.11) of type (7.12).  $\square$

A corollary of Lemma 5 is

**Proposition 42** *Suppose a function  $\psi$  defined on  $\Omega$  satisfies the estimate*

$$|\psi(w)| \leq C|w - \hat{w}|^{-N}.$$

Then

$$|P_m(\text{id} - \mu T_m)^{-1} \psi(w)| \leq C_m C |w - \hat{w}|^{1-N} \quad (8.9)$$

at  $m \geq N + 6$ , the constant  $C_m$  depends on  $m$ .

**Proof.** We follow the method explained before formulation of Lemma 5. We can write  $P_m(\text{id} - \mu T_m)^{-1} \psi$  as the sum

$$(P_m + P_m \mu T_m + P_m \mu T_m \mu T_m + \dots) \psi. \quad (8.10)$$

Recall that  $\mathcal{P}_w(\omega)$  is the kernel of  $P_m$

$$\mathcal{P}_w(\omega) = \frac{1}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^m$$

and denote by  $K_m(w, \omega)$  the kernel of  $T_m$

$$K_m(w, \omega) = \frac{1}{w - \omega} \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^m \left( \frac{1}{w - \omega} + \frac{m}{w - \hat{\omega}} \right).$$

In each term of sum (8.10) we can change the order of integration, for example,

$$\begin{aligned}
P_m \mu T_m \mu T_m \psi(w) &= \int \mathcal{P}_w(\omega) \mu(\omega) \int K_m(\omega, \omega_1) \mu(\omega_1) \int K_m(\omega_1, t) \psi(t) dS_t dS_{\omega_1} dS_\omega = \\
&= \int \psi(t) \int \mu(\omega_1) K_m(\omega_1, t) \int \mathcal{P}_w(\omega) \mu(\omega) K_m(\omega, \omega_1) dS_\omega dS_{\omega_1} dS_t = \\
&= \int \psi(t) \tilde{T}_m \mu \tilde{T}_m \mu \mathcal{P}_w(t) dS_t.
\end{aligned}$$

We see that sum (8.10) equals to

$$\int \psi(t) (\text{Id} - \tilde{T}_m \mu)^{-1} \mathcal{P}_w(t) dS_t.$$

By Lemma 5 at  $m \geq N+6$ , we can represent the expression  $(\text{Id} - \tilde{T}_m \mu)^{-1} \mathcal{P}_w(t)$  as a function of type (7.12) for  $l \geq N$ ,  $m \geq l+6$ . Therefore,

$$\frac{1}{\pi} \int_{\Omega} g_w(\omega) \psi(\omega) \left( \frac{\omega - \hat{\omega}}{w - \hat{\omega}} \right)^l \frac{dS_\omega}{w - \omega}$$

is another form of sum (8.10) and, by Proposition 30, this integral satisfies estimate (8.9).  $\square$

Now we have all instruments to prove Theorem 2'.

Let  $F$  be a quasiconformal local homeomorphism with the complex dilatation  $\mu$  defined on  $\Omega_t$ . Then  $f_1 = (\log F_w)_z = F_{ww}/F_w$  satisfies the equation

$$(f_1)_{\bar{w}} = \mu(f_1)_w + \mu_w(f_1) + \mu_{ww}. \tag{8.11}$$

From the other hand, if we have a solution to this equation we can find a  $\mu$ -quasiholomorphic function  $F$  such that  $f_1 = F_{ww}/F_w$ . Indeed, if we define  $\tilde{f}_1 = \mu f_1 + \mu_w$ , then equation (8.11) implies  $\tilde{f}_{1\bar{w}} = \tilde{f}_w$  and we can define the function

$$g(w) = \int_0^w f_1 ds + \tilde{f}_1 d\bar{s}.$$

Here the integral doesn't depend on a way. Now we have

$$(e^g)_{\bar{w}} = \tilde{f}_1 e^g = (\mu f_1 + \mu_w) e^g = (\mu e^g)_w.$$

Hence, we can define

$$F(w) = \int_0^w e^g ds + \mu e^g d\bar{s}. \quad (8.12)$$

This map is  $\mu$ -quasiholomorphic and  $F_{ww}/F_w = f_1$ .

A generalization of equation (8.11) is the equation for  $f_k = (f_1)_{w^{k-1}}$ :

$$(f_k)_{\bar{w}} = \mu(f_k)_w + P_k, \quad P_k = \mu_w f_k + (P_{k-1})_w. \quad (8.13)$$

Here  $P_j$ ,  $1 \leq j \leq k$  are defined by induction functions from  $f_1, \dots, f_k$ , where

$$f_j = (f_{j-1})_w, \quad j = 2, \dots, k. \quad (8.14)$$

Suppose now that we have a solution to equation (8.13) with some functions  $f_1, \dots, f_k$  not necessary satisfying relations (8.14). We define

$$\tilde{f}_j = \mu f_j + P_{j-1}, \quad 1 \leq j \leq k, \quad (8.15)$$

From (8.13) follows  $(\tilde{f}_k)_w = (f_k)_{\bar{w}}$  and the function

$$\int_0^w f_k dt + \tilde{f}_k d\bar{t}$$

is well-defined (i.e., it doesn't depend on a way of integration). Suppose now that we have the relations

$$f_{j-1} = \int_0^w f_j dt + \tilde{f}_j d\bar{t}, \quad 2 \leq j \leq k. \quad (8.16)$$

Then, by induction, these functions are well-defined and relations (8.14) are satisfied. Also, by induction, we obtain the relations

$$(f_j)_{\bar{w}} = \mu(f_j)_w + P_j, \quad 1 \leq j \leq k. \quad (8.17)$$

Thus to obtain a solution to equation (8.13) with functions  $f_j$  satisfying relations (8.14), (8.17) it is enough to satisfy equations (8.13), (8.15), and (8.16). We shall consider these equations as a system and we shall find a solution to this system satisfying estimates of Theorem 2'.

**Remark.** It seems, the important case is  $k = 1$ . The equations for higher derivatives we consider mainly for completeness.

**Proposition 43** . *In conditions of Theorem 2' there exists a solution to system(8.13), (8.15), (8.16) satisfying estimates*

$$|f_j(w)| \leq Cb|w - \hat{w}|^{-j}, 1 \leq j \leq k. \quad (8.18)$$

**Proof.** We can write  $P_k$  as

$$P_k = \sum_{i=1}^k n_{ki} \mu_{w^i} f_{k+1-i} + \mu_{w^{k+1}}, \quad (8.19)$$

where  $n_{ik}$  are integer.

We solve system (8.13), (8.15), (8.16) by an iteration method. On the first step we solve the equation

$$(f_{k1})_{\bar{w}} = \mu(f_{k1})_w + \mu_{w^{k+1}}. \quad (8.20)$$

We have the solution to this equation

$$f_{k1} = P_m(\text{id} - \mu T_m)^{-1} \mu_{w^{k+1}}. \quad (8.21)$$

Since we have the estimate  $|\mu_{w^{k+1}}(w)| \leq b|w - \hat{w}|^{-k-1}$ , we obtain by Proposition 42

$$|f_{k1}(w)| \leq Cb|w - \hat{w}|^{-k} \quad (8.22)$$

with some uniform  $C$ .

Now we define the iterations

$$f_{k,i+1} = P_m(\text{id} - \mu T_m)^{-1} P_k(f_{ki}, \dots, f_{1i}), \quad (8.23)$$

$$\tilde{f}_{j,i+1} = \mu f_{ji} + P_{j-1}(f_{ji}, \dots, f_{1i}), f_{j-1,i+1}(w) = \int_0^w f_{j,i+1} ds + \tilde{f}_{j,i+1} d\bar{s}, 1 \leq j \leq k. \quad (8.24)$$

By induction, applying representation (8.19), we can see that

$$|f_{ji}(w)| \leq Cb|w - \hat{w}|^{-j}.$$

Here  $C$ , seems, can depend on  $i$  but, in fact, iteration process (8.23), (8.24) converges in  $C_k^0 \times C_{k-1}^0 \times \dots \times C_1^0$  (see Definition 2). Indeed,

$$f_{k,i+1} - f_{ki} = P_m(\text{id} - \mu T_m)^{-1} [P_k(f_{ki}, \dots, f_{1i}) - P_k(f_{k,i-1}, \dots, f_{1,i-1})].$$

By (8.19),

$$\|P_k(f_{ki}, \dots, f_{1i}) - P_k(f_{k,i-1}, \dots, f_{1,i-1})\|_{0,k+1} \leq Cb(\|f_{ki} - f_{k,i-1}\|_{0,k} + \dots + \|f_{1i} - f_{1,i-1}\|_{0,1})$$

and, hence, analogously to (8.22),

$$\|f_{k,i+1} - f_{ki}\|_{0,k} \leq Cb(\|f_{ki} - f_{k,i-1}\|_{0,k} + \dots + \|f_{1i} - f_{1,i-1}\|_{0,1}). \quad (8.25)$$

From (8.24) follows

$$\|f_{k,i+1} - f_{ki}\|_{0,j} \leq Cb(\|f_{ki} - f_{k,i-1}\|_{0,k} + \dots + \|f_{1i} - f_{1,i-1}\|_{0,1}). \quad (8.26)$$

The constant  $C$  in these inequalities doesn't depend on  $i$ . Thus at small enough  $b$  the iterations converge.  $\square$

In particular,  $f_1$  is a solution to equation (8.11) satisfying the estimate  $|f_1(w)| \leq C|w - \hat{w}|^{-1}$ . Formula (8.12) defines then a  $\mu$ -quasiholomorphic function  $F$  such that  $|F_{w^2}/F_w(w)| \leq Cb/|w - \hat{w}|$  with some uniform  $C$ . It is easy to obtain estimates for other derivatives of  $F$ . We have

$$\frac{F_{w\bar{w}}}{F_w} = \mu_w + \mu \frac{F_{w^2}}{F_w}$$

and we get the estimate  $Cb/|w - \hat{w}|$ . Also,

$$\frac{F_{\bar{w}^2}}{F_w} = \frac{(\mu F_w)_{\bar{w}}}{F_w} = \mu_{\bar{w}} + \mu \frac{F_{w\bar{w}}}{F_w}$$

and we again obtain the estimate  $Cb/|w - \hat{w}|$ . Also, applying estimates (8.18) we get

$$|(f_1)_{(j)}(w)| \leq Cb|w - \hat{w}|^{1-|(j)|}, \quad |(j)| \leq k - 1.$$

By integration we obtain the estimate

$$|\log F_w(w)| \leq cb |\log(|w - \hat{w}|)|$$

with some uniform  $c$ . I.e.,

$$|w - \hat{w}|^{cb} \leq |F_w(w)| \leq |w - \hat{w}|^{-cb}.$$

It is estimate (4.42).

**Proposition 44**  *$F$  is a homeomorphism.*



**Proof.** In conditions of Theorem 2'  $\Omega = h(D)$ , where  $h$  is holomorphic and satisfies estimates (4.38) (we omit here the index  $t$ ). The map  $F \circ h$  is quasiholomorphic on  $D$  with the Beltramy coefficient  $\tilde{\mu}(z) = \mu \circ h(z)h_z(z)/\overline{h_z(z)}$ . We have at  $|(j)| = 1$

$$\begin{aligned} |\partial_{(j)}\tilde{\mu}| &\leq |\partial_{(j)}\mu \circ h \cdot h_z \cdot h_z/\overline{h_z} + \mu \circ h(h_{z^2}/\overline{h_z} + h_z\overline{h_{z^2}}/\overline{h_z^2})| \leq \\ &\leq c(b + b\varepsilon)|z - \bar{z}^{-1}|. \end{aligned}$$

Analogously, at  $|(j)| = 2$

$$|\partial_{(j)}\tilde{\mu}| \leq c(b + b\varepsilon)|z - \bar{z}^{-1}|^{-2}$$

with some uniform  $c$ . We get for the normal map  $f^{\tilde{\mu}}$  the estimates of Lemma 1 with small coefficients. From the other hand,

$$F \circ h = \tilde{h} \circ f^{\tilde{\mu}}$$

with some holomorphic  $\tilde{h}$ . Denote  $\tilde{g} = h \circ (f^{\tilde{\mu}})^{-1}$ . From Lemma 1 follows

$$|\tilde{g}_z| \geq c, |(\tilde{g}_{\bar{z}}| \leq Cb, \partial_{(j)}\tilde{g})(z)| \leq cb|z - \bar{z}^{-1}|^{-1}, |(j)| = 2.$$

for some uniform  $c, C$ . We see that  $\tilde{h} = F \circ \tilde{g}$  satisfies the estimate

$$\left| \frac{\tilde{h}_{z^2}}{\tilde{h}_z} \right| (z) \leq 10 \sup_{|(j)|=1, |(s)|=2} \left| \frac{\partial_{(s)}F \circ \tilde{g} \cdot (\partial_{(j)}\tilde{g})^2}{F_w \circ \tilde{g} \cdot \tilde{g}_z} (z) + \frac{\partial_{(j)}F \circ \tilde{g} \cdot \partial_{(s)}\tilde{g}}{F_w \circ \tilde{g} \cdot \tilde{g}_z} (z) \right| \leq \frac{cb}{|z - \bar{z}^{-1}|}.$$

By the criterium of univalence (for example [Pom])  $\tilde{h}$  is an univalent function if  $|(\tilde{h}_{z^2}/\tilde{h}_z)(z)| \leq 1/(1 - |z|^2)$ . This condition is satisfied and, hence,  $F$  is a homeomorphism.  $\square$

**Proof of estimates for derivatives with respect to parameters.**

In what follows some constants can depend on  $b$  and we shall supply these constants by the subscript  $b$ .

We at first consider equation (8.20). We can write solution (8.21) as  $f_{k1} = P_m h$ , where  $h$  satisfies the equation

$$h = \mu T_m h + \mu_w^k.$$

For a derivative with respect to a parameter  $t$  we get the equation

$$h_t = \mu T_m h_t + \mu_t T_m h + \mu(T_m)_t \hat{g}_t h + \mu_w^{k,t},$$

where  $\hat{g}(\omega)$  is the function  $\omega \mapsto \hat{\omega}$  and  $(T_m)_t$  is the operator with the kernel  $(K_m)_t$  and  $K_m$  is the kernel of  $T_m$ .  $(K_m)_t$  is a sum of items of the types

$$\frac{1}{(w-\omega)^2} \frac{1}{w-\hat{\omega}} \left( \frac{\omega-\hat{\omega}}{w-\hat{\omega}} \right)^l, \frac{1}{w-\omega} \frac{1}{(w-\hat{\omega})^2} \left( \frac{\omega-\hat{\omega}}{w-\hat{\omega}} \right)^l, \quad (8.27)$$

where  $l = m - 1$  or  $l = m$ . According to (6.1) and (5.15), we have estimate  $|\hat{g}_t(\omega)| \leq M_b |\omega - \hat{\omega}|^{-n'}$  for some  $c$  and  $n'$ . From the other hand, from the equation  $h = (\text{Id} - \mu T_m)^{-1} \mu_{w^k}$  follows  $\|h\|_{p,k} \leq Cb$  for some uniform  $C$ .

The operators with kernels (8.27) are of the types considered in Proposition 34. We obtain

$$\|(T_m)_t \hat{g}_t h\|_{p, n'+3} \leq CM_b b$$

for  $m$  great enough,  $p \geq 2$  and close enough to 2. From the other hand,  $\mu_{w^k}$  and  $\mu_t T_m h$  also belong to  $L_N^p(\Omega)$  with some, maybe new,  $N$ . We get for  $h_t$  the equation

$$h_t - \mu T_m h_t = H,$$

where  $H$  belongs to  $L_N^p(\Omega)$  for some  $N$  and has an uniform norm. Applying Proposition 35, we see that the operator  $\text{Id} - \mu T_m$  is invertible in  $L_N^p(\Omega)$  for  $m$  great enough and  $b = \|\mu\|_{C^0}$  small enough. Thus  $h_t$  belongs to  $L_N^p(\Omega)$ . From the other hand,

$$(f_{k1}(w))_t = P_m h_t + (P_m)_t \hat{g}_t h,$$

where  $(P_m)_t$  is the operator with the kernel

$$\frac{m}{w-\omega} \left( \frac{\omega-\hat{\omega}}{w-\hat{\omega}} \right)^{m-1} \left( \frac{\omega-\hat{\omega}}{(w-\hat{\omega})^2} - \frac{1}{w-\hat{\omega}} \right).$$

Since  $\hat{g}_t$  and  $h_t$  belong to  $L_N^p(\Omega)$ , we obtain

$$|f_{k1}(w)| \leq C(\|h_t\|_{p,N} + \|h \hat{g}_t\|_{p,N}) \|\mathcal{P}_{wN}\|_q,$$

where  $p^{-1} + q^{-1} = 1$  and  $\mathcal{P}_{wN}$  is the function

$$\mathcal{P}_{wN}(\omega) = [(\mathcal{P}_w)(\omega) + ((\mathcal{P}_w)_t(\omega))(\omega - \hat{\omega})^{-N}].$$

But

$$\|\mathcal{P}_{wN}\|_q \leq C|w - \hat{w}|^{-(N+1)}$$

and we obtain

$$|(f_{k1}(w))_t| \leq C(\|h_t\|_{p,N} + \|h\hat{g}_t\|_{p,N})|w - \hat{w}|^{-(N+1)} \leq CM_b|w - \hat{w}|^{-(N+1)} \quad (8.28)$$

with some uniform  $C$ .

We obtain estimates for higher derivatives of  $h$  and  $f_1$  by induction. At differentiation there appears derivatives of the function  $\hat{g}$  and operators with kernels of the types considered in Proposition 34.

Analogously to (8.28) we obtain the estimate

$$|(f_{k1}(w))_{0,(l)}| \leq CM_b|w - \hat{w}|^{-N}, \quad |(l)| \leq L$$

with some new  $N$  and  $M_b$  if  $m$  is great enough.

Consider now iterations (8.23), (8.24).

We see that  $f_{k,i+1} - f_{ki} = P_m \Delta h_{ki}$ , where  $\Delta h_{ki}$  satisfies the equation

$$\Delta h_{ki} = \mu T_m \Delta h_{ki} + P_k(f_{ki}, \dots, f_{1i}) - P_k(f_{k,i-1}, \dots, f_{1,i-1}) \quad (8.29)$$

Also,

$$\tilde{f}_{j,i+1} - \tilde{f}_{ji} = \mu(f_{ji} - f_{j,i+1}) + P_{j-1}(f_{ji}, \dots, f_{1i}) - P_{j-1}(f_{j,i-1}, \dots, f_{1,i-1}), \quad (8.30)$$

$$f_{j-1,i+1}(w) - f_{j-1,i}(w) = \int_0^w (f_{j,i+1} - f_{ji}) ds + (\tilde{f}_{j,i+1} - \tilde{f}_{ji}) d\bar{s}, \quad 1 \leq j \leq k. \quad (8.31)$$

For the derivative  $(\Delta h_{ki})_s$  we get the equation

$$\begin{aligned} (\Delta h_{ki})_t - \mu T_m (\Delta h_{ki})_t &= \mu_t T_m \Delta h_{ki} + \mu (T_m)_t \Delta h_{ki} + \\ &+ [P_k(f_{ki}, \dots, f_{1i})]_t - [P_k(f_{k,i-1}, \dots, f_{1,i-1})]_t. \end{aligned}$$

From (8.19) follows that we can write the difference  $[P_k(f_{ki}, \dots, f_{1i})]_t - [P_k(f_{k,i-1}, \dots, f_{1,i-1})]_t$  as a sum of terms of the types

$$p_{kj}(f_{ji} - f_{j,i-1}), \quad p'_{kj}(f_{ji} - f_{j,i-1})_s,$$

where  $p_{kj} = n_{kj}(\mu_{w^{k+1-j}})_s$  and  $p'_{kj} = n_{kj}\mu_{w^{k+1-j}}$

We have the estimates

$$\|p_{kj}\|_{0,N} \leq M_b \quad (8.32)$$

for some  $N$  and  $M_b$  and

$$\|p'_{kj}\|_{0,j} \leq C \quad (8.33)$$

for some uniform  $C$ .

By (8.30), (8.31), applying (8.25), (8.26), we get inductively

$$\|\Delta h_{ki}\|_{p,k+1} \leq Cb^i.$$

We also have from (8.25), (8.26)

$$\|f_{ki} - f_{k,i-1}\|_{0,k} \leq C(ab)^i$$

for some uniform  $C$  and  $a$ .

Denote by  $G_i$  the sum  $\mu_t T_m \Delta h_{ki} + \mu(T_m)_t \Delta h_{ki} + \sum p_{ki}(f_{ji} - f_{j,i-1})$ . Then

$$\|G_i\|_{p,N} \leq M_b(ab)^i$$

for some uniform  $N$ ,  $M_b$  and  $a$ . We get from (8.29)

$$\begin{aligned} (f_{k,i+1} - f_{ki})_t &= (P_m)_t \Delta h_{ki} + P_m(\text{id} - \mu T_m)^{-1} G_i + \\ &+ P_m(\text{id} - \mu T_m)^{-1} \sum p'_{kj}(f_{ji} - f_{j,i-1})_t \end{aligned} \quad (8.34)$$

Analogously to (8.28) we obtain

$$|((P_m)_t \Delta h_{ki} + P_m(\text{id} - \mu T_m)^{-1} G_i)(w)| \leq M_b(ab)^i |w - \hat{w}|^{-N} \quad (8.35)$$

for some uniform  $N$ ,  $M_b$  and  $a$ . Estimate now the term

$$P_m(\text{id} - \mu T_m)^{-1} \sum p'_{kj}(f_{ji} - f_{j,i-1})_t \quad (8.36)$$

in (8.34). Suppose that

$$\|(f_{ji} - f_{j,i-1})_t\|_{0,N-k+j} \leq M_{bi} \quad (8.37)$$

with the same  $N$  as in (8.35). Then, applying (8.33), we get

$$\sum \|p'_{kj}(f_{ji} - f_{j,i-1})_t\|_{0,N+1} \leq cM_{bi}b$$

with the same  $N$  and  $M_{bi}$  and some  $c$  independent of  $i$ . Applying Proposition 42. we obtain that term (8.36) has the estimate by modulus

$$cbM_{bi}|w - \hat{w}|^{-N} \quad (8.38)$$

with some uniform  $c$  independent of  $b$  and  $i$ .

By (8.34), (8.35), and (8.38), we get

$$|(f_{k,i+1} - f_{ki})_t| \leq (M_b(ab)^i + cbM_{bi})|w - \hat{w}|^{-N}. \quad (8.39)$$

Now, by (8.24), applying estimates (8.32) and (8.33), it isn't difficult to obtain

$$|(f_{j,i+1} - f_{ji})_t| \leq (M_b(ab)^i + cbM_{bi})|w - \hat{w}|^{-N-j+k}, \quad 1 \leq j \leq k-1. \quad (8.40)$$

with some  $c$  and  $a$  independent of  $i$ . It is an estimate of type (8.37). Modifying, if necessary,  $a$  and  $c$  in (8.39) we can suppose that these constants are the same as in (8.40). We get the estimate

$$|(f_{j,i+1} - f_{ji})_t| \leq (M_b(Ab)^i + AbM_{bi})|w - \hat{w}|^{-N-j+k}, \quad 1 \leq j \leq k, \quad (8.41)$$

where  $A = \max\{a, c\}$ .

Modifying  $N$ , if necessary, we can suppose that  $f_{k1} \in C_N^0$  with the same  $N$  as in (8.35). Then  $f_{j1} \in C_{N-k+j}^0$ . We put  $f_{j,0} = 0$ ,  $1 \leq j \leq k$  and  $M_{b1} = \max \|f_{j1}\|_{0,N-k+j}$ ,  $1 \leq j \leq k$ . Using (8.41), we obtain inductively

$$|(f_{j,i+1} - f_{ji})_s| \leq (Ab)^i(M_{b1} + iM_b)$$

At small enough  $b$  the iterations converge in  $C_{N-k+j}^0$ .

We can obtain estimates for higher derivatives with respect to the parameters by the same method. Instead of (8.34) we obtain the equation for differences  $(f_{k,i+1} - f_{ki})_{0,(l)}$  with multi-index  $(l)$

$$(f_{k,i+1} - f_{ki})_{0,(l)} = H_i + P_m(\text{id} - \mu T_m)^{-1} \sum p'_{kj}(f_{ji} - f_{j,i-1})_{0,(l)},$$

where for  $H_i$  we have the estimate  $M_b(ab)^i$  with some  $M_b$  and  $a$ . Other modifications are obvious.  $\square$ .

Thus we finished the proof of Theorem 2' and, hence, Theorem 2.

Remark. It seems, we can prove the estimates for derivatives with respect to parameters not applying the  $L^p$ -estimates of Section 6. For it we need in generalization of the estimates of Section 7 on operators of the types  $(P_m)_t$  and  $(T_m)_t$  and similar ones, containing  $t$ -derivatives. However the  $L^p$ -estimates can be useful, and it is of some interest that we have estimates  $(\text{dist}(w, \partial\Omega))^{-N}$  for the growth in  $L^p$  also.

## REFERENCES

- [Ah] L. Ahlfors, *Lectures on Quasiconformal Mappings*, University Lecture Series 38. Providence, (AMS), (2006).
- [AhB] L. Ahlfors, L. Bers, *Riemann's Mapping Theorem for Variable Metrics*, Ann. Math. (2) 72 (1960), 385-404.
- [As] K. Astala, T. Iwaniec, G. Martin, *Elliptic Partial Differential equations and Quasiconformal Mappings in the Plane* Princeton University Press, Princeton, New Jersey, (2009).
- [Br1] M. Brunella, *Feuilletages Holomorphes sur les Surfaces Complexes Compactes*, Ann. Scient. Ec. Norm. Sup., 4e serie, 30 (1997), 569-594.
- [Br2] M. Brunella, *Plurisubharmonic Variation of of the Leafwise Poincare Metric*, Internat. J. Math. 14 (2003), 139-151.
- [Br3] M. Brunella, *Uniformisation of Foliations by Curves*, Holomorphic Dynamical Systems, Lecture Notes in Math., Springer-Verlag, (2012), 105-165.
- [CDFG] S. Calsamiglia, B. Deroin, S. Frankel, A. Guillot, *Singular sets of holonomy maps for algebraic foliations*, J. Eur. Math. Soc. 15 (2013), n. 3, 1067-1099.
- [DNS] T.C. Dinh, V.A. Nguyen, N. Sibony, *Entropy for Hyperbolic Riemann Surface Laminations I*, Frontiers in complex dynamics. In celebration of John Milnors 80th birthday. NJ: Princeton University Press. 569-592 (2014).
- [Gl1] A. A. Glutsyuk, *Hyperbolcty of the Leaves of a Generic One-dimensional Holomorphic Foliations on a Nonsingular Projective Algebraic Variety*, Proceedings of V.A.Steklov Inst. of Math., V. 213, (1997), 90-111.
- [Gl2] A. A. Glutsyuk, *On Simultaneous Uniformisation and Local Nonuniformisability*, C. R. Math. Acad. Sci. Paris, 334, No.6 (2002), 489-494.
- [Il1] Yu. S. Ilyashenko, *Foliations by Analytic Curves*, Mat. Sb, 88 (1972), 558-577.

- [Il2] Yu. S. Ilyashenko, *Covering Manifolds for Analytic Families of Leaves of Foliations by Analytic Curves*, Topological Meth. in Nonlinear Analysis, 11 (1998), 361-373.
- [Il3] Yu. S. Ilyashenko, *Persistence Theorems and Simultaneous Uniformization*, Proceedings of V.A.Steklov Inst. of Math., V.254, (2006), 184-200.
- [Le] O. Lehto, K. Virtanen, *Quasiconformal Mapping in the Plane* Springer-Verlag, Berlin-New York, (1971).
- [LN] A. Lins-Neto, *Uniformization and the Poincare Metric on the leaves of a Foliation by Curves*, Bol. Soc. Bras. Mat., Nova Ser. 31, No. 3 (2000), 351-366.
- [Pom] S. Pommerenke, *Univalent Functions* Vandenhoeck and Ruprecht, Gottingen, (1975).
- [RS] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. Volume 2* Academic Press, (1975).
- [Sh1] A.A. Shcherbakov, *Metrics and Smooth Uniformisation of Leaves of Holomorphic Foliations*, Moscow Math. J., 11, No.1 (2011), 157-178.
- [Sh1] A.A. Shcherbakov, *Almost Complex Structure on Universal Coverings of Foliations* Trans. of the Moscow Math. Soc., (2015), 137-179.
- [V] A. Verjovsky, *A Uniformization Theorem for Holomorphic Foliations*, Contemporary Math., 58, part III (1987), 233-245.