#### Abstract

The operator that first truncates to a neighborhood of the origin in the spectral domain then truncates to a neighborhood of the origin in the spatial domain is investigated in the case of Boolean cubes. This operator is self adjoint on a space of bandlimited signals. The eigenspaces of this iterated projection operator are studied and are shown to depend fundamentally on the neighborhood structure of the cube when regarded as a metric graph with path distance equal to Hamming distance. **keywords:** Boolean cube, Slepian sequence, time and band limiting, graph Laplacian,

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# Spatio-spectral limiting on hypercubes: eigenspaces

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The main result of this work identifies eigenspaces of joint spatio-spectral limiting on Boolean hypercubes. Corresponding distributions of eigenvalues will be described in other work. Spatial limiting refers, broadly, to truncation of a function to a neighborhood of a point. Spectral limiting refers to restriction to certain eigenmodes. The theory of joint spatio-spectral limiting on  $\mathbb{R}$ —the theory of time and band limiting—was developed in a series of works by Landau, Slepian and Pollak [14, 15, 20–22] that appeared in the Bell Systems Tech. Journal in the 1960s. This theory first identified the eigenfunctions of time- and band-limiting operators, compositions  $P_{\Omega}Q_T$  where  $(Q_T f)(t) = \mathbb{1}_{[-T,T]}(t) f(t)$  and  $(P_{\Omega}f)(t) = (Q_{\Omega/2}\widehat{f})^{\vee}(t)$  (here  $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(u) e^{-2\pi i u \xi} d\xi$ , [22], and eventually quantified the distribution of eigenvalues of  $P_{\Omega}Q_T$  [16]. Extensions to  $\mathbb{R}^n$  and the compact-discrete setting were developed by Slepian [20,21] and finite dimensional versions on  $\mathbb{Z}_N$  by Grünbaum [9,10] and by Xu and Chamzas [28], cf., [13]. Since 2000, besides refined study of eigenfunctions of time and band limiting, e.g., [2-5, 17-19, 27, 30], a variety of applications of timeand band-limiting methods to communication channel modeling, e.g., [8,29], multiband signals [7], irregular sampling [1], and super-resolution limits [6], have been developed. These developments all make important use of Euclidean harmonic analysis (or analogous methods on the circle, the integers, and the integers modulo N).

More recently, aspects of the theory developed in classical settings have been developed in more general settings such as the 2-sphere by Simons et al. [23, 26], locally

compact abelian groups by Zhu and Wakin [31] and, at least in an experimental sense, finite graphs G = (V, E), see Tsitsvero et al. [24, 25]. It is unreasonable to hope for an explicit description of eigenvectors of joint spatiospectral limiting on a graph Gunless G has a particular structure with ample symmetry, such as a when G is the Cayley graph of a finite abelian group, see [12]. Our main result, Cor. 9 identifies the eigenspaces of spatio-spectral limiting operators on hypercubes, which are identified with Cayley graphs of  $\mathbb{Z}_2^N$ .

## 1 The Boolean Hypercube, space limiting and band limiting

The Boolean hypercube  $\mathcal{B}_N$  is the set  $\{0,1\}^N$ . It can be regarded as the Cayley graph of the group of  $(\mathbb{Z}_2)^N$  with symmetric generators  $e_i$  having entry 1 in the *i*th coordinate and zeros in the other N-1 coordinates. With componentwise addition modulo one, vertices corresponding to elements of  $(\mathbb{Z}_2)^N$  share an edge precisely when their difference in  $(\mathbb{Z}_2)^N$  is equal to  $e_i$  for some *i*. It will be convenient to index vertices of  $\mathcal{B}_N$  by subsets S of  $\{1, \ldots, N\}$  according to coordinates having bit-value one. That is, if  $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \{0, 1\}^N$  has r nonzero bit indices  $\beta_1, \ldots, \beta_r$  in  $\{1, \ldots, N\}$ , we identify  $S = \{\beta_1, \ldots, \beta_r\}$  as those indices such that  $\epsilon_{\beta_i} = 1$ . Distance between vertices is defined by Hamming distance—the number of differing coordinates—which is equal to the path distance when each edge has unit weight. The unnormalized graph Laplacian of  $\mathcal{B}_N$  is the matrix L, thought of as a function on  $\mathcal{B}_N \times \mathcal{B}_N$ , with  $L_{SS} = N$ and  $L_{RS} = -1$  if  $R \sim S$ , that is, R and S are nearest neighbors—they differ in a single coordinate—and  $L_{RS} = 0$  otherwise. Thus L = NI - A where A is the adjacency matrix  $A_{RS} = 1$  if  $R \sim S$  and  $A_{RS} = 0$  otherwise. It will be convenient to carry out our analysis on the unnormalized Laplacian, though the normalized Laplacian  $\mathcal{B}_N$  is  $\mathcal{L} = L/N$  is preferred for analysis asymptotic in N. We denote by |S| the number of elements of S, equivalently, the number of ones in the element of  $\mathcal{B}_N$  determined by S. The eigenvalues of L are  $0, 2, \ldots, 2N$  where 2K has multiplicity  $\binom{N}{K}$ . The corresponding eigenvectors, which together form the graph (and group) Fourier transform, are the Hadamard vectors defined by  $H_S(R) = (-1)^{|R \cap S|}$ . The normalized Hadamard vectors  $\bar{H}_S = H_S/2^{N/2}$  together make the Fourier transform unitary. Lemma 1, which states that  $H_S$  is an eigenvector of L with eigenvalue 2|S|, is well known both in group theory and graph theory. A combinatorial proof compares intersection parities  $|R \cap S| \mod 2$ and  $|P \cap S| \mod 2$  among  $P \sim R$ , e.g., [12].

#### **Lemma 1** $H_S$ is an eigenvector of L with eigenvalue 2|S|.

We denote by H the matrix with columns  $H_S, S \subset \{1, \ldots, N\}$ . Up to an indexing of the columns and a factor  $2^{N/2}$ , H is the Hadamard matrix of order N obtained by taking the N-fold tensor product of the matrix  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . On  $\mathbb{R}$ , the timelimiting operator is  $(Q_T f)(t) = \mathbb{1}_{[-T,T]}(t)f(t)$  which cuts off f in a neighborhood of the origin. In  $\mathcal{B}_N$ , cutting off in a (symmetric) neighborhood of the origin—the vertex corresponding to  $(0, \ldots, 0) \in \mathbb{Z}_2^N$ —means multiplying by the characteristic function of a Hamming ball of a given radius. The closed Hamming ball B(0,r) of radius r consists of those  $R \in \mathcal{B}_N$ such that  $|R| \leq r$ . The Hamming sphere  $\Sigma_r$  consists of those vertices having Hamming distance r from the origin. They are indexed by the r-element subsets of  $\{1, \ldots, N\}$ and thus have  $\binom{N}{r}$  vertices. Consequently, B(0, K) has  $\dim(K) = \sum_{k=0}^{K} \binom{N}{k}$  vertices. On  $\mathcal{B}_N$ , one denotes  $Q = Q_K$  by  $(Qf)(S) = \mathbb{1}_{B(0,K)}f(S)$ . On  $\mathbb{R}$ , it is typical to

On  $\mathcal{B}_N$ , one denotes  $Q = Q_K$  by  $(Qf)(S) = \mathbb{1}_{B(0,K)}f(S)$ . On  $\mathbb{R}$ , it is typical to denote the  $\Omega$ -bandlimiting operator  $(P_\Omega f)(t) = (\widehat{f}(\xi)\mathbb{1}_{[-\Omega/2,\Omega/2]}(\xi))^{\vee}$  where  $\widehat{f}$  denotes the Fourier transform of f. By analogy one can define a bandlimiting operator  $P = P_K$ by  $P = \overline{H}Q\overline{H}$ . Equivalently,  $P_K$  is the projection onto the span of those  $H_S$  with  $|S| \leq K$ . By analogy with Paley–Wiener spaces on  $\mathbb{R}$ , we refer to the span of  $\{H_S : |S| \leq K\}$ as the Boolean *Paley–Wiener* space BPW<sub>K</sub>, which has dimension dim(K).

In this work we seek to characterize the eigenspaces of the joint projection operator  $P_{K_1}Q_{K_2}P_{K_1}$  which is the  $\mathcal{B}_N$ -analogue of the *time- and band-limiting operator*  $P_\Omega Q_T P_\Omega$  on  $\mathbb{R}$ . To simplify the presentation somewhat we will restrict to the case of fixed  $K_1 = K_2 = K$ . The eigenvalues of  $P_\Omega Q_T P_\Omega$  on  $\mathbb{R}$  are non-degenerate, as are the eigenvalues of their finite dimensional  $(\mathbb{Z}_N)$  analogues [9,22]. These facts are tied to the simple linear geometry of  $\mathbb{R}$ . In contrast, the eigenvalues of  $P_K Q_K P_K$  on  $\mathcal{B}_N$  have high multiplicity, namely  $\binom{N}{k} - \binom{N}{k-1}$ ,  $0 \leq k \leq K$ , and there are (K+1-k) distinct eigenvalues with such multiplicity. Summation by parts gives

$$\sum_{k=0}^{K} (K+1-k) \left( \binom{N}{k} - \binom{N}{k-1} \right) = \sum_{k=0}^{K} \binom{N}{k} = \dim(K).$$

Thus the eigenspaces of PQP provide a decomposition of  $BPW_K$ . This decomposition will be done in the frequency domain by identifying certain spaces of vectors that are invariant under the adjacency matrix A, and expressing the bandlimiting operator as a polynomial in A. Any vector in this space is described in terms of at most N + 1coefficients and the eigenvectors of PQP are determined by certain eigencoefficients. As such, the eigenspace problem is reduced from a problem of finding eigenvectors of a matrix of size on the order of  $2^N$  to that of finding eigenvectors of a matrix of size on the order of N.

Before addressing this problem for PQP, we will solve the corresponding problem for a certain second-order difference operator analogue of the so-called *prolate differential operator* on  $\mathbb{R}$ ,

$$PDO = \frac{\mathrm{d}}{\mathrm{d}t}(t^2 - T^2)\frac{\mathrm{d}}{\mathrm{d}t} + (\pi\Omega)^2 t^2$$
(1)

that commutes with the operator that truncates [-1, 1] and bandlimits to  $[-\Omega/2, \Omega/2]$ . Up to dilation, the eigenfunctions of PDO (and hence of time and bandlimiting) are the so-called prolate spheroidal wave functions (PSWFs) [11,22]. The structure of PDO allows for efficient numerical computation of the PSWFs. It is reasonable to seek a parallel route to identify eigenvectors of PQP in the Boolean case.

The identity  $(\frac{d}{dt}f)^{\wedge}(\xi) = 2\pi i \xi \hat{f}(\xi)$  means that, up to a constant, differentiation  $\sqrt{-\Delta}$ , is conjugation of multiplication by  $\xi$ , the corresponding eigenvalue, with the Fourier transform. The operator  $D = \bar{H}T\bar{H}$  that conjugates the diagonal matrix T whose diagonal elements are square roots of eigenvalues of L, by H can thus be regarded, up to normalization, as a Boolean analogue of d/dt while T can also be

regarded as an analogue, again up to normalization, of multiplication by t. The Boolean difference operator

(BDO) 
$$D(\alpha I - T^2)D + \beta T^2$$
 (2)

then can be regarded as a Boolean analogue of PDO in (1) where  $\alpha$  and  $\beta$  play normalizing roles.

We will show that the eigenspaces of BDO are identified at the coefficient level with spaces of eigenvectors of a certain tri-diagonal matrix. Unlike the case of the real line, BDO and PQP do not commute [12]. Nonetheless, BPDO and PQP share invariant subspaces that enables similar analysis for each, but the eigenspaces of PQP are more complicated because the coefficient matrix in question is not tri-diagonal.

#### 1.1 Dyadic lexicographic order

In order to represent linear operators on  $\ell^2(\mathcal{B}_N)$  one needs a way of listing the elements of  $\mathcal{B}_N$  in a particular order. One way is to start with  $S \sim (\epsilon_0, \ldots, \epsilon_{N-1})$  and associate  $n(S) = \sum_{j=0}^{N-1} \epsilon_j 2^j$ . However, to describe operations on  $\mathcal{B}_N$  relative to Hamming distance it is preferable to work with an order that respects Hamming distance to the origin. The dyadic lexicographic order is defined on elements of  $\mathcal{B}_N$  as follows. Recall that R is identified with a subset of  $\{1, \ldots, N\}$  so that the elements of R are those  $\beta$  such that  $\epsilon_\beta = 1$  when  $R \sim (\epsilon_0, \ldots, \epsilon_{N-1})$ . The dyadic lexicographic order " $\leq$ " stipulates  $R \leq S$  if |R| < |S| and, when |R| = |S| if, in the smallest bit index  $\beta$  in which R and S differ, one has  $\beta \in S \setminus R$ . In this ordering, the adjacency matrix Acan be represented as a block symmetric matrix with nonzero entries restricted to the blocks corresponding to products of Hamming spheres  $\Sigma_r \times \Sigma_{r\pm 1}$ .



Figure 1: Adjacency matrix for N = 8 in dyadic lexicographic order.

## 2 Hamming sphere analysis of adjacency

#### 2.1 Compositions of outer and inner adjacency maps

The adjacency matrix A of  $\mathcal{B}_N$  can be written  $A = A_+ + A_-$  where  $A_- = A_+^T$  and  $A_+$  is lower triangular when expressed in dyadic lexicographic order. Thus  $A_+$  maps data on r-spheres to data on (r+1)-spheres while  $A_-$  maps from r-spheres to (r-1)-spheres. We may refer to  $A_+$  and  $A_-$  as the respective *outer* and *inner adjacency maps*.

The space  $\ell^2(\Sigma_r)$  of vectors supported on  $\Sigma_r$  has an orthogonal decomposition

$$\ell^2(\Sigma_r) = A_+ \ell^2(\Sigma_{r-1}) \oplus \mathcal{W}_r$$

where  $W_r$  is the orthogonal complement of  $A_+\ell^2(\Sigma_{r-1})$  inside  $\ell^2(\Sigma_r)$ . Applying such a decomposition at each scale, one obtains a type of multiscale decomposition:

$$\ell^2(\Sigma_r) = A_+ \ell^2(\Sigma_{r-1}) \oplus \mathcal{W}_r = \dots = A_+^r \mathcal{W}_0 \oplus A_+^{r-1} \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_r.$$

Since the origin is a singleton,  $\mathcal{W}_0$  is simply the constants at the origin.

The outer and inner adjacencies do not commute. The following theorem quantifies composition of  $A_{-}$  with a power of  $A_{+}$ .

**Theorem 2** Let  $W \in W_r$  and k such that r + k < N. Then

$$A_{-}A_{+}^{k+1}W = \left[ (N-2r) + (N-2(r+1)) + \dots + (N-2(r+k)) \right] A_{+}^{k}W.$$

We denote by m(r,k) the multiplier of  $A^k_+W$ , that is,

$$m(r,k) = \left[ (N-2r) + (N-2(r+1)) + \dots + (N-2(r+k)) \right].$$
(3)

The proof uses the following lemma whose proof will be given after that of Thm. 2.

**Lemma 3** Let  $C = [A_-, A_+] = A_-A_+ - A_+A_-$  be the commutator of  $A_-$  and  $A_+$ . Then for each r, the restriction of C to  $\Sigma_r$  is multiplication by N - 2r.

**Proof of Thm. 2.** We first prove the cases k = 0 and k = 1. The general case is then proved by induction on k for each fixed r. The case k = 1 thus does not require separate proof, but we include that case to illustrate how the orthogonality condition defining  $W_r$  propagates through powers of  $A_+$ . For k = 0 the claim is that  $A_-A_+W = (N - 2r)W$ . If  $R \in \Sigma_r$  then the value  $(A_-A_+W)(R) = \sum_{S \sim \sim R} W_S$ , counting multiplicity, where  $S \sim \sim R$  means that there is a path of length two from S to R through a vertex in  $\Sigma_{r+1}$ , and the multiplicity of the value  $W_S$  is the number of such distinct paths. Equivalently, it is the number of common neighbors that R and S share in  $\Sigma_{r+1}$ . Since  $R \in \Sigma_r$ , it has N - r neighbors in  $\Sigma_{r+1}$ , one for each bit index not in R. Denote by  $\beta \in R$  a bit index element of  $\{1, \ldots, N\}$  and let  $(R \cup \{\gamma\} \setminus \{\beta\})$  denote the element of  $\Sigma_r$  obtained by replacing  $\beta$  by  $\gamma$ . If  $R = \{\beta_1, \ldots, \beta_r\}$  then

$$(A_{-}A_{+}W)(R) = (N-r)W + \sum_{i=1}^{r} \sum_{\beta \notin R} W(R \cup \{\beta\} \setminus \{\beta_i\})$$

$$\tag{4}$$

where the first term counts those paths of the form  $R \mapsto R \cup \{\gamma\} \mapsto R$  where  $\gamma$  is one of the N-r elements of  $\{1, \ldots, N\} \setminus R$ , and the second term counts those paths through  $\Sigma_{r+1}$  that originate from vertices in  $\Sigma_r$  of the form  $R \cup \{\beta\} \setminus \{\beta_i\}$  where  $\beta \notin R$ , and terminate at R. The sum on the right in (4) can also be written

$$\sum_{i=1}^{r} [W(R) + \sum_{\beta \notin R} W(R \cup \{\beta\} \setminus \{\beta_i\})] - rW(R) = -rW(R)$$

where, in the last equation we used the fact that  $W(R) + \sum_{\beta \notin R} W(R \cup \{\beta\} \setminus \{\beta_i\}) = 0$ because it is the inner product of W with  $A_+ \delta_{R \setminus \{\beta_i\}}$ . Here,  $\delta_S$  is the vertex function taking value one at S and zero elsewhere, so  $A_+ \delta_{R \setminus \{\beta_i\}}$  takes value one at each  $\Sigma_r$ neighbor of  $R \setminus \{\beta_i\}$ , that is, at R and each  $R \cup \{\beta\} \setminus \{\beta_i\}, \beta \notin R$ , and zero elsewhere. We conclude that

$$(A_-A_+W)(R) = (N-2r)W(R)$$

for each  $R \in \Sigma_r$  as claimed, when  $W \in \mathcal{W}_r$ . This proves the case k = 0.

Next we consider the case k = 1. Again let  $W \in W_r$ . The vector  $A_-A_+^2$  is supported on  $\Sigma_{r+1}$ . We will use  $S \in \Sigma_{r+1}$  to denote a generic target at which  $A_-A_+^2 W$ will be evaluated. The values  $(A_-A_+^2 W)(S)$  are the values of W that originate at some  $R \in \Sigma_r$  and terminate at S through a path of length three that passes through  $\Sigma_{r+2}$ . The values can be organized as

$$(A_{-}A_{+}^{2}W)(S) = \sum_{b \notin S} (A_{+}^{2}W)(S \cup \{b\}) = 2 \sum_{b \notin S} \sum_{\{b_{1}, b_{2}\} \subset S \cup \{b\}} W((S \cup \{b\}) \setminus \{b_{1}, b_{2}\})$$
  
$$= 2 \sum_{b \notin S} \sum_{\beta \in S} W((S \setminus \{\beta\}) + 2 \sum_{b \notin S} \sum_{\{b_{1}, b_{2}\} \subset S \cup \{b\}, b_{i} \neq b} W((S \cup \{b\}) \setminus \{b_{1}, b_{2}\}) = I + II$$
  
(5)

where the sum in I accounts for the cases in which one of the  $b_i$  is equal to the bit index b in the outer sum, and in II, neither of the  $b_i$  is equal to b. The factor two accounts for the fact that each endpoint of  $A^2_+$  is attained through two paths. In general, a vertex value W(R),  $R \in \Sigma_r$  contributes to  $(A^k_+W)(S)$ ,  $S \in \Sigma_{r+k}$  with multiplicity k! which is the number of paths over which the value at R is transmitted to S, one for each permutation of the k elements in  $S \setminus R$ .

Each inner sum in term I gives the value  $(A_+W)(S)$  and there are N-r-1 values, one for each  $b \notin S$ . Multiplying by 2 gives  $2(N-r-1)(A_+W)(S)$  for the term I. By adding and subtracting terms of the form  $W(S \setminus \{\beta\})$ , the sum in the term II is

$$\begin{split} \sum_{b \notin S} \sum_{\{b_1, b_2\} \subset S \cup \{b\}, b_i \neq b} W((S \cup \{b\}) \setminus \{b_1, b_2\}) &= \sum_{\{b_1, b_2\} \subset S} \sum_{b \notin S} W((S \cup \{b\}) \setminus \{b_1, b_2\}) \\ &= \sum_{\{b_1, b_2\} \subset S} \left( \left[ W(S \setminus \{b_1\}) + W(S \setminus \{b_2\}) + \sum_{b \notin S} W(S \cup \{b\} \setminus \{b_1, b_2\}) \right] \\ &- W(S \setminus \{b_1\}) - W(S \setminus \{b_2\}) \right) = - \sum_{\{b_1, b_2\} \subset S} W(S \setminus \{b_1\}) + W(S \setminus \{b_2\}) \end{split}$$

where we have used the fact that

$$W(S \setminus \{b_1\}) + W(S \setminus \{b_2\}) + \sum_{b \notin S} W((S \cup \{b\}) \setminus \{b_1, b_2\}) = 0$$

since it is equal to the inner product of W with  $A_+\delta_{S\setminus\{b_1,b_2\}}$  and  $S\setminus\{b_1,b_2\}$  is in  $\Sigma_{r-1}$ .

In the last sum, each term  $W(S \setminus \{b_i\})$  occurs once for each other element of S, that is, it occurs r times. Since the vertex values that contribute to S under  $A_+$  are precisely those from vertices in  $\Sigma_r$  obtained by deleting a single element of S, one has

$$\sum_{b \notin S} \sum_{\{b_1, b_2\} \subset S \cup \{b\}, b_i \neq b} W((S \cup \{b\}) \setminus \{b_1, b_2\}) = -r(A_+W)(S) \,.$$

Counting this twice gives the sum II in (5) and adding it to the sum I gives a total

$$(A_{-}A_{+}^{2}W)(S) = 2(N-r-1)(A_{+}W)(S) - 2r(A_{+}W)(S)$$
  
= [(N-2r) + (N-2(r+1))](A\_{+}W)(S)

as claimed. This proves the case k = 1 of Thm. 2.

The proof of Thm. 2 can be completed now by induction on k. Suppose that  $A_{-}A_{+}^{k} = m(r, k-1)A_{+}^{k-1}$  has been established on  $\mathcal{W}_{r}$ . We have, for  $W \in \mathcal{W}_{r}$ 

$$\begin{aligned} A_{-}A_{+}^{k+1}W &= (A_{-}A_{+}A_{+}^{k} - A_{+}A_{-}A_{+}^{k} + A_{+}A_{-}A_{+}^{k})W \\ &= CA_{+}^{k}W + A_{+}A_{-}A_{+}^{k})W = (N - 2(k+r))A_{+}^{k}W + m(r,k-1)A_{+}^{k}W \\ &= m(r,k)A_{+}^{k}W \end{aligned}$$

where we used Lem. 3, the induction hypothesis, and the fact that m(r, k - 1) + (N - 2(r + k)) = m(r, k). This completes the proof of Thm. 2.

**Proof of Lem. 3.** The argument is similar to that used in the case k = 0 of the proof of Thm. 2. For  $R \in \Sigma_r$ , the value  $(A_-A_+V)(R)$  is the sum of the values V(P),  $P \in \Sigma_r$  such that there is a two-edge path from P to R that passes through a vertex in  $\Sigma_{r+1}$ . This sum can be expressed as

$$(A_{-}A_{+}V)(R) = (N-r)V(R) + \sum_{\beta \notin R, b \in R} V(R \cup \{\beta\} \setminus \{b\})$$

where the first term counts the number of elements not in R, corresponding to paths from R to a vertex in  $\Sigma_{r+1}$  and back to R, and the second counts all other paths, which necessarily come from one-bit substitutions of bits in R by bits not in R.

On the other hand, the value  $(A_+A_-V)(R)$  is the sum of the values V(P),  $P \in \Sigma_r$ such that there is a two-edge path from P to R that passes through a vertex in  $\Sigma_{r-1}$ . This sum can be expressed as

$$(A_+A_-V)(R)=r\,V(R)+\sum_{\beta\notin R,\,b\in R}V((R\setminus\{b\})\cup\{\beta\})$$

where the first term counts the number of paths from R to  $\Sigma_{r-1}$  and back to R (which is the number of ways to delete an element of R) and the second counts the number of paths that first delete an element of R then add an element not in R. Subtracting the two terms gives

$$(CV)(R) = (A_{-}A_{+}V)(R) - (A_{+}A_{-}V)(R) = (N - 2r)V(R)$$

This proves the lemma.  $\blacksquare$ 

As a consequence of Theorem 2, the action of any sequence of powers of  $A_+$  and  $A_-$  can be computed. For example, if  $W \in \mathcal{W}_r$  then

$$A_{-}^{2}A_{+}^{k+1}W = m(r,k)A_{-}A_{+}^{k}W = m(r,k)m(r,k-1)A_{+}^{k-1}W.$$

In general, if  $B = A_{-}^{\tau_p} A_{+}^{\sigma_p} \cdots A_{-}^{\tau_1} A_{+}^{\sigma_1}$  then for  $W \in \mathcal{W}_r$ , BW is defined on  $\Sigma_q$  where  $q = r + (\sigma_1 + \cdots + \sigma_p) - (\tau_1 + \cdots + \tau_p)$  and is a multiple of  $A_{+}^{q-r}W$  if  $q \ge r$ . However, if in any of the partial compositions of powers of  $A_{+}$  and  $A_{-}$  defining B, the number of applications of  $A_{+}$  exceed those of  $A_{-}$  by more than N-r or the number of applications of  $A_{-}$  exceeds those of  $A_{+}$ , then BW = 0.

Higher order commutators of  $A_{-}$  and  $A_{+}$  simplify as follows. Again, it suffices to consider a vector V supported in  $\Sigma_{r}$  for some r. In this case, CV = (N - 2r)V and  $A_{-}CV = (N - 2r)A_{-}V$  whereas  $A_{-}V$  is supported in  $\Sigma_{r-1}$  so  $CA_{-}V = (N - 2(r - 1))A_{-}V$  by Lem. 3. Consequently,

$$[A_{-}, C] = (A_{-}C - CA_{-})V = (N - 2r)A_{-}V - (N - 2(r - 1))A_{-}V = -2A_{-}V \text{ and}$$
$$[A_{+}, C] = (A_{+}C - CA_{+})V = (N - 2r)A_{+}V - (N - 2(r + 1))A_{+}V = 2A_{+}V.$$

It follows from Lem. 3 that  $[A_+, [A_-, C]] = -2[A_+, A_-] = 2C$  and  $[A_-, [A_+, C]] = 2[A_-, A_+] = 2C$  as well. In particular, any higher order commutators involving  $A_-$  and  $A_+$  reduce to multiples of  $A_-$ ,  $A_+$  and C themselves, or vanish as in the case  $[A_-, [A_-, C]] = -2[A_-, A_-] = 0$ .

#### 2.2 Projection onto $W_r$

Theorem 2 can be used to compute the projection of an arbitrary element of  $\ell^2(\Sigma_r)$ , that is, a vertex function  $V_r$  on  $\mathcal{B}_N$  supported in  $\Sigma_r$ , onto the space  $\mathcal{W}_r$  of vectors orthogonal to  $A_+(\ell^2(\Sigma_{r-1}))$ . This is done iteratively: first one computes the orthogonal complement of vectors of the form  $A_+^r V_0$  where  $V_0$  is supported in  $\Sigma_0$  (this is the same as subtracting the average of  $V_r$  since  $A_+^r V_0$  is constant on  $\Sigma_r$ ). Call this orthogonal projection  $V_r^{(0)}$ . Next one projects  $V_r^{(0)}$  onto the orthogonal complement of  $A_+^{r-1}(\mathcal{W}_1)$ and continues until one has subtracted orthogonal projections onto  $\mathcal{W}_\ell$  for each  $\ell < r$ . The result is a decomposition

$$V_r = \sum_{k=0}^r A_+^{r-k} W_k, \quad W_k \in \mathcal{W}_k.$$

Working backwards, one finds that

$$W_0 = A_-^r V_r / (m(0, r-1)m(0, r-2) \cdots m(0, 0))$$

where m(r, k) is as in (3). Similarly,

$$W_1 = A_{-}^{r-1}(V_r - A_{+}^r W_0) / (m(1, r-2)m(1, r-3) \cdots m(1, 0))$$

etc. The projections must be computed iteratively starting with the innermost projections in order to ensure that components extracted at each step are orthogonal to components from smaller spheres.

**Observation.** The columns of the matrix  $P_{\mathcal{W}_r}$  of the projection onto  $\mathcal{W}_r$  in the standard basis appear to form a Parseval frame for  $\mathcal{W}_r$ . The norm of each column is  $1 - \binom{N}{r-1} / \binom{N}{r}$  and the inner product of columns R and S (equal to  $P_{\mathcal{W}_r}(R, S)$  since  $P_{\mathcal{W}_r}^2 = P_{\mathcal{W}_r}$ ) depends only on  $|R \cap S|$ .



Figure 2: Sample columns of the matrix of the projection onto  $\mathcal{W}_r$ , N = 8, r = 3.

## **3** Eigenspaces of Boolean difference operators

Define  $\mathcal{V}_r$  to consist of those vectors V such that

$$V = \sum_{k=0}^{N-r} c_k A_+^k W; \quad W \in \mathcal{W}_r.$$

That is, V is a sum of multiples of nonnegative powers of  $A_+$  applied to a fixed vector  $W \in \mathcal{W}_r$  and so V is a fixed multiple of  $A_+^{\rho-r}W$  on  $\Sigma_{\rho}$ ,  $\rho \geq r$ . Theorem 4 states that on  $\mathcal{V}_r$ , conjugation of the Boolean difference operator BDO in (2) by the normalized Hadamard matrix  $\overline{H}$ , is equivalent to multiplying the spherical coefficients  $[c_0, \ldots, c_{N-r}]$  of  $V \in \mathcal{V}_r$  by a certain tri-diagonal matrix M of size N - r + 1.

Define HBDO to be the conjugation of BDO by  $\overline{H}$ , that is HBDO =  $2^{-N}HBDOH$ . Since  $D = \overline{H}T\overline{H}$ , HBDO can be written as  $T(\alpha - L)T + \beta L$  where, as before, T is the diagonal matrix with entries  $T_{RR} = \sqrt{2r}$  when |R| = r. Consider now the action of HBDO on a vector  $V = \sum_{k=0}^{N} V_k$  in  $\mathcal{V}_r$  where  $V_k$  is the restriction of V to  $\Sigma_k$ . Since L = NI - A, it maps  $V_k$  into a multiple by N on  $\Sigma_k$  and parts  $A_{\pm}V_k$  supported in  $\Sigma_{k\pm 1}$ . One has

$$(\text{HBDOV}_k)_k = (T(\alpha - NI)T + \beta NI)V_k = [2k(\alpha - N) + \beta N]V_k.$$

The contribution of  $HBDOV_k$  to other spheres comes from the part

$$TAT - \beta A = (TA_+T - \beta A_+) + TA_-T - \beta A_-$$

which expresses the parts that map  $V_k$  into  $\Sigma_{k+1}$  and  $\Sigma_{k-1}$  respectively. One has

$$(TA_{+}T - \beta A_{+})V_{k} = (2\sqrt{k(k+1)} - \beta)A_{+}V_{k} \text{ and} (TA_{-}T - \beta A_{-})V_{k} = (2\sqrt{k(k-1)} - \beta)A_{-}V_{k}.$$

Since  $V_k = c_k A_+^{k-r} V_r, V_r \in \mathcal{W}_r$ ,

$$(TA_{+}T - \beta A_{+})V_{k} = c_{k}((2\sqrt{k(k+1)} - \beta)A_{+})A_{+}^{k-r}V_{r}$$
$$= c_{k}(2\sqrt{k(k+1)} - \beta)A_{+}^{k+1-r}V_{r} = (c_{k}/c_{k+1})(2\sqrt{k(k+1)} - \beta)V_{k+1}$$

if  $k + 1 \leq N$ . If k = N then this term vanishes. Similarly,

$$(TA_{-}T - \beta A_{-})V_{k} = c_{k}(2\sqrt{k(k-1)} - \beta)A_{-}A_{+}^{k-r}V_{r}$$
  
=  $c_{k}(2\sqrt{k(k-1)} - \beta)m(r, k-r-1)A_{+}^{k-1-r}V_{r}$   
=  $(c_{k}/c_{k-1})(2\sqrt{k(k-1)} - \beta)m(r, k-r-1)V_{k-1}$ 

where, as before, m(r, k - r - 1) is defined by (3) when  $k - 1 \ge r$ . If k = r then this term vanishes.

Putting these pieces together, one considers the coefficient of  $V_k$  in the image under HBDO. The calculations above indicate that HBDO is represented on  $\mathcal{V}_r$  by a matrix  $M_r(k,\ell)$  of size N-r+1 acting on the vectors of coefficients  $[c_0,\ldots,c_{N-r}]^T$  of powers  $A^k_+$  of  $W \in \mathcal{W}_r$ . One then considers the mapping as a *discrete* mapping on the coefficients of the vectors  $A^{k-r}_+V_r$ . It follows from the calculations above that  $M_r(k,\ell)$ can be expressed as the lower-right minor  $(M_r(k,\ell) = M(k+r,\ell+r))$  of the size N+1tri-diagonal matrix M with entries:

$$M(k,\ell) = \begin{cases} (2\sqrt{\ell(\ell-1)} - \beta)m(r,\ell-1-r); \ k = \ell - 1 \ge r \\ 2\ell(\alpha - N) + \beta N; k = \ell \ge r \\ 2\sqrt{\ell(\ell+1)} - \beta; k = \ell + 1; r \le \ell < N \\ 0, \text{ else }. \end{cases}$$
(6)

We have proved the following.

**Theorem 4** If  $V \in \mathcal{V}_r$ ,  $V = \sum_{k=0}^{N-r} c_k A_+^k W$ , then HBDOV  $= \sum_{k=0}^{N-r} d_k A_+^k W$  where  $\mathbf{d} = M_r \mathbf{c}$  where  $\mathbf{c} = [c_0, \dots, c_{N-r}]^T$ .

If  $\mathbf{c} = [c_0, \ldots, c_{N-r}]^T$  is an eigenvector of  $M_r$  then  $\sum c_k A_+^{k-r} V_r$  ( $V_r \in \mathcal{W}_r$ ) is an eigenvector of HBDO having the same eigenvalue.

**Corollary 5** Any eigenvector of HBDO has the form  $V = \sum_{k=0}^{N-r} c_k A_+^k W$  where **c** is an eigenvector of the matrix  $M_r$ . Since  $W_r$  has dimension  $\binom{N}{r} - \binom{N}{r-1}$ , for  $\alpha, \beta$  such that  $M_r$  is nondegenerate, HBDO has N - r + 1 eigenspaces of dimension  $\binom{N}{r} - \binom{N}{r-1}$ .

The theorem and corollary apply to HBDO without regard to the parameters  $\alpha, \beta$ which can be connected to bandlimiting properties. In [12] it was shown that when  $\alpha = \beta = 2\sqrt{K(K-1)}$ , the bandlimiting operator  $P_K$  commutes with BDO. Since conjugation of  $P_K$  by  $\overline{H}$  is  $Q_K$ , this means that HBDO commutes with  $Q_K$  when  $\alpha = \beta = 2\sqrt{K(K-1)}$  in HBDO. In this case, one can define the reduced space  $\mathcal{V}_{r,K}$ of vectors of the form  $\sum_{k=0}^{K-r} c_k A_+^k W$ ,  $W \in \mathcal{W}_r$  and the reduced matrix  $M_{r,K}$ , the restriction of  $M_r$  to its principal minor of size K + 1 - r. One has the following.

**Proposition 6** For each N and  $r \leq K < N$  with  $\alpha = \beta = \sqrt{K(K-1)}$ , the tridiagonal matrix  $M_{r,K}$  of size (K+1-r) has eigenvalues equal to those of the conjugation of HBDO by  $Q_K$  applied to the space  $\mathcal{V}_{r,K}$ .

Table 1: Matrix  $M = M_{8,3,2}^{\text{HBDO}}$  of HBDO on  $\mathcal{V}_r$  with (N, K, r) = (8, 3, 2)

51.1384	-8.1169	0	0	0	0	0
-2.0292	48.9948	0	0	0	0	0
0	0	46.8513	12.0964	0	0	0
0	0	2.0161	44.7077	16.1050	0	0
0	0	0	4.0262	42.5641	0	0
0	0	0	0	6.0333	40.4205	-48.2306
0	0	0	0	0	8.0384	38.2769

## 4 Eigenspaces of PQP

As before we assume that  $Q = Q_K$  is multiplication by the characteristic function of the closed ball of radius K centered at the origin. Then  $\overline{H}Q\overline{H} = P$  and conjugation of PQP by  $\overline{H}$  leaves the operator QPQ. We study the eigenspaces of the latter.

**Lemma 7**  $A_+$  and  $A_-$  map  $\mathcal{V}_r$  to itself.

For  $W \in \mathcal{W}_r$ ,  $A_+$  maps  $\sum_{k=0}^{N-r} c_k A_+^k W$  to  $\sum_{k=0}^{N-r-1} c_k A_+^{k+1} W = \sum_{k=1}^{N-r} c_{k-1} A_+^k W$ which is an element of  $\mathcal{V}_r$  whose coefficient of  $W = A_+^0 W$  is zero while  $A_-$  maps  $\sum_{k=0}^{N-r} c_k A_+^k W$  to  $\sum_{k=1}^{N-r} c_k m(r,k-1) A_+^{k-1} W = \sum_{k=0}^{N-r-1} c_{k+1} m(r,k) A_+^k W$  which is an element of  $\mathcal{V}_r$  whose coefficient of  $A_+^{N-r} W$  is zero. The lemma implies that  $A = A_+ + A_-$  preserves  $\mathcal{V}_r$  and so does any polynomial p(A) in A. **Proposition 8** The spectrum-limiting operator  $P = P_K$  can be expressed as a polynomial p(A) of degree N.

**Proof.** By Lem. 1, the Hadamard vectors are eigenvectors of L and hence of A, with  $AH_R = (N - 2|R|)H_R$ . One can then express P in terms of A simply by forming the Lagrange interpolating polynomial that maps the eigenvalues of A to those of P (which are one or zero). Recall that the Lagrange interpolating polynomial that maps  $x_k$  to  $y_k$ ,  $k = 0, \ldots, N$  is  $\sum p_k$  where  $p_k(x) = y_k \prod_{j=0, j \neq k}^{N} \frac{x - x_j}{x_k - x_j}$ . We choose p so that  $p(A)H_R = H_R$  if  $|R| \leq K$  and  $p(A)H_R = 0$  if |R| > K. Since  $AH_R = (N - 2|R|)H_R$  this means one should have p(N - 2r) = 1 if  $0 \leq r \leq K$  and p(N - 2r) = 0 if r > K. Therefore, set

$$p_k = \prod_{j=0, j \neq k}^{N} \frac{x - (N - 2j)}{2(j - k)}; \qquad p(x) = \sum_{k=0}^{K} p_k \tag{7}$$

Then P = p(A) as verified on the Hadamard basis. This proves the proposition.

As a consequence of the proposition, the space  $\mathcal{V}_r$  is invariant under P. The action of P on  $\mathcal{V}_r$  can be quantified by a matrix  $M_{P,r}$  of size (N-r+1) as follows. Given any  $W \in \mathcal{W}_r$ , one can write  $P(A_+^k W) = \sum_{\ell=0}^{N-r} M_{P,r}(k,\ell) A_+^\ell W$  in which  $M_{P,r}(k,\ell)$  is independent of W. Then  $M_{P,r}(k,\ell)$  is the  $(k,\ell)$ th entry of  $M_{P,r}$  and if  $V = \sum_{\ell=0}^{N-r} c_\ell A_+^\ell W$  $(W \in \mathcal{W}_r)$  then  $PV = \sum_{k=0}^{N-r} d_k A_+^k W$  where  $d_k = \sum_{k=0}^{M-r} M_{P,r}(k,\ell) c_\ell$ . In particular, if  $[c_0, \ldots, c_{N-r}]^T$  is a  $\lambda$ -eigenvector of  $M_{P,r}$  then V is a  $\lambda$ -eigenvector of P.

The eigenvectors of PQP have the form HV where V is an eigenvector of QPQwhere  $P = P_K$  and  $Q = Q_K$  for the same K. Eigenvectors of QPQ in  $\mathcal{V}_r$  can be obtained from those of the matrix  $M_{QPQ,r}$  which is just the (K - r + 1)-principal minor of  $M_{P,r}$ . This is because if  $V \in \mathcal{V}_r$ ,  $V = \sum_{k=0}^{N-r} c_k A_+^k W$ ,  $W \in \mathcal{W}_r$  then QV = $\sum_{k=0}^{K-r} c_k A_+^k W$ : Q kills those terms supported in  $\Sigma_\rho$  where  $\rho > K$ . Then PQV = $\sum_{k=0}^{K-r} d_k A_+^k W$  where  $d_k = \sum_{\ell=0}^{K-r} M_{P,r}(k,\ell)c_\ell$  and, finally,  $QPQV = \sum_{k=0}^{K-r} d_k A_+^k W$ for the same **d**. That, is, the coefficients of the powers of  $A_+^k W$  that appear in QPQVare obtained by applying the principal minor of  $M_{P,r}$  to the coefficients of  $A_+^\ell W$  for  $\ell \leq K - k$ . These observations give us the following.

**Corollary 9** If  $[c_0, \ldots c_{K-r}]^T$  is a  $\lambda$ -eigenvector of the principal minor of size (K - r + 1) of the matrix  $M_{P,r}$  described above then  $V = \sum_{k=0}^{K-r} c_k A_+^k W$ ,  $W \in W_r$ , is a  $\lambda$ -eigenvector of QPQ and HV is a  $\lambda$ -eigenvector of PQP.

Computing the eigenspaces of PQP then boils down to computing the eigenvectors of the principal minors of the matrices  $M_{P,r}$ .

In what follows, we outline a formal procedure to compute the matrices  $M_{P,r}$ . First, one expresses P in terms of the Lagrange interpolation polynomial p(A) as defined in (7). Next, one observes that the matrix of a power of A operating on  $\mathcal{V}_r$  is the power of the matrix of A operating on  $\mathcal{V}_r$ , that is,  $M_{A^n,r} = (M_{A,r})^n$ ,  $n = 0, 1, 2, \ldots$  Consequently, one can replace each occurrence of  $A - (N - 2j)I_{2^N}$  in (7) by  $M_{A,r} - (N - 2j)I_{N-r+1}$  to obtain the matrix  $M_{P,r}$ . One can represent the actions of  $A_+$  and  $A_-$  on  $\mathcal{V}_r$  as coefficient mappings in the proof of Lem. 7. In particular,  $A_+$  acts simply as a *shift* since

$$A_{+} \sum_{k=0}^{N-r} c_{k} A_{+}^{k} W = \sum_{k=0}^{N-r} c_{k} A_{+}^{k+1} W = \sum_{k=1}^{N-r} c_{k-1} A_{+}^{k} W$$

where the upper index of the sum remains N-r because if  $W \in \mathcal{W}_r$  then  $A^{N-r+1}_+W = 0$ . One can thus represent  $A_+$  through the matrix of size N-r+1 that maps  $[c_0, \ldots, c_{N-r}]$  to  $[0, c_0, \ldots, c_{N-r-1}]$ . This is just the matrix  $M_{A_+}$  on  $\mathbb{C}^{N-r+1}$  having ones on the diagonal below the main diagonal and zeros elsewhere.

On the other hand, for  $W \in \mathcal{W}_r$ ,

$$A_{-}\sum_{k=0}^{N-r} c_{k}A_{+}^{k}W = \sum_{k=0}^{N-r} c_{k}m(r,k-1)A_{+}^{k-1}W = \sum_{k=0}^{N-r-1} c_{k+1}m(r,k)A_{+}^{k}W.$$

The matrix of  $A_-$  thus maps  $[c_0, \ldots, c_{N-r}]$  to  $[m(r, 0)c_1, \ldots, m(r, N-r-1)c_{N-r}, 0]$ . This is just the matrix  $M_{A_-}$  having the value  $M_{A_-}(k, k+1) = m(r, k)$  on the diagonal above the main diagonal and zeros elsewhere.

One can write the matrix of A on  $\mathbb{C}^{N-r+1}$  as  $M_A = M_{A_+} + M_{A_-}$ . The actions of powers of A on  $\mathcal{V}_r$  can then be expressed in terms of the corresponding powers of  $M_A$  on  $\mathbb{C}^{N+1-r}$  and the bandlimiting operator  $P = P_K$  acting on  $\mathcal{V}_r$  can be expressed as the matrix  $M_{P_r}$  obtained substituting  $M_A$  for A in the interpolating polynomial p(A) in Prop. 8. We summarize how to compute the action of the operator QPQ on  $\mathcal{V}_r$ , assuming  $M_{P_r}$  has nondegenerate eigenvalues, as follows. Once the eigenspaces of QPQ are computed by identifying the eigenspaces of QPQ lying in  $\mathcal{V}_r$  for each r, a complete eigenspace decomposition of PQP is obtained by conjugating with the normalized Hadamard matrix.

#### Algorithm to compute the matrix and eigenspaces of QPQ on $\mathcal{V}_r$

Step 1: Write the coefficient matrix  $M_{A_+}$  of  $A_+$  as the matrix of size (N-r+1) with ones on the diagonal below the main diagonal and zeros elsewhere. Write the matrix  $M_{A_-}$  of  $A_-$  as the matrix of size (N-r+1) with entries  $M_{A_-}(k, k+1) = m(r, k)$ ,  $k = 0, \ldots N - r$ , and zeros elsewhere. The matrix  $M_A$  of A is  $M_{A_+} + M_{A_-}$ .

Step 2: Compute the coefficient matrices  $M_{P,r}$  by substituting  $M_{A,r} - (N-2j)I_{N-r+1}$  for each occurrence of  $A - (N-2j)I_{2^N}$  in (7).

Step 3: Estimate the eigenvalues and eigenvectors of  $M_{P,r}$  using standard numerical methods. Each eigenspace has dimension  $\binom{N}{r} - \binom{N}{r-1}$ . A basis for each corresponding eigenspace can be identified starting with a basis for  $\mathcal{W}_r$  and forming the expansions  $\sum_{k=0}^{K-r} c_k A_+^k W$  where  $[c_0, \ldots, c_{K-r}]^T$  is one of the eigenvectors of  $M_{P,r}$  and W is one of the basis vectors for  $\mathcal{W}_r$ .

**Observation.** The adjacency matrix A has norm N and the norm of  $M_{P,r}$  scales like  $N^{\alpha N}$  for some  $\alpha > 1/2$ .  $M_{P,r}$  is poorly conditioned: matlab returns Inf for the condition number of  $M_{P,1}$  when  $N \ge 10$  and  $K \le 4$ .

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0.3437	0.1562	0.0156	-0.0052	-0.0013	0.0003	0.0002	1
0.9375	0.5000	0.0937	0.0000	-0.0026	0.0000	0.0003	1
0.9375	0.9375	0.4062	0.0938	0.0078	-0.0026	-0.0013	1
-3.7500	0.0000	1.1250	0.5000	0.0938	0.0000	-0.0052	1
-11.2500	-3.7500	1.1250	1.1250	0.4062	0.0938	0.0156	1
22.5000	0.0000	-3.7500	0.0000	0.9375	0.5000	0.1562	1
112.5000	22.5000	-11.2500	-3.7500	0.9375	0.9375	0.3438	1
0	0	0	0	0	0	0	1

Table 2: Matrix  $M = M_{8,3,1}$  of P on  $\mathcal{V}_r$  with (N, K, r) = (8, 3, 1)

Table 3: Matrix  $M = M_{8,3,2}$  of P on  $\mathcal{V}_r$  with (N, K, r) = (8, 3, 2)

0.3125	0.1875	0.0312	-0.0104	-0.0078	0.0029	0.0005
0.7500	0.5000	0.1250	-0.0000	-0.0104	0.0029	0.0006
0.7500	0.7500	0.3750	0.1250	0.0312	0.0015	0
-1.5000	-0.0000	0.7500	0.5000	0.1875	0.0032	-0.0030
-4.5000	-1.5000	0.7500	0.7500	0.3125	0.0938	0.0121
0	0	0	0	0	0.4964	0.1754
0	0	0	0	0	1.0357	0.4503

Table 4: Matrix  $M = M_{8,3,3}$  of P on  $\mathcal{V}_r$  with (N, K, r) = (8, 3, 3)

0.2500	0.2500	0.1250	0.0530	0.0234	0.0012
0.5000	0.5000	0.2500	0.0559	0.0250	0.0007
0.5000	0.5000	0.2500	0.1255	0.0014	-0.0021
0	0	0	0.4929	0.0311	0.0112
0	0	0	1.1723	0.7162	0.1409
0	0	0	1.0173	2.6598	0.3792



Figure 3: Eigenvectors of PQP, N = 8, K = 3, r = 2. The dotted curves are plots of two different elements W of  $\mathcal{W}_r$  obtained by projecting a delta in  $\Sigma_r$  onto  $\mathcal{W}_r$ . The dashed curves are horizontal shifts of eigenvectors V of QPQ of the form  $\sum c_k A^k_+ W$  where  $c_k$  form the principal eigenvector of  $M_{r,K}$ . The solid curves are shifted multiples of HV where H is the Hadamard matrix. They are the corresponding eigenvectors of PQP.

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